Present Certainty Equivalent Evaluation of Financial Hedging vs. Operational Hedging

November 2, 2009

1 Introduction

One could argue that all firms should make decision based on risk-neutral measures. As widely observed and highlighted by examples above, firms take uncertainty into account in both short and long term decisions Smith and Nau 1995 describe in detail the rational behind risk-sensitive decision making, they note that the decision makers are firm's managers, the managers has their own risk profiles, thus even the large public company are risk-sensitive.

Other reasons for risk aversion and impropriety of NPV include

(1) incomplete market

(2) consumer may not diversify

(3) other cost related to the variation of cash flow, including bankruptcy cost etc..

2 Literature Review

Smith 1998 (page 1698) categorizes the elements of Certainty Equivalence as risk premium and delay premium. Note that utility of NPV method used by Bouakiz and Sobel (1992) ignores the delay premium.

Smith and Nau (1995) show that for complete market (every risk is tradable), the following three methods, NPV, real option, and decision analysis are equivalent. That is, the optimal decisions, optimal trading strategies, and optimal financing strategies are the same. This is because tradability allows for risk neutral treatment. They note, however, that in case of non-tradable risk, like the demand in our case, NPV and real options approaches cannot be naively applied, but instead, use of decision analysis with integrated evaluation is more appropriate. What Smith and Nau label as decision analysis is actually identical in spirit to our approach and can be labeled as a special case of Epstein-Zin framework (to be verified???).

Fama (1970) uses time additive utility functions to study multi-period optimal consumption decisions. Blanchard and Mankiw (1988) use the same additive utility function to study certainty equivalence in consumption. They use both quadratic and exponential utility functions (as examples?).

3 Basic Model

Our basic model differs from some of the risk-averse literature in that we explicitly consider the beliefs and preferences of the MRN. We attribute beliefs and preferences to this MRN, as if it was privately owned and operated by a single owner/manager (note adding references). We will discuss the implications of this assumption in the conclusion section.

Similarly to Smith and Nau (1995), the MRN's beliefs are captured by his subjective probability of the random sources, while his preference is captured by a non-decreasing concave utility function. MRN's goal is to maximize his expected utility for consumptions. Smith and Nau use an additive exponential utility function to represent the decision maker's preferences.

The cash consumption takes place at the end of each period. The cash consumed is a result of (a) operation cash flow and (b) cash flow from security trading. and its quantity is determined by the operation cash flow and the security trading cash flow. The operational cash flow is affected by two decisions: the upfront allocation of capacity and production and transshipment quantities in each period. The security trading cash flow is the result of the MRN's trading of securities in the financial market.

The financial market has two traded securities, a risk free bond and a foreign currency. To simplify the notation and analysis, we assume that the foreign currency and bond pay the same dividend rate r. (Note: can we generalize to a different dividend rate in foreign currency???). The MRN's trading strategies are defined in space of borrowing and lending the risk free bond and the foreign currency in each period. In practice, while all MRNs do borrow and lend of risk-free bonds, not all of them engage in trading of risky foreign currency in order to hedge the exchange rate risk. That is, not all are involved in financial hedging.

Once the operational decisions and trading strategies are decided, the randomness comes from two sources: the product demand and the exchange rate uncertainty. We express the exchange rate as the value of the foreign currency in domestic currency. Without loss of generality, the random demand is assumed to be independent of the exchange rate because any correlation can be decomposed into a perfectly correlated source and an independent source. The price of the foreign currency measured in domestic currency is modeled as a

Markovian process, consistent with the model in Hull (1998).

Smith (1998) shows that without financial hedging, but with optimal trading in the risk-free bond, maximizing MRN's expected consumption utility is equivalent to the maximizing MRN's Present Certainty Equivalent Value in a general setting. We proceed to show that it is true in our setting without financial hedging.

4 Present Certainty Equivalent Valuation without Financial Hedging

4.1 Definition of Present Certainty Equivalent Value

The Present Certainty Equivalent Value of an uncertain cash flow is a deterministic cash value in the initial period such that the maximum expected utility generated in all periods by it in combination with bond investment is the same as the utility generated by the uncertain cash flow with bond investment. To operationalize this definition, we use he following notation:

- $n \in \{0, 1, \cdots, N\}$: period index
- X_n: period n uncertain cash flow that represents the decision maker's private belief
- x_n: realization of X_n
- β_n: cumulative shares of risk-free bond held in period n, after realization of X_n
- U_n: utility function that evaulates the decision maker's preferences for consumptions in periods n to N.

- $\mathcal{U}_n$: the maximum expected utility generated by the cash flow combined with bond investment for periods n to N.

- $PCEV_n$: the present certainty equivalent value in period n for cash flows in period n to N.

Note that the cumulative bond investment β_n is a strategy that depends on the realization of cash flow in the past. Furthermore $\beta_N = 0$ implies that there is no consumption after period N and all the borrowing has to be repaid in full in period N. In each period, the sequence of events is that MRN

1 receives income cash.

2 enters financial trade.

3 receives trade cash.

4 consumes the net cash.

For clarity of exposition we first assume that the risk-free bond pays 0 dividend if held for any period of time. This implies that the price of the bond is 1 is each period.

Definition 1 *The maximum expected utility for future uncertain cash flows and a given initial bond holding of β_{n-1} that carries over from the past, is defined as follows:*

$$\begin{aligned}&\mathcal{U}_n(X_n, \cdots, X_N|\beta_{n-1})\\&= \max_{\beta_n, \cdots, \beta_{N-1}} \mathrm{E}[U_n(X_n + \beta_{n-1} - \beta_n, \cdots, X_i + \beta_{i-1} - \beta_i, \cdots, X_N + \beta_{N-1})]\end{aligned} \tag{1}$$

Definition 2 *The present certainty equivalent value (PCEV) is defined as a* *constant such that,*

$$\mathcal{U}_n(PCEV_n(X_n, \cdots, X_N|\beta_{n-1}), 0, \cdots, 0|\beta_{n-1}) = \mathcal{U}_n(X_n, \cdots, X_N|\beta_{n-1}) \quad (2)$$

In general, calculating PCEV based on the above two definitions may be not easy. Therefore, we first assume that utility function is time additive, that is,

$$U_n(x_n, \cdots, x_N) = \sum_{i=n}^{N} u(x_n) \quad (3)$$

For time additive utility functions, (1) becomes:

$$\mathcal{U}_n(X_n, \cdots, X_N|\beta_{n-1}) = \max_{\beta_n, \cdots, \beta_{N-1}, \beta_N = 0} \sum_{i=n}^{N} \mathrm{E}[u(X_i + \beta_{i-1} - \beta_i)] \quad (4)$$

We further assume that the utility function has constant absolute risk aversion, that is,

$$u(x) = -\exp(-x) \quad (5)$$

This is exponential utility function and it has the following well-known properties which will be used repeatedly.

Proposition 3 *The exponential utility function u satisfies:*
(i) $u(x + y) = -u(x)u(y)$
(ii) The derivative $u'(x) = -u(x)$
(iii)The inverse $u^{-1}(-xy) = u^{-1}(x) + u^{-1}(y)$

To study PCEV of a stream of uncertain cash flows, we first consider properties of an uncertain cash flow in one period.

Lemma 4 *If the utility function is time additive and exponential, then*

$$\mathcal{U}_n(X_n, 0, \cdots, 0|\beta_{n-1}) = (N-n+1)\mathrm{E}_{X_n}[u(\frac{X_n+\beta_{n-1}}{N-n+1})] \tag{6}$$

The corresponding optimal bond shares held in each periods are

$$\beta_i = \frac{(N-i)(X_n+\beta_{n-1})}{N-n+1} \quad \forall i \in \{n, \cdots, N-1\} \tag{7}$$

Proof. According to the sequence of events, the decision of β_i ($\forall i \geq n$) is made after the realization of income cash X_n. Thus, the maximum utility problem becomes

$$\mathrm{E}\big[\max_{\beta_n, \cdots, \beta_{N-1}, \beta_N = 0}\{u(X_n+\beta_{n-1}-\beta_n) + \sum_{i=n+1}^{N} u(\beta_i - \beta_{i+1})\}|X_n\big]$$

The first order optimality condition is

$$\begin{aligned} u'(X_n+\beta_{n-1}-\beta_n) &= u'(\beta_n - \beta_{n+1}) \\ u'(\beta_i - \beta_{i+1}) &= u'(\beta_{i+1}-\beta_{i+2}) \quad \forall i \in \{n, \cdots, N-2\} \end{aligned}$$

This system equation has a unique solution of bond shares in Equation (7). Thus, these bond shares held in each period are optimal and the maximum utility is obtained. ■

Intuitively if the decision maker has some income cash in the initial period, then consuming this cash in equal installment in all periods maximizes his total consumption utility. Since this intuition holds true for the constant cash, e.g., $PCEV_n$, in the initial period, we have

Lemma 5 *The $PCEV_n$ of an uncertain cash in initial period n is*

$$PCEV_n(X_n, 0, \cdots, 0|\beta_{n-1}) = (N-n+1)u^{-1}\big(\mathrm{E}[u(\frac{X_n}{N-n+1})]\big) \tag{8}$$

and the $PCEV_n$ is independent of initial wealth β_{n-1}.

