CMT107 : Visual Computing - Exercise Sheet 1

Vectors and Matrices

1) Which of the following pairs of vectors **a** and **b** are equal? (Multiple choice)

a) **a** and **b** have the same length, the same direction, and are at the same position

b) **a** and **b** are both unit vectors, and have the same direction, but are at different positions

c) **a** and **b** are both zero vectors, and are at the same position, but have unknown directions

d) The tail of **a** is the head of **b**, and vice versa.

2) Give a general representation of the vector **u** by a linear combination of *n* vectors $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$. In which case is the representation unique? If the representation is unique, are **u**, $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ linearly independent, or linearly dependent?

3) What is the dimension of a vector space? What is a basis of a vector space?

4) What are a vector space, an affine space, a Euclidean space, and a Euclidean affine space?

5) Which of the following operations are meaningful (defined) on an affine space? And if the operation is defined, what is the result of the operation?

a) Point-point addition

b) Point-point subtraction

c) Point-vector addition

d) Point-vector subtraction

e) Vector-vector addition

f) Vector-vector subtraction

g) Inner product

h) Outer product

6) Calculate the length of the projection of vector u $=[1\ 2\ -2]^T$ onto the vector v $= [4\ 2\ 3]^T$.

7) The vector $\mathbf{w} = [1, 3, 2]^T$ is decomposed as the sum of the two orthogonal vectors **u** and **v**, where **u** is parallel to the vector $\mathbf{z}=[2, 1, -1]^T$. Calculate **u** and **v**.

8) Given two vectors $\mathbf{a} = [3\ 4\ 0]^T$ and $\mathbf{b} = [1\ 1\ -2]^T$, compute the cross product between **a** and **b** and $\sin\theta$, where θ is the angle between **a** and **b**.

9) Take 2D Euclidean space as an example, show that the angle θ between two vector **u** and **v** can be calculated by $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

10) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are 3D vectors, show that . $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$

11) Given a 2x3 matrix **A**, prove that $\mathbf{AA}^{\mathrm{T}}$ is a symmetric matrix.

12) Let **i**, **j**, and **k** are orthonormal basis vectors in a 3D Euclidean affine space, and let two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are represented by $\mathbf{v}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{v}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$. Show that the inner product and cross product of $\mathbf{v}_1$ and $\mathbf{v}_2$ are $\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$ and $\mathbf{v}_1 \times \mathbf{v}_2 = (y_1z_2 - z_1y_2)\mathbf{i} + (z_1x_2 - x_1z_2)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k}$, respectively.

Transformations and Projection

1) List three different reference frames used in graphics, and briefly describe their location of origin and for which objects they are used.

2) Rotating a 2D vector **a** = $[x\ y]^T$ by an angle ϕ counter-clockwise, we get a vector **b** = $[x'\ y']^T$. show that

$x' = x \cdot \cos\phi - y \cdot \sin\phi$

$y' = x \cdot \sin\phi + y \cdot \cos\phi$

3) Show that translation does NOT satisfy the following property of linear transformation.

$$\mathbf{T}(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = s_1\mathbf{T}(\mathbf{v}_1) + s_2\mathbf{T}(\mathbf{v}_2),\ s_1, s_2 \in \boldsymbol{R}$$

4) Explain how to convert standard 3D Cartesian coordinates (x, y, z) to homogeneous coordinates and how to convert homogeneous coordinates to standard 3D Cartesian coordinates.

5) Show how to perform a 2D rotation about an arbitrary point. Provide a matrix in homogeneous coordinates for each step in the operation.

6) Show how to perform a 3D rotation about an arbitrary axis. Again, give matrices in homogeneous coordinates for each step in the operation.

7) Convert the following coordinates of 3D points to homogeneous coordinates

a) (x, y, z), b) (x, 0, z), c) (0, 0, 0), d) (∞, 0, 0)

8) Convert the following homogeneous coordinates to 2D point coordinates, if the conversion is meaningful. Otherwise, explain why they cannot be converted.

a) (x, y, z), b) (x, 0, z), c) (x, y, 0), d) (0, 0, 0)

9) Transformation of object in one reference frame can be equivalently represented by a transformation of the reference frame. A vector $\mathbf{v}=[x, y, z]^T$ in a reference frame OXYZ is rotated around the X-axis by angle α, and then translated with an offset $\mathbf{t}=[tx, ty, tz]^T$. First, give the homogenous matrix representation of these transformations. Then, give the equivalent matrix of the reference frame transformations instead of the object transformation.