Chapter 9 Asymptotic Distance Distributions for Regular and Irregular LDPC Codes Ensembles

9.1 Introduction

So far we have known that low-density parity-check (LDPC) attracted a great deal of attention recently due to their impressive performance under iterative decoding. However, there is no complete and concrete understanding of the structure of LDCP codes so far, and knowledge of such characteristics as the minimum distance and distance distribution could definitely facilitate our analysis of the best possible performance of such codes in various noisy channels. Moreover, accurate information about the possible distance distributions provides estimates on the gap between performance of these codes under maximum likelihood and BP-based iterative decoding algorithms.

Recall that the average distance distribution (or weight enumerator function) of turbo codes is evaluated in a statistical rather than deterministic view by taking all the possible interleaving patterns into account. We will encounter the similar scenarios in evaluation of the average distance distributions of LDPC codes because parity-check matrix of an LDPC code is chosen based on a semirandom method, whose degree distributions for variable and check nodes, associated with columns and rows of the parity-check matrix, is only finalized by a given pair of distribution functions.

In this section, we will deal with the problem of estimation of the average distance distributions for the ensembles of regular and irregular LDPC codes in a statistical view.

First, classical regular LDPC codes ensemble is considered with all columns and rows of parity-check matrices with given weight. Then we extend these results to irregular LDPC codes with a given degree distributions for variable and check nodes.

9.2 Distance distribution and Average Ensemble Distance Distribution of Linear Block Codes

9.2.1 Distance distribution of a linear block code

Let $C^{(n)}$ be an *ensemble* of block codes of length *n* defined by parity-check matrix of size $m \times n$. For a code $C \in C^{(n)}$ we define the distance distribution as an (n+1)-vector

$$B(C)=(B_0(C), B_1(C), \ldots\ldots, B_n(C)) \tag{9-1a}$$

where

$$B_i(C) = \left|\{\mathbf{c} \in C : wt(\mathbf{c}) = i\}\right| \tag{9-1b}$$

where $wt(\mathbf{c})$ is the Hamming weight of the codeword $\mathbf{c}$. Clearly, $B_0(C)$=1 as there is only one all-zero codeword in a linear block code.

Example 9.1: Consider a systematic (7, 4) binary Hamming code C with the following parity-check matrix, which is previously used in the for the input-redundancy weight-enumerating function (IRWEF).

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{9-2}$$

By exhaustively searching all the codewords with Hamming weights through the parity-check matrix, we can easily express the distance distribution vector of the code as

$$B(C)=(B_0(C),\ B_1(C),\ B_2(C),\ B_3(C),\ B_4(C),\ B_5(C),\ B_6(C),\ B_7(C))$$

$$=(1, 0, 0, 7, 7, 0, 0, 1) \tag{9-3}$$

9.2.2 Average ensemble distance distribution of a linear block codes family

Generically, the average ensemble distance distribution of linear block codes is

$$B(C^{(n)})=(B_0(C^{(n)}),\ B_1(C^{(n)}),\ \ldots\ldots,\ B_n(C^{(n)})) \tag{9-4a}$$

where

$$B_i(C^{(n)})=\frac{1}{\left|C^{(n)}\right|}\sum_{C\in C^{(n)}} B_i(C) \tag{9-4b}$$

Example 9.2: Let $C^{(5)}$ be an *ensemble* of two (5, 3) block codes C_1 and C_2, whose corresponding parity-check matrices H_1 and H_2 are given as

$$H_1=\begin{bmatrix}1 & 1 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\end{bmatrix} \tag{9-5a}$$

$$H_2=\begin{bmatrix}1 & 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0 & 1\end{bmatrix} \tag{9-5b}$$

Based on the matrix H_1 the distance distribution vector of the block code C_1 is

$$B(C_1)=(B_0(C_1),\ B_1(C_1),\ B_2(C_1),\ B_3(C_1),\ B_4(C_1),\ B_5(C_1))$$

$$=(1, 0, 2, 4, 1, 0) \tag{9-6a}$$

Based on the matrix H_2 the distance distribution vector of the block code C_2 is

$$B(C_2)=(B_0(C_2),\ B_1(C_2),\ B_2(C_2),\ B_3(C_2),\ B_4(C_2),\ B_5(C_2))$$

$$=(1, 0, 3, 3, 0, 1) \tag{9-6b}$$

Therefore, according to (9-4) the average distance distribution of block codes ensemble $C^{(5)}$ is

$$B(C^{(5)})=(B_0(C^{(5)}),\ B_1(C^{(5)}),\ B_2(C^{(5)}),\ B_3(C^{(5)}),\ B_4(C^{(5)}),\ B_5(C^{(5)}))$$

$$=\frac{1}{2}((1+1), (0+0), (2+3), (4+3), (1+0), (0+1))$$

$$=(1, 0, 5/2, 7/2, 1/2, 1/2) \tag{9-7}$$

Note that the notion of average ensemble distance distribution can be similarly apply to concatenated codes family that adopts block or convolutional codes as component codes in a parallel and serial fashion.

9.3 Average Distance Distribution of Regular LDPC Codes Ensemble

In order to well express the theoretical analysis for the average distance distribution, we will unify some key notations that are used throughout in this section. Note that these notations may have the same meaning as those characterized by the different symbols in other chapters and section.

9.3.1 Preliminaries:

For regular LDPC codes, consider the ensemble of all $m\times n$ (0, 1)-matrices with $m<n$, and having all row sums equal k and column sums equal l. In other words, for every matrix $A=[a_{i,j}]$, $i=1, 2, \dots, m$, $j=1, 2, \dots, n$, from this ensemble we have

$$\sum_{j=1}^{n} a_{i,j} = k \qquad \text{for every } i = 1, 2, \ldots, m \tag{9-8a}$$

$$\sum_{i=1}^{m} a_{i,j} = l \qquad \text{for every } j = 1, 2, \ldots, n \tag{9-8b}$$

Computing the total number of ones in the matrices in two ways, i.e., by rows and by column, we conclude that $mk = nl$. Let

$$\alpha = \frac{m}{n} = \frac{l}{k} \qquad 0 < \alpha \le 1 \tag{9-9}$$

We denote the described ensemble, whose matrix satisfies the constraints for row and column sums by (9-8a) and (9-8b), respectively, by $\Lambda_n^{k,\alpha}$. Let $\omega = \theta n$, $0 < \theta < 1$, and denote the subset of the matrices from $\Lambda_n^{k,\alpha}$ having an even sum of the first ω elements in every row as $\Lambda_{n,\theta}^{k,\alpha}$. In other words

$$\sum_{j=1}^{\omega} a_{i,j} \in \{0, 2, 4, \ldots\} \qquad \text{for every } i = 1, 2, \ldots, m \tag{9-10}$$

This condition yields that

$$(\alpha k)(\theta n) = lm \equiv 0 \bmod 2 \tag{9-11}$$

An alternative description of the matrices in this subset is that the componentwise modulo-2 sum of their first ω columns is the all-zero column vector of dimension m. Note that this implies that the following vector

$$1^w 0^{n-w} = \underbrace{1 \;\; \ldots\ldots \;\; 1}_{\omega}\underbrace{0 \;\; \ldots\ldots \;\; 0}_{n-\omega} \tag{9-12}$$

must be a valid codeword of (l, k)-regular LDPC of Hamming weight ω. Let

$$P_{n,\theta}^{k,\alpha} = \frac{\left|\Lambda_{n,\theta}^{k,\alpha}\right|}{\left|\Lambda_{n}^{k,\alpha}\right|} \tag{9-13}$$

Thus for sufficiently long block lengths the average ensemble distance distribution regarding the Hamming distance ω ($0 \le \omega \le n$) is approximated as

$$B_{\omega}(\Lambda_{n}^{k,\alpha}) \approx P_{n,\theta}^{k,\alpha} \tag{9-14}$$

In this section, our central task is to estimate the number of such matrices $\left|\Lambda_{n,\theta}^{k,\alpha}\right|$ in the total number $\left|\Lambda_{n}^{k,\alpha}\right|$ and obtain the crucial ratio $P_{n,\theta}^{k,\alpha}$, or $\left(\log P_{n,\theta}^{k,\alpha}\right)/n$ as block length tends to infinity.

9.3.2 Two fundamental lemmas: Let $K=(k_1, k_2, \ldots, k_n)$ and $L=(l_1, l_2, \ldots, l_n)$, where k_i and l_i are nonnegative integers, and let $N_n^{K,L}$ represent the ensemble of square $n \times n$ matrices with row sums k_i and column sums l_i.

The following two lemma are crucial and fundamental for derivation of the average distance distribution of LDPC codes ensemble. The interested reader can refer to [70] for the proofs of the lemmas. Note that unless stated otherwise we will use natural logarithms throughout in this section.

Lemma 9.1 [70]: Let $n \to +n$, and

$$\max_{1 \le i \le n} \{k_i, l_i\} \le (\log n)^{\frac{1}{4}-\varepsilon}, \qquad \varepsilon > 0 \tag{9-15}$$

$$\left|\{i : k_i = 0 \quad and \quad l_i = 0\}\right| = O(\log n) \tag{9-16}$$

Then, for $\delta > 0$

$$\left|\Lambda_n^{K,L}\right| = \frac{\left(\sum_{i=1}^{n} k_i\right)!}{\prod_{i=1}^{n} k_i!\, l_i!} \exp\left(\frac{-1}{2\left(\sum_{i=1}^{n} k_i\right)^2}\left(\sum_{i=1}^{n} k_i\left(k_i - 1\right)\sum_{i=1}^{n} l_i\left(l_i - 1\right)\right)\right)$$

$$\times\left(1 + o\left(n^{-1+\delta}\right)\right) \tag{9-17}$$

In 1977, Good and Crook [71] showed that Lemma 9.1 is still valid even without the condition (9-16). Hence, it is quite straightforward to generalize it to arbitrary rectangular matrices. Again, let $K = (k_1, k_2, \ldots, k_m)$ and $L = (l_1, l_2, \ldots, l_n)$, $\Lambda_{m,n}^{K,L}$ be the ensemble of rectangular $m \times n$ matrices $m < n$, with row sums k_i, $i = 1, 2, \ldots, m$, and column sums l_j, $j = 1, 2, \ldots, n$.