Proof. As a special case of Lemma 4, the maximum utility of $PCEV_n$ in the initial period n is

$$\mathcal{U}_n(PCEV_n, 0, \cdots, 0|\beta_{n-1}) = (N-n+1)u(\frac{PCEV_n + \beta_{n-1}}{N-n+1})$$

From (2), we have

$$\begin{aligned} u(\frac{PCEV_n + \beta_{n-1}}{N-n+1}) &= \mathrm{E}[u(\frac{X_n + \beta_{n-1}}{N-n+1})] \\ -u(\frac{PCEV_n}{N-n+1})u(\frac{\beta_{n-1}}{N-n+1}) &= -\mathrm{E}[u(\frac{X_n}{N-n+1})u(\frac{\beta_{n-1}}{N-n+1})] \end{aligned}$$

The last equality follows from (i) in Proposition 3. Finally because β_{n-1} is deterministic, canceling its utility in both sides of the equation concludes the proof. ■

This lemma implies that the $PCEV_n(X_n, 0, \cdots, 0|\beta_{n-1})$ is independent of β_{n-1}. Therefore, we will write $PCEV_n$ without condition on β_{n-1}. This Lemma and the utility function Proposition (i) and (iii) immediate imply the following:

Lemma 6 *If δ is a deterministic cash flow, then*

$$PCEV_n(X_n + \delta, 0, \cdots, 0) = PCEV_n(X_n, 0, \cdots, 0) + \delta$$

With this δ-property, we are ready to generalize the independence property to multiple future uncertain cash flows and to develop a recursive procedure for computing its PCEV.

Lemma 7 *If the cash flow in each period is independent, then $PCEV_n$ is independent of β_{n-1} and can be computed recursively, that is,*

$$PCEV_n(X_n, \cdots, X_N|\beta_{n-1}) = PCEV_n(X_n, \cdots, X_N) \tag{9}$$

$$PCEV_n(X_n, \cdots, X_N|\beta_{n-1}) = PCEV_n(X_n, 0, \cdots, 0) + PCEV_{n+1}(X_{n+1}, \cdots, X_N) \tag{10}$$

Proof. : The Lemma is true when $n = N$. Now suppose they are true for all $n \leq N$, we proceed to show both statements are true for $n-1$. We start with the second statement, which at $n-1$ becomes:

$$PCEV_{n-1}(X_{n-1}, X_n, \cdots, X_N|\beta_{n-2}) = PCEV_{n-1}(X_{n-1}, 0, \cdots, 0) + PCEV_n(X_n, \cdots, X_N)$$

It is equivalent to show that the maximum utilities generated by the two sides of the above equations are equal. We start with the maximum utility generated by the left side. By definition Eq. (1) for $n-1$, this maximum utility for any given bond shares β_{n-2} is:

$$\begin{aligned}
&\mathcal{U}_{n-1}(PCEV_{n-1}(X_{n-1}, \cdots, X_N|\beta_{n-2}), 0, \ldots, 0|\beta_{n-2})\\
&= \mathcal{U}_{n-1}(X_{n-1}, \cdots, X_N|\beta_{n-2}) = \max_{\beta_{n-1}, \cdots, \beta_{N-1}, \beta_N = 0} \mathrm{E}[\sum_{i=n-1}^{N} u(X_i + \beta_{i-1} - \beta_i)]\\
&= \max_{\beta_{n-1}}\{\mathrm{E}_{X_{n-1}}[u(X_{n-1} + \beta_{n-2} - \beta_{n-1}) + \max_{\beta_n, \cdots, \beta_{N-1}, \beta_N = 0}\{\mathrm{E}[\sum_{i=n}^{N} u(X_i + \beta_{i-1} - \beta_i)]\}]\}\\
&= \max_{\beta_{n-1}}\{\mathrm{E}_{X_{n-1}}[u(X_{n-1} + \beta_{n-2} - \beta_{n-1}) + \mathcal{U}_n(X_n, \cdots, X_N|\beta_{n-1})]\} \text{by Eq. (1)}\\
&= \max_{\beta_{n-1}}\{\mathrm{E}_{X_{n-1}}[u(X_{n-1} + \beta_{n-2} - \beta_{n-1}) + \mathcal{U}_n(PCEV_n(X_n, \cdots, X_N), 0, \cdots, 0)|\beta_{n-1})]\}\\
&= \max_{\beta_{n-1}, \cdots, \beta_{N-1}, \beta_N = 0}\{\mathrm{E}[u(X_{n-1} + \beta_{n-2} - \beta_{n-1})\\
&+ u(PCEV_n(X_n, \cdots, X_N) + \beta_{n-1} - \beta_n) + \sum_{i=n+1}^{N} u(\beta_{i-1} - \beta_i)]\}
\end{aligned}$$

where the last equality follows from the definition Eq. (1) again and the second to the last equality from the definition Eq. (2) and induction assumption that $PCEV_n$ is independent of β_{n-1}. Using variable substitution $\beta'_{n-1} = \beta_{n-1} + PCEV_n(X_n, \cdots, X_N)$, the above expression becomes

$$\begin{aligned}
&= \max_{\beta'_{n-1},\cdots,\beta_{N-1},\beta_N=0} \{\mathrm{E}[u(X_{n-1} + PCEV_n(X_n, \cdots, X_N) + \beta_{n-2} - \beta'_{n-1}) \\
&\quad +u(\beta'_{n-1} - \beta_n) + \sum_{i=n+1}^{N} u(\beta_{i-1} - \beta_i)]\} \\
&= \mathcal{U}_{n-1}(X_{n-1} + PCEV_n(X_n, \cdots, X_N), 0, \cdots, 0|\beta_{n-2}) \\
&= \mathcal{U}_{n-1}(PCEV_{n-1}(X_{n-1}, 0, \cdots, 0) + PCEV_n(X_n, \cdots, X_N), 0 \cdots, 0|\beta_{n-2})
\end{aligned}$$

The second to the last equality follows from the Eq. (1). The last equality follows from the delta property in Lemma 6 because $PCEV_n$ is a deterministic cash flow, which concludes the induction proof of the second statement. The truth of the first statement for $n-1$ follows immediately from the truth of the second statement. ■

This lemma immediately imply

Theorem 8 *If the uncertain cash flow is independent in each period, then*

$$PCEV_n(X_n, \cdots, X_N) = \sum_{i=n}^{N} PCEV_i(X_i, 0, \cdots, 0)$$

4.2 Markov Modulated Cash Flow Example for Two Periods

In this section we consider the cash flow to be a Markov Process. Since the future cash flow's distribution depends on realization of the past cash flow, both the maximum utility and the PCEV depend on the past cash flow realization. Since the cash flow is a Markov process, these values depend only on the cash flow realization in the immediate past period. We have to change maximum utility definition Eq. (1) and PCEV definition Eq. (2) slightly to reflect this dependency as follow:

Similar to Eq. (1), the maximum utility is

$$\begin{aligned}
&\mathcal{U}_n(X_n, \cdots, X_N | x_{n-1}, \beta_{n-1}) \\
&= \max_{\beta_n, \cdots, \beta_{N-1}} \mathrm{E}[U_n(X_n + \beta_{n-1} - \beta_n, \cdots, X_i + \beta_{i-1} - \beta_i, \cdots, X_N + \beta_{N-1}) | X_{n-1} = x_{n-1}]
\end{aligned} \tag{11}$$

Similar to Eq. (2), the present certainty equivalent value definition becomes:

$$\mathcal{U}_n(PCEV_n(X_n, \cdots, X_N | x_{n-1}, \beta_{n-1}), 0, \cdots, 0 | \beta_{n-1}) = \mathcal{U}_n(X_n, \cdots, X_N | x_{n-1}, \beta_{n-1}) \tag{12}$$

We first study a two-period problem so that we can see the structural properties that will be generalized to N periods. Let $N = 1$. We face two cash flows X_0, X_1, where X_1's distribution is a function of X_0.

4.2.1 $PCEV_1(X_1 | x_0, \beta_0(x_0))$

By definition Eq. (11), the maximum utility generated by PCEV is

$$\mathcal{U}_1(PCEV_1(X_1 | x_0, \beta_0(x_0)) | \beta_0(x_0)) = u(PCEV_1(X_1 | x_0, \beta_0(x_0)) + \beta_0(x_0))$$

and similarly the maximum utility generated by the uncertain cash flow X_1 is,

$$\begin{aligned}
\mathcal{U}_1(X_1 | x_0, \beta_0(x_0)) &= \mathrm{E}_{X_1}[u(X_1 + \beta_0(x_0)) | X_0 = x_0] \\
&= \mathrm{E}_{X_1}[-u(X_1) u(\beta_0) | X_0 = x_0] \\
&= -\mathrm{E}_{X_1}[u(X_1) | X_0 = x_0] u(\beta_0(x_0) | X_0 = x_0)
\end{aligned}$$

By the PCEV definition Eq. (12), we have:

$$PCEV_1(X_1|x_0, \beta_0(x_0)) + \beta_0(x_0) = u^{-1}(\mathrm{E}_{X_1}[u(X_1)|X_0 = x_0]) + \beta_0(x_0)$$

Therefore the PCEV again does not depend on the bond investment carry-over from the past and we have:

$$PCEV_1(X_1|x_0) = u^{-1}(\mathrm{E}_{X_1}[u(X_1)|X_0 = x_0]) \tag{13}$$

4.2.2 $PCEV_0(X_0, 0|\beta_{-1})$

Now we discuss a single cash flow at time 0 with initial wealth β_{-1}. Again by definition Eq. (11), the utility generated by the certain cash flow is:

$$\begin{aligned}\mathcal{U}_0(PCEV_0(X_0, 0|\beta_{-1}), 0|\beta_{-1}) &= \max_{\beta_0}\{u(PCEV_0(X_0, 0|\beta_{-1}) + \beta_{-1} - \beta_0) + u(\beta_0)\} \\ &= 2u((PCEV_0(X_0, 0|\beta_{-1}) + \beta_{-1})/2)\end{aligned}$$

Again by Eq. (11), the utility generated by the single uncertain cash flow is:

$$\begin{aligned}\mathcal{U}_0(X_0, 0|\beta_{-1}) &= \max_{\beta_0(X_0)}\{\mathrm{E}[u(X_0 + \beta_{-1} - \beta_0(X_0) + u(\beta_0(X_0))]\} \\ &= 2\mathrm{E}[u((X_0 + \beta_{-1})/2)] \\ &= -2\mathrm{E}[u(X_0/2)]u(\beta_{-1}/2)\end{aligned}$$

The last inequality follows from independency of β_{-1} on X_0 and the property (i) of the exponential utility function, and the second to the last inequality from the first order optimality condition and property (ii) of the exponential utility function.

