2: Course Policies

(a) Yes. Each student must write up their own solutions.

(b) No. She credited Dan.

(c) No. She wrote her own solution and made a citation.

(d) Yes. Frank directly copied solutions from Grace.

(e) No. She wrote her own solution and made a citation.

3: Gradient Descent Doesn't Go Nuts with Ill-Conditioning

$$\vec{w_t} = \vec{w}_{t-1} - \eta\,(F^T(F\vec{w}_{t-1} - \vec{y}))$$

$$= (I - \eta F^T F)\,\vec{w}_{t-1} + \eta F^T \vec{y}$$

$$\|\vec{w_t}\|_2 = \|(I - \eta F^T F)\vec{w}_{t-1} + \eta F^T \vec{y}\|_2$$

$$\leq \|(I - \eta F^T F)\vec{w}_{t-1}\|_2 + \|\eta F^T \vec{y}\|_2$$

$$I - \eta F^T F \xlongequal{SVD} U\Sigma V^T = (I - \eta F^T F)^T = V\Sigma U^T$$

$$\|(I - \eta F^T F)\vec{w}_{t-1}\|_2 = \sqrt{\vec{w}_{t-1}^{\,T}\, U\Sigma V^T V\Sigma U^T \vec{w}_{t-1}}$$

$$= \sqrt{\vec{w}_{t-1}^{\,T}\Sigma^2 \vec{w}_{t-1}} = \sqrt{\sum_{i=1}^{n} \sigma_i^2 w_i^2}$$

$$\leq |\sigma_{max}| \cdot \|\vec{w}_{t-1}\|_2$$

The gradient descent cannot diverge, so

$(I - \eta F^T F)^t = U\Sigma^t V^T$ don't diverge, $|\sigma_{max}| \leq 1$

$$\|F^T\vec{y}\|_2 = \sqrt{\vec{y}^T F F^T \vec{y}} \xlongequal{F^T = U'\Sigma' V'^T} \sqrt{\vec{y}^T \Sigma'^2 \vec{y}} \leq \alpha \|y\|_2$$

So $\|\vec{w_t}\|_2 \leq \|(I - \eta F^T F)\vec{w}_{t-1}\|_2 + \|\eta F^T \vec{y}\|_2$

$$\leq \|\vec{w}_{t-1}\|_2 + \eta\alpha \|y\|_2$$

4: Regularization from the Augmentation Perspective

$$\|\hat{y} - \hat{X}w\|_2^2 = \left\| \begin{bmatrix} y \\ 0_d \end{bmatrix} - \begin{bmatrix} X \\ I \end{bmatrix} w \right\|_2^2$$

the OLS solution $= \left([X^T I^T] \begin{bmatrix} X \\ I \end{bmatrix} \right)^{-1} [X^T I^T] \begin{bmatrix} y \\ 0_d \end{bmatrix}$

$$= (X^T X + I^T I)^{-1} X^T y = (X^T X + \Sigma^{-1})^{-1} X^T y$$

5: Vector Calculus Review

1、(a) $\frac{\partial}{\partial x_i}(x^T c) = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} x_k c_k\right) = c_i$

$$\frac{\partial}{\partial x}(x^T c) = \left[\frac{\partial}{\partial x_1}(x^T c) \ \cdots \ \frac{\partial}{\partial x_n}(x^T c)\right]$$

$$= [c_1 \ \ c_2 \ \cdots \ c_n] = c^T$$

(b) $\frac{\partial}{\partial x}\|x\|^2 = [2x_1 \ \ 2x_2 \ \cdots \ 2x_n] = 2x^T$

(c) $\frac{\partial}{\partial x_i}(Ax) = \frac{\partial}{\partial x_i}\left(\begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}\right)$

$$= \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \qquad \frac{\partial}{\partial x}(Ax) = \left[\frac{\partial}{\partial x_1}(Ax) \cdots \frac{\partial}{\partial x_n}(Ax)\right]$$

$$\frac{\partial}{\partial x}(Ax) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