Lemma 9.2: Let $n \to +n$, and

$$\max\left\{\max_{1 \le i \le m} k_i,\ \max_{1 \le j \le n} l_j\right\} \le \left(\log n\right)^{\frac{1}{4} - \varepsilon}, \qquad \varepsilon > 0 \tag{9-18}$$

Then, for $\delta > 0$

$$\left|\Lambda_{m,n}^{K,L}\right| = \frac{\left(\sum_{i=1}^{n} k_i\right)!}{\prod_{i=1}^{n} k_i! \prod_{j=1}^{n} l_j!} \exp\left(\frac{-1}{2\left(\sum_{i=1}^{m} k_i\right)^2}\left(\sum_{i=1}^{m} k_i\left(k_i - 1\right)\sum_{j=1}^{n} l_j\left(l_j - 1\right)\right)\right)$$

$$\times\left(1 + o\left(n^{-1+\delta}\right)\right) \tag{9-19}$$

Proof: To prove this lemma by using the conclusion obtained by Lemma 9.1, we let

$$k_{m+1}=k_{m+2}=\ldots\ldots=k_n=0 \tag{9-20}$$

Clearly, (9-18) implies (9-15), and then in this case $\left|\Lambda_{m,n}^{K,L}\right|$ becomes

$$\left|N_{m,n}^{K,L}\right|=\left|N_n^{K,L}\right|=\frac{\left(\sum_{i=1}^{n}k_i\right)!}{\coprod_{i=1}^{n}k_i!l_i!}\exp\left(\frac{-1}{2\left(\sum_{i=1}^{n}k_i\right)^2}\left(\sum_{i=1}^{n}k_i\left(k_i-1\right)\sum_{i=1}^{n}l_i\left(l_i-1\right)\right)\right)\left(1+o\left(n^{-1+\delta}\right)\right)$$

$$=\frac{\left(\sum_{i=1}^{m}k_i+\sum_{i=m+1}^{n}k_i\right)!}{\left(\coprod_{i=1}^{m}k_i!\right)\left(\prod_{i=m+1}^{n}k_i!\right)\left(\prod_{i=1}^{n}l_i!\right)}\exp\left(\frac{-1}{2\left(\sum_{i=1}^{m}k_i+\sum_{i=m+1}^{n}k_i\right)^2}\left(\left(\sum_{i=1}^{m}k_i\left(k_i-1\right)+\sum_{i=m+1}^{n}k_i\left(k_i-1\right)\right)\sum_{i=1}^{n}l_i\left(l_i-1\right)\right)\right)$$

$$\times\left(1+o\left(n^{-1+\delta}\right)\right)=\frac{\left(\sum_{i=1}^{m}k_i\right)!}{\coprod_{i=1}^{m}k_i!\prod_{i=1}^{n}l_i!}$$

$$\exp\left(\frac{-1}{2\left(\sum_{i=1}^{m}k_i\right)^2}\left(\sum_{i=1}^{m}k_i\left(k_i-1\right)\sum_{i=1}^{n}l_i\left(l_i-1\right)\right)\right)\left(1+o\left(n^{-1+\delta}\right)\right) \tag{9-21}$$

Thus, the theorem is proved. □

9.3.3 The central theorem for evaluation of average distance distribution of regular LDPC codes ensemble and simulation results

In this part the following important theorem [72], regarding the asymptotical average distance distribution of regular LDPC codes ensemble, is presented and the details of the proof for the theorem is provided.

Theorem 9.1: Let t be the (only) positive root of the following equation

$$\frac{(1+t)^{k-1}+(1-t)^{k-1}}{(1+t)^{k}+(1-t)^{k}}=1-\theta \tag{9-22}$$

Then, for $0<\theta<1$ and k even

$$\lim_{n\to+\infty}\frac{1}{n}\log P_{n,\theta}^{k,\alpha}=\alpha\log\left(\frac{(1+t)^{k}+(1-t)^{k}}{2t^{\theta k}}\left((1-\theta)^{1-\theta}\theta^{\theta}\right)^{k}\right) \tag{9-23a}$$

$$P_{n,\theta}^{k,\alpha}=P_{n,1-\theta}^{k,\alpha} \tag{9-23b}$$

and for k odd

$$\lim_{n\to+\infty}\frac{1}{n}\log P_{n,\theta}^{k,\alpha}=\begin{cases}\alpha\log\left(\frac{(1+t)^{k}+(1-t)^{k}}{2t^{\theta k}}\left((1-\theta)^{1-\theta}\theta^{\theta}\right)^{k}\right) & if\ 0<\theta\le\frac{k-1}{k}\\ -\infty & otherwise\end{cases} \tag{9-24}$$

□

Proof: The proof of the theorem is accomplished by the following steps, where also contains several other necessary lemmas and corollaries that are used to support the main theorem.

1) The Case of Even k: Let $\mathbf{A}\in\Lambda_{n,\theta}^{k,\alpha}$. For a fixed ω, the matrix $\mathbf{A}$ is naturally partitioned into two submatrices $\mathbf{A}^{left}$ and $\mathbf{A}^{right}$ of size $m\times\omega$ and $m\times(n-\omega)$, respectively, which consist of the first ω columns and the last $n-\omega$ columns of $\mathbf{A}$. Let m_i be the number of rows in $\mathbf{A}^{left}$ with sums equal to i, where $i\in\{0, 2, 4, \ldots, k\}$. Consequently, $\mathbf{A}^{right}$ has m_i rows with sums $k-i$, and the following equalities are valid:

$$m_0+m_2+m_4+\ldots\ldots+m_k=\alpha n \tag{9-25a}$$

$$2m_2 + 4m_4 + \ldots\ldots + km_k = \alpha k\theta n \tag{9-25b}$$

Obviously, $m_i \geq 0$. The first constraint simply means that the number of all rows of different sums is equal to number of the parity-check equations, while the second constraint counts the number of ones in the submatrix $\mathbf{A}^{left}$.

We denote the set of all possible matrices $\mathbf{A}^{left}$ by $L_{n,\theta}^{k,\alpha}$ and the set of all possible matrices $\mathbf{A}^{right}$ by $R_{n,\theta}^{k,\alpha}$. Clearly, we get

$$\left|\Lambda_{n,\theta}^{k,\alpha}\right| = \sum \binom{\alpha n}{m_0, m_2, \ldots, m_k} \left|L_{n,\theta}^{k,\alpha}\right| \left|R_{n,\theta}^{k,\alpha}\right| \tag{9-26a}$$

where the sum is taken over all solutions m_0, m_2, …, m_k of (9-25a) and (9-25b), and

$$\binom{\alpha n}{m_0, m_2, \ldots, m_k} = \frac{(\alpha n)!}{\prod_{i=0}^{k/2} m_{2i}!} \tag{9-26b}$$

Therefore, the value of $\left|\Lambda_{n,\theta}^{k,\alpha}\right|$ can be finally specified if we can evaluate the values of $\left|L_{n,\theta}^{k,\alpha}\right|$ and $\left|R_{n,\theta}^{k,\alpha}\right|$, respectively. The following lemma will provide the estimates of them.

A: Definition of $M_{n,\theta}^{k}$

Lemma 9.3: The following holds:

$$\left|L_{n,\theta}^{k,\alpha}\right| = g_L(n) \frac{(\alpha k\theta n)!}{(\alpha k)!^{\theta n}\, 2!^{m_2}\, 4!^{m_4} \ldots k!^{m_k}} \tag{9-27a}$$

where for n sufficiently large

$$\frac{1}{2}\exp\left(-\frac{k(k\alpha - 1)}{2\theta}\right) \leq g_L(n) \leq 2\exp\left(-\frac{k\alpha - 1}{2}\right) \tag{9-27b}$$

and

$$\left|R_{n,\theta}^{k,\alpha}\right| = g_R(n)\,\frac{\left(\alpha k(1-\theta)n\right)!}{(\alpha k)!^{(1-\theta)n}\,2!^{m_{k-2}}\,4!^{m_{k-4}}\ldots k!^{m_0}} \tag{9-28a}$$

where for n sufficiently large

$$\frac{1}{2}\exp\left(-\frac{k(k\alpha-1)}{2(1-\theta)}\right) \le g_R(n) \le 2\exp\left(-\frac{k\alpha-1}{2}\right) \tag{9-28b}$$

□

The proof of the theorem can be referred to the Appendix L. For $n \to +\infty$ we use notation $a_n \overset{\log}{\sim} b_n$ if $\log a_n \sim \log b_n$, i.e., $\lim\limits_{n\to+\infty} \log a_n / \log b_n = 1$. We say that a_n and b_n are logarithmically equivalent.

By the Lemma 9.2 and (9-26), we compute $\left|\Lambda_{n,\theta}^{k,\alpha}\right|$ using the identities as follows

$$\frac{\left(\alpha k\theta n\right)!}{\left(\alpha k\right)!^{\theta n}}\frac{\left(\alpha k(1-\theta)n\right)!}{\left(\alpha k\right)!^{(1-\theta)n}} = \frac{\left(\alpha k\theta n\right)!\left(\alpha k(1-\theta)n\right)!}{\left(\alpha k\right)!^{n}} \tag{9-29a}$$

$$\frac{\left(\alpha n\right)!}{m_0!m_2!m_4!\ldots\ldots m_k!}\left(\frac{1}{0!^{m_0}\,2!^{m_2}\,4!^{m_4}\ldots\ldots(k-2)!^{m_{k-2}}\,k!^{m_k}}\right)\left(\frac{1}{0!^{m_k}\,2!^{m_{k-2}}\,4!^{m_{k-4}}\ldots\ldots(k-2)!^{m_2}\,k!^{m_0}}\right)$$

$$= \frac{\left(\alpha n\right)!}{m_0!\left(0!k!\right)^{m_0} m_2!\left(2!(k-2)!\right)^{m_2}\ldots\ldots m_{k-2}!\left((k-2)!2!\right)^{m_{k-2}} m_k!\left(k!0!\right)^{m_k}} \tag{9-29b}$$

Thus, $\left|\Lambda_{n,\theta}^{k,\alpha}\right|$ is expressed as

$$\left|\Lambda_{n,\theta}^{k,\alpha}\right| \overset{\log}{\sim} \frac{\left(\alpha k\theta n\right)!\left(\alpha k(1-\theta)n\right)!}{\left(\alpha k\right)!^{n}}$$

$$\times \sum \frac{(\alpha n)!}{m_0!(0!k!)^{m_0}\, m_2!(2!(k-2)!)^{m_2} \,......m_{k-2}!((k-2)!2!)^{m_{k-2}}\, m_k!(k!0!)^{m_k}} \tag{9-30a}$$

where the summation is over all m_0, m_2, …, m_k satisfying both (9-25a) and (9-25b).

The set of matrices $L_{n,\theta}^{k,\alpha}$ is equal to the set $\Lambda_n^{k,\alpha}$ in case of $\theta=1$ while $m_i=n\alpha$ for $i=k$. Thus, according to (9-27a) $\left|\Lambda_n^{k,\alpha}\right|$ can be expressed as

$$\left|\Lambda_n^{k,\alpha}\right| \overset{\log}{\sim} \frac{(nk\alpha)!}{(k!)^{n\alpha}(\alpha k)!^n} \tag{9-30b}$$

for $n\to+\infty$. Based on (9-13), (9-30a) and (9-30b) $P_{n,\theta}^{k,\alpha}$ is

$$P_{n,\theta}^{k,\alpha} = \left|\Lambda_{n,\theta}^{k,\alpha}\right| / \left|\Lambda_n^{k,\alpha}\right| \overset{\log}{\sim} \frac{(\alpha k\theta n)!(\alpha k(1-\theta)n)!}{(\alpha kn)!}$$

$$\times \sum \left(\frac{(\alpha n)!}{m_0!m_2!......m_k!} \left(\frac{k!}{0!k!}\right)^{m_0} \left(\frac{k!}{2!(m-2)!}\right)^{m_2} ......\left(\frac{k!}{(k-2)!2!}\right)^{m_{k-2}} \left(\frac{k!}{k!0!}\right)^{m_k} \right)$$

$$= \frac{1}{\binom{nk\alpha}{n\theta k\alpha}} \sum \binom{\alpha n}{m_0, m_2, ..., m_k} \binom{k}{2}^{m_2} \binom{k}{4}^{m_4} ......\binom{k}{k-2}^{m_{k-2}} \tag{9-31}$$

Apparently $P_{n,\theta}^{k,\alpha} = P_{n,1-\theta}^{k,\alpha}$ and thus (9-23b) is proved. It is also easy to show that

$$P_{n,\theta}^{k,\alpha} \overset{\log}{\sim} P_{n\alpha,\theta}^{k,1} \tag{9-32}$$

By (9-32), it suffices to accomplish the calculations for $\alpha=1$. Thus, we simply denote $P_{n,\theta}^{k,1}$ by $P_{n,\theta}^{k}$. Let $M_{n,\theta}^{k}$ be

$$M_{n,\theta}^{k}=\max\binom{n}{m_0,m_2,......,m_k}\binom{k}{2}^{m_2}\binom{k}{4}^{m_4}\cdots\cdots\binom{k}{k-2}^{m_{k-2}} \tag{9-33}$$

where the maximum is over all m_0, m_2, …, m_k fulfilling (9-25) with α=1, namely for square matrix $n\times n$

$$m_0+m_2+m_4+\cdots\cdots+m_k=n \tag{9-34a}$$

$$2m_2+4m_4+\cdots\cdots+km_k=k\theta n \tag{9-34b}$$

Using the natural entropy [72]

$$H(\theta)=\lim_{n\to+\infty}\frac{1}{n}\log\binom{n}{\theta n}=-\theta\log\theta-(1-\theta)\log(1-\theta) \tag{9-35}$$

Hence, it yields

$$\log\binom{nk}{n\theta k}\sim knH(\theta) \tag{9-36a}$$

Equivalently

$$\binom{nk}{n\theta k}^{-1}\sim\exp\left(-knH(\theta)\right) \tag{9-36b}$$

The following lemma gives the estimate of $P_{n,\theta}^{k}$ for $n\to+\infty$.

