Supplemental Materials: Additive Separability and Excess Unobserved Heterogeneity: A Test for Hicks-neutral Productivity Shocks in U.S. Manufacturing Industry

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Notation. $C(x,v) = E(Y|X = x, V = v)$, $\Delta_i = \widehat{h}(V_i) - h(V_i)$, $f^*(x,v) = f(x)f(v)/f(x,v)$, where $f(\cdot)$ denotes marginal/joint densities. $t_i = \mathbf{1}\{X_i \in \mathcal{X}_0\}$. Remember that under $\mathbb{H}_0$, $Y = m_1(X) + h(V) + \epsilon$. Also let $\sum_{i,j}^N \equiv \sum_{i=1}^N \sum_{j=1}^N$, $\sum_{i,j,k}^N \equiv \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N$, $\sum_{j>i}^N \equiv \sum_{i=1}^{N-1} \sum_{j>i}^N$ and $\sum_{k>j>i}^N \equiv \sum_{i=1}^{N-2} \sum_{j>i}^{N-1} \sum_{k>j}^N$. Let $K_{i,j}^X = K_h(X_i - X_j) = k(X_i - X_j/h)/h$, $K_{i,\cdot}^X = K_h(X_i - x)$ and similar for other variables. I suppress superscript (or subscript) p for quantile(and trimming) indicators.

1 Intermediate Lemmas

Lemma A 1 U-statistic. Serfling (2009). *A "mth-order" U-statistic of the form*

$$U_N \equiv \binom{N}{m}^{-1} \sum_{i_1=1}^{N-m+1} \sum_{i_2>i_1}^{N-m+2} \cdots \sum_{i_m>i_{m-1}}^{N} p_N(W_{i_1}, \ldots, W_{i_m})$$

where p_N is a symmetric in $W_{i_1}, \ldots, W_{i_m}$. Suppose that $\mathbb{E}||p_N(W_i, W_j)||^2 = o(N^{m-1})$. Also define

$$\begin{aligned} r_N(W_i) &\equiv \mathbb{E}[p_N(W_{i_1}, \ldots, W_{i_m})|W_{i_1}] \\ \theta_N &\equiv \mathbb{E}[r_N(W_i)] = \mathbb{E}[p_N(W_{i_1}, \ldots, W_{i_m})] \\ \widehat{U}_N &\equiv \theta_N + mN^{-1} \sum_{i=1}^{N} [r_N(W_{i_1}) - \theta_N] \end{aligned}$$

where θ_N is assumed to exist and $\widehat{U}_N$ is called the "projection" of U_N. Then

$$U_N - \widehat{U}_N = o_p(N^{-1/2})$$

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Lemma A 2 Recursive Bias-reducing Conditional Mean Estimators**.** *Suppose* $E|(Y|U)|^2 < \infty$*,* $\inf_{\mathcal{U}} f(u) > 0$ *where* $f(\cdot)$ *is the density function of* U *that is everywhere positive,* $d < 4$*. For each* $u \in \mathcal{U}$*, the bias correction estimator* $\widehat{C}(u) = \widehat{E}(Y|U = u)$ *is defined in Eq.* (4.2)*, then it follows that*

$$a). \qquad |\widehat{C}(u) - C(u)| = f_U(u)^{-1} \left| N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U (\widetilde{Y}_i - C(u)) \right| + o_p(N^{-1/2})$$

$$b). \qquad N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U (\widetilde{Y}_i - C(u)) = N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \epsilon_i + O(h^4)$$

where $\widetilde{Y}_i = Y_i - \widehat{\delta}_i(u)$*,* $\widehat{\delta}_i(u) = \widehat{C}_0(U_i) - \widehat{C}_0(u)$ *and* $\epsilon_i = Y_i - C(U_i)$*.* $\widehat{C}_0(\cdot)$ *is the preliminary conditional expectation estimator in Eq.* (4.1)*.*

Proof. See Shen and Klein (2017) for detailed proof. For a). one must show that

$$DC = \left| \frac{N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \widetilde{Y}_i}{\widehat{f}_U(u)} - C(u) \right| \left| \frac{\widehat{f}_U(u)}{f_U(u)} - 1 \right| = o_p(N^{-1/2})$$

Under the well-known nonparametric rate, one would have

$$|\widehat{f}_U(u) - f_U(u)| = O_p(h^2 + N^{-1/2} h^{-d/2}); \left| N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \widetilde{Y}_i - f_U(u) C(u) \right| = O_p(h^2 + N^{-1/2} h^{-d/2})$$