By PCEV definition Eq. (12), we can find the PCEV:

$$\begin{aligned}PCEV_0(X_0, 0|\beta_{-1}) + \beta_{-1} &= 2\{u^{-1}(-\mathrm{E}[u(X_0/2)]u(\beta_1/2))\} \\ &= 2u^{-1}(\mathrm{E}[u(X_0/2)]) + \beta_{-1}\text{by Property (iii)}\end{aligned}$$

Therefore we have the PCEV value independent of past investment (initial wealth) and specifically,

$$PCEV_0(X_0, 0) = 2u^{-1}(\mathrm{E}[u(X_0/2)]) \tag{14}$$

4.2.3 $PCEV_0(X_0, X_1|\beta_{-1})$

Finally we study the PCEV of two uncertain cash flows with initial wealth. We start with definition Eq. (11) for PCEV,

$$\begin{aligned}\mathcal{U}_0(PCEV_0(X_0, X_1|\beta_{-1}), 0|\beta_{-1}) &= \max_{\beta_0}\{u(PCEV_0(X_0, X_1|\beta_{-1}) + \beta_{-1} - \beta_0) + u(\beta_0)\} \\ &= 2u((PCEV_0(X_0, X_1|\beta_{-1}) + \beta_{-1})/2)\end{aligned}$$

Now we use definition Eq. (11) for the two uncertain cash flows,

$$\begin{aligned}\mathcal{U}_0(X_0, X_1|\beta_{-1}) &= \max_{\beta_0(X_0)}\{\mathrm{E}_{X_0,X_1}[u(X_0 + \beta_{-1} - \beta_0(X_0) + u(X_1 + \beta_0(X_0))]\} \\ &= \max_{\beta_0(X_0)}\{\mathrm{E}_{X_0}[u(X_0 + \beta_{-1} - \beta_0(X_0)) + \mathrm{E}_{X_1}[u(X_1 + \beta_0(X_0))|X_0]]\} \\ &= \max_{\beta_0(X_0)}\{\mathrm{E}_{X_0}[u(X_0 + \beta_{-1} - \beta_0(X_0)) + u(PCEV_1(X_1|X_0) + \beta_0(X_0))]\} \\ &= 2\mathrm{E}_{X_0}[u((X_0 + PCEV_1(X_1|X_0) + \beta_{-1})/2)] \\ &= -2\mathrm{E}_{X_0}[u((X_0 + PCEV_1(X_1|X_0))/2)]u(\beta_{-1}/2)\end{aligned}$$

Using the PCEV definition Eq. (12), we have:

$$PCEV_0(X_0, X_1|\beta_{-1}) + \beta_{-1} = 2u^{-1}(\mathrm{E}_{X_0}[u((X_0 + PCEV_1(X_1|X_0))/2)]) + \beta_{-1}$$

Thus we have the PCEV is independent of initial wealth, and we further have,

$$
\begin{aligned}
PCEV_0(X_0, X_1) = &\quad 2u^{-1}(\mathrm{E}_{X_0}[u((X_0 + PCEV_1(X_1|X_0))/2)]) \\
= &\quad PCEV_0(X_0 + PCEV_1(X_1|X_0), 0) \text{by Eq. (14)}
\end{aligned} \tag{15}
$$

4.3 PCEV of Markov Stochastic Cash Flow

We want to show the definitions in Eq. (11) and Eq. (12), and the time additive utility function imply first the PCEV value is independent of initial wealth and second the PCEV value can be computed recursively. In order to study these properties of a stream of cash flows, we start with a single cash flow.

Lemma 9 *If utility function is time additive exponential, then*

$$
PCEV_n(X_n, 0, \cdots, 0|x_{n-1}, \beta_{n-1}) = (N-n+1)u^{-1}(\mathrm{E}_{X_n}[u(\frac{X_n}{N-n+1})|X_{n-1} = x_{n-1}]) \tag{16}
$$

$$
PCEV_n(X_n, 0, \cdots, 0|x_{n-1}) = PCEV_n(X_n, 0, \cdots, 0|x_{n-1}, \beta_{n-1}) \tag{17}
$$

Proof. For any cash flow Y_n that may depend on x_{n-1}, we have by Eq. (11),

$$
\mathcal{U}_n(Y_n, 0, \cdots, 0|x_{n-1}, \beta_{n-1}) = -(N-n+1)\mathrm{E}_{Y_n}[u(\frac{Y_n}{N-n+1})|X_{n-1} = x_{n-1}])u(\frac{\beta_{n-1}}{N-n+1}) \tag{18}
$$

The equality follows from the Property (i) and (ii) of the exponential utility function and the first order optimality condition.

Since Y_n is arbitrary, We can let $Y_n = PCEV_n(X_n, 0, \cdots, 0|x_{n-1}, \beta_{n-1})$. Since it is a deterministic cash flow, the above expression is reduced to

$$
-(N-n+1)u(\frac{PCEV_n(X_n, 0, \cdots, 0|x_{n-1}, \beta_{n-1})}{N-n+1})u(\frac{\beta_{n-1}}{N-n+1})
$$

Now letting $Y_n = X_n$ and using Eq. (12) imply the first statement. The second statement follows from the first statement immediately. ■

Now we are ready to study the PCEV of a stream of cash flows:

Theorem 10 *If the decision maker's preference is represented by the time additive utility function, then*
(1) $PCEV_n$ is independent of β_{n-1} and

$$\begin{aligned}(2) & PCEV_n(X_n, \cdots, X_N | x_{n-1}, \beta_{n-1}) \\ & = PCEV_n(X_n + PCEV_{n+1}(X_{n+1}, \cdots, X_N | X_n), 0, \cdots, 0 | x_{n-1}, \beta_{n-1}) \quad (19)\end{aligned}$$

Proof. We prove above statements by induction. Truth for $n = N$ is easily verified by previous Lemma because it is a single cash flow case. Now suppose truth holds for $n + 1$, we proceed to show the truth of the second statement for n first.

By Eq.(11) and Eq.(12), the maximum utility generated by the PCEV in the left side of the second statement is:

$$
\begin{aligned}
& \mathcal{U}_n(X_n,\cdots,X_N|x_{n-1},\beta_{n-1}) \\
= & \max_{\beta_n(X_n),\cdots,\beta_{N-1}(X_{N-1})} \mathrm{E}[U_n(X_n+\beta_{n-1}-\beta_n,\cdots,X_i+\beta_{i-1}-\beta_i,\cdots,X_N+\beta_{N-1})|X_{n-1}=x_{n-1}] \\
= & \max_{\beta_n(X_n),\cdots,\beta_{N-1}(X_{N-1}),\beta_N=0} \mathrm{E}[\sum_{i=n}^{N} u(X_i+\beta_{i-1}-\beta_i)|X_{n-1}=x_{n-1}] \\
= & \max_{\beta_n(X_n)} \big\{\mathrm{E}_{X_n}[\{u(X_n+\beta_{n-1}-\beta_n) \\
& + \max_{\beta_{n+1}(X_{n+1}),\cdots,\beta_{N-1}(X_{N-1}),\beta_N=0} \mathrm{E}[\sum_{i=n+1}^{N} u(X_i+\beta_{i-1}-\beta_i)|X_n]\}|X_{n-1}=x_{n-1}]\big\} \\
= & \max_{\beta_n(X_n)} \big\{\mathrm{E}_{X_n}[\{u(X_n+\beta_{n-1}-\beta_n)+\mathcal{U}_{n+1}(X_{n+1},\cdots,X_N|X_n,\beta_n)\}|X_{n-1}=x_{n-1}]\big\} \\
= & \max_{\beta_n(X_n)} \big\{\mathrm{E}_{X_n}[\{u(X_n+\beta_{n-1}-\beta_n) \\
& +\mathcal{U}_{n+1}(PCEV_{n+1}(X_{n+1},\cdots,X_N|X_n,\beta_n),0,\cdots,0|\beta_n\}|X_{n-1}=x_{n-1}]\big\} \\
= & \max_{\beta_n(X_n),\beta_{N-1},\beta_N=0} \big\{\mathrm{E}_{X_n}[\{u(X_n+\beta_{n-1}-\beta_n) \\
& +u(PCEV_{n+1}(X_{n+1},\cdots,X_N|X_n,\beta_n)+\beta_n-\beta_{n+1})+\sum_{i=n+2}^{N} u(\beta_{i-1}-\beta_i)\}|X_{n-1}=x_{n-1}]\big\}
\end{aligned}
$$

By induction hypothesis $PCEV_{n+1}$ is independent of β_n, we can use the variable substitution $\beta'_n=\beta_n+PCEV_{n+1}(X_{n+1},\cdots,X_N|X_n,\beta'_n)$, the above expression then becomes:

$$
\mathcal{U}_n(X_n+PCEV_{n+1}(X_{n+1},\cdots,X_N|X_n,\beta'_n),0,\cdots,0|x_{n-1},\beta_{n-1})
$$

The definition Eq.(12) concludes the proof of the second statement. The truth of the first statement follows from the second statement and Lemma 9. ■

The independence requires the property of time additive and exponential property, but recursing requires the Markov property and time additive.