Lemma 9.4: Prove that $P_{n,\theta}^{k}\overset{\log}{\sim}\exp\left(-knH(\theta)\right)M_{n,\theta}^{k}$ □

The proof of the theorem can be referred to the Appendix L.

B: Computation of $\log M_{n,\theta}^{k}$

From (9-25b) with α=1 m_k is evaluated as

$$m_k = \theta n - \frac{2}{k} m_2 - \frac{4}{k} m_4 - \frac{6}{k} m_6 - \ldots\ldots - \frac{k-2}{k} m_{k-2} := \mu_k \tag{9-37}$$

For matrix $\mathbf{A}^{right}$ of size $m \times (n-\omega)$, (9-25b) with $\alpha = 1$ is transferred into

$$km_0 + (k-2)m_2 + (k-4)m_4 + \ldots\ldots + 2m_{k-2} = k(1-\theta)n \tag{9-38a}$$

Hence, m_0 is

$$m_0 = (1-\theta)n - \frac{k-2}{k} m_2 - \frac{k-4}{k} m_4 - \frac{k-6}{k} m_6 - \ldots\ldots - \frac{2}{k} m_{k-2} := \mu_0 \tag{9-38b}$$

Recall that Stirling's approximation for sufficiently large n is

$$n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n} = \sqrt{2\pi}\,\frac{n^{n+1/2}}{e^n} \tag{9-39a}$$

It logarithmic equivalence is

$$\log n! \approx \frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log n - n$$

$$\sim n \log n - n \tag{9-39b}$$

From (9-37), (9-38b) and (9-33), we have

$$\log M_{n,\theta}^{k} = \max \log \left(\frac{n!}{m_0!m_2!\ldots\ldots m_k!} \binom{k}{2}^{m_2} \binom{k}{4}^{m_4} \ldots\ldots \binom{k}{k-2}^{m_{k-2}} \right)$$

$$\sim \max \left\{ n\log n - n - (\mu_0 \log \mu_0 - \mu_0) - (m_2 \log m_2 - m_2) - (m_4 \log m_4 - m_4) \right.$$

$$- \ldots\ldots - (m_{k-2} \log m_{k-2} - m_{k-2}) - (\mu_k \log \mu_k - \mu_k) + m_2 \log \binom{k}{2} + m_4 \log \binom{k}{4}$$

$$+\ldots\ldots+m_{k-2}\log\binom{k}{k-2}\Big\}$$

$$=\max\Big\{n\log n-\mu_0\log\mu_0-m_2\log m_2-m_4\log m_4-\ldots\ldots-m_{k-2}\log m_{k-2}$$

$$-\mu_k\log\mu_k+m_2\log\binom{k}{2}+m_4\log\binom{k}{4}+\ldots\ldots+m_{k-2}\log\binom{k}{k-2}\Big\} \tag{9-40}$$

Let the partial derivative with respect to m_2 be zero, it yields

$$-\left(\frac{\partial\mu_0}{\partial m_2}\log\mu_0+\frac{\partial\mu_0}{\partial m_2}\right)-\left(\log m_2+1\right)-\left(\frac{\partial\mu_k}{\partial m_2}\log\mu_k+\frac{\partial\mu_k}{\partial m_2}\right)+\log\binom{k}{2}=0 \tag{9-41}$$

After simplifications, it becomes

$$\frac{k-2}{k}\log\mu_0-\log m_2+\frac{2}{k}\log\mu_k+\log\binom{k}{2}=0 \tag{9-42a}$$

Similarly, we derive the partial derivatives with respect to m_4, m_6, ……, m_{k-2} and set a system of equations to be zero as

$$\frac{k-4}{k}\log\mu_0-\log m_4+\frac{4}{k}\log\mu_k+\log\binom{k}{4}=0 \tag{9-42b}$$

$$\frac{k-6}{k}\log\mu_0-\log m_6+\frac{6}{k}\log\mu_k+\log\binom{k}{6}=0 \tag{9-42c}$$

………………

$$\frac{k-2i}{k}\log\mu_0-\log m_{2i}+\frac{2i}{k}\log\mu_k+\log\binom{k}{2i}=0 \tag{9-42d}$$

………………

$$\frac{2}{k}\log\mu_0 - \log m_{k-2} + \frac{2}{k}\log\mu_k + \log\binom{k}{k-2} = 0 \tag{9-42e}$$

Solving the system of the first and ith equation, namely, (9-42a) and (9-42d), to find μ_0 and μ_k for every i=2, 3, …., $k/2-1$, we obtain

$$\log\mu_0 = \frac{1}{i-1}\left[\log\left(m_2\binom{k}{2}^{-1}\right)^i + \log\left(\binom{k}{2i}m_{2i}^{-1}\right)\right] \tag{9-43a}$$

$$\log\mu_k = \frac{1}{2(i-1)}\left[\log\left(\binom{k}{2}m_2^{-1}\right)^{k-2i} + \log\left(m_{2i}\binom{k}{2i}^{-1}\right)^{k-2}\right] \tag{9-43b}$$

Namely,

$$\mu_0 = \left[m_2\binom{k}{2}^{-1}\right]^{\frac{i}{i-1}}\left[\binom{k}{2i}m_{2i}^{-1}\right]^{\frac{1}{i-1}} \tag{9-44a}$$

$$\mu_k = \left[\binom{k}{2}m_2^{-1}\right]^{\frac{k-2i}{2(i-1)}}\left[m_{2i}\binom{k}{2i}^{-1}\right]^{\frac{k-2}{2(i-1)}} \tag{9-44b}$$

Applying (9-44a) to (9-38b) we find

$$\frac{k-2}{k}m_2 + \frac{k-4}{k}m_4 + \frac{k-6}{k}m_6 + \ldots\ldots + \frac{2}{k}m_{k-2}$$

$$= (1-\theta)n - \mu_0 = (1-\theta)n - \left[m_2\binom{k}{2}^{-1}\right]^{\frac{i}{i-1}}\left[\binom{k}{2i}m_{2i}^{-1}\right]^{\frac{1}{i-1}} \tag{9-45a}$$

Applying (9-44b) to (9-37) we obtain

$$\frac{2}{k}m_2+\frac{4}{k}m_4+\frac{6}{k}m_6+\ldots\ldots+\frac{k-2}{k}m_{k-2}$$

$$=\theta n-\mu_k=\theta n-\left[\binom{k}{2}m_2^{-1}\right]^{\frac{k-2i}{2(i-1)}}\left[m_{2i}\binom{k}{2i}^{-1}\right]^{\frac{k-2}{2(i-1)}} \tag{9-45b}$$

Let

$$\frac{m_4}{m_2}=t_4,\quad \frac{m_6}{m_2}=t_6,\ \ldots\ldots,\ \frac{m_{k-2}}{m_2}=t_{k-2} \tag{9-46}$$

Then, (9-45a) becomes

$$\frac{k-2}{k}+\frac{k-4}{k}t_4+\frac{k-6}{k}t_6+\ldots\ldots+\frac{2}{k}t_{k-2}$$

$$+\binom{k}{2i}^{1/(i-1)}\binom{k}{2}^{-i/(i-1)}\left(t_{2i}\right)^{-1/(i-1)}=\frac{(1-\theta)n}{m_2} \tag{9-47}$$

Let

$$C_{k,n}=\binom{k}{2i}^{-1/(i-1)}\binom{k}{2}^{i/(i-1)}\left(t_{2i}\right)^{1/(i-1)} \tag{9-48a}$$

From (9-47) we see that $C_{k,n}$ does not depend on i. Thus,

$$t_{2i}=C_{k,n}^{i-1}\binom{k}{2i}\binom{k}{2}^{-1} \tag{9-48b}$$

From (9-46) and (9-48b) it follows that to solve the equations (9-42) we need to find $C_{k,n}$ and m_2. Therefore, (9-45b) is rewritten by using (9-46) as

$$\frac{2}{k}+\frac{4}{k}t_4+\frac{6}{k}t_6+\ldots\ldots+\frac{k-2}{k}t_{k-2}$$

$$+\binom{k}{2}^{\frac{k-2i}{2(i-1)}}\binom{k}{2i}^{-\frac{k-2}{2(i-1)}}\left(t_{2i}\right)^{\frac{k-2}{2(i-1)}}=\frac{\theta n}{m_2} \tag{9-49}$$

Dividing (9-49) by (9-47) and using (9-48b), we obtain

$$C_{k,n}^{k/2}+\sum_{i=1}^{k/2-1}\left[\binom{k-1}{2i}-g\binom{k-1}{2i-1}\right]C_{k,n}^{k/2-i}-g=0 \tag{9-50}$$

Considering the following binomial identities

$$\left(\sqrt{x}+1\right)^{k-1}+\left(\sqrt{x}-1\right)^{k-1}=2\sum_{i=0}^{k/2-1}\binom{k-1}{2i}x^{k/2-i}x^{-1/2} \tag{9-51a}$$

Thus,

$$\frac{\sqrt{x}}{2}\left(\left(\sqrt{x}+1\right)^{k-1}+\left(\sqrt{x}-1\right)^{k-1}\right)-x^{k/2}$$

$$=\sum_{i=0}^{k/2-1}\binom{k-1}{2i}x^{k/2-i}-x^{k/2}=\sum_{i=1}^{k/2-1}\binom{k-1}{2i}x^{k/2-i} \tag{9-51b}$$

and

$$\left(\sqrt{x}+1\right)^{k-1}-\left(\sqrt{x}-1\right)^{k-1}=2\sum_{i=1}^{k/2}\binom{k-1}{2i-1}x^{k/2-i}$$

$$=2\sum_{i=1}^{k/2-1}\binom{k-1}{2i-1}x^{k/2-i}+2 \tag{9-52a}$$

Thus,

$$\frac{1}{2}\left(\left(\sqrt{x}+1\right)^{k-1}-\left(\sqrt{x}-1\right)^{k-1}\right)-1$$

$$=\sum_{i=1}^{k/2-1}\binom{k-1}{2i-1}x^{k/2-i} \tag{9-52b}$$

Let

$$t=\sqrt{C_{k,n}} \tag{9-53}$$

Based on (9-51)–(9-53), the left-side of (9-50) can be rewritten as

$$C_{k,n}^{k/2}+\sum_{i=1}^{k/2-1}\left[\binom{k-1}{2i}-g\binom{k-1}{2i-1}\right]C_{k,n}^{k/2-i}-g$$

$$=t^k+\frac{t}{2}\left[(t+1)^{k-1}+(t-1)^{k-1}\right]-t^k-g\left\{\frac{1}{2}\left[(t+1)^{k-1}-(t-1)^{k-1}\right]-1\right\}-g$$

$$=\frac{t}{2}\left[(t+1)^{k-1}+(t-1)^{k-1}\right]-\frac{g}{2}\left[(t+1)^{k-1}-(t-1)^{k-1}\right] \tag{9-54}$$