From this, let $c_f = \sup_u \widehat{f}(u)$,

$$\begin{aligned} \left| \frac{N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \widetilde{Y}_i}{\widehat{f}_U(u)} - C(u) \right| \quad & \le c_f \left\{ \left| N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \widetilde{Y}_i - f_U(u) C(u) \right| + C(u) \left| \widehat{f}_U(u) - f_U(u) \right| \right\} \\ & = O_p(h^2 + N^{-1/2} h^{-d/2}) \end{aligned}$$

Then by Cauchy-Schwartz inequality, we have

$$DC \le \sqrt{ \left(\frac{N^{-1} \sum_{i=1}^{N} K_{i,\cdot}^U \widetilde{Y}_i}{\widehat{f}_U(u)} - C(u) \right)^2 \left(\frac{\widehat{f}_U(u)}{f_U(u)} - 1 \right)^2 } = O(h^4 + N^{-1} h^{-d} + N^{-1/2} h^{2-d/2})$$

Suppose $h = O(N^{-r})$, to make sure $\sqrt{N} DC = o_p(N^{-1/2})$ hold, $1/8 < r < 1/2d$ and generally requires $d < 4$.

For b). To begin with, first add and subtract $K_{i,\cdot}^U \delta_i(u)$ and use the fact $\delta_i(u) = C(U_i) -$

$C(u)$,

$$\begin{aligned} N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(\widetilde{Y}_i - C(u)) &= N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(Y_i - \delta_i(u) - C(u) - \widehat{\delta}_i(u) + \delta_i(u)) \\ &= N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(Y_i - C(U_i) - \widehat{\delta}_i(u) + \delta_i(u)) \\ &= N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}\epsilon_i - N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u)) \end{aligned}$$

And it is true that

$$\sqrt{N}^{-1/2}\sum_{i=1}^{N} K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u)) = o_p(N^{-1/2})$$

To see this, remember $\delta_i(u) - \delta_i(u) = O$

$$E\left[N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u))\right] = E\left[K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u))\right] = O(h^2)O(h^2) = O(N^{-4r})$$

$$\text{Var}\left[N^{-1}\sum_{i=1}^{N} K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u))\right] = N^{-1}\text{Var}\left[K^U_{i,\cdot}(\widehat{\delta}_i(u) - \delta_i(u))\right] = O(N^{-2}h^{d+1})$$

As long as $1/8 < r < 1/(d+1)$, b). will hold. See Shen and Klein (2017) for higher order bias reduction. □

Lemma A 3. *Suppose that $R(\cdot)$ is any measurable function defined on $\mathbb{R}^d$ with continuous and bounded second derivatives. t_i is the quantile or trimming indicator defined in Eq. (4.9) and the density function $f_X(\cdot)$ satisfies Assumption A-2. For any $x_0 \in \mathcal{X}$, it is true that*

$$E\left[t_i^p R(X_i) K_h(X_i - x_0)\right] = t^p(x_0)R(x_0)f(x_0) + O(h^2)$$

where $t^p(x_0) \equiv \mathbf{1}\{x_0 \in \mathcal{X}_0\}$.

Proof. Define the upper bound $q_L \equiv q_X(\frac{p-1}{P})$ and lower bound $q_U \equiv q_X(\frac{p}{P})$,

$$\begin{aligned} E\left[t_i^p R(X_i) K_h(X_i - x_0)\right] &= \int_{q_U}^{q_L} k(\frac{x - x_0}{h})R(x)f(x)dx \\ &= \int_{\frac{q_U - x_0}{h}}^{\frac{q_L - x_0}{h}} R(x_0 + uh)f(x_0 + uh)k(u)du \\ &= \int_{\frac{q_U - x_0}{h}}^{\frac{q_L - x_0}{h}} \left[R(x_0)f(x_0) + h(R(x_0)f(x_0))'u + h^2(R(x)f(x))''|_{x^+}u^2/2\right] k(u)du \\ &= t^p(x_0)R(x_0)f(x_0) + \left[R(x_0)f(x_0)\right]' h\int_{\frac{q_U - x_0}{h}}^{\frac{q_L - x_0}{h}} uk(u)du + O(h^2) \end{aligned}$$

The second term is $o(h^2)$. □

2 Additional Asymptotic Proofs

Lemma 3. D_N^g**.** *Suppose that Assumption A.1-A.6 hold and under* $\mathbb{H}_0$*, then* D_N^g *in Eq.* (5.4) *can be written as the following,*

$$D_N^g = N^{-1}\sum_{i=1}^{N}\{t_i f^*(X_i, V_i)\epsilon_i + E(t)h(V_i)\} + o_p(N^{-1/2})$$