5 PCEV with Financial Hedging of a Financial Asset

In this section we consider the uncertainty of the decision maker's cash flow comes only from a financial security that is traded in the market. At the beginning of each period, the security price is realized through market trading, but its future price is uncertain depending on realization of uncertain future events. A typical example is the value of another county's currency that is traded in the currency exchange market.

Since the financial security future price is the only source of uncertainty and it is traded in a market, the decision maker can increase his utility by financial hedging that is to buy or sell the security in the market. When a well functioning financial market is in equilibrium, there is no arbitrage opportunities in trading strategy because any arbitrage opportunity vanishes when it is exploited. If the financial market consists of a risky financial security and a risk free bond and has no arbitrage, there exits a risk neutral measure such that all derivatives (contingent claims) on the risky security can be priced at present by taking expectation under this risk neutral measure. Formally for two period,

Proposition 11 *If there is no arbitrage opportunity in the trading of the risky security and the risk free bond return is 0, then there exits a risk neutral measure M such that:*

(a) $\mathrm{E}_0^M[S_1|S_0] = S_0$

(b) Let $C_{01}(S_0, S_1)$ be any derivative that is written on the security at time 0 and expires at time 1, then its price is $\mathrm{E}_0^M[C_{01}(S_0, S_1)|S_0]$ at period 0.

With this property of the security market, we proceed to study the properties of the present certainty equivalent value with financial hedging. Financial hedging is defined as any activity in the financial market that involves buying and selling the risky security, in combination with or without the risk free bond trading.

5.1 A Single Uncertain Cash Flow - One Period Problem

We start by considering the simplest case of a single uncertain cash flow in period 1 because the present time 0's cash flow is certain. In this simple case, the decision maker has cash flow 0 at period 0 and at period 1 cash flow X_1 that is a function of the security's price S_1. The present certainty equivalent value of this cash flow $(0, X_1(S_1))$ is defined through the maximum utility that the decision maker generates in combination with financial hedging. In analogy to Equation 1 the maximum utility is

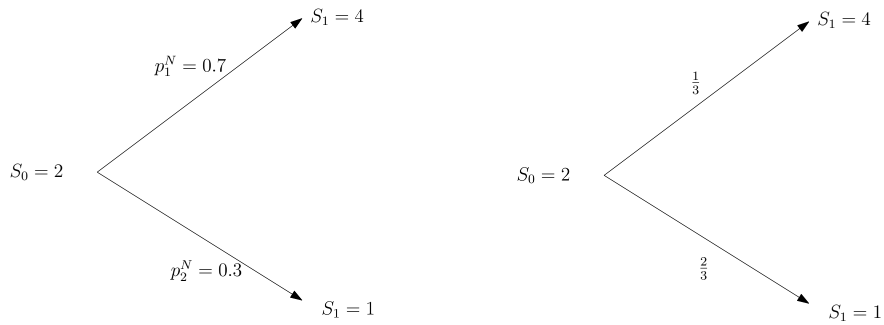
$$\begin{aligned}
&\mathcal{U}_0^N(0, X_1(S_1)|\beta_{-1}, S_0) \\
&= \max_{\beta_0(S_0), C_1(S_0,S_1)} \mathrm{E}_0^N\big[U\big(\beta_{-1} - \beta_0(S_0) - \mathrm{E}_0^M[C_{01}(S_0, S_1)|S_0], \\
&X_1(S_1) + C_{01}(S_0, S_1) + \beta_0(S_0)\big)|S_0\big]
\end{aligned} \tag{20}$$

Now in analogy to Equation 2, the $PCEV_0$ with financial hedging is defined as,

$$\mathcal{U}_0^N(PCEV_0^N, 0|\beta_{-1}, S_0) = \mathcal{U}_0^N(0, X_1(S_1)|\beta_{-1}, S_0) \tag{21}$$

In Equation 20, the first expectation is taken with respect to information at time 0 and natural probability of the security uncertainty, while the second expectation with respect to market risk neutral probability. To distinguish between the two different probability, we let N be the superscript standing for natural probability measure.

We now study whether the present certainty equivalent value is the same



(a) Natural probability measure (b) Market risk neutral measure

Figure 1: a risky asset price

under the two different probability measures in a simplest example. In this example, the decision maker's two period cash flow is 0 in period 0 and S_1 in period 1.

Figure 1 shows a risky asset whose value evolves from a deterministic $S_0 = 2$ in period 0 to an uncertain S_1 in period 1. Under natural probability measure (Figure 1 (a)), when the probability for the asset price to go up 0.7 or down is 0.3, it is evident that the asset price has a positive average return because the $\mathrm{E}_0^N(S_1|S_0) = 3.1 > S_0$.

5.1.1 Maximum Utility under Risk Neutral and Natural Probability Measures

By Equation 20, the maximum utility of the simple cash flow is:

$$\begin{aligned}\mathcal{U}_0^N(0, S_1|\beta_{-1}, S_0) = \max_{\beta_0(S_0), C_{01}(S_0,S_1)} \mathrm{E}_0^N\big[U\big(&\beta_{-1} - \beta_0 - E_0^M[C_{01}(S_0,S_1)|S_0], \\ &S_1 + C_1(S_0,S_1) + \beta_0(S_0)\big)|S_0\big]\end{aligned}$$

Now we solve the above optimization problem for the example given in Figure 1. Since there are only two realizations of the asset price, for notation clarity, let

$$x_1 = C_{01}(2,4), x_2 = C_{01}(2,1), \beta_{-1} = 0$$

Further more, the decision variable β_0 is redundant and can be ignored. When utility function is additive exponential, these notation reduces the problem to be

$$\max_{x_1,x_2}\{u(-1/3x_1 - 2/3x_2) + p_1^N u(4+x_1) + p_2^N u(1+x_2)\}$$

Since the objective function is concave, the first order condition suffices for solving optimal contract,

$$-1/3u'(-1/3x_1 - 2/3x_2) + p_1^N u'(4+x_1) = 0 \tag{22}$$

$$-2/3u'(-1/3x_1 - 2/3x_2) + p_2^N u'(1+x_2) = 0 \tag{23}$$

Since the utility is exponential, we have $u'(x_1) = -u(x_1)$, thus the above equations become

$$1/3u(-1/3x_1 - 2/3x_2) = p_1^N u(4+x_1) \tag{24}$$

$$2/3u(-1/3x_1 - 2/3x_2) = p_2^N u(1+x_2) \tag{25}$$

And the maximum utility is

$$\mathcal{U}_0^N(0, S_1|S_0, 0) = 2u(-1/3x_1 - 2/3x_2) = 2u(-(p_1^M x_1 + p_2^M x_2))$$

These optimal contract depends on the probability measure, and consequently the corresponding maximum utility clearly depends on the probability measure too. We first proceed to analyze the optimal contract and maximum utility when the natural probability measure coincides with the risk neutral probability measure because it is easier to solve.

Proposition 12 *If the $N = M$, then $x_1 = -3$, $x_2 = 0$.*

This solution represents the decision maker shorts a call option on the asset at time 0: if the asset value turns out to be 4, he will pay 3 to the counter party; else he pays nothing. The value of this call option is 1 under risk neutral measure. With this call option, the decision maker consumes 1$ in period 0 and 1$ in period 1, regardless of the realization of the risky asset price.

What would happen if the Natural probability measure is different from the risk neutral measure:

Proposition 13 *If $p_1^N = 0.7$ and $p_2^N = 0.3$, then $x_1 = -2.55$ and $x_2 = -0.40$*

This financial transaction can be viewed as the decision maker shorts a digital call option that pays the counter party 2.15$ if the asset value goes up to 4$ and 0$ otherwise at time 1. He also borrows 0.40$. This combination of transactions, allows him to consume equally between period 0 and period 1.

It is evident from this example, the optimal financial hedging contract for the maximum utility satisfies the following property.

$$p_i^M u(-\sum_{j=1}^{2} p_j^M x_j) = p_i^N u(s_i + x_i) \quad \forall i \in \{1, 2\} \tag{26}$$

The left side of the equation is the multiplication of the utility in period 0 times the risk neutral probability of an event. The right side of the equation is the expected utility under the same event of the natural probability measure. The optimal consumption cash flow equalizes them.

Under the optimal consumption cash flow, the maximum utility is,

$$\mathcal{U}_0^N(0, S_1|\beta_{-1}, S_0) = 2u(-\sum_{j=1}^{2} p_j^M x_j) \tag{27}$$

This maximum utility implies that in each period 0 and 1, the utility is the same.

5.1.2 Present Certainty Equivalent Value

In this section we show that independent of the probability measure, the present certainty equivalent value of the cash flow $(0, S_1)$ is $\mathrm{E}_0^M[S_1]$, the present expected value of S_1 under risk-neutral measure. We start with the study of the maximum utility under a present certainty equivalent value under a natural probability measure.