Hence, let (9-54) be zero as (9-50) we have

$$\frac{t\left((t+1)^{k-1}+(t-1)^{k-1}\right)}{(t+1)^{k-1}-(t-1)^{k-1}}=g=\frac{\theta}{1-\theta} \tag{9-55a}$$

Equivalently

$$\frac{t\left((t+1)^{k-1}+(t-1)^{k-1}\right)}{(t+1)^{k-1}-(t-1)^{k-1}}+1$$

$$=\frac{t(t+1)^{k-1}+(t+1)^{k-1}+t(t-1)^{k-1}-(t-1)^{k-1}}{(t+1)^{k-1}-(t-1)^{k-1}}$$

$$=\frac{(t+1)^k+(t-1)^k}{(t+1)^{k-1}-(t-1)^{k-1}}=\frac{1}{1-\theta} \tag{9-55b}$$

Since $k-1$ is odd, it finally results in

$$\frac{(1+t)^{k-1}+(1-t)^{k-1}}{(1+t)^{k}+(1-t)^{k}}=1-\theta \tag{9-55c}$$

By (9-48b) and (9-53)

$$t_{2i}=\binom{k}{2i}\binom{k}{2}^{-1}t^{2(i-1)} \tag{9-56}$$

and

$$\binom{k}{2i}^{1/(i-1)}\binom{k}{2}^{-i/(i-1)}\left(t_{2i}\right)^{-1/(i-1)}$$

$$=\binom{k}{2i}^{1/(i-1)}\binom{k}{2}^{-i/(i-1)}\binom{k}{2i}^{-1/(i-1)}\binom{k}{2}^{1/(i-1)}t^{-2}=\binom{k}{2}^{-1}t^{-2} \tag{9-57}$$

Meanwhile, we have for i=1, 2, ……, $k/2-1$

$$\frac{k-2i}{k}t_{2i}=\frac{k-2i}{k}\frac{\binom{k}{2i}}{\binom{k}{2}}t^{2i}\,t^{-2}$$

$$=\binom{k}{2}^{-1}t^{-2}\,\frac{k-2i}{k}\,\frac{k!}{(k-2i)!(2i)!}t^{2i}$$

$$=\binom{k}{2}^{-1}t^{-2}\binom{k-1}{2i}t^{2i} \tag{9-58}$$

Using (9-57) and (9-58), the left-side of (9-47) can be rewritten as

$$\binom{k}{2}^{-1} t^{-2} \sum_{i=1}^{k/2-1} \binom{k-1}{2i} t^{2i} + \binom{k}{2}^{-1} t^{-2}$$

$$= \binom{k}{2}^{-1} t^{-2} \sum_{i=0}^{k/2-1} \binom{k-1}{2i} t^{2i}$$

$$= \frac{1}{2}\binom{k}{2}^{-1} t^{-2} \left((1+t)^{k-1} + (1-t)^{k-1} \right) \tag{9-59}$$

Therefore, it yields

$$\frac{1}{2}\binom{k}{2}^{-1} t^{-2} \left((1+t)^{k-1} + (1-t)^{k-1} \right) = \frac{(1-\theta)n}{m_2} \tag{9-60a}$$

Alternatively,

$$m_2 = \frac{2(1-\theta)\binom{k}{2} t^2 n}{(1+t)^{k-1} + (1-t)^{k-1}} \tag{9-60b}$$

By (9-46), (9-56) and (9-60b)

$$m_{2i} = m_2 t_{2i} = \frac{2(1-\theta)\binom{k}{2i} t^{2i} n}{(1+t)^{k-1} + (1-t)^{k-1}} \tag{9-61}$$

Using (9-57), (9-56), (9-60b) and (9-61), (9-44a) and (9-44b) can be rewritten as

$$\mu_0 = \left[m_2 \binom{k}{2}^{-1} \right]^{\frac{i}{i-1}} \left[\binom{k}{2i} m_{2i}^{-1} \right]^{\frac{1}{i-1}} = m_2 \left(\frac{m_{2i}}{m_2} \right)^{-1/(i-1)} \binom{k}{2i}^{1/(i-1)} \binom{k}{2}^{-i/(i-1)}$$

$$= m_2\, t_{2i}^{-1/(i-1)} \binom{k}{2i}^{1/(i-1)} \binom{k}{2}^{-i/(i-1)} = \frac{2(1-\theta)\binom{k}{2} t^2 n}{(1+t)^{k-1} + (1-t)^{k-1}}\, t^{-2} \binom{k}{2}^{-1}$$

$$=\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}} \tag{9-62a}$$

and

$$\mu_k=\left[\binom{k}{2}m_2^{-1}\right]^{\frac{k-2i}{2(i-1)}}\left[m_{2i}\binom{k}{2i}^{-1}\right]^{\frac{k-2}{2(i-1)}}=m_2\left(\frac{m_{2i}}{m_2}\right)^{\frac{k-2}{2(i-1)}}\binom{k}{2}^{\frac{k-2i}{2(i-1)}}\binom{k}{2i}^{-\frac{k-2}{2(i-1)}}$$

$$=m_2\binom{k}{2i}^{\frac{k-2}{2(i-1)}}\binom{k}{2}^{-\frac{k-2}{2(i-1)}}t^{k-2}\binom{k}{2}^{\frac{k-2i}{2(i-1)}}\binom{k}{2i}^{-\frac{k-2}{2(i-1)}}$$

$$=\frac{2(1-\theta)\binom{k}{2}t^2n}{(1+t)^{k-1}+(1-t)^{k-1}}t^{k-2}\binom{k}{2}^{-1}=\frac{2(1-\theta)t^kn}{(1+t)^{k-1}+(1-t)^{k-1}} \tag{9-62b}$$

C: Determination of $\log M_{n,\theta}^k$

After obtaining μ_0 and μ_k, we now return to (9-40) in evaluation of $\log M_{n,\theta}^k$, whose computation consists of the two steps as follows

Step 1: Computation of $-\mu_0\log\mu_0$ and $-\mu_k\log\mu_k$ based on (9-62a) and (9-62b)

$$-\mu_0\log\mu_0=-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\left[\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}+\log n\right]$$

$$=-\frac{2(1-\theta)n\log n}{(1+t)^{k-1}+(1-t)^{k-1}}-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}} \tag{9-63a}$$

$$-\mu_k\log\mu_k=-\frac{2(1-\theta)t^kn}{(1+t)^{k-1}+(1-t)^{k-1}}\left[\log\frac{2(1-\theta)t^k}{(1+t)^{k-1}+(1-t)^{k-1}}+\log n\right]$$

$$=-\frac{2(1-\theta)t^k n\log n}{(1+t)^{k-1}+(1-t)^{k-1}}-\frac{2(1-\theta)t^k n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)t^k}{(1+t)^{k-1}+(1-t)^{k-1}} \tag{9-63b}$$

Step 2: Computation of $-\sum_{i=1}^{k/2-1} m_{2i}\log m_{2i}$ and $\sum_{i=1}^{k/2-1} m_{2i}\log\binom{k}{2i}$ based on (9-61)

$$-m_2\log m_2 - m_2\log m_2 - m_6\log m_6 - \ldots\ldots - m_{k-2}\log m_{k-2}$$

$$=-\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}} \tag{9-64a}$$

$$m_2\log\binom{k}{2}+m_4\log\binom{k}{4}+m_2\log\binom{k}{2}+\ldots\ldots+m_{k-2}\log\binom{k}{k-2}$$

$$=\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\binom{k}{2i} \tag{9-64b}$$

□

According to (9-63) and (9-64), we obtain $\log M_{n,\theta}^k$ as

$$\log M_{n,\theta}^k \sim -n\log n-\frac{2(1-\theta)n\log n}{(1+t)^{k-1}+(1-t)^{k-1}}-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$-\frac{2(1-\theta)t^k n\log n}{(1+t)^{k-1}+(1-t)^{k-1}}-\frac{2(1-\theta)t^k n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)t^k}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$-\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$+\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\binom{k}{2i} \tag{9-65}$$

We will first compute the coefficient at $n\log n$ in (9-65) because it may dominate $\log M_{n,\theta}^{k}$ as $n\to+\infty$.

$$1-\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}-\frac{2(1-\theta)t^k}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$+\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}\left(1+t^k-\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}\right)$$

$$=1-\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}$$

$$=1-\frac{(1-\theta)\left((1+t)^k+(1-t)^k\right)}{(1+t)^{k-1}+(1-t)^{k-1}}=0 \tag{9-66}$$

The last expression in (9-66) is due to (9-55c). We now turn to the remaining terms in the two summations in (9-65)

$$-\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)\binom{k}{2i}t^{2i}}{(1+t)^{k-1}+(1-t)^{k-1}}+\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\binom{k}{2i}$$

$$=-\sum_{i=1}^{k/2-1}\frac{2(1-\theta)\binom{k}{2i}t^{2i}n}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)t^{2i}}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$=-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=1}^{k/2-1}\binom{k}{2i}t^{2i}\log\frac{2(1-\theta)t^{2i}}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$=-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=1}^{k/2-1}\binom{k}{2i}t^{2i}\left(\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}+2i\log t\right) \tag{9-67}$$

Notice the following equalities

$$1+t^k+\sum_{i=1}^{k/2-1}\binom{k}{2i}=\sum_{i=0}^{k/2}\binom{k}{2i} \tag{9-68a}$$

$$2i\binom{k}{2i}=2i\frac{k!}{(2i)!(k-2i)!}$$

$$=k\,\frac{(k-1)!}{(2i-1)!(k-2i)!}=k\binom{k-1}{2i-1} \tag{9-68b}$$

Then, by using (9-67) and (9-68), $\log M_{n,\theta}^{k}$ can be further equivalent to

$$\log M_{n,\theta}^{k}\sim-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=1}^{k/2-1}2i\binom{k}{2i}t^{2i}\log t$$

$$=-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}$$

$$-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}(k\log t)\sum_{i=1}^{k/2-1}\binom{k-1}{2i-1}t^{2i}$$

$$=-\frac{2(1-\theta)n}{(1+t)^{k-1}+(1-t)^{k-1}}\begin{pmatrix}\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}\\ +(k\log t)\sum_{i=1}^{k/2-1}\binom{k-1}{2i-1}t^{2i}\end{pmatrix} \tag{9-69}$$

Note that

$$\sum_{i=0}^{k/2}\binom{k}{2i}t^{2i}=\frac{(1+t)^{k}+(1-t)^{k}}{2} \tag{9-70a}$$

$$\sum_{i=1}^{k/2}\binom{k-1}{2i-1}t^{2i}=t\frac{(1+t)^{k-1}-(1-t)^{k-1}}{2} \tag{9-70b}$$

Thus, $\log M_{n,\theta}^{k}$ in (9-69) can be further expressed as

$$\log M_{n,\theta}^{k}\sim-(1-\theta)n\begin{bmatrix}\frac{(1+t)^{k}+(1-t)^{k}}{(1+t)^{k-1}+(1-t)^{k-1}}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}\\ +\frac{(1+t)^{k-1}-(1-t)^{k-1}}{(1+t)^{k-1}+(1-t)^{k-1}}kt\log t\end{bmatrix}$$

$$=-(1-\theta)n\left[\frac{1}{1-\theta}\log\frac{2(1-\theta)}{(1+t)^{k-1}+(1-t)^{k-1}}+\frac{\theta}{1-\theta}k\log t\right]$$

$$=n\left[\log\frac{(1+t)^{k-1}+(1-t)^{k-1}}{2(1-\theta)t^{k\theta}}\right] \tag{9-71}$$

where the last two expressions in right-side of (9-71) is by using (9-55c). Again, by (9-55c) equivalently

$$\frac{(1+t)^{k-1}+(1-t)^{k-1}}{1-\theta}=(1+t)^{k}+(1-t)^{k} \tag{9-72}$$

Finally, we have

$$\log M_{n,\theta}^{k} \sim n\log\frac{(1+t)^{k}+(1-t)^{k}}{2t^{k\theta}} \tag{9-73}$$

