CMT117 Exercise Questions: Propositional Logic

Question 1. Let $L = \{p, q\}$ and let v be the valuation such that $v(p) = T$ and $v(q) = F$. For each of the following sentences, determine whether or not v satisfies that sentence.

(i) $q \to p$

(ii) $\neg p \to \neg q$

(iii) $p \vee (q \wedge p)$

(iv) $(q \wedge p) \vee q$

(v) $p \to (p \to \neg p)$

(vi) $p \leftrightarrow (p \vee q)$

(vii) $\neg p \wedge (\neg q \to \neg p)$

(viii) $\neg(q \to \neg p) \leftrightarrow \neg(p \to \neg q)$

Question 2. For *(i)*-*(ii)*, assume $L = \{p, q, r\}$ and let A stand for the sentence

$$(\neg p \to (q \vee \neg r)) \wedge (p \to \neg q).$$

(i) Determine, using the truth-table method or otherwise, whether the following are true, justifying your answer in each case:

(1) $A \models p \vee q$

(2) $p \vee q \models A$

(3) $\top \models A$

(ii) Write down 2 sentences $B_1, B_2 \in SL$ such that $A \models B_1$ and $A \models B_2$. These 2 sentences should be such that $B_1 \not\equiv B_2, A \not\equiv B_1$ and $A \not\equiv B_2$.

Question 3. Assuming $L = \{p, q, r\}$. Determine whether or not the following hold, justifying your answer in each case:

(1) $\neg p \vee (q \to r) \in Cn(\{\neg\neg r, p \vee q\})$.

(2) $\top \in Cn(\neg p \to (q \wedge r))$.

Question 4. Consider the following two examples of reasoning in natural language. In each one, translate (roughly) each of the 4 statements in the example into a sentence of propositional logic, and then determine whether they are examples of valid inference.

(i)
I will either grow up to be president or become a screenwriter
In order to become a screenwriter, I have to be good at writing
I am not good at writing

$\therefore$ I will grow up to be president

(ii)
If I am hungry, I eat cereal
If I eat cereal, I have milk
I am not hungry

$\therefore$ I will not have milk

Question 5. Suppose $L = \{p, q, r\}$. Recall that, for any formula A, the set of models of A is denoted by $Mod(A)$. Assuming we can represent each model as a sequence abc of Ts and Fs where a, b, c denote the truth-values of p, q, r respectively,:

(i) Write down $Mod(p)$.
(ii) Write down $Mod((p \vee r) \to (\neg p \wedge q))$.
(iii) Find a formula B such that $Mod(B) = \{TFT, TFF, FTT, FTF, FFF\}$

Before the next question we make the following definition:

Definition 1. Let $L = \{p_1, p_2, \ldots, p_n\}$ and $A \in SL$. We say A is a sentence in *full disjunctive normal form (DNF)* if it is written in the form

$$(c_{11} \wedge c_{12} \wedge \ldots \wedge c_{1n}) \vee (c_{21} \wedge c_{22} \wedge \ldots \wedge c_{2n}) \vee \ldots \vee (c_{m1} \wedge c_{m2} \wedge \ldots \wedge c_{mn})$$

for some $m \geq 0$, where, for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, c_{ij} is either p_j or $\neg p_j$.

In other words, A is in full DNF if it takes the form of a *disjunction* in which each of the m disjuncts is a *conjunction* in which every p.v. (not counting the special p.v.s $\top, \bot$) occurs exactly once as a conjunct, either as itself or as its negation. For example if $L = \{p, q, r\}$ then the following are two sentences in full DNF:

$$(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$$

$$\neg p \wedge \neg q \wedge \neg r$$

Note we adopt the convention that an *empty* disjunction is identified with $\bot$. So $\bot$ is a special case of sentence in full DNF (i.e., corresponding to the case $m = 0$).

Question 6. Assume $L = \{p, q, r\}$.

(i) For each of the following sentences, give a sentence in full DNF that is logically equivalent to it. (Note these are the same sentences as parts *(i)*-*(ii)* in Question 5 above).

(1) p

(2) $(p \vee r) \to (\neg p \wedge q)$

(ii) By informally sketching a procedure to convert any sentence to one in full DNF that is logically equivalent to it, show that any sentence is logically equivalent to one that uses only connectives $\wedge, \vee$ and $\neg$.

Question 7. *(i)*. Write down an informal proof of the following fact:

For any 2 sentences $A, B \in SL$: $A \models B$ iff $Cn(B) \subseteq Cn(A)$

Hint: See slide 51 of the Propositional Logic lecture slides for examples of the kind of proof required here. Also, to prove an "iff" statement such as this it is usually helpful to split the proof into two separate "halves": (1) the "if" (or "right-to-left") part, and (2) the "only if" (or "left-to-right") part. In this case you would first show (1) $A \models B$ given the assumption $Cn(B) \subseteq Cn(A)$ and then the converse direction (2) $Cn(B) \subseteq Cn(A)$ on the assumption $A \models B$.

(ii). Use the fact just proved in part *(i)* to write down an informal proof of the following fact:

For any 2 sentences $A, B \in SL$: $A \equiv B$ iff $Cn(A) = Cn(B)$

(i.e., the fact is saying that 2 sentences are logically equivalent iff the have the same set of logical consequences).

Question 8. In video 5 we looked at the king's first trial in Smullyan's "Lady or Tiger" story (see slide 48 and the associated files on Learning Central). Let's continue the story with the second trial.

> *And so, the first prisoner saved his life and made off with the lady. The signs on the doors were then changed, and new occupants for the rooms were selected accordingly. This time the signs read as follows:*
>
> *I: AT LEAST ONE OF THESE ROOMS CONTAINS A LADY*
>
> *II: A TIGER IS IN THE OTHER ROOM*
>
> *"Are the statements on the signs true?" asked the second prisoner.*
>
> *"They are either both true or both false," replied the king.*
>
> *Which room should the prisoner pick?*

(i) Write down the information given in this second trial in the form of a sentence in propositional logic, and use it to determine which room contains the lady.

To complete the story, here is the king's third trial.

In this trial, the king explained that, again, the signs were either both true or both false. Here are the signs:

I: EITHER A TIGER IS IN THIS ROOM OR A LADY IS IN THE OTHER ROOM

II: A LADY IS IN THE OTHER ROOM

Does the first room contain a lady or a tiger? What about the other room?

(ii) Write down the information given in this third trial in the form of a sentence in propositional logic, and use it to determine which room contains the lady.

NOTE: More practice questions can be found on the website of the Huth/Ryan book: http://www.cs.bham.ac.uk/research/projects/lics/tutor/chap1/questions.html