CMT117 Exercises: Natural Deduction, Completeness Theorem and Horn Logic

Question 1. Use the proof rules of Natural Deduction to prove the following are valid:

(a) $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

(b) $p \wedge q \vdash q \wedge p$

(c) $p \to (p \to q), p \vdash q$

(d) $p \vdash q \to (p \wedge q)$

(e) $(p \to r) \wedge (q \to r) \vdash (p \wedge q) \to r$

(f) $q \to r \vdash (p \to q) \to (p \to r)$

(g) $p \to q, r \to s \vdash (p \vee r) \to (q \vee s)$

(h) $p \vee q \vdash r \to ((p \vee q) \wedge r)$

(i) $(p \vee (q \to p)) \wedge q \vdash p$

(j) $p \to q, r \to s \vdash (p \wedge r) \to (q \wedge s)$

(k) $p \vee (p \wedge q) \vdash p$

(l) $\neg p \vee \neg q \vdash \neg(p \wedge q)$

(m) $\neg p, p \vee q \vdash q$

(n) $p \vee q, \neg q \vee r \vdash p \vee r$

(o) $p \wedge \neg p \vdash \neg(r \to q) \wedge (r \to q)$

(p) $\neg(\neg p \vee q) \vdash p$

Question 2. Show the following sentences are *theorems* in propositional logic. (See slide 82 from the Propositional Logic slides for the definition of “theorem”.)

(a) $(p \wedge q) \to p$

(b) $q \to (p \to (p \to (q \to p)))$

(c) $(p \to q) \to ((r \to s) \to ((p \wedge r) \to (q \wedge s)))$ (*Hint:* you may be able to ‘recycle’ and augment a proof from Question 1).

Question 3. Consider the following new proof rule, called RAA (for *Reductio ad Absurdum*, from Latin):

$$\frac{\boxed{\begin{array}{c}\neg A \\ \vdots \\ \bot\end{array}}}{A}\ \textit{(RAA)}$$

Note RAA is similar to $\neg i$, except the negation is in a different place.

(i) Show RAA can be derived from the other Natural Deduction rules.

(ii) Proponents of intuitionistic logic reject the rule RAA. Can you explain why? (*Hint:* see Propositional Logic slide 81).

Question 4. Prove the following mathematical statement (see Propositional Logic slides 88-92) by induction

$$\text{For every integer } i \text{ such that } i \geq 1,\ \ 1 + 2 + \cdots i = \frac{i(i+1)}{2}$$

Question 5. Show that there does **not** exist a proof using the Natural Deduction rules of the following:

$$p \to q, s \to t \vdash (p \vee s) \to (q \wedge t)$$

Hint: You may make use of the Completeness Theorem (specifically the Soundness Theorem).

Question 6. Assume $L = \{p, q, r\}$

(i) Which of the following sentences are Horn clauses:

(a) q

(b) $\neg p$

(c) $(p \wedge \neg r) \to \bot$

(d) $(q \wedge r) \to \neg p$

(e) $r \to \top$

(f) $(r \wedge q) \to p$

(ii) Which of the following sentences are Horn sentences:

(a) $q \to r$

(b) $(p \to \bot) \wedge ((q \wedge r) \to \bot)$

(c) $q \wedge r$

(d) $((p \wedge \neg q) \to r) \wedge (r \to q)$

(e) $(q \to r) \vee (p \to r)$

(f) $p \to (q \to r)$

(iii) For each of the sentences from part *(ii)* above that *are* Horn sentences, write down one Horn consequence of that sentence, that is not equal to the sentence itself.

Question 7. Let $L = \{p, q, r\}$ and let us write valuations as triples denoting the truth-values of p, q, r respectively (e.g., TFT is the valuation v such that $v(p) = T, v(q) = F$ and $v(r) = T$). For each of the following pairs of valuations v_1 and v_2, write down their consensus $v_1 \sqcap v_2$:

(a) $v_1 = TTT$, $v_2 = FTT$.

(b) $v_1 = FTF$, $v_2 = TFT$.

(c) $v_1 = FFF$, $v_2 = TFF$.

Question 8. For each of the following sets of valuations, determine whether they are closed under consensus. For the cases in which they are not closed under consensus, find the smallest set of valuations containing that set which *is* closed under consensus.

(a) $\{TFT, FTT, TTF\}$

(b) $\{TTT, FFF\}$

(c) $\{TFF\}$

Question 9. Assume again $L = \{p, q, r\}$. For each of the following sentences A in SL, determine whether they can be expressed as Horn sentences. That is, determine whether there is some Horn sentence B such that $A \equiv B$. In case they *can* be so expressed, try to find an appropriate equivalent Horn sentence.

(a) $A = (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$

(b) $A = \neg r \wedge (p \leftrightarrow \neg q)$