According to the Lemma 9.4 and (9-32), we have

$$\log P_{n,\theta}^{k,\alpha} \sim \log P_{n\alpha,\theta}^{k} \tag{9-74a}$$

and

$$P_{n\alpha,\theta}^{k} \overset{\log}{\sim} \exp\left(-knH(\theta)\right)M_{n\alpha,\theta}^{k} \tag{9-74b}$$

Therefore, it results in

$$\begin{aligned}\log P_{n,\theta}^{k,\alpha} &\sim \ln\left[\exp\left(-kn\alpha H(\theta)\right)\right]+\log M_{n\alpha,\theta}^{k}\\ &= -kn\alpha H(\theta)+n\alpha\log\frac{(1+t)^{k}+(1-t)^{k}}{2t^{k\theta}}\end{aligned} \tag{9-75}$$

Averaging by n, we conclude

$$\begin{aligned}\frac{1}{n}\log P_{n,\theta}^{k,\alpha} &\sim -k\alpha H(\theta)+\alpha\log\frac{(1+t)^{k}+(1-t)^{k}}{2t^{k\theta}}\\ &= \alpha\log\left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{k}+\alpha\log\frac{(1+t)^{k}+(1-t)^{k}}{2t^{k\theta}}\\ &= \alpha\log\left(\frac{(1+t)^{k}+(1-t)^{k}}{2t^{k\theta}}\left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{k}\right)\end{aligned} \tag{9-76}$$

Thus, we prove the conclusion (9-23a) for k even.

2) The Case of Odd k:

For the similar analysis and process, we can show the similar conclusion for k odd, i.e.,

$$\lim_{n\to+\infty}\frac{1}{n}\log P_{n,\theta}^{k,\alpha}=\begin{cases}\alpha\log\left(\dfrac{(1+t)^k+(1-t)^k}{2t^{\theta k}}\left((1-\theta)^{1-\theta}\theta^{\theta}\right)^k\right) & if\ 0<\theta\le\dfrac{k-1}{k}\\ -\infty & otherwise\end{cases}\tag{9-77}$$

For more details of the derivations, such as the study of the equation (9-22) in relation to the associated unique root, and of $P_{n,\theta}^{k,\alpha}$ as a function of θ, etc., the interested reader can further refer to [72]. Finally, we prove the entire theorem. □

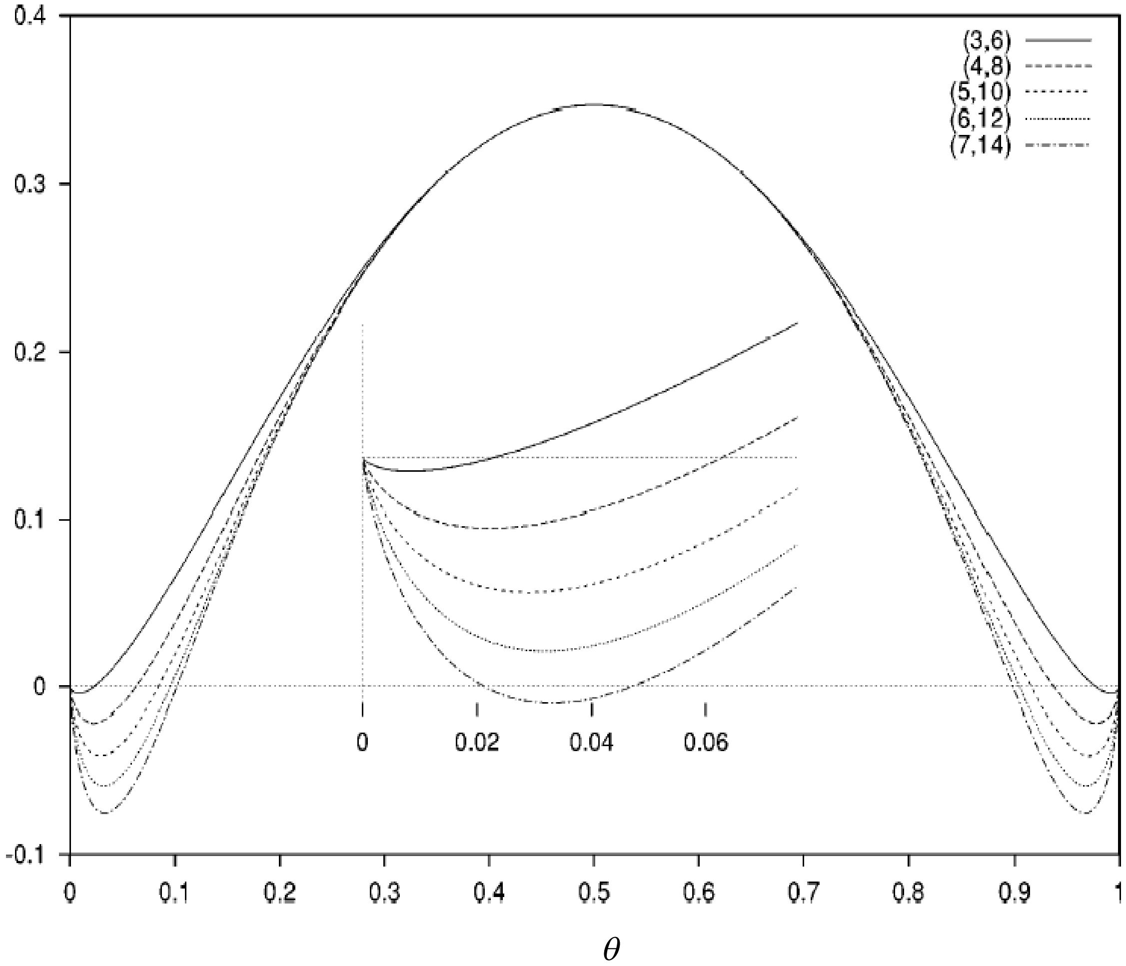

Fig. 9.1 Asymptotic normalized distance distributions for various (d_v, d_c) rate-1/2 regular LDPC codes.

The normalized distance distributions are simulated by (9-1b), (9-13), (9-23) and (9-24) and the results [72] are shown in Fig. 9.1 for various (d_v, d_c) rate-1/2 regular LDPC codes, where the horizontal axis represents the normalized distance θ associated with Hamming distance θn, and the vertical axis stands for the asymptotic ratio of the number of codewords of distance θ to the number of all codewords, respectively. Statistically, we can clearly observe that the largest proportion of codewords has the normalized distance around 0.5.

9.4 Average Distance Distribution of Irregular LDPC Codes Ensemble

So far we have derived the formula of average distance distribution of regular LDPC codes ensemble, now in this section we will focus the problem of estimation of the average distance distribution in ensembles of irregular LDPC codes, which perform better in terms of BER than the regular LDPC codes under iterative decoding [6][7]. An explicit formula describing the average distance distributions, which can be regarded as an extension to those of regular LDPC codes, is derived theoretically in the subsequent part of this section.

9.4.1 Preliminaries:

The following necessary preliminaries are ready to derive the formula of average distance distribution of irregular LDPC codes ensemble. We consider ensemble $C_{m,n}$ of irregular LDPC codes associated with ensemble $H_{m,n}$ of parity-check matrices. The codes are defined by $m \times n$ matrices from $H_{m,n}$, and thus have rate R at least $1-m/n$. Let h and g

be nonnegative integers independent of *n*, and the proportional ratios v_1, v_2, ……, v_g, η_1, η_2, ……., η_h satisfy the following conditions

$$\sum_{i=1}^{g} v_i = 1 \qquad \varepsilon_1,\ \varepsilon_2,\ \ldots\ldots,\ \varepsilon_g \in (0,1] \tag{9-78a}$$

$$\sum_{i=1}^{h} \eta_i = 1 \qquad \eta_1,\ \eta_2,\ \ldots\ldots,\ \eta_g \in (0,1] \tag{9-78b}$$

where (9-78a) and (9-78b) corresponds to the rows and columns of the parity-check matrix $H_{m,n}$, respectively. In addition, we assume that the numbers $\varepsilon_i m$ for i=1, 2, …, g and $\eta_j n$ for j=1, 2, …, h are integers. Let the following serial of intervals of *integer* numbers be defined:

$$I_i = \left[1 + \sum_{l=1}^{i-1} \varepsilon_l m,\quad \sum_{l=1}^{i} \varepsilon_l m\right] \qquad i = 1,\ 2,\ \ldots,\ g \tag{9-79a}$$

$$J_j = \left[1 + \sum_{l=1}^{j-1} \eta_l n,\quad \sum_{l=1}^{j} \eta_l n\right] \qquad j = 1,\ 2,\ \ldots,\ h \tag{9-79b}$$

Actually, an integer number $k_1 \in I_i$ means that the k_1th row of an $m \times n$ parity-check (0, 1)-matrix H ($H = [h_{i,j}] \in H_{m,n}$) belongs to the ith horizontal block of H, while an integer number $k_2 \in J_j$ implies that the k_2th column of H belongs to the jth vertical block of H. The horizontal and vertical sub-blocks are defined such that for every $i \in I_l$

$$\sum_{j=1}^{n} h_{i,j} = r_l \tag{9-80a}$$

and for every $j \in J_l$

$$\sum_{i=1}^{m} h_{i,j} = s_l \tag{9-80b}$$

which means that the number of ones is the same for each row and column in the same horizontal and vertical sub-block. In other words, we partition the rows of the matrix H into g strips, where the ith strip has $\varepsilon_i m$ rows. Also, we partition all the columns into h strips, where the jth strip has $\eta_j n$ column. A matrix belongs to the defined ensemble $H_{m,n}$ of parity-check matrices if the row sums of the rows belonging to the l th (horizontal) strip are r_l for l=1, 2, …, g, and the column sums of the columns belonging to the lth (vertical) strip are s_l for l=1, 2, …, h.

Note that the above definition is a generalized form for regular LDPC codes, whose parity-check matrices can be partitioned into one horizontal and vertical strip since all row sums and columns sums are equal for all the rows and for all the columns.

Let θ be a number in the interval (0, 1). Given a matrix $H \in H_{m,n}$, similar as for regular LDPC codes descried in the last subsection, we may find the submatrices consisting of $\omega = \theta n$ columns of the matrix H having even row sums. Obviously, it is equal to the number of codewords of weight θn in the code defined by the matrix H. Let us define a class of matrices $H_{m,n,\theta} \subset H_{m,n}$ made up of all matrices such that the sum of the first θn entries in each row is even. This means that the vector given by (9-12) is a valid codeword of weight ω. We will aim at determining the average of this number in the ensemble $C_{m,n}$ in the sequel, namely the calculation of b_θ

$$b_\theta := \frac{1}{n}\log P \tag{9-81a}$$

where P is the ratio defined as

$$P = \frac{\left|H_{m,n,\theta}\right|}{\left|H_{m,n}\right|} \tag{9-81b}$$

We start with the following lemma that allows to estimate the number of matrices with given row and column profiles.

Lemma 9.5: Let M_n be an ensemble of $m \times n$ binary matrices

$$m = \gamma n \qquad 0 < \gamma \leq 1 \tag{9-82}$$

such that their row sums $r^{(i)}$, $i=1, 2,\ldots, m$, and column sums $s^{(j)}$, $j=1, 2,\ldots, n$, satisfy

(1) All $r^{(i)}$'s and $s^{(j)}$'s are bounded, namely there exists a positive constant c bounding from above all the row and column sums.