By Equation 20, the maximum utility of the present certainty equivalent value $PCEV_0^N$ is:

$$\begin{aligned}
&\mathcal{U}_0^N(PCEV_0^N, 0|\beta_{-1}, S_0) \\
&= \max_{\beta_0(S_0), C_{01}(S_0,S_1)} \mathrm{E}_0^N[U(\beta_{-1} + PCEV_0^N - \beta_0(S_0) - E_0^M[C_{01}(S_0, S_1)|S_0], \\
&C_{01}(S_0, S_1) + \beta_0(S_0))|S_0]
\end{aligned}$$

Now we solve the above optimization problem similarly for the example given in Figure 1. Since there are only two realizations of the asset price, for notation clarity, let

$$y_1 = C_{01}(2, 4), y_2 = C_{01}(2, 1), \beta_{-1} = 0$$

And similarly β_0 is a redundant decision variable, thus the optimization problem becomes,

$$\max_{y_1,y_2}\{u(PCEV_0^N - \sum_{j=1}^{2} p_j^M y_j) + p_1^N u(y_1) + p_2^N u(y_2)\}$$

The first order optimality condition is,

$$p_i^M u(PCEV_0^N - \sum_{j=1}^{2} p_j^M y_j) = p_i^N u(y_i) \quad \forall i \in \{1,2\} \tag{28}$$

And the maximum utility is

$$\mathcal{U}_0^N(PCEV_0^N, 0|\beta_{-1}, S_0) = 2u(PCEV_0^N - \sum_{j=1}^{2} p_j^M y_j) \tag{29}$$

By definition of the certainty equivalent, the maximum utility must equal. Equations (27), (29), (26) and (28) imply:

$$PCEV_0^N - \sum_{j=1}^{2} p_j^M y_j = -\sum_{j=1}^{2} p_j^M x_j \tag{30}$$

$$s_i + x_i = y_i \quad \forall i \in \{1,2\} \tag{31}$$

Therefore we have

$$PCEV_0^N = \sum_{j=1}^{2} p_j^M s_j = \mathrm{E}_0^M[S_1|S_0]$$

Lemma 14 *If the cash flow is* $(0, S_1)$ *where* S_1 *is the value of a tradable market security at time 1, then the present certainty equivalent value of the cash flow is its expected value under risk neutral measure, independent of the natural probability measure.*

In summary, while the optimal trading strategy to maximize the utility depends on the probability measures, the certainty equivalent value depends on the risk neutral measures only. My next work is to extend the results to a more general cash flow.

5.1.3 Cash flow as an arbitrary function of the security

In this section we extend the results of the previous section to cash flow as a function of the security, let the general cash flow in period 0 and 1 be

$$(0, X_1(S_1))$$

If we allow that in Equation (31) s_i is replaced by $X_1(s_i)$, then the following Lemma follows from the same analysis as in the previous subsection.

Lemma 15 *The present certainty equivalent value of cash flow* $(0, X_1(S_1))$ *is independent of the probability measure and*

$$PCEV_0^N(0, X_1(S_1)) = \mathrm{E}_0^M[X_1(S_1)|S_0]$$

Proof. We start with definition Equation 20. Since the contract contains the bond trading, the bond decision is redundant, the maximum utility is reduced to

$$\begin{aligned}
&\mathcal{U}_0^N(0, X_1(S_1)|\beta_{-1}, S_0) \\
&= \max_{C_{01}(S_0,S_1)} \mathrm{E}_0^N\big[U\big(\beta_{-1} - \mathrm{E}_0^M[C_{01}(S_0,S_1)|S_0], X_1(S_1) + C_{01}(S_0,S_1)\big)|S_0\big] \\
&= \max_{C'_{01}(S_0,S_1)} \mathrm{E}_0^N\big[U\big(\beta_{-1} - \mathrm{E}_0^M[C'_{01}(S_0,S_1) - X_1(S_1)|S_0], C'_{01}(S_0,S_1)\big)|S_0\big] \\
&= \mathcal{U}_0^N(\mathrm{E}_0^M[X_1(S_1)|S_0], 0)|\beta_{-1}, S_0)
\end{aligned}$$

Where the second equality is due to variable substitution $C'_{01}(S_0,S_1) = X_1(S_1) + C_{01}(S_0,S_1)$ and the last equality is due to the definition of the maximum utility.

The proof is concluded by Equation (21). ■

This proof sheds some lights on how to generalize the result into more general cash flow and time periods. The proof makes use of only the property of the price of derivatives under risk neutral measure and the definition of the present certainty equivalent value, it does not require any property of utility function. Thus this result is very general, it is true for risk-averse decision maker, risk-seeking decision maker, and any consumption preference of the decision maker.

To understand the intuition behind this proof, we first note that by definition the present certainty equivalent value of a stream of future cash flows is an amount of cash at present time that generates the same maximum utility as the stream of future cash flows. In another words, the consumer is indifferent between the present certainty equivalent value now and the stream of future cash flows. It implies that the present certainty equivalent value is the cash value of the stream of future cash flows.

Second when the stream of future cash flows are the function of the underlying security only, they are contingent claims on the underlying security, thus they can be replicated using a portfolio of derivatives of the underlying security. Thus the present certainty equivalent value (the cash price) of the stream of future cash flows equals the cash value of its replicating portfolio of derivatives.

Thirdly the cash value of each derivative is its expected value under risk-neutral measure under complete market. Thus the cash value of each derivative is independent of probability measure. The cash value of the replicating portfolio of derivatives is independent of probability measure.

Thus the certainty equivalent value of the stream of future cash flows is independent of the probability measure because it equals the cash value of the replicating portfolio of derivatives.

We now make give a second proof using the property of the property of the consumption cash flow at the maximum utility.

Assumption 16 *The resulted optimal consumption cash flow of problem in equation (1) is unique, i.e.,*

$$\mathrm{E}U(c_0, \cdots, c_N) = \mathrm{E}U(c'_0, \cdots, c'_N) \quad \textit{iff} \quad c'_i = c_i \; \forall i \in \{0, \cdots, N\} \tag{32}$$

where the c_i and c'_i are optimal consumption cash flow for any income cash flow.

Lemma 17 *If $\mathcal{U}(0, S_1|S_0) = \mathcal{U}(PCEV, 0|S_0)$ and let the optimal derivatives be*

$$C^*_{01}(S_0, S_1) = \arg \max_{C_{01}(S_0,S_1)} \mathrm{E}^N \big[U\big(- \mathrm{E}^M_0[C_{01}(S_0, S_1)|S_0], S_1 + C_{01}(S_0, S_1)\big)|S_0\big] \tag{33}$$

$$C'_{01}(S_0, S_1) = \arg \max_{C_{01}(S_0,S_1)} \mathrm{E}^N \big[U\big(PCEV - \mathrm{E}^M_0[C_{01}(S_0, S_1)|S_0], C_{01}(S_0, S_1)\big)|S_0\big] \tag{34}$$

then

$$(a)\; C'_{01}(S_0, S_1) = C^*_{01}(S_0, S_1) + S_1 \tag{35}$$

$$(b)\; PCEV = E^M_0[S_1|S_0] \tag{36}$$

Proof. Since the maximum utilities are equal at the optimal consumption cash flow, Assumption 16 implies the second period consumption cash flow must equal under both maximization problem, i.e., part (a) is true.

Since the first period consumption cash flow also must be equal, we have

$$PCEV = \mathrm{E}^M_0[C'_{01}(S_0, S_1) - C^*_{01}(S_0, S_1)|S_0]$$

Combining it with part (a) completes proof of part (b). ■

This intuition can be further understood by an argument similar to the arbitrage pricing. Suppose the PCEV of a cash flow is different from the price of the cash flow's replication derivatives, then the maximum utilities generated by the PCEV and the cash flow will be different, which contradicts to the definition of the PCEV.

To show this, we first consider that the PCEV is strictly greater than the price. In this case, we can use this PCEV to create another cash flow that has more cash in period 0 than the cash flow and in all other periods has identical cash. Thus this new cash flow must generate more utility than the original cash flow. However, this new cash flow must generate no more utility than the PCEV because it is one feasible income cash flow generated by the PCEV. Thus, the PCEV generates more utility than the original cash flow, which contradict the definition of the PCEV.

Second if the PCEV is strictly smaller than the price of the replicating derivative of portfolio of the cash flow. Then we can transform the replicating portfolio into its price at time 0 by selling the derivatives in market. Since the price is strictly greater than the PCEV, this price must generate strictly more utility than the PCEV, again this contradicts the definition of PCEV.

In summary when PCEV is differ from the price of the replicating derivatives of the cash flow and we trade the derivatives in the market, it causes contradiction to the definition of the PCEV. Thus, the PCEV of a cash flow is the price of its replicating derivatives.

5.1.4 PCEV of an income cash flow $(0, C_1(S_1))$ where the S_1 has three realizations

We now extend the PCEV result to a multiple security price realization for a two period problem. We can trade the derivatives of the security to maximize our utility. We first define the underlying security and its derivatives.

Let the security price in period 0 is $S_0 = s_0$ and in period 1 have 3 possible realizations. The probability of realization under risk neutral measure is,

$$p_i^M = \text{prob}^M(S_1 = s_i)\ \forall i \in \{1, 2, 3\}$$

This must satisfy,

$$\sum_{i=1}^{3} p_i^M s_i = s_0$$

Let the natural probability of the security price realization to be

$$p_i^N = \text{prob}^N(S_1 = s_i)\ \forall i \in \{1, 2, 3\}$$

Under the natural probability measure, the expected value of the security price in period 1 does not have to be its price in period 0.

