1. Let $X_1, \cdots, X_n$ be iid from the Beta distribution $Beta(\alpha = 2, \beta = 2)$, which has pdf

$$f(x) = 6 \cdot x \cdot (1 - x) \text{ for } x \in [0, 1] \text{ and } f(x) = 0 \text{ outside } [0, 1].$$

Let $F(x)$ be the cdf of $Beta(\alpha = 2, \beta = 2)$, and let. $\widehat{F}_n(x)$ be the empirical cdf (EDF) based on the sample $X_1, \cdots, X_n$.

(a) Find (by integrating the density) the cdf $F(x)$.

(b) Plot the curve $F(x)$ using R over the range $[0, 1]$.

(c) For a given $x = 0.3$, and $n = 100$, what is the mean (expectation) of the EDF $\widehat{F}_n(x)$?

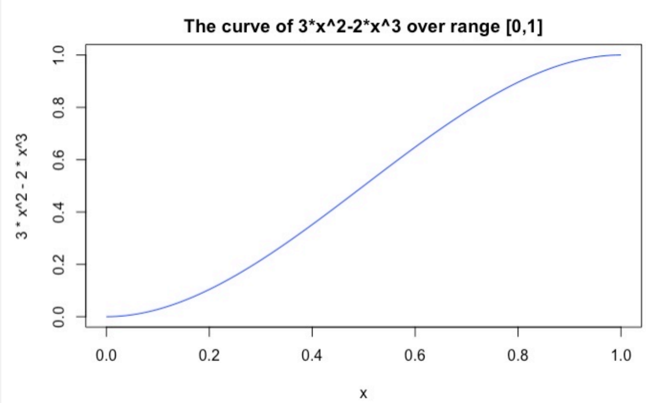
(d) For a given $x = 0.3$, and $n = 100$, what is the variance of the EDF $\widehat{F}_n(x)$?

(a) $f(x) = 6x(1-x) = 6x - 6x^2$

$F(x) = \int_{-\infty}^{\infty} f(x)\, dx = 3x^2 - 2x^3$ for $0 < x \leq 1$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 3x^2 - 2x^3 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

(b)



(c) $E(\hat{F_n}(x)) = E(I(X \leq x)) = F(x)$

$E(\hat{F_n}(0.3)) = F(0.3) = 3 \times 0.3^2 - 2 \times 0.3^3 = 0.216$

(d) $var(\hat{F_n}(x)) = \frac{F(x)(1-F(x))}{n} = \frac{F(0.3) \times (1 - F(0.3))}{100} = 0.00169$

2.

2. Let U be a uniform random variable over $[0, 1]$. Define $W = -4 \log U$.

(a) Show (by finding the cdf $P(W \leq x)$) that W has the same distribution as the exponential distribution $Exp(0.25)$.

(b) Use **rexp()** to simulate at least 2000 realizations from the exponential distribution $Exp(0.25)$. Plot the EDF of your realizations.

(c) Use **runif()** to simulate at least 2000 realizations of $W = -4 \log U$.
Plot the EDF of your realizations, on the same plot (in a contrasting color)

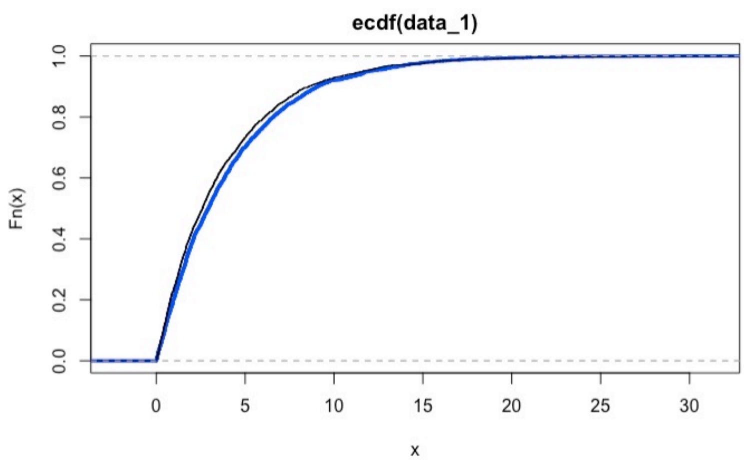
(a) $P(W \leq x) = P(-4 \log u \leq x)$

$= P(u \geq e^{-\frac{1}{4}x})$

$= 1 - P(u < e^{-\frac{1}{4}x})$

$= 1 - e^{-\frac{1}{4}x}$

$= 1 - \exp(-\frac{1}{4}x)$



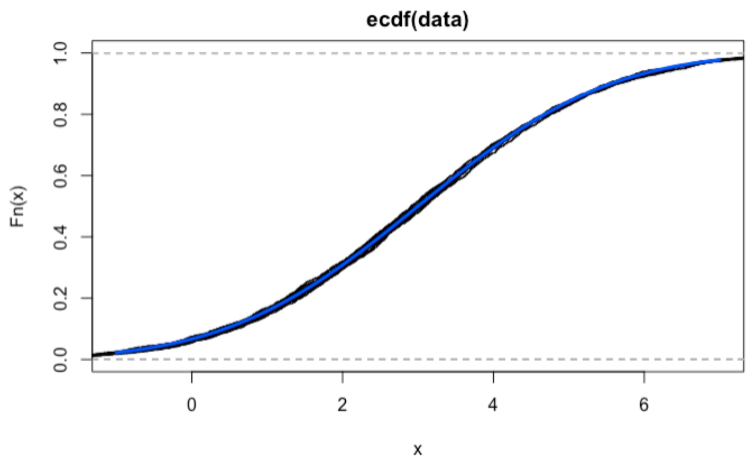
3. Use **rnorm()** to generate 2000 data points from $N(3, 2^2)$, the Normal distribution with mean 3 and variance $2^2 = 4$. Careful: I write it like this because texts use variance, but R uses stdev.