(2) All column sums are strictly positive. Then

$$\left|M_n\right| \overset{\log}{\sim} \frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}!\right)\left(\prod_{j=1}^{n} s^{(j)}!\right)} \tag{9-83}$$

where $a_n \overset{\log}{\sim} b_n$ means that $\lim_{n\to+\infty} \log a_n / \log b_n = 1$.

Proof: According to a modification for rectangular matrices of the Lemma 9.2 due to the O'Neil's principle [70]

$$\left|M_n\right| \sim \frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}!\right)\left(\prod_{j=1}^{n} s^{(j)}!\right)}$$

$$\times \exp\left(\frac{-1}{2\left(\sum_{i=1}^{m} r^{(i)}\right)^2} \left(\sum_{i=1}^{m} r^{(i)}\left(r^{(i)}-1\right)\right)\left(\sum_{j=1}^{n} s^{(j)}\left(s^{(j)}-1\right)\right) \right) \tag{9-84}$$

By the conditions of the Lemma

$$\sum_{i=1}^{m} r^{(i)} = \sum_{j=1}^{n} s^{(i)} \geq n \tag{9-85}$$

and by (9-82)

$$\frac{-1}{2\left(\sum_{i=1}^{m} r^{(i)}\right)^2} \left(\sum_{i=1}^{m} r^{(i)}\left(r^{(i)}-1\right)\right)\left(\sum_{j=1}^{n} s^{(j)}\left(s^{(j)}-1\right)\right)$$

$$\leq \frac{1}{2n^2} mc^2 \, nc^2 = \frac{\gamma c^4}{2} = c_1 \tag{9-86}$$

Clearly, the exponent in (9-84) is between two constants, e^{-c_1} and 1. It is easy to follow by (9-85) that

$$\frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}!\right)\left(\prod_{j=1}^{n} s^{(j)}!\right)} \geq \frac{n!}{c!^{(1+\gamma)n}} \to +\infty \tag{9-87}$$

Hence, $|M_n|$ tends to infinity with n, and (9-83) is proved. □

9.4.2 Counting the Number of Matrices from $H_{m,n,\theta}$ for the Horizontal Blocks of Equal Sizes

In this part we only deal with a special case for the ensemble $H_{m,n}$ where $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_g = 1/g$. The following identity holds for the nonzero entries in the matrix $H_{m,n}$, which is counted by the row and column prospects.

$$m\sum_{i=1}^{g}\frac{r_i}{g}=\frac{m}{g}\sum_{i=1}^{g}r_i=n\sum_{j=1}^{h}\eta_j s_j \tag{9-88}$$

Let us choose a matrix $A\in H_{n,m,\theta}$ that implies that the sum of the first $\omega=\theta n$ entry in each row is even. The matrix A is comprised of g horizontal blocks and h vertical blocks, such that the row sums in each horizontal block and column sums in each vertical block are the same for all the rows and for all the columns. Given the parameter θ we can specify q such that

$$\theta\in[\eta_1+\eta_2+\ldots..+\eta_{q-1},\ \eta_1+\eta_2+\ldots..+\eta_{q-1}+\eta_q) \tag{9-89}$$

Note that the left sum is set to be zero in case of $q=1$. Suppose that among the rows of the ith horizontal block there are $m_v^{(i)}$ rows such that the even sum of their first ω entries equals v, $v=0, 2, \ldots, r_i-\pi(r_i)$, π is the parity function

$$\pi(a)=\begin{cases}0 & a\ \ is\ \ even\\ 1 & otherwise.\end{cases} \tag{9-90}$$

The matrix A is thus portioned into two parts, matrices A^{left} and A^{right}, containing correspondingly the first ω and the last $n-\omega$ columns of A. The probability (or ratio) that an arbitrary matrix taken from the ensemble $H_{n,m}$ belongs to $H_{m,n,\theta}$ is

$$P=\frac{\sum\left(\prod_{i=1}^{g}\binom{m/g}{m_0^{(i)},m_2^{(i)},\ldots..,m_{r_i-\pi(r_i)}^{(i)}}\left|\Lambda^{left}\right|\left|\Lambda^{right}\right|\right)}{\left|H_{m,n,\theta}\right|} \tag{9-91}$$

where Λ^{left} and Λ^{right} represent the ensembles of all possible matrices A^{left} and A^{right}, respectively, and the summation is over all $m_v^{(i)}$, $v=0, 2, \ldots, r_i-\pi(r_i)$, $i=1, 2, \ldots, g$, satisfying the following requirements

$$\sum_{v=0,2,\ldots,r_i-\pi(r_i)} m_v^{(i)} = \frac{m}{g} \qquad i=1, 2, \ldots, g \tag{9-92a}$$

$$\sum_{i=1}^{g} \sum_{v=2,\ldots,r_i-\pi(r_i)} v m_v^{(i)} = n(s_1\eta_1 + s_2\eta_2 + \ldots + s_{q-1}\eta_{q-1} + s_q(\theta-\eta_1-\eta_2-\ldots-\eta_{q-1})) \tag{9-92b}$$

By (9-83) we can calculate quantities of $\left|\Lambda^{left}\right|$, $\left|\Lambda^{right}\right|$ and $\left|H_{m,n}\right|$ as follows:

$$\left|\Lambda^{left}\right| \overset{\log}{\sim} \frac{\left(n\left(s_1\eta_1+s_2\eta_2+\ldots+s_{q-1}\eta_{q-1}+s_q\left(\theta-\eta_1-\eta_2-\ldots-\eta_{q-1}\right)\right)\right)!}{\prod_{i=1}^{g}\left(2!^{m_2^{(i)}}\,4!^{m_4^{(i)}}\ldots\left(r_i-\pi\left(r_i\right)\right)!^{m_{r_i-\pi(r_i)}^{(i)}}\right)} \times\left(s_1!^{\eta_1 n}\,s_2!^{\eta_2 n}\ldots s_{q-1}!^{\eta_{q-1} n}\,s_q!^{\left(\theta-\eta_1-\eta_2-\ldots-\eta_{q-1}\right)n}\right)^{-1} \tag{9-93a}$$

$$\left|\Lambda^{right}\right| \overset{\log}{\sim} \frac{\left(n\left(s_q\left(\eta_1+\eta_2+\ldots+\eta_q-\theta\right)+s_{q+1}\eta_{q+1}+s_{q+2}\eta_{q+2}+\ldots+s_h\eta_q\right)\right)!}{\prod_{i=1}^{g}\left(\left(r_i\right)!^{m_0^{(i)}}\left(r_i-2\right)!^{m_2^{(i)}}\ldots\left(\pi\left(r_i\right)\right)!^{m_{r_i-\pi(r_i)}^{(i)}}\right)} \times\left(s_q!^{\left(\eta_1+\eta_2+\ldots+\eta_q-\theta\right)n}\,s_{q+1}!^{\eta_{q+1}n}\,s_{q+2}!^{\eta_{q+2}n}\ldots s_h!^{\eta_h n}\right)^{-1}$$

$$= \frac{\left(n\left(s_q\left(\eta_1+\eta_2+\ldots+\eta_q-\theta\right)+s_{q+1}\eta_{q+1}+s_{q+2}\eta_{q+2}+\ldots+s_h\eta_q\right)\right)!}{\prod_{i=1}^{g}\left(\left(r_i\right)!^{m_0^{(i)}}\left(r_i-2\right)!^{m_2^{(i)}}\ldots\left(2+\pi\left(r_i\right)\right)!^{m_{r_i-\pi(r_i)-2}^{(i)}}\right)}$$

$$\times\left(s_q!^{(\eta_1+\eta_2+.....+\eta_q-\theta)n}\; s_{q+1}!^{\eta_{q+1}n}\; s_{q+2}!^{\eta_{q+2}n}\;....s_h!^{\eta_h n}\right)^{-1} \tag{9-93b}$$

$$\left|H_{m,n}\right| \overset{\log}{\sim} \frac{\left(n\left(s_1\eta_1+s_2\eta_2+.....+s_h\eta_h\right)\right)!}{s_1!^{\eta_1 n}\, s_2!^{\eta_2 n}\,.....s_h!^{\eta_h n}\, r_1!^{m/g}\, r_2!^{m/g}\,.....r_g!^{m/g}} \tag{9-93c}$$

To further evaluate the expression (9-91), we should first calculate the following preliminary results

$$\left(n\left(s_1\eta_1+s_2\eta_2+....+s_{q-1}\eta_{q-1}+s_q\left(\theta-\eta_1-\eta_2-....-\eta_{q-1}\right)\right)\right)!$$

$$\times\left(n\left(s_q\left(\eta_1+\eta_2+....+\eta_q-\theta\right)+s_{q+1}\eta_{q+1}+s_{q+2}\eta_{q+2}+....+s_h\eta_q\right)\right)!$$

$$\times\left(\left(n\left(s_1\eta_1+s_2\eta_2+.....+s_h\eta_h\right)\right)!\right)^{-1}$$

$$=\binom{n\left(s_1\eta_1+s_2\eta_2+.....+s_h\eta_h\right)}{n\left(s_1\eta_1+s_2\eta_2+....+s_{q-1}\eta_{q-1}+s_q\left(\theta-\eta_1-\eta_2-....-\eta_{q-1}\right)\right)}^{-1} \tag{9-94a}$$

$$\prod_{i=1}^{g}\left(2!^{m_2^{(i)}}\;4!^{m_4^{(i)}}\;....\left(r_i-\pi\left(r_i\right)\right)!^{m^{(i)}_{r_i-\pi(r_i)}}\right)\prod_{i=1}^{g}\left(\left(r_i\right)!^{m_0^{(i)}}\left(r_i-2\right)!^{m_2^{(i)}}\;....\left(2+\pi\left(r_i\right)\right)!^{m^{(i)}_{r_i-\pi(r_i)-2}}\right)$$

$$=\prod_{i=1}^{g}\left(0!^{m_0^{(i)}}\;2!^{m_2^{(i)}}\;4!^{m_4^{(i)}}\;....\left(r_i-\pi\left(r_i\right)\right)!^{m^{(i)}_{r_i-\pi(r_i)}}\right)$$

$$\times\prod_{i=1}^{g}\left(\left(r_i\right)!^{m_0^{(i)}}\left(r_i-2\right)!^{m_2^{(i)}}\;.....\left(2+\pi\left(r_i\right)\right)!^{m^{(i)}_{r_i-\pi(r_i)-2}}\left(\pi\left(r_i\right)\right)!^{m^{(i)}_{r_i-\pi(r_i)}}\right)$$

$$=\prod_{i=1}^{g}\left(\left(0!\left(r_i\right)!\right)^{m_0^{(i)}}\left(2!\left(r_i-2\right)!\right)^{m_2^{(i)}}\;.....\left(\left(r_i-\pi\left(r_i\right)\right)!\,\pi\left(r_i\right)!\right)^{m^{(i)}_{r_i-\pi(r_i)}}\right) \tag{9-94b}$$

$$\left(s_1!^{\eta_1 n}\, s_2!^{\eta_2 n}\,....s_{q-1}!^{\eta_{q-1}n}\, s_q!^{(\theta-\eta_1-\eta_2-...-\eta_{q-1})n}\right)\left(s_q!^{(\eta_1+\eta_2+.....+\eta_q-\theta)n}\, s_{q+1}!^{\eta_{q+1}n}\, s_{q+2}!^{\eta_{q+2}n}\,....s_h!^{\eta_h n}\right)$$

$$= s_1!^{\eta_1 n} s_2!^{\eta_2 n} \ldots s_q!^{\eta_q n} \ldots s_h!^{\eta_h n} = \prod_{i=1}^{g} r_i!^{m_0^{(i)}+m_2^{(i)}+\ldots\ldots+m_{r_i-\pi(r_i)}^{(i)}} \tag{9-94c}$$

By (9-94a), (9-94b) and (9-94c), (9-91) can be further expressed as

$$P \overset{\log}{\sim} \begin{pmatrix} n\left(s_1\eta_1 + s_2\eta_2 + \ldots\ldots + s_h\eta_h\right) \\ n\left(s_1\eta_1 + s_2\eta_2 + \ldots. + s_{q-1}\eta_{q-1} + s_q\left(\theta - \eta_1 - \eta_2 - \ldots. - \eta_{q-1}\right)\right) \end{pmatrix}^{-1}$$

$$\times \sum \prod_{i=1}^{g} \begin{pmatrix} m/g \\ m_0^{(i)}, m_2^{(i)}, \ldots\ldots, m_{r_i-\pi(r_i)}^{(i)} \end{pmatrix} \begin{pmatrix} r_i \\ 2 \end{pmatrix}^{m_2^{(i)}} \begin{pmatrix} r_i \\ 4 \end{pmatrix}^{m_4^{(i)}} \ldots\ldots \begin{pmatrix} r_i \\ r_i - \pi(r_i) \end{pmatrix}^{m_{r_i-\pi(r_i)}^{(i)}} \tag{9-95}$$

where the summation is taken over all numbers $m_v^{(i)}$ that satisfy the requirements by (9-92a) and (9-92b).