Proof. Recall that D_N^g and by definition,

$$\begin{aligned}
D_N^g &= N^{-1}\sum_{i=1}^{N} t_i[\widehat{g}(X_i) - g(X_i)] \\
&= \frac{1}{N(N-1)}\sum_{i,j}^{N} t_i\widehat{C}(X_i, V_j) - \frac{1}{N}\sum_{i=1}^{N}\int t_i C(X_i, v)dF(v) \\
&= \frac{1}{N(N-1)}\sum_{i,j}^{N} t_i\left[\widehat{C}(X_i, V_j) - C(X_i, V_j)\right] + \frac{1}{N(N-1)}\sum_{i,j}^{N} t_i\left[C(X_i, V_j) - \int C(X_i, v)dF(v)\right] \\
&\equiv D_N^{g1} + D_N^{g2}
\end{aligned}$$

The third equality follows by adding and subtracting $\sum_{i,j}^{N} t_i C(X_i, V_j)/N(N-1)$.

Part a), for D_N^{g1}, by Lemma A 2, it is true that

$$\widehat{C}(X_i, V_j) - C(X_i, V_j) = \frac{1}{N-2}\sum_{k\neq i,j}^{N} K_{k,i}^X K_{K,j}^V[Y_k - C(X_i, V_j)]/f(X_i, V_j) + o_p(N^{-1/2})$$

Substitute this into D_N^{g1} and rewrite it as $D_N^{g1} = \widetilde{D}_N^{g1} + o_p(N^{-1/2})$, so from now on it suffices to work with $\widetilde{D}_N^{g1}$ defined below,

$$\begin{aligned}
\widetilde{D}_N^{g1} &= \frac{1}{N(N-1)(N-2)}\sum_{i,j,k}^{N}\frac{t_i}{f(X_i, V_j)}K_{k,i}^X K_{k,j}^V\left[Y_k - C(X_i, V_j)\right] \\
&= \binom{N}{3}^{-1}\sum_{k>j>i}^{N}\sum_{l=1}^{6}\delta_{g1l}/6
\end{aligned}$$

To represent $\widetilde{D}_N^{g1}$ as a third-order U-statistic and where

$$\delta_{g11} = \frac{t_i}{f(X_i, V_j)} K_{k,i}^X K_{k,j}^V \left[Y_k - C(X_i, V_j)\right]; \delta_{g12} = \frac{t_i}{f(X_i, V_k)} K_{j,i}^X K_{j,k}^V \left[Y_j - C(X_i, V_k)\right]$$
$$\delta_{g13} = \frac{t_j}{f(X_j, V_i)} K_{k,j}^X K_{k,i}^V \left[Y_k - C(X_j, V_i)\right]; \delta_{g14} = \frac{t_k}{f(X_k, V_i)} K_{j,k}^X K_{j,i}^V \left[Y_j - C(X_k, V_i)\right]$$
$$\delta_{g15} = \frac{t_k}{f(X_k, V_j)} K_{i,k}^X K_{i,j}^V \left[Y_i - C(X_k, V_j)\right]; \delta_{g16} = \frac{t_j}{f(X_j, V_k)} K_{i,j}^X K_{i,k}^V \left[Y_i - C(X_j, V_k)\right]$$

Moreover, by Lemma A 3,

$$E(\delta_{g11}|X_i) = O(h^4); E(\delta_{g12}|X_i) = O(h^4)$$
$$E(\delta_{g13}|V_i) = O(h^4); E(\delta_{g14}|V_i) = O(h^4)$$
$$E(\delta_{g15}|W_i) = t_i f^*(X_i, V_i)[Y_i - C(X_i, V_i)] + O(h^4)$$
$$E(\delta_{g16}|W_i) = t_i f^*(X_i, V_i)[Y_i - C(X_i, V_i)] + O(h^4)$$

Proving the above results is nothing hard but tedious. To conserve space, I only show one item and the others follow the same line of reasoning. Now take $E(\delta_{g15}|W_i)$ as an example,

$$\begin{aligned} E(\delta_{g15}|W_i) &= \int \frac{t(x)}{f(x,v)} K(X_i - x) K(V_i - v) \left[Y_i - C(x,v)\right] f(x) f(v) dx dv \\ &= \int_{\{u_1 : t(X_i + u_1 h) = 1\}} f^*(X_i + u_1 h, V_i + u_2 h) \left[Y_i - C(X_i + u_1 h, V_i + u_2 h)\right] k(u_1) k(u_2) du_1 du_2 \\ &= t_i f^*(X_i, V_i)[Y_i - C(X_i, V_i)] + O(h^4) \end{aligned}$$

Note that the second equality holds by the transformation of variables, letting $x = X_i + u_1 h$ and $v = V_i + u_2 h$. The third equality follows from the Taylor expansion on h around $u_1 = u_2 = 0$.