(d) $x^T A x = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$
$+ a_{21}x_1x_2 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$
$+ \cdots + a_{n1}x_1x_n + a_{n2}x_2x_n +$
$\cdots + a_{nn}x_n^2$

$$\frac{\partial}{\partial x_i}(x^T A x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{ni}x_n$$

$$= x^T \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} + x^T \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = x^T\left((A^T)_{\to i} + A_{\to i}\right)$$

$$\frac{\partial}{\partial x}(x^T A x) = x^T(A^T + A)$$

(e) $A = A^T$, A is symmetric

6: ReLU Elbow Update under SGD

(a)

(i)

$$e = \frac{-b}{\omega}$$

(ii)

$$\frac{\partial l}{\partial \phi} = \phi - y$$

(iii)

$$\frac{\partial l}{\partial \omega} = \frac{\partial l}{\partial \phi}\frac{\partial \phi}{\partial \omega} = \begin{cases} (\phi - y)x, & \text{if } \omega x + b > 0 \\ 0, & \text{otherwise} \end{cases}$$

(iv)

$$\frac{\partial l}{\partial b} = \frac{\partial l}{\partial \phi}\frac{\partial \phi}{\partial b} = \begin{cases} \phi - y, & \text{if } \omega x + b > 0 \\ 0, & \text{otherwise} \end{cases}$$

(b)

(i) The slope and elbow won't change.

$$\omega_{i+1} = \omega_i - \lambda \frac{\partial l}{\partial \omega} = \omega_i$$
$$b_{i+1} = b_i - \lambda \frac{\partial l}{\partial\ b} = b_i$$

(ii) The slope will go downward. When $b > 0$, the elbow will move to the left if $x > \frac{\omega}{b}$, will move to the right if $x < \frac{\omega}{b}$. When $b < 0$, the elbow will move to the right if $x > \frac{\omega}{b}$, will move to the left if $x < \frac{\omega}{b}$.

$$\omega_{i+1} = \omega_i - \lambda \frac{\partial l}{\partial \omega} = \omega_i - \lambda x$$
$$b_{i+1} = b_i - \lambda \frac{\partial l}{\partial\ b} = b_i - \lambda$$

(iii) The slope will go upward. The elbow will move to the right.

$$\omega_{i+1} = \omega_i - \lambda \frac{\partial l}{\partial \omega} = \omega_i - \lambda x$$
$$b_{i+1} = b_i - \lambda \frac{\partial l}{\partial\ b} = b_i - \lambda$$

(iv) The slope will go upward. When $b > 0$, the elbow will move to the left if $x < \frac{\omega}{b}$, will move to the right if $x > \frac{\omega}{b}$. When $b < 0$, the elbow will move to the right if $x < \frac{\omega}{b}$, will move to the left if $x > \frac{\omega}{b}$

$$\omega_{i+1} = \omega_i - \lambda \frac{\partial l}{\partial \omega} = \omega_i - \lambda x$$
$$b_{i+1} = b_i - \lambda \frac{\partial l}{\partial\ b} = b_i - \lambda$$

(c)

$$e_i = \frac{-b_i}{w_i^{(1)}}$$

(d)

$$e_i = \begin{cases} \frac{-(b_i - \lambda(\hat{f}(x) - y) w_i^{(2)})}{w_i^{(1)} - \lambda(\hat{f}(x) - y) w_i^{(2)} x}, & \text{if } \omega_i^{(1)} x + b_i > 0 \\ \frac{-b_i}{\omega_i^{(1)}}, & \text{otherwise} \end{cases}$$

7: Using PyTorch to Learn the Color Organ

(a) 202Ω

(b) 213Ω

(c) It converges to the same value. lr = 1e7 causes training to diverge. lr = 1e5 causes training to converge quickly.

(d) 341Ω

(e) loss fn = lambda x, y: (x - (0.3 + 0.7 * y)) ** 2

(f) 30Ω

(g) Learned resistor values are $R_{low} = 40\Omega, R_{high} = 155\Omega$. The circuit will converge within 100000 steps. It takes 30000 more steps to converge. When resistor values far away from the solution, the gradients are small and loss surface is quite flat.

(h) Yes. Yes.

(i) The time is shorter.

8: Homework Process and Study Group

(a) Nothing.

(b)I finished homework0 by myself.

(c)4hrs.