Let us define

$$s = \eta_1 s_1 + \eta_1 s_1 + \ldots\ldots + \eta_h s_h \tag{9-96a}$$

$$\overline{\theta} = \frac{s_1\eta_1 + s_2\eta_2 + \ldots. + s_{q-1}\eta_{q-1} + s_q\left(\theta - \eta_1 - \eta_2 - \ldots. - \eta_{q-1}\right)}{s} \tag{9-96b}$$

Evidently, $0 < \tilde{\theta} < 1$, and furthermore (9-95) can be simply formulated as

$$P \overset{\log}{\sim} \begin{pmatrix} ns \\ n\overline{\theta}s \end{pmatrix}^{-1} \Sigma_0 \tag{9-97a}$$

where Σ_0 is the sum from (9-95)

$$\Sigma_0 = \sum \prod_{i=1}^{g} \begin{pmatrix} m/g \\ m_0^{(i)}, m_2^{(i)}, \ldots\ldots, m_{r_i-\pi(r_i)}^{(i)} \end{pmatrix} \begin{pmatrix} r_i \\ 2 \end{pmatrix}^{m_2^{(i)}} \begin{pmatrix} r_i \\ 4 \end{pmatrix}^{m_4^{(i)}} \ldots\ldots \begin{pmatrix} r_i \\ r_i - \pi(r_i) \end{pmatrix}^{m_{r_i-\pi(r_i)}^{(i)}} \tag{9-97b}$$

Hence, by using in the new notations (9-96a) and (9-96b) the constraints (9-92a) and (9-92b) can be rewritten as follows

$$\sum_{v=0,2,\ldots,r_i-\pi(r_i)} m_v^{(i)} = \frac{m}{g} \qquad i=1,\, 2,\, \ldots,\, g \tag{9-98a}$$

$$\sum_{i=1}^{g} \sum_{v=2,\ldots,r_i-\pi(r_i)} v m_v^{(i)} = ns\overline{\theta} \tag{9-98b}$$

By (9-98a)

$$m_{r_i-\pi(r_i)}^{(i)} = \frac{m}{g} - \sum_{v=0,2,\ldots,r_i-\pi(r_i)-2} m_v^{(i)} \tag{9-99}$$

Thus, (9-98b) yields the following expression

$$\sum_{i=1}^{g}\left(\sum_{v=2,\ldots,r_i-\pi(r_i)-2} v m_v^{(i)} + \left(r_i-\pi\left(r_i\right)\right)\left(\frac{m}{g} - \sum_{v=0,2,\ldots,r_i-\pi(r_i)-2} m_v^{(i)}\right)\right) = ns\tilde{\theta} \tag{9-100}$$

Form (9-100) we get

$$\sum_{i=1}^{g} m_0^{(i)}\left(r_i-\pi\left(r_i\right)\right) = -\sum_{i=1}^{g} \sum_{v=2,4,\ldots,r_i-\pi(r_i)-2} \left(\left(r_i-\pi\left(r_i\right)\right)-v\right) m_v^{(i)}$$

$$+\frac{m}{g}\sum_{i=1}^{g}\left(r_i-\pi\left(r_i\right)\right) - ns\tilde{\theta} \tag{9-101}$$

Since

$$\sum_{i=1}^{g} m_0^{(i)}\left(r_i-\pi\left(r_i\right)\right) \geq 0 \tag{9-102}$$

It is necessary that

$$\frac{m}{g}\sum_{i=1}^{g}\left(r_i-\pi\left(r_i\right)\right) \geq ns\tilde{\theta} \tag{9-103}$$

namely,

$$\overline{\theta} \leq \frac{m}{n} \frac{1}{sg} \sum_{i=1}^{g} \left(r_i - \pi\left(r_i\right) \right) = \frac{\gamma}{sg} \sum_{i=1}^{g} \left(r_i - \pi\left(r_i\right) \right) \tag{9-104}$$

By (9-88) and (9-96a), it yields

$$s = \frac{\gamma}{g} \sum_{i=1}^{g} r_i \tag{9-105}$$

Substituting s in (9-104) by (9-105), we get

$$\overline{\theta} \leq 1 - \frac{\sum_{i=1}^{g} \pi\left(r_i\right)}{\sum_{i=1}^{g} r_i} \tag{9-106}$$

Analogous with the case of regular LDPC codes, we denote

$$M_{n,\theta} = \mathbf{max} \left(\prod_{i=1}^{g} \binom{m/g}{m_0^{(i)}, m_2^{(i)}, \ldots\ldots, m_{r_i-\pi(r_i)}^{(i)}} \binom{r_i}{2}^{m_2^{(i)}} \binom{r_i}{4}^{m_4^{(i)}} \cdots\cdots \binom{r_i}{r_i - \pi\left(r_i\right)}^{m_{r_i-\pi(r_i)}^{(i)}} \right) \tag{9-107}$$

where the maximum is taken over all $m_v^{(i)}$ satisfying (9-98a) and (9-98b). By (9-36) and taking into account that Σ_0 in (9-97a) consists of polynomial summands in n, thus it yields

$$P \overset{\log}{\sim} \exp\left(-nsH\left(\overline{\theta}\right)\right) M_{n,\theta} \tag{9-108}$$

where the asymptotics for $M_{n,\theta}$ will be computed under the constraints (9-98a) and (9-98b).

9.4.3 The Basic System of Equations for Computing the Asymptotics of $M_{n,\theta}$

By (9-98a) we have

$$m_0^{(i)} = \frac{m}{g} - m_2^{(i)} - m_4^{(i)} - \ldots\ldots - m_{r_i-\pi(r_i)}^{(i)} \qquad i=1,\, 2,\, \ldots,\, g-1 \tag{9-109a}$$

and

$$m_0^{(g)} + m_{r_g-\pi(r_g)}^{(g)} = \frac{m}{g} - m_2^{(g)} - m_4^{(g)} - \ldots\ldots - m_{r_g-\pi(r_g)-2}^{(g)} \tag{9-109b}$$

By (9-98b)

$$\begin{aligned}\left(r_g - \pi(r_g)\right) m_{r_g-\pi(r_g)}^{(g)} &= ns\bar{\theta} - \left(\sum_{j=1}^{g-1} \sum_{v=2,\ldots,r_j-\pi(r_j)} v m_v^{(j)}\right) - 2m_2^{(g)} - 4m_4^{(g)} \\ &\quad - \ldots\ldots - \left(r_g - \pi(r_g) - 2\right) m_{r_g-\pi(r_g)-2}^{(g)}\end{aligned} \tag{9-110}$$

We always assume in the sequel that $r_g \geq 2$, and hence $r_g > \pi(r_g)$. By (9-110) we obtain

$$\begin{aligned}k_g := m_{r_g-\pi(r_g)}^{(g)} &= \frac{ns\bar{\theta}}{r_g - \pi(r_g)} - \frac{1}{r_g - \pi(r_g)}\left(\sum_{j=1}^{g-1} \sum_{v=2,\ldots,r_j-\pi(r_j)} v m_v^{(j)}\right) \\ &\quad - \frac{2}{r_g - \pi(r_g)} m_2^{(g)} - \frac{4}{r_g - \pi(r_g)} m_4^{(g)} - \ldots\ldots - \frac{r_g - \pi(r_g) - 2}{r_g - \pi(r_g)} m_{r_g-\pi(r_g)-2}^{(g)}\end{aligned} \tag{9-111a}$$

By (9-109b) and (9-110)

$$\begin{aligned}\bar{k}_0^{(g)} := m_0^{(g)} &= \frac{m}{g} - m_{r_g-\pi(r_g)}^{(g)} - m_2^{(g)} - m_4^{(g)} - \ldots\ldots - m_{r_g-\pi(r_g)-2}^{(g)} \\ &= \frac{m}{g} - \frac{ns\bar{\theta}}{r_g - \pi(r_g)} + \frac{1}{r_g - \pi(r_g)}\left(\sum_{j=1}^{g-1} \sum_{v=2,\ldots,r_j-\pi(r_j)} v m_v^{(j)}\right) + \left(\frac{2}{r_g - \pi(r_g)} - 1\right) m_2^{(g)} \\ &\quad + \left(\frac{4}{r_g - \pi(r_g)} - 1\right) m_4^{(g)} + \ldots\ldots + \left(\frac{r_g - \pi(r_g) - 2}{r_g - \pi(r_g)} - 1\right) m_{r_g-\pi(r_g)-2}^{(g)}\end{aligned} \tag{9-111b}$$

and by (9-109a) we denote

$$k_0^{(i)} := m_0^{(i)} = \frac{m}{g} - m_2^{(i)} - m_4^{(i)} - \ldots\ldots - m_{r_i-\pi(r_i)}^{(i)} \qquad i=1,\, 2,\, \ldots,\, g-1 \tag{9-111c}$$

Based on the above expressions k_g, $\overline{k}_0^{(g)}$ and $k_0^{(i)}$ are independent of v. Using the phenomenal Stirling approximation (9-39b) and the constraint (9-98a), (9-107) can be further expressed in its logarithmic form as

$$\log M_{n,\theta} = \mathbf{max}\left\{\sum_{i=1}^{g}\log\left[\binom{m/g}{m_0^{(i)}, m_2^{(i)}, \ldots\ldots, m_{r_i-\pi(r_i)}^{(i)}}\binom{r_i}{2}^{m_2^{(i)}}\binom{r_i}{4}^{m_4^{(i)}}\ldots\ldots\binom{r_i}{r_i-\pi(r_i)}^{m_{r_i-\pi(r_i)}^{(i)}}\right]\right\}$$

$$\sim \max\left\{\sum_{i=1}^{g}\left[\left(\frac{m}{g}\log\left(\frac{m}{g}\right)-\frac{m}{g}\right)-\sum_{v=0,2,\ldots,r_i-\pi(r_i)}\left(m_v^{(i)}\log m_v^{(i)} - m_v^{(i)}\right)+\sum_{v=2,4,\ldots,r_i-\pi(r_i)} m_v^{(i)}\log\binom{r_i}{v}\right]\right\}$$

$$= \max\left\{\sum_{i=1}^{g}\left[\frac{m}{g}\log\left(\frac{m}{g}\right)-\sum_{v=0,2,\ldots,r_i-\pi(r_i)} m_v^{(i)}\log m_v^{(i)} + \sum_{v=2,4,\ldots,r_i-\pi(r_i)} m_v^{(i)}\log\binom{r_i}{v}\right]\right\} \tag{9-112}$$