A derivative of the security is defined by its payoff at period 1

$$D = d_i \text{ with probability } p_i^N, \quad \forall i \in \{1, 2, 3\}$$

and its price at period 0 is its expected value under risk neutral measure

$$\mathrm{E}_0^M[D] = \sum_{i=1}^{3} [p_i^M d_i]$$

Now since the cash flow in the period 1 is a function of the security price, thus it has three realizations corresponding to the three realizations of the security price in period 1.

$$\forall i \in \{1, 2, 3\}\ \text{prob}^N(C_1 = c_i) = p_i^N$$

Since the derivatives are available in the market, the maximum utility generated by cash flow $(0, C_1(S_1))$ is

$$\begin{aligned}\mathcal{U}(0, C_1(S_1)) &= \max_D \left\{ E_0^N [U(0 - \mathrm{E}_0^M[D], C_1(S_1) + D)] \right\} \\ &= \max_{d_i, \forall i} \left\{ \sum_{i=1}^{3} [p_i^N U(-\sum_{i=1}^{3} p_i^M d_i, c_i + d_i)] \right\} \end{aligned} \tag{37}$$

Lemma 18 *If D^* is optimal to Equation (37), then*

$$\bar{D}^* = D^* + C_1(S_1)$$

is optimal to

$$\max_{\bar{D}} \left\{ E_0^N [U(\mathrm{E}_0^M[C_1(S_1)] - \bar{D}, \bar{D})] \right\}$$

and

$$PCEV(0, C_1(S_1)) = \mathrm{E}_0^M[C_1(S_1)]$$

Proof. Using variable substitution $\bar{D} = D + C_1(S_1)$, the problem in Equation (37) is equivalent to

$$\max_{\bar{d}_i, \forall i} \left\{ \sum_{j=1}^{3} [p_j^N U(\mathrm{E}_0^M[C_1(S_1)] - \sum_{i=1}^{3} p_i^M \bar{d}_i, \bar{d}_i)] \right\}$$

Thus if D^* is optimal to the original problem, then $\bar{D}^*$ is optimal to the above problem. Furthermore, the maximum utilities of these two problems are equal, i.e.,

$$\mathcal{U}(0, C_1(S_1)) = \mathcal{U}(\mathrm{E}_0^M[C_1(S_1)], 0)$$

This completes the proof of the PCEV. ■

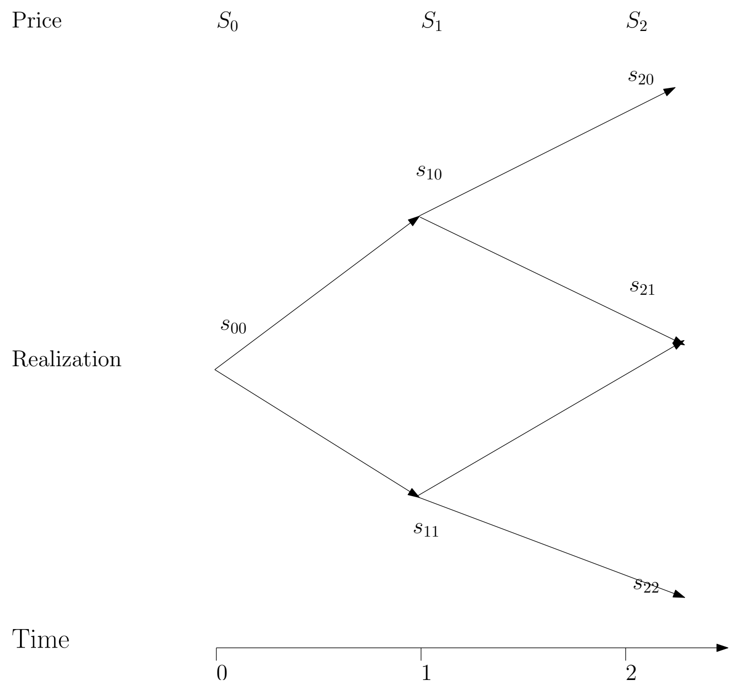


Figure 2: Two period price evolution of a security

5.2 PCEV of Three-Period Cash Flow

In this section we extend the result into three periods, the realizations of the risky security price and the related contracts are show in Figure 2.

In each period the security price goes up with risk neutral probability p_1 and goes down with risk neutral probability p_2.

5.2.1 PCEV of $(0, 0, t(S_2))$

When we face uncertainty future cash flow $(0, 0, t(S_2)$, we can use four derivatives to change the income cash flow to consumption cash flows. They are respectively,

- x_1, x_2: the payoff of derivatives that starts at time 0 and expires at time

1

- u_1, u_2, u_3: the payoff of derivatives that starts at time 0 and expires at time 2
- y_1, y_2: the payoff of derivatives that starts at time 1 and expires at time 2 and if the security price is s_{10} in period 1.
- z_1, z_2: the payoff of derivatives that starts at time 1 and expires at time 2 and if the security price is s_{11} in period 1.

There are total four consumption paths in this example, the consumption cash flow a given set of derivative contracts x, y, z, u are

path	1	2	3
1	$-x_1p_1 - x_2p_2 - u_1p_1^2 - 2u_2p_1p_2 - u_3p_2^2$	$x_1 - y_1p_1 - y_2p_2$	$y_1 + u_1 + t_1$
2	the same as path 1	the same as path 1	$y_2 + u_2 + t_2$
3	the same as path 1	$x_2 - z_1p_1 - z_2p_2$	$z_1 + u_2 + t_2$
4	the same as path 1	the same as path 3	$z_2 + u_3 + t_3$

Table 1: consumption cash flow with derivative x, y, z, u

We use variable substitution to get the PCEV of the given three period uncertain cash flow.

$$y_1' = y_1 + u_1 + t_1, y_2' = y_2 + u_2 + t_2, z_1' = z_1 + u_2 + t_2, z_2' = z_2 + u_3 + t_3$$

The consumption cash flow becomes

path	1	2	3
1	$-x_1p_1 - x_2p_2 - u_1p_1^2 - 2u_2p_1p_2 - u_3p_2^2$	$x_1 - (y_1' - t_1 - u_1)p_1 - (y_2' - t_2 - u_2)p_2$	y_1'
2	the same as path 1	the same as path 1	y_2'
3	the same as path 1	$x_2 - (z_1' - t_2 - u_2)p_1 - (z_2' - t_3 - u_3)p_2$	z_1'
4	the same as path 1	the same as path 3	z_2'

Table 2: consumption cash flow with derivative x, y', z', u

Finally we make variable substitution on x

$$x_1' = x_1-(y_1'-t_1-u_1)p_1-(y_2'-t_2-u_2)p_2, \quad x_2' = x_2-(z_1'-t_2-u_2)p_1-(z_2'-t_3-u_3)$$

The consumption cash flow becomes

path	1	2	3
1	$-x_1'p_1 - x_2'p_2 - (y_1'-t_1)p_1^2 - (y_2'+z_1'-2t_2)p_1p_2 - (z_2'-t_3)p_2^2$	x_1'	y_1'
2	the same as path 1	the same as path 1	y_2'
3	the same as path 1	x_2'	z_1'
4	the same as path 1	the same as path 2	z_2'

Table 3: consumption cash flow with derivative x', y', z'

Through those variable substitutions, we have two observations:
(1) we have moved the uncertain cash flow $t(S_2)$ into a deterministic cash at time 0. Thus, we have

$$PCEV_0(0,0,t(S_2)) = \mathrm{E}_0^M[t(S_2)|S_0]$$

(2) we find that the contract u is redundant, therefore we do not need to enter derivative contract across two or more periods for us to maximize the consumption utility. This redundancy is due to the fact that any such derivative can be replicated by the contracts in adjacent periods, in the above example, the contract u can be replicated by the following:

$$\bar{x}_1 = u_1p_1 + u_2p_2, \bar{x}_2 = u_2p_1 + u_3p_2$$

$$\bar{y}_1 = u_1, \bar{y}_2 = u_2, \bar{z}_1 = u_2, \bar{z}_2 = u_3$$

Next we formalize the above two observations. Since the bond shares are embedded in the derivative contract, they are redundant when we buy deriva-

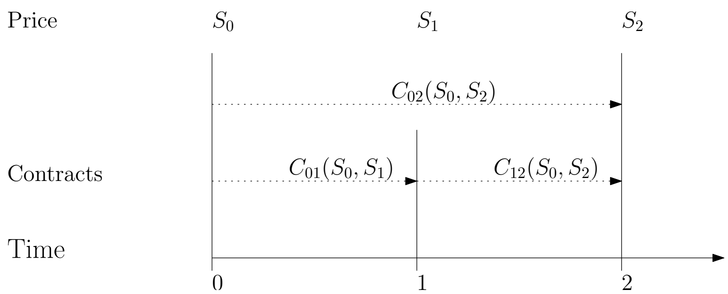


Figure 3: Three contracts to change the consumption cash flow

tives. Thus, the decision maker has three derivative contracts to manipulate his consumption cash flows: the first one changes the cash flow between time 0 and time 1, the second one changes the cash flow between time 1 and time 2, and the third changes the cash flow between time 0 and time 2, as illustrated in Figure 3. The decision maker's utility maximization problem becomes

$$
\begin{aligned}
&\mathcal{U}_0^N(0, 0, t(S_2)|S_0) \\
&= \max_{C_{01}(S_0,S_1), C_{02}(S_0,S_2), C_{12}(S_1,S_2)} \mathrm{E}_0^N[U(-E_0^M[C_{01}(S_0, S_1)|S_0] - E_0^M[C_{02}(S_0, S_2)|S_0], \\
&C_{01}(S_0, S_1) - E_1^M[C_{12}(S_1, S_2)|S_1], S_2 + C_{02}(S_0, S_2) + C_{12}(S_1, S_2)|S_0]
\end{aligned}
$$