By Assumption A.1-A.3, it is true that $\sum_{l=1}^N E|g_{1l}|^2 = O(1) = o(N^2)$ as every multiplicative term is bounded. Therefore, the standard second order U-statistic applies,

$$\begin{aligned} \widetilde{D}_N^{g1} &= N^{-1} \sum_{i=1}^N t_i f^*(X_i, V_i)[Y_i - C(X_i, V_i)] + O(h^4) + o_p(N^{-1/2}) \\ &= N^{-1} \sum_{i=1}^N t_i f^*(X_i, V_i) \epsilon_i + o_p(N^{-1/2}) \end{aligned}$$

Part b), for D_N^{g2}, one can also rewrite it as a second-order U-statistic,

$$\begin{aligned} D_N^{g2} &= \frac{1}{N(N-1)} \sum_{j,i}^N t_i \left[C(X_i, V_j) - \int C(X_i, v) dF(v)\right] \\ &= \binom{N}{2}^{-1} \sum_{j>i}^N (\delta_{g21} + \delta_{g22})/2 \end{aligned}$$

where in particular

$$\delta_{g21} = t_i \left[C(X_i, V_j) - \int C(X_i, v) dF(v) \right]; \delta_{g22} = t_j \left[C(X_j, V_i) - \int C(X_j, v) dF(v) \right]$$

It is obvious that $E(\delta_{g21}|W_i) = 0$. But for δ_{g22}, it can be shown that

$$\begin{aligned} E(\delta_{g22}|V_i) &= E_X[t(X)C(X, V_i)] - E\left[t(X)Y f^*(X, V)\right] \\ &= E(t)h(V_i) \end{aligned}$$

where E_X is the expectation taken with respect to only X. The second equality is true only under $\mathbb{H}_0$. Also by Assumption A.1-A.3, we have $E|g_{21}|^2 + E|g_{22}|^2 = O(1) = o(N)$, then the standard U-statistic theorem implies that

$$\widetilde{D}_N^{g2} = N^{-1} \sum_{i=1}^{N} E(t)h(V_i) + o_p(N^{-1/2})$$

Finally, combine $\widetilde{D}_N^{g1}$ and $\widetilde{D}_N^{g2}$, then Lemma 3 follows that

$$D_N^g = N^{-1} \sum_{i=1}^{N} \{t_i f^*(X_i, V_i)\epsilon_i + E(t)h(V_i)\} + o_p(N^{-1/2})$$

□

Lemma 4 and Lemma 5 use U-statistic theorem to analysis $N^{-1} \sum_{i=1}^N \widehat{E}(\Delta(V)|X_i)$.

Lemma 4. **Δ_i.** *Given X_i and $V_i = v$, let $\Delta(v) = \widehat{h}(v) - h(v)$, then it follows that*

$$\Delta(v) = \Delta_1(v) + \Delta_2(v) + \Delta_3 + o_p(N^{-1/2})$$

where

$$\begin{aligned} \Delta_1(v) &= \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i) K_{i,\cdot}^V}{f(X_i, v)} [Y_i - C(X_i, v)] \\ \Delta_2 &= \frac{1}{N} \sum_{i=1}^{N} m_1(X_i) - E[m_1(X)] \\ \Delta_3 &= E(Y) - \overline{Y} \end{aligned}$$

Proof. Given $V_i = v \in \mathcal{V}$, recall that $\Delta(v) = \widehat{h}(v) - h(v)$ with $\widehat{h}(\cdot)$ in Eq. (4.4) and $h(\cdot)$ in Eq. (3.4). Following the similar argument in Lemma 3 $\Delta(v)$ can be decomposed into three components,

$$\Delta(v) = \underbrace{N^{-1} \sum_{i=1}^{N} \left[\widehat{C}(X_i, v) - C(X_i, v) \right]}_{\Delta_1(v)} + \underbrace{\left[\frac{1}{N} \sum_{i=1}^{N} C(X_i, v) - \int C(x, v) dF(x) \right]}_{\Delta_2(v)} + \underbrace{E(Y) - \overline{Y}}_{\Delta_3}$$

