Classify Critical Points

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Project Description

An important problem in multivariable calculus is to find the extrema of a multivariable function. Extrema is the most basic feature of the graph of a function, includes maxima, minima, and saddle points. ClassifyCriticalPoints can find the critical points of a multivariable function and classify them.

Functions

SelectReal

`SelectReal[rules,vars]` selects from the list of lists of rules whose rules involves only real numbers.

```
SelectReal[rules_List,vars_List]:= Select[rules, Element[vars/.#, Reals]&]
```

HessianMatrix

`HessianMatrix[f,{vars}]` computes the Hessian matrix of the function `f` with respect to the variables `vars`

```
HessianMatrix[f_,vars_]:= Outer[D[f, #1, #2]&, vars, vars]
```

ClassifyCriticalPoints

```
ClassifyCriticalPoints[fun_,vars_List]:=
    Module[{soln,hmat,hmatn,decide,detlist1,detlist2},
    soln = SelectReal[Solve[Thread[D[fun,#]&/@vars==Table[0,{Length[vars]}]],vars],vars];
    hmat = HessianMatrix[fun,vars];
    hmatn[x_]:= hmat[[1;;x,1;;x]];
    detlist1 = Table[Det[hmatn[i]],{i,1,Length[vars]}];
    detlist2 = Table[(-1)^i Det[hmatn[i]],{i,1,Length[vars]}];
    decide[detlist1_List,detlist2_List]:=
            If[And@@Map[N[#]>0&,detlist1]==True,"Local Minimum",
            If[And@@Map[N[#]>0&,detlist2]==True,"Local Maximum",
            If[Or@@Map[N[#]==0&,detlist1]==True,"Inconclusive",
            "Saddle Point"]]];
    {decide[detlist1,detlist2],vars}/.soln
]
```

Notes

This project was inspired when I was taking Multivariable Calculus class in my freshman year. An important problem in multivariable calculus is to find the extrema of a multivariable function.

Here is how my professor introduced the necessary condition for extrema:

Stationary Point, Critical Point

A **stationary point** is a point where the gradient vanishes. A **critical point** is a stationary point or a point in the domain of the function where the gradient fails to exist.

Theorem

Suppose f is a differentiable function. If f has an extremum at an interior point (x_0, y_0) of its domain, then $\nabla f(x_0, y_0) = (0, 0)$.

The most usual way to classify critical points

Using the **second derivative test** to find the extrema:

Suppose that f has continuous second order partial derivatives and that f has a critical point at (x_0, y_0).

If $\partial_{x\,x} f(x_0, y_0)\, \partial_{y\,y} f(x_0, y_0) - \left(\partial_{x\,y} f(x_0, y_0)\right)^2 > 0$ and $\partial_{x\,x} f(x_0, y_0) > 0$, then f has a local min at $(0, 0)$.

If $\partial_{x\,x} f(x_0, y_0)\, \partial_{y\,y} f(x_0, y_0) - \left(\partial_{x\,y} f(x_0, y_0)\right)^2 > 0$ and $\partial_{x\,x} f(x_0, y_0) < 0$, then f has a local max at $(0, 0)$.

If $\partial_{x\,x} f(x_0, y_0)\, \partial_{y\,y} f(x_0, y_0) - \left(\partial_{x\,y} f(x_0, y_0)\right)^2 < 0$, then $(0, 0)$ is a saddle point of f.

A homework problem a Find and classify critical points

Find and classify the critical points of the function $f(x, y) = 8\,x\,y\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$.

Solution:

The gradient of f is

```
Clear[f, gradf, x, y]
```

$$\texttt{f[x\_, y\_] := } 8\,x\,y\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

```
gradf[x_, y_] = Grad[f[x, y], {x, y}]
```

$$\left\{-\frac{2\,x^2\,y}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + 8\,y\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}},\ -\frac{8\,x\,y^2}{9\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + 8\,x\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}\right\}$$

The critical points are

```
cpts = Solve[gradf[x, y] == {0, 0}]
```

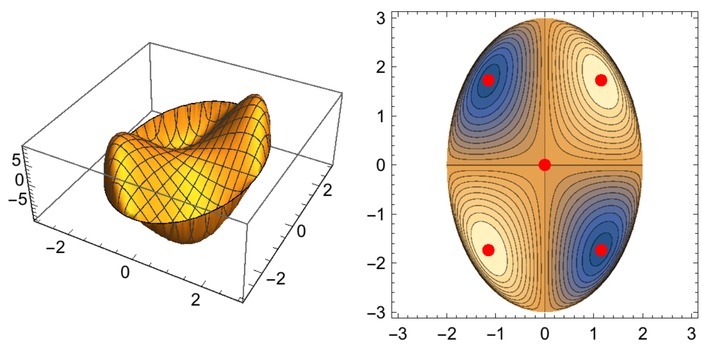
$$\left\{\{x \to 0,\ y \to 0\},\ \left\{x \to -\frac{2}{\sqrt{3}},\ y \to -\sqrt{3}\right\},\ \left\{x \to \frac{2}{\sqrt{3}},\ y \to -\sqrt{3}\right\},\ \left\{x \to -\frac{2}{\sqrt{3}},\ y \to \sqrt{3}\right\},\ \left\{x \to \frac{2}{\sqrt{3}},\ y \to \sqrt{3}\right\}\right\}$$

```
pts = {x, y} /. cpts
```

$$\left\{\{0, 0\},\ \left\{-\frac{2}{\sqrt{3}}, -\sqrt{3}\right\},\ \left\{\frac{2}{\sqrt{3}}, -\sqrt{3}\right\},\ \left\{-\frac{2}{\sqrt{3}}, \sqrt{3}\right\},\ \left\{\frac{2}{\sqrt{3}}, \sqrt{3}\right\}\right\}$$