Since the bond holding can be incooporated into the contracts, the problem is implified to

$$
\begin{aligned}
&\mathcal{U}_0^N(0, 0, t(S_2)|S_0) \\
&= \max_{C_{01}(S_0,S_1), C_{02}(S_0,S_2), C_{12}(S_1,S_2)} \mathrm{E}_0^N\big[U\big(- \mathrm{E}_0^M[C_{01}(S_0, S_1)|S_0] - \mathrm{E}_0^M[C_{02}(S_0, S_2)|S_0], \\
&C_{01}(S_0, S_1) - \mathrm{E}_1^M[C_{12}(S_1, S_2)|S_1], t(S_2) + C_{02}(S_0, S_2) + C_{12}(S_1, S_2)\big)|S_0\big]
\end{aligned}
$$

Further more it can be shown that the contract beween time 0 and time 2 is redundant because we can use variable substitution as follow:

$$C'_{12}(S_1, S_2) = C_{12}(S_1, S_2) + C_{02}(S_0, S_2)$$

$$C'_{01}(S_0, S_1) = C_{01}(S_0, S_1) + \mathrm{E}_1^M[C_{02}(S_0, S_2)|S_1]$$

Then we have

$$\begin{aligned}\mathrm{E}_0^M[C'_{01}(S_0, S_1)|S_0] &= \mathrm{E}_0^M[C_{01}(S_0, S_1)|S_0] + E_0^M[\mathrm{E}_1^M[C_{02}(S_0, S_2)|S_1]|S_0] \\ &= \mathrm{E}_0^M[C_{01}(S_0, S_1)|S_0] + \mathrm{E}_0^M[C_{02}(S_0, S_2)|S_0]\end{aligned}$$

The second equation is due to the fact that S_0 is a function of S_1. And we also have

$$C_{01}(S_0, S_1) - \mathrm{E}_1^M[C_{12}(S_1, S_2)|S_1] = C'_{01}(S_0, S_1) - \mathrm{E}_1^M[C'_{12}(S_1, S_2)|S_1]$$

Thus, the maximum utility problem is reduced to

$$\begin{aligned}\mathcal{U}_0^N(0, 0, t(S_2)|S_0) = \max_{C'_{01}(S_0,S_1), C''_{12}(S_1,S_2)} \mathrm{E}_0^N\big[U\big(&-\mathrm{E}_0^M[C'_{01}(S_0, S_1)|S_0], C'_{01}(S_0, S_1) \\ &- \mathrm{E}_1^M[C'_{12}(S_1, S_2)|S_1], t(S_2) + C'_{12}(S_1, S_2)\big)|S_0\big]\end{aligned} \tag{38}$$

Lemma 19 *The certainty equivalent value of three periods is*

$$PCEV_0^N(0, 0, t(S_2)|S_0) = \mathrm{E}_0^M[t(S_2)|S_0]$$

Proof. By Equation (38) and variable substitution $C''_{12}(S_1, S_2) = C'_{12}(S_1, S_2) + t(S_2)$

$$
\begin{aligned}
&\mathcal{U}_0^N(0, 0, t(S_2)|S_0) \\
&= \max_{C'_{01}(S_0,S_1), C''_{12}(S_1,S_2)} \mathrm{E}_0^N \big[U\big(- \mathrm{E}_0^M[C'_{01}(S_0, S_1)|S_0], \\
&C'_{01}(S_0, S_1) - \mathrm{E}_1^M[C''_{12}(S_1, S_2) - t(S_2)|S_1], C''_{12}(S_1, S_2)\big)|S_0\big] \\
&= \max_{C''_{01}(S_0,S_1), C''_{12}(S_1,S_2)} \mathrm{E}_0^N \big[U\big(\mathrm{E}_0^M[t(S_2)|S_0] - \mathrm{E}_0^M[C''_{01}(S_0, S_1)|S_0], \\
&C''_{01}(S_0, S_1) - \mathrm{E}_1^M[C''_{12}(S_1, S_2)|S_1], C''_{12}(S_1, S_2)\big)|S_0\big]
\end{aligned}
$$

where the second equation is by variable substitution $C''_{01}(S_0, S_1) = C'_{01}(S_0, S_1) + E_1^M[t(S_2)|S_1]$ and $E_0^M[E_1^M[t(S_2)|S_1]|S_0] = \mathrm{E}_0^M[S_2|S_0]$ because S_0 is a function of S_1. And finally the definition of the present certainty equivalent value concludes the proof. ■

5.3 Multiple Period Cash Flow

We start with definition of the maximum utility for a give cash flow under financial hedging. The financial hedging contract can be defined as follow,

Definition 20 *A hedging contract $C_{ij}(S_i, S_j)$ is an agreement between a decision maker and the other party, which becomes effective at time i and expires at time j; It represents that the decision maker pays the other party a cash price $P_{ij}(C_{ij})$ at time i and receives from the other party a security price dependent cash $C_{ij}(S_i, S_j)$ at time j.*

Proposition 21 *Under complete market assumption, the hedging contract cash price at time i is*

$$P_{ij}(C_{ij}) = E_i^M[C_{ij}(S_i, S_j)|S_i] \quad \forall i < j. \tag{39}$$

The maximum utility under probability measure $\boldsymbol{N}$ of a cash flow that is a function of the security price can then be expressed as follow,

$$\begin{aligned}
&\mathcal{U}_n^{\boldsymbol{N}}(X_n(S_n), \cdots, X_N(S_N)|S_n) \qquad (40)\\
&= \max_{C_{ij}(S_i,S_j)\forall n\le i<j, n<j\le N} \mathrm{E}_i^{\boldsymbol{N}}\big[U\big(-\sum_{i=n+1}^{N} P_{ni}(C_{nk}) + X_n(S_n),\\
&-\sum_{i=n+2}^{N} P_{n+1,i}(C_{n+1,i}) + X_{n+1}(S_{n+1}) + C_{n,n+1}(S_n, S_{n+1}), \cdots,\\
&X_N(S_N) + \sum_{i=n}^{N-1} C_{iN}(S_i, S_N)\big)|S_n\big]
\end{aligned}$$

The present certainty equivalent value under probability measure $\boldsymbol{N}$ is then defined as

$$\mathcal{U}_n^{\boldsymbol{N}}(PCEV_n^{\boldsymbol{N}}(X_n(S_n), \cdots, X_N(S_N)|S_n), 0, \cdots, 0) = \mathcal{U}_n^{\boldsymbol{N}}(X_n(S_n), \cdots, X_N(S_N)|S_n) \qquad (41)$$

These definitions are parallel to definitions Equation (1) and (2) where the bond trading strategy is replaced by derivative trading on the security. We next demonstrate the present certainty equivalent value under any utility function.

Since there are many derivative to achieve an optimal consumption cash flow, we first show that some derivative contracts are redundant.

Lemma 22 *If the security price is a Markov process, the cash flow of any derivative contracts of two or more periods can be replicated by the cash flow of derivative contracts of adjacent periods.*

Proof. It is equivalent to show that for a given derivative $C_{ij}(S_i, S_j), \forall j \ge i+2$, we can construct a series of derivatives $C_{k,k+1}(S_k, S_{k+1}), \forall i \le k < j$ to replicate its cash flow. We prove this by induction, consider the premise $j = i + 2$ first.

The cash flow generated by the $C_{i,i+2}(S_i, S_{i+2})$ is in Table 4. To generate the

Table 4: Cash flow generated by derivative $C_{i,i+2}(S_i, S_{i+2})$

period	cash flow
i	$-\mathrm{E}_i^M[C_{i,i+2}(S_i, S_{i+2})\vert S_i]$
$i+1$	0
$i+2$	$C_{i,i+2}(S_i, S_{i+2})$

same cash flow in period i, we buy the following derivative in period i

$$C_{i,i+1}(S_i, S_{i+1}) = \mathrm{E}_{i+1}^M[C_{i,i+2}(S_i, S_{i+2})|S_{i+1}]$$

because we pay for this derivative at period i the price

$$-\mathrm{E}_i^M[\mathrm{E}_{i+1}^M[C_{i,i+2}(S_i, S_{i+2})|S_{i+1}]|S_0] = -\mathrm{E}_i^M[C_{i,i+2}(S_i, S_{i+2})|S_i]$$

The equality follows from the Markov property.

To generate the same cash flow in period $i+2$, we buy the following derivatives in period $i+1$

$$C_{i+1,i+2}(S_{i+1}, S_{i+2}) = C_{i,i+2}(S_i, S_{i+2}|S_{i+1})$$

where $C_{i,i+2}(S_i, S_{i+2}|S_{i+1})$ represents payoff in period $i+2$ of contract $C_{i,i+2}$ if the current period is $i+1$. Now by construction, this derivative generate the same cash flow as $C_{i,i+2}$ in period $i+2$. To complete the proof of premise, we need only to show that the cash flow in period $i+1$ is 0 under those two replicating derivatives. The cash flow in period $i+1$ is

$$
\begin{aligned}
&C_{i,i+1}(S_i, S_{i+1}) - \mathrm{E}^M_{i+1}[C_{i+1,i+2}(S_{i+1}, S_{i+2})|S_{i+1}] \\
&= C_{i,i+1}(S_i, S_{i+1}) - E^M_{i+1}[C_{i,i+2}(S_i, S_{i+2})|S_{i+1}] = 0
\end{aligned}
$$

Thus, we have completed the proof for $j = i+2$. Now assume that the Lemma is true for any $j \geq i+2$, we proceed to show it is true for $j+1$.