By Lemma A 2, $\Delta_1(v) = \widetilde{\Delta}_1(v) + o_p(N^{-1/2})$ where $\widetilde{\Delta}_1(v)$ is defined below,

$$\begin{aligned}\widetilde{\Delta}_1(v) &= \frac{1}{N(N-1)} \sum_{i,j}^{N} \frac{K_{j,i}^X K_{j,\cdot}^V}{f(X_i, v)} [Y_j - C(X_i, v)] \\ &= \binom{N}{2}^{-1} \sum_{j>i}^{N} (d_{ij} + d_{ji})/2\end{aligned}$$

where in particular

$$d_{ij} = \frac{K_{j,i}^X K_{j,\cdot}^V}{f(X_i, v)} [Y_j - C(X_i, v)]; d_{ji} = \frac{K_{i,j}^X K_{i,\cdot}^V}{f(X_j, v)} [Y_i - C(X_j, v)]$$

It is straightforward to show that

$$E(d_{ij}|W_i) = O(h^4); E(d_{ji}|W_i) = \frac{f(X_i) K_{i,\cdot}^V}{f(X_i, v)} [Y_i - C(X_i, v)] + O(h^4)$$

The second moment condition $E|d_{ij}|^2 + E|d_{ji}|^2 = O(h^{-4})$ holds trivially according to Assumption A.1-A.3. So by Lemma A 1, we have

$$\Delta_1(v) = \widehat{\Delta}_1(v) + o_p(N^{-1/2})$$

Note that under $\mathbb{H}_0$, $\Delta_2 = N^{-1} \sum_i^N m_1(X_i) - E[m_1(X)]$. So it is true that $\Delta(v) = \widehat{\Delta}_1(v) + \Delta_2(v) + \Delta_3 + o_p(N^{-1/2})$. □

Lemma 5. **D_N^h.** *Suppose that Assumption A.1-A.6 hold and under $\mathbb{H}_0$, then D_N^h in Eq.* (5.6) *can be written as the following,*

$$D_N^h = N^{-1} \sum_{i=1}^{N} E(t|V_i) f^*(X_i, V_i) \epsilon_i + o_p(N^{-1/2})$$

Proof. To begin with, a result implied from Lemma 4 states that $E(\widehat{\Delta}|X_i) = o_p(N^{-1/2})$. As this can be seen from below,

$$E(\widehat{\Delta}|X_i) = E[\widehat{\Delta}_1(V)|X_i] + E(\Delta_2|X_i) + E(\Delta_3|X_i) + o_p(N^{-1/2})$$

Also, it is not hard to see the following from Lemma 4 that given X_i,

$$\begin{aligned}E[\widehat{\Delta}_1(V)|X_i] &= E\left[\frac{f(X_j) K_{i,j}^V}{f(X_j, V_i)} (Y_j - C(X_j, V_i)] \Big| X_i\right] = O(h^4) \\ E(\Delta_2|X_i) &= N^{-1}[m_1(X_j) - E(m_1(X))] = O_p(N^{-1}) \\ E(\Delta_3|X_i) &= -E[\overline{Y} - E(Y)|X_i] = -N^{-1}[E(Y|X_i) - E(Y)] = O_p(N^{-1})\end{aligned}$$

Therefore, D_N^h can be further decomposed into four components like below,

$$D_N^h = \underbrace{N^{-1}\sum_{i=1}^N t_i[\widehat{E}(\widehat{\Delta}_1(V)|X_i) - E(\widehat{\Delta}_1(V)|X_i)]}_{D_N^{h1}} + \underbrace{N^{-1}\sum_{i=1}^N t_i[\widehat{E}(\Delta_2(V)|X_i) - E(\Delta_2(V)|X_i)]}_{D_N^{h2}}$$
$$+ \underbrace{N^{-1}\sum_{i=1}^N t_i[\widehat{E}(\Delta_3|X_i) - E(\Delta_3|X_i)]}_{D_N^{h3}} + o_p(N^{-1/2})$$

In what follows, only the first three components need to be analyzed separately.