   (a) Plot the EDF curve **within** $[-1, 7]$.

   (b) Repeat the above procedure 10 times to generate another 10 EDF curves from the same distribution and with the same sample size.

   (c) Plot the new 10 EDF curves over the range $[-1, 7]$

   (d) Use **pnorm()** to find tha actual cdf of $N(3, 2^2)$ and superimpose this curve on your plot (in a contrasting color).



4. The centered Laplace (double exponential) distribution has density $f(x) = (2\beta)^{-1}e^{-|x|/\beta}$ on $-\infty < x < \infty$ where $\beta > 0$.

   (a) Derive the cdf of Laplace(β).
   Hint: It is symmetric so $F(x) = 1 - F(-x)$, and the tail probability for negative x is easiest.

   (b) Derive the inverse cdf of Laplace(β). Hint: solve $F(x) = u$, for the uniform U, $0 \le U \le 1$.

   (c) Implement the inverse cdf method, using **runif()** to sample from Laplace(1). Save 2000 samples drawn using your function.

   (d) Note that realizations from Laplace distribution are simply exponential realizations y then becoming $\pm y$ each with probability (1/2). Use **rexp()** and *either* **rbinom(2000,1,0.5)** *or* **rbinom(1, 2000, 0.5)** to generate 2000 realizations from the Laplace distribution with $\beta = 1$.

   (e) Use the function **qqplot()** to compare empirical quantiles of the two samples.

(a) $f(x) = (2\beta)^{-1} e^{-|x|/\beta}$

$$F(x) = \frac{1}{2\beta}\int e^{-\frac{|x|}{b}}\,dx$$

$$= \frac{1}{2\beta}\frac{x}{|x|}\int e^{-\frac{|x|}{b}}\cdot\frac{|x|}{x}\,dx$$

$$= \frac{x}{2\beta|x|}\int \frac{x e^{-\frac{|x|}{b}}}{|x|}\,dx \qquad u = -\frac{|x|}{b}$$

$$\frac{du}{dx} = -\frac{x}{\beta|x|}$$

$$dx = -\frac{\beta|x|}{x}\,du$$

$$= \frac{x}{2\beta|x|}\left(-\beta\int e^{u}\,du\right)$$

$$= \frac{x}{2\beta|x|}\left(-\beta e^{-\frac{|x|}{\beta}}\right)$$

$$= \frac{x e^{-\frac{|x|}{\beta}}}{2|x|}$$

For $x<0$, $F(x) = \frac{1}{2}e^{-\frac{x}{\beta}}$. And since $F(x) = 1 - F(-x)$, $F(x) = 1 - \frac{1}{2}e^{-\frac{x}{\beta}}$ for $x \geq 0$.

So $$F(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{\beta}}, & x \le 0 \\ 1-\frac{1}{2}e^{-\frac{x}{\beta}}, & x > 0 \end{cases}$$

(b) When $x \le 0$, $u = F(x) = \frac{1}{2}e^{-\frac{x}{\beta}}$

$$\log(2u) = -\frac{x}{\beta}$$

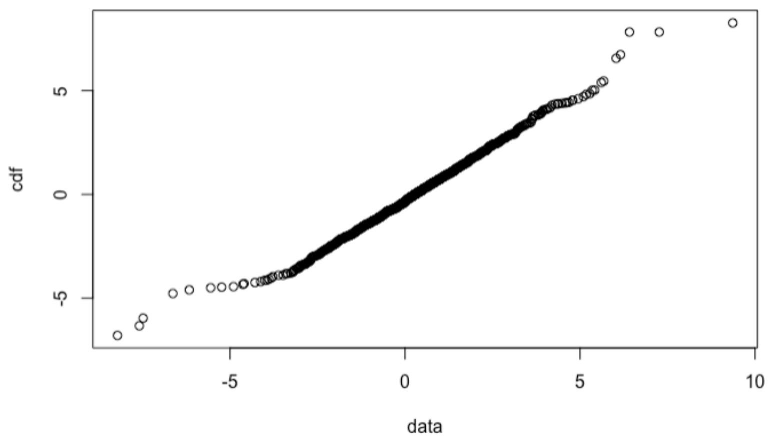
$$x = -\beta\log(2u)$$

When $x > 0$, $u = F(x) = 1 - \frac{1}{2}e^{-\frac{x}{\beta}}$

$$1-u = \frac{1}{2}e^{-\frac{x}{\beta}}$$

$$\log(2-2u) = -\frac{x}{\beta}$$

$$x = -\beta\log(2-2u)$$



cdf
5
0
-5
-5
0
5
10
data