First we apply the same steps as the case of $i+2$, we have that the contract $C_{i,j+1}(S_i, S_{j+1})$ can be replicated by contract $C_{i,j}(S_i, S_j)$ and $C_{j,j+1}(S_j, S_j+1)$. Applying the induction hypothesis on the contract C_{ij} completes the proof. ■

Lemma 22 reduces the number of derivative contracts and we focus on the derivatives that have one period valid lift, it allows us to derive the PCEV and optimal consumption cash flow in a clear way.

Theorem 23 *The PCEV of cash flows that are the functions of an underlying security is their total expected value under risk neutral measure, i.e.,*

$$
PCEV_n(X_n(S_n), \cdots, X_N(S_N)|S_n) = \mathrm{E}^M_n[\sum_{i=n}^{N} X_i(S_i)|S_n]. \tag{42}
$$

Proof. From Proposition 21 and Lemma 22, the maximum utility in Equation (40) becomes

$$
\max_{C_{i,i+1}(S_i,S_{i+1})\forall n \leq i < N} \mathrm{E}^{\boldsymbol{N}}_n[U(-\mathrm{E}^M_n[C_{n,n+1}(S_n, S_{n+1}|S_n] + X_n(S_n), \cdots,
$$

$$
C_{i-1,i}(S_{i-1}, S_i) + X_i(S_i) - \mathrm{E}^M_i[C_{i,i+1}(S_i, S_{i+1})|S_i], \cdots, C_{N-1,N}(S_{N-1}, S_N) + X_N(S_N)|S_n]
$$

Applying the following variable substitution

$$C'_{i-1,i} = C_{i-1,i} + \mathrm{E}_i^M[\sum_{k=i}^{N} X_k(S_k)|S_i] \quad \forall n \leq i \leq N$$

The maximum utility becomes,

$$\max_{C'_{i,i+1}(S_i,S_{i+1})\forall n \leq i < N} \mathrm{E}_n^{\boldsymbol{N}}[U(-E_n^M[-\mathrm{E}_n^M[C'_{n,n+1}(S_n, S_{n+1})|S_n] + \mathrm{E}_n^M[\sum_{i=n}^{N} X_i(S_i)|S_n],$$

$$\cdots, C'_{i-1,i}(S_{i-1}, S_i) - E_i^M[C'_{i,i+1}(S_i, S_{i+1}), \cdots, C'_{N-1,N}(S_{N-1}, S_N))|S_n]$$

The definition of PCEV according to Equation (41) completes the proof. ■

We now proceed to study the optimal consumption stream and its optimal trading strategies. We assume that the utility function is additive exponential.

Lemma 24 *If the utility function is additive exponential and the natural probability measure coincides with risk neutral measure, the optimal consumption cash flows is the same in each period and under reach random realization.*

Proof. Theorem 23 implies that a stream of optimal consumption cash flows of a stream of income cash flows are the same as the optimal consumption cash flows of the income cash flows' present certainty equivalent value. Thus it suffices if we have formally for a consumption from period n to N

$$\mathcal{U}_n(PCEV, 0, \cdots, 0) = (N - n + 1)u(\frac{PCEV}{N - n + 1}) \tag{43}$$

$$CF_i = \frac{PCEV}{N - n + 1} \forall n \leq i \leq N \tag{44}$$

where CF_i is the optimal consumption cash flow in period i. We prove this by induction.

We start with the premises that $n = N - 1$, then the utility maximization problem when $S_{N-1} = s_i$ is

$$\max_{C_{N-1,N}(S_{N-1},S_N)} \{u(PCEV - \mathrm{E}^M_{N-1}[C_{N-1,N}(S_{N-1},S_N)|S_{N-1} = s_i])$$
$$+ \mathrm{E}^M_{N-1}[u(C_{N-1,N}(S_{N-1},S_N))|S_{N-1} = s_i]\}$$

For convenience, let

$$CF_{ij} = C_{N-1,N}(S_{N-1} = s_i, S_N = s_j), \quad P_{ij} = \mathrm{Prob}(S_{N-1} = s_i, S_N = s_j)$$

Thus, the one period problem becomes,

$$\max_{CF_{ij}} \{u(PCEV - \sum_j (P_{ij}CF_{ij})) + \sum_j P_{ij}u(CF_{ij})\}$$

The first order optimality condition implies

$$P_{ij}u(PCEV - \sum_j (P_{ij}F_{ij})) = P_{ij}u(CF_{ij})\forall j$$

Therefore,

$$CF_{ij} = PCEV/2, \forall j$$

and the maximum utility becomes

$$2u(PCEV/2)$$

Now suppose the theorem is true for $n+1$, we next show its truth for n.

By Lemma 22, we apply Equation (40) to the PCEV cash flow at n and let

$S_n = s_i$.

$$\max_{C_{i,i+1}(S_i,S_{i+1}),\forall n\leq i<N} \mathrm{E}_n^M[u_n(PCEV - \mathrm{E}_n^M[C_{n,n+1}(S_n,S_{n+1})|S_n]) + \cdots + u_i(C_{i-1,i}(S_{i-1},S_i) - \mathrm{E}_i^M[C_{i,i+1}(S_i,S_{i+1})|S_i]) + \cdots + u_N(C_{N-1,N}(S_{N-1},S_N))|S_n]$$

Since the lemma is true for optimal consumption from period n to N, applying the induction hypothesis to the above problem, we have

$$\max_{C_{n,n+1}(S_n=s_i,S_{n+1})} \mathrm{E}_n^M[u_n(PCEV - \mathrm{E}_n^M[C_{n,n+1}(S_n,S_{n+1})|S_n=s_i]) + (N-n)u(\frac{C_{n,n+1}(S_n,S_{n+1})}{N-n})|S_n=s_i]$$

For convenience, lets define

$$CF_{ij} = \frac{C_{n,n+1}(S_n=s_i,S_{n+1}=s_j)}{N-n}, P_{ij} = \mathrm{Prob}(S_n=s_i,S_{n+1}=s_j)$$

Then the problem becomes,

$$\max_{CF_{ij}}\{u(PCEV-(N-n)\sum_j[P_{ij}CF_{ij}]) + \sum_j P_{ij}u(CF_{ij})\}$$

The first order necessary condition implies that

$$P_{ij}u(PCEV-(N-n)\sum_j[P_{ij}CF_{ij}]) = P_{ij}u(CF_{ij}), \forall j$$

Therefore, we have

$$PCEV-(N-n)CF_{ij} = CF_{ij}\forall j$$

The optimal $CF_{ij} = PCEV/(N-n+1)$ and substituting it back to the objective function completes the proof. ■

We now proceed to study the optimal consumption cash flow and maximum utility under the natural probability measure.

Lemma 25 *Under natural probability measure and the optimal trading, the optimal consumption cash flow and maximum utilities for periods n to N are*

$$\mathcal{U}_n(PCEV_n, 0, \cdots, 0) = (N-n+1)u\big(\frac{PCEV}{N-n+1} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{p_{ik}^M})]\big) \quad (45)$$

$$CF_{ij} = \alpha + \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M}) \quad (46)$$

$$\alpha = \frac{PCEV}{N-n+1} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})] \quad (47)$$

where the risk neutral probability P_{ik}^M = prob $(S_j = s_i, S_{j+1} = s_k)$ for all $n \leq j < N$ and P_{ik}^N is the natural probability.

Proof. We prove by induction. We first prove the premise, i.e., $n = N - 1$. The utility maximization problem under natural probability is

$$\max_{CF_{ij}} \{u(PCEV - \sum_j (P_{ij}^M CF_{ij})) + \sum_j P_{ij}^N u(CF_{ij})\}$$

The first order optimality condition implies

$$P_{ij}^M u(PCEV - \sum_k (P_{ik}^M CF_{ik})) = P_{ij}^N u(CF_{ij}) \forall j$$
$$u(PCEV - \sum_k (P_{ik}^M CF_{ik})) = u(CF_{ij} - \ln(P_{ij}^N / P_{ij}^M))$$

Therefore,

$$CF_{ij} = \alpha + \ln(\frac{P_{ij}^{\boldsymbol{N}}}{P_{ij}^M}), \forall j \quad \alpha = (PCEV - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})])/2$$

Suppose the lemma is true for $n-1$, we want to show it is true for n. By induction hypothesis we have in period n

$$\max_{C_{n,n+1}(S_n=s_i,S_{n+1})} \mathrm{E}_n^N[u(PCEV - \mathrm{E}_n^M[C_{n,n+1}(S_n,S_{n+1})|S_n=s_i]) \\ + (N-n)u(\frac{C_{n,n+1}(S_n,S_{n+1})}{N-n} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})])|S_n=s_i]$$

Let $CF_{ij} = \frac{C_{n,n+1}(S_n=s_i,S_{n+1}=s_j)}{N-n}$, the above problem becomes

$$\max_{CF_{ij}}\{u(PCEV - (N-n)\sum_j [P_{ij}^M CF_{ij}]) + \sum_j P_{ij}^N u(CF_{ij} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M}))\}$$

The first order optimality condition implies that

$$\begin{aligned} P_{ij}^M u(PCEV - (N-n)\sum_j [P_{ij}^M CF_{ij}]) &= P_{ij}^N u(CF_{ij} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})) \\ u(PCEV - (N-n)\sum_j [P_{ij}^M CF_{ij}]) &= u(CF_{ij} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})] - \ln(P_{ij}^N/P_{ij}^M)) \end{aligned}$$

Thus let $\alpha = CF_{ij} - \sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})] - \ln(P_{ij}^N/P_{ij}^M)$, we have

$$PCEV - (N-n)\alpha - (N-n+1)\sum_k [P_{ik}^M \ln(\frac{P_{ik}^{\boldsymbol{N}}}{P_{ik}^M})] = \alpha$$

Solving for α and substituting back complete the proof. ■