Here is a plot and a contour plot of the function together with the critical points.

```
GraphicsRow[{Plot3D[f[x, y], {x, -3, 3}, {y, -3, 3}], ContourPlot[f[x, y], {x, -3, 3},
   {y, -3, 3}, Contours → 20, Epilog → {Red, PointSize[0.04], Point /@ pts}]}]
```



we can confirm with the second derivative test. First we compute the Hessian:

```
(hessian[x_, y_] = HessianMatrix[f[x, y], {x, y}]) // MatrixForm
```

$$\begin{pmatrix} -\frac{x^3 y}{2\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)^{3/2}} - \frac{6 x y}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} & -\frac{2 x^2 y^2}{9\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)^{3/2}} - \frac{2 x^2}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} - \frac{8 y^2}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} + 8\sqrt{1 \\ -\frac{2 x^2 y^2}{9\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)^{3/2}} - \frac{2 x^2}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} - \frac{8 y^2}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} + 8\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}} & -\frac{8 x y^3}{81\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)^{3/2}} - \frac{8 x y}{3\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} \end{pmatrix}$$

Evaluating the Hessian at the critical point we obtain

```
hcpts = hessian[x, y] /. cpts
```

$$\left\{\{\{0, 8\}, \{8, 0\}\},\ \left\{\left\{-16\sqrt{3}, -\frac{16}{\sqrt{3}}\right\}, \left\{-\frac{16}{\sqrt{3}}, -\frac{64}{3\sqrt{3}}\right\}\right\},\ \left\{\left\{16\sqrt{3}, -\frac{16}{\sqrt{3}}\right\}, \left\{-\frac{16}{\sqrt{3}}, \frac{64}{3\sqrt{3}}\right\}\right\},\ \left\{\left\{16\sqrt{3}, -\frac{16}{\sqrt{3}}\right\}, \left\{-\frac{16}{\sqrt{3}}, \frac{64}{3\sqrt{3}}\right\}\right\},\ \left\{\left\{-16\sqrt{3}, -\frac{16}{\sqrt{3}}\right\}, \left\{-\frac{16}{\sqrt{3}}, -\frac{64}{3\sqrt{3}}\right\}\right\}\right\}$$

```
Map[MatrixForm, hcpts]
```

$$\left\{\begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}, \begin{pmatrix} -16\sqrt{3} & -\frac{16}{\sqrt{3}} \\ -\frac{16}{\sqrt{3}} & -\frac{64}{3\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 16\sqrt{3} & -\frac{16}{\sqrt{3}} \\ -\frac{16}{\sqrt{3}} & \frac{64}{3\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 16\sqrt{3} & -\frac{16}{\sqrt{3}} \\ -\frac{16}{\sqrt{3}} & \frac{64}{3\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -16\sqrt{3} & -\frac{16}{\sqrt{3}} \\ -\frac{16}{\sqrt{3}} & -\frac{64}{3\sqrt{3}} \end{pmatrix}\right\}$$

```
Map[Det, hcpts]
```

{-64, 256, 256, 256, 256}

So the first critical point is saddle point, and max or min are the rest three. Examining the elements in the first row and column of the Hessian matrices we see that the second and the last one are max, and the third one and fourth one are min

At first, I solve this extrema kind of problems based on above procedure. After I took linear algebra and vector calculus, I wrote the following function, which is to generalize it to work for n-variables function.

```
ClassifyCriticalPoints[f[x, y], {x, y}]
```

$$\left\{\{\text{Inconclusive}, \{0, 0\}\}, \left\{\text{Local Maximum}, \left\{-\frac{2}{\sqrt{3}}, -\sqrt{3}\right\}\right\}, \left\{\text{Local Minimum}, \left\{\frac{2}{\sqrt{3}}, -\sqrt{3}\right\}\right\}, \left\{\text{Local Minimum}, \left\{-\frac{2}{\sqrt{3}}, \sqrt{3}\right\}\right\}, \left\{\text{Local Maximum}, \left\{\frac{2}{\sqrt{3}}, \sqrt{3}\right\}\right\}\right\}$$

Test

```
Clear[f, x, y, z]
f[x_, y_, z_] := x^4 + y^4 + z^4 - 4 x y z
```

```
ClassifyCriticalPoints[f[x, y, z], {x, y, z}]
```

{{Local Minimum, {-1, -1, 1}}, {Local Minimum, {-1, 1, -1}},
{Inconclusive, {0, 0, 0}}, {Local Minimum, {1, -1, -1}}, {Local Minimum, {1, 1, 1}}}

Source

Definition, theorem and problem are from my advisor Dennis Schneider's mulitivariable calculus textbook. It is not officially published.