In order to find the maximum asymptotical value of $\log M_{n,\theta}$, equating to zero the partial derivatives with respect to $m_{2v}^{(i)}$, $v=1,\, 2, \ldots,\, (r_i-\pi(r_i))/2$, $i=1,\, 2, \ldots,\, g$, of the sum from (9-112) (derivatives in all but excluding $m_{r_g-\pi(r_g)}^{(g)}$). From (9-111a), (9-111b) and (9-111c), we get the following derivatives:

1) For $m_{2v}^{(i)}$, $i=1,\, 2, \ldots,\, g-1$, $v=1,\, 2,\, \ldots,\, (r_i-\pi(r_i))/2$

$$-\frac{\partial\left(k_0^{(i)}\log\left(k_0^{(i)}\right)\right)}{\partial m_{2v}^{(i)}}-\frac{\partial\left(m_{2v}^{(i)}\log\left(m_{2v}^{(i)}\right)\right)}{\partial m_{2v}^{(i)}}-\frac{\partial\left(\overline{k}_0^{(g)}\log\left(\overline{k}_0^{(g)}\right)\right)}{\partial m_{2v}^{(i)}}$$

$$-\frac{\partial\left(k_g \log\left(k_g\right)\right)}{\partial m_{2v}^{(i)}}+\log\binom{r_i}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\binom{r_g}{r_g-\pi(r_g)}=0 \tag{9-113}$$

The left-side of (9-113) is

$$\log\left(k_0^{(i)}\right)+1-\log\left(m_{2v}^{(i)}\right)-1-\frac{2v}{r_g-\pi(r_g)}\log\left(\overline{k}_0^{(g)}\right)-\frac{2v}{r_g-\pi(r_g)}$$

$$+\frac{2v}{r_g-\pi(r_g)}\log\left(k_g\right)+\frac{2v}{r_g-\pi(r_g)}+\log\binom{r_i}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\binom{r_g}{r_g-\pi(r_g)} \tag{9-114}$$

Thus, the equations of derivatives are

$$\log\left(k_0^{(i)}\right)-\log\left(m_{2v}^{(i)}\right)-\frac{2v}{r_g-\pi(r_g)}\log\left(\overline{k}_0^{(g)}\right)+\frac{2v}{r_g-\pi(r_g)}\log\left(k_g\right)$$

$$+\log\binom{r_i}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\binom{r_g}{r_g-\pi(r_g)}=0 \tag{9-115}$$

2) For $m_{2v}^{(g)}$, v=1, 2, …., $(r_i-\pi(r_i))/2-1$

$$-\frac{\partial\left(\overline{k}_0^{(g)}\log\left(\overline{k}_0^{(g)}\right)\right)}{\partial m_{2v}^{(g)}}-\frac{\partial\left(m_{2v}^{(g)}\log\left(m_{2v}^{(g)}\right)\right)}{\partial m_{2v}^{(g)}}-\frac{\partial\left(k_g \log\left(k_g\right)\right)}{\partial m_{2v}^{(g)}}$$

$$+\log\binom{r_g}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\binom{r_g}{r_g-\pi(r_g)}=0 \tag{9-116}$$

The left-side of (9-116) is

$$-\left(\frac{2v}{r_g-\pi(r_g)}-1\right)\log\left(\overline{k}_0^{(g)}\right)-\left(\frac{2v}{r_g-\pi(r_g)}-1\right)-\log\left(m_{2v}^{(g)}\right)-1+\frac{2v}{r_g-\pi(r_g)}\log\left(k_g\right)$$

$$+\frac{2v}{r_g-\pi(r_g)}+\log\binom{r_g}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\left(\frac{r_g}{r_g-\pi(r_g)}\right) \tag{9-117}$$

Thus, the equations of derivatives are

$$-\log\left(m_{2v}^{(g)}\right)+\frac{2v}{r_g-\pi(r_g)}\log\left(k_g\right)+\left(1-\frac{2v}{r_g-\pi(r_g)}\right)\log\left(\overline{k}_0^{(g)}\right)$$

$$+\log\binom{r_g}{2v}-\frac{2v}{r_g-\pi(r_g)}\log\left(\frac{r_g}{r_g-\pi(r_g)}\right) \tag{9-118}$$

Therefore, (9-115) and (9-118) are the equations of derivatives for finding $m_{2v}^{(i)}$, v=1, 2, …., $(r_i-\pi(r_i))/2$, i=1, 2, …, g, and then computing $\log M_{n,\theta}$.

9.4.4 Solution of the System of Derivative Equations

The following theorem offers the solution of the system of derivative equations

Theorem 9.2: The system of derivative equations (9-115) and (9-118) has the unique solution given by

$$m_{2v}^{(i)}=\frac{2\gamma n}{g}\frac{\binom{r_i}{2v}t^{2v}}{(1+t)^{r_i}+(1-t)^{r_i}} \tag{9-119}$$

where v=0, 1, 2, …., $(r_i-\pi(r_i))/2$, i=1, 2, …, $g-1$, and t is the unique positive root of

$$\sum_{i=1}^{g}r_i\frac{(1+t)^{r_i-1}+(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}=(1-\overline{\theta})\sum_{i=1}^{g}r_i \tag{9-120}$$

The proof for the theorem can be referred to the Appendix M. □

9.4.5 Calculation of Asymptotics for $M_{n,\theta}$ and P

By (9-112) and (9-119) we have

$$\log M_{n,\theta} \sim \sum_{i=1}^{g}\left\{\frac{\gamma n}{g}\log\frac{\gamma n}{g} - \sum_{v=0}^{(r_i-\pi(r_i))/2} m_{2v}^{(i)}\log\left(m_{2v}^{(i)}\binom{r_i}{2v}^{-1}\right)\right\}$$

$$=\gamma n\log n+\gamma n\log\frac{\gamma}{g}-\sum_{i=1}^{g}\frac{2\gamma n}{g\left((1+t)^{r_i}+(1-t)^{r_i}\right)}$$

$$\times\sum_{v=0}^{(r_i-\pi(r_i))/2}\binom{r_i}{2v}t^{2v}\ \log\left(\frac{2\gamma n}{g}\frac{t^{2v}}{(1+t)^{r_i}+(1-t)^{r_i}}\right)$$

$$=\frac{2\gamma n}{g}\left(\frac{g}{2}\log n+\frac{g}{2}\log\frac{\gamma}{g}-\sum_{i=1}^{g}\frac{1}{(1+t)^{r_i}+(1-t)^{r_i}}\right.$$

$$\left.\times\left(\log\frac{2\gamma n/g}{(1+t)^{r_i}+(1-t)^{r_i}}\sum_{v=0}^{(r_i-\pi(r_i))/2}\binom{r_i}{2v}t^{2v}+(\log t)\sum_{v=0}^{(r_i-\pi(r_i))/2}2t\binom{r_i}{2v}t^{2v}\right)\right) \quad \text{(9-121)}$$

Using the following identities

$$\sum_{v=0}^{(r_i-\pi(r_i))/2}\binom{r_i}{2v}t^{2v}=\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2} \quad \text{(9-122a)}$$

$$\sum_{v=0}^{(r_i-\pi(r_i))/2}2v\binom{k}{2v}t^{2v}=t\sum_{v=0}^{(r_i-\pi(r_i))/2}2v\binom{r_i}{2v}t^{2v-1}$$

$$=t\left[\sum_{v=0}^{(r_i-\pi(r_i))/2}\binom{r_i}{2v}t^{2v}\right]'=t\left(\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right)'$$

$$=r_i t\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{2} \quad \text{(9-122b)}$$

Thus, we obtain

$$-\sum_{i=1}^{g}\frac{1}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}\log\frac{2\gamma n/g}{(1+t)^{r_i}+(1-t)^{r_i}}\sum_{v=0}^{(r_i-\pi(r_i))/2}\binom{r_i}{2v}t^{2v}$$

$$=-\sum_{i=1}^{g}\frac{1}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}\frac{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}{2}\log\frac{2\gamma n/g}{(1+t)^{r_i}+(1-t)^{r_i}}$$

$$=-\frac{g}{2}\log\frac{\gamma}{g}-\frac{g}{2}\log n-\frac{1}{2}\sum_{i=1}^{g}\log\frac{2}{(1+t)^{r_i}+(1-t)^{r_i}} \tag{9-123a}$$

$$-\sum_{i=1}^{g}\frac{1}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}\left(\log t\right)\sum_{v=0}^{(r_i-\pi(r_i))/2}2t\binom{r_i}{2v}t^{2v}$$

$$=-\sum_{i=1}^{g}\frac{1}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}\left(\log t\right)r_i t\frac{\left(1+t\right)^{r_i-1}-\left(1-t\right)^{r_i-1}}{2}$$

$$=-\frac{1}{2}t\log t\sum_{i=1}^{g}r_i\frac{\left(1+t\right)^{r_i-1}-\left(1-t\right)^{r_i-1}}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}} \tag{9-123b}$$

Therefore, (9-121) is

$$\log M_{n,\theta}\sim-\frac{\gamma n}{g}\left(\sum_{i=1}^{g}\log\frac{2}{(1+t)^{r_i}+(1-t)^{r_i}}+t\left(\log t\right)\sum_{i=1}^{g}r_i\frac{\left(1+t\right)^{r_i-1}-\left(1-t\right)^{r_i-1}}{\left(1+t\right)^{r_i}+\left(1-t\right)^{r_i}}\right) \tag{9-124a}$$

By (M–69b) in the Appendix M

$$\log M_{n,\theta}\sim\frac{\gamma n}{g}\left(-\frac{gs\overline{\theta}}{\gamma}\log t+\sum_{i=1}^{g}\log\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right) \tag{9-124b}$$

By (9-105)

$$\log M_{n,\theta}\sim\frac{\gamma n}{g}\left(-\overline{\theta}\left(\log t\right)\sum_{i=1}^{g}r_i+\sum_{i=1}^{g}\log\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right) \tag{9-124c}$$

By (9-105) and (9-108)

$$\log P \sim \frac{\gamma n}{g}\sum_{i=1}^{g}\left(-H\left(\overline{\theta}\right)r_i - \overline{\theta}r_i \log t + \log\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right) \tag{9-125}$$

Finally, we conclude that if $\overline{\theta}$ satisfies the condition in (9-106), then

$$\lim_{n\to+\infty}\frac{1}{n}\log P = \frac{\gamma}{g}\sum_{i=1}^{g}\left(-H\left(\overline{\theta}\right)r_i - \overline{\theta}r_i \log t + \log\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right) \tag{9-126}$$

9.4.6 Asymptotic of P in the General Case of Arbitrary Sizes of Horizontal Blocks

It is easy to extend the conclusion (9-126) for equal sizes of horizontal blocks to the general case of arbitrary sizes of horizontal blocks. The reader can refer to [73] for more details with respect to the derivations. In this general case the asymptotic of P is derived by slight modifications of (12.456), i.e.,

$$\lim_{n\to+\infty}\frac{1}{n}\log P = \gamma\sum_{i=1}^{g}\left(-H\left(\overline{\theta}\right)\varepsilon_i r_i - \overline{\theta}\left(\log t\right)\varepsilon_i r_i + \varepsilon_i \log\frac{(1+t)^{r_i}+(1-t)^{r_i}}{2}\right) \tag{9-127}$$

Clearly, (9-127) is regarded as a generalized form of (9-126) that is obtained by simply letting $\varepsilon_i = 1/g$, $i=1, 2, \ldots, g$, in (9-127).