For D_N^{h1}, we can represent it as a third-order U-statistic,

$$\begin{aligned}
D_N^{h1} &= \frac{1}{N(N-1)}\sum_{i,j}^N t_i f(X_i)^{-1}K_{i,j}^X[\widehat{\Delta}_{1j} - E(\widehat{\Delta}_1|X_i)] + o_p(N^{-1/2}) \\
&= \frac{1}{N(N-1)}\sum_{i,j}^N t_i f(X_i)^{-1}K_{i,j}^X\widehat{\Delta}_{1j} + o_p(N^{-1/2}) \\
&= \frac{1}{N(N-1)(N-2)}\sum_{i,j,k}^N \frac{t_i f(X_k)K_{i,j}^X K_{k,j}^V}{f(X_i)f(X_k,V_j)}[Y_k - C(X_k,V_j)] + o_p(N^{-1/2}) \\
&= \binom{N}{3}^{-1}\sum_{k>j>i}^N\sum_{l=1}^6 h_{1l}/6 + o_p(N^{-1/2})
\end{aligned}$$

The first equality holds as we can remove the random denominator of $\widehat{E}(\widehat{\Delta}_1|X_i)$ according to Lemma A 2. The second equality is because of the fact $E[\widehat{\Delta}_1|X_i] = O(h^4)$. Substitution of $\widehat{\Delta}_1$ gives the third equality. And where in particular,

$$\begin{aligned}
h_{11} &= \frac{t_i f(X_k)K_{i,j}^X K_{k,j}^V}{f(X_i)f(X_k,V_j)}[Y_k - C(X_k,V_j)]; h_{12} = \frac{t_i f(X_j)K_{i,k}^X K_{j,k}^V}{f(X_i)f(X_j,V_k)}[Y_j - C(X_j,V_k)] \\
h_{13} &= \frac{t_j f(X_k)K_{j,i}^X K_{k,i}^V}{f(X_j)f(X_k,V_i)}[Y_k - C(X_k,V_i)]; h_{14} = \frac{t_k f(X_j)K_{k,i}^X K_{j,i}^V}{f(X_k)f(X_j,V_i)}[Y_j - C(X_j,V_i)] \\
h_{15} &= \frac{t_k f(X_i)K_{k,j}^X K_{i,j}^V}{f(X_k)f(X_i,V_j)}[Y_i - C(X_i,V_j)]; h_{16} = \frac{t_j f(X_i)K_{j,k}^X K_{i,k}^V}{f(X_j)f(X_i,V_k)}[Y_i - C(X_i,V_k)]
\end{aligned}$$

It is easy to see that $E(h_{1l}|W_i) = O(h^4)$ for $l = \{1,2,3,4\}$ and

$$E(h_{15}|W_i) = E(h_{16}|W_i) = E[t(X)|V_i]f^*(X_i,V_i)[Y_i - C(X_i,V_i)] + O(h^4)$$

It is also true that $\sum_{l=1}^6 E|h_{1l}|^2 = O(h^{-4}) = o(N^2)$ by Assumption A-5. Then standard

U-statistic theorem follows,

$$D_N^{h11} = N^{-1}\sum_{i=1}^{N} E[t(X)|V_i]f^*(X_i,V_i)\epsilon_i + o_p(N^{-1/2})$$

For D_N^{h2}, we can also represent it as a third-order U-statistic,

$$\begin{aligned}
D_N^{h2} &= N^{-1}\sum_{i=1}^{N} t_i[\widehat{E}(\Delta_2|X_i) - E(\Delta_2|X_i)] \\
&= N^{-1}\sum_{i,j}^{N} t_i f(X_i)^{-1}K_{i,j}^X[\Delta_{2j} - E(\Delta_2|X_i)] + o_p(N^{-1/2}) \\
&= N^{-1}\sum_{i,j}^{N} t_i f(X_i)^{-1}K_{i,j}^X[m_1(X_j) - m_1(X_i)]/N + o_p(N^{-1/2}) \\
&= o_p(N^{-1/2})
\end{aligned}$$

The second equality holds as we can remove the random denominator of $\widehat{E}(\Delta_2|X_i)$ according to Lemma A 2. Substitution of Δ_2 gives the third equality.

For D_N^{h13},

$$\begin{aligned}
D_N^{h13} &= -N^{-1}\sum_{i=1}^{N} t_i[\widehat{E}(\Delta_3|X_i) - E(\Delta_3|X_i)] \\
&= -N^{-1}\sum_{i,j}^{N} t_i f(X_i)^{-1}K_{i,j}^X[Y_j/N - E(Y|X_i)/N] + o_p(N^{-1/2}) \\
&= o_p(N^{-1/2})
\end{aligned}$$

So, combining all above, D_N^h follows.

□

References

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