1. 3.1.1

Proposition 3.2.1 (2nd order sufficiency conditions)

Assume that f and h are both C^2.

Let $x^* \in \mathbb{R}^n$, $\lambda^* \in \mathbb{R}^m$ satisfy $\quad \nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \qquad \nabla_\lambda \mathcal{L}(x^*, \lambda^*) = 0$

$$y' \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*)\, y > 0 \qquad \forall y \neq 0 \text{ with } \nabla h(x^*)' y = 0$$

Then x^* is a strict local minimum of f subject to $h(x) = 0$.

In fact, $\exists\ \gamma > 0$ and $\varepsilon > 0$ s.t. $\quad f(x) \geq f(x^*) + \frac{\gamma}{2} \|x - x^*\|^2 \ \forall x$ with $h(x) = 0$ and $\|x - x^*\| < \varepsilon$

(a) $\quad f(x) = \|x\|^2 \qquad h(x) = \sum_{i=1}^{n} x_i - 1$

$$\mathcal{L}(x, \lambda) = f(x) + \lambda h(x) = \sum x_i^2 + \lambda\left(\sum x_i - 1\right)$$

By Lagrange multiplier thm, $\nabla \mathcal{L}(x, \lambda) = 0$ will give us the critical pts.

$$\nabla \mathcal{L}(x, \lambda) = \nabla\left(\sum x_i^{*2} + \lambda^*\left(\sum x_i^* - 1\right)\right) = 0$$

$$2x_i^* + \lambda^* = 0$$

$$x_i^* = \frac{-\lambda^*}{2}$$

We have $\quad \sum x_i^* = -\frac{n\lambda^*}{2} = 1 \quad \Leftrightarrow \lambda^* = -\frac{2}{n} \qquad x_i = +\frac{\cancel{2}}{n} \cdot \frac{1}{\cancel{2}} = \frac{1}{n}$

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot & 2 \end{pmatrix} \qquad y' \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*)\, y = 2 \sum y_i^2 \geq 0 \quad \forall y \in V(x^*)$$

By proposition, x^* is a local minimum.

(b) $\quad f(x) = \sum_{i=1}^{n} x_i \qquad h(x) = \|x\|^2 - 1$

$$\mathcal{L}(x, \lambda) = f(x) + \lambda h(x) = \sum x_i + \lambda\left(\sum x_i^2 - 1\right)$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$$

$$\nabla\left(\sum x_i^* + \lambda^*\left(\sum x_i^{*2} - 1\right)\right) = 0$$

$$1 + 2\lambda^* x_i^* = 0$$

$$x_i^* = -\frac{1}{2\lambda^*} \qquad x_i^{*2} = \frac{1}{4\lambda^{*2}}$$

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 2\lambda^* & 0 & \cdots & 0 \\ 0 & 2\lambda^* & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot & 2\lambda^* \end{pmatrix} \qquad y' \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*)\, y = 2\lambda \sum y_i^2 \geq 0 \quad \forall y \in V(x^*)$$

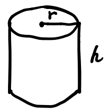
By proposition, x^* is a local minimum.

(a) $\lambda^* = -\frac{\sqrt{n}}{2} \qquad x_i^* = \frac{1}{\sqrt{n}} \qquad y' \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) \Rightarrow \quad y = -\sqrt{n} \sum y_i^2 \leq 0$

2. Industrial Design

A cylindrical can is to hold 4 inch^3 juice.

The cost per inch^2 of top + bottom is twice cost / square of side



top + bottom area = $2 \cdot \pi \cdot r^2$

side area = $2\pi r \cdot h$

minimize $S(r,h) = 2\pi r^2 + 2\pi r h$ subject to $\pi r^2 \cdot h = 4$

$\mathcal{L}(r,h,\lambda) = 2\pi r^2 + 2\pi r h - \lambda(\pi r^2 h - 4)$

$\nabla \mathcal{L}(r,h,\lambda) = \vec{0}$ solve,

$r = \sqrt[3]{\frac{2}{\pi}}$ $\quad h = 2\sqrt[3]{\frac{2}{\pi}}$ $\quad \lambda = 2^{2/3}\pi^{1/3}$

3. Duality

$$\min_{A'x \geq b} c'x \Leftrightarrow \max_{A\mu = c, \mu \geq 0} b'\mu$$

Proof: $(\Rightarrow)$

the dual function is given by

$$q(\mu) = \min_{x \in \mathbb{R}^n} \{ f(x) + \mu g(x) \} = \min_{x \in \mathbb{R}^n} \{ c'(x) + \mu(b - A'x) \} = \min_{x \in \mathbb{R}^n} \{ (c' - \mu A')x + \mu b \}$$

$$= \begin{cases} b'\mu & \text{if } c = \mu A \\ -\infty & \end{cases}$$

$\Leftrightarrow$ $\max q(\mu)$ subject to $\mu \geq 0$ or equivalently $\max_{A\mu = c, \mu \geq 0} b'\mu$

$(\Leftarrow)$

$$p(x) = \max_{\mu \geq 0} \{ f(\mu) + x\, g(\mu) \} = \max_{\mu \geq 0} \{ b'\mu + x(c' - A\mu) \} = \max_{\mu \geq 0} \{ (b' - A)\mu + c'x \}$$

$$= \begin{cases} c'x & \text{if } b - A'x \leq 0 \\ \infty & \end{cases}$$

4. 4.2.1

minimize $f(x) = \frac{1}{2}(x_1^2 - x_2^2) - 3x_2$ subject to $x_2 = 0$

(a) Calculate optimal solution and LM.

$$\mathcal{L}(x,\lambda) = \frac{1}{2}(x_1^2 - x_2^2) - 3x_2 + \lambda x_2$$
$$= \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + (\lambda - 3)x_2$$

$$\frac{\partial L}{\partial x_1} = x_1 = 0$$
$$\frac{\partial L}{\partial x_2} = -x_2 + \lambda - 3 = 0$$

$\vec{x}^* = (0,0)$ $\lambda^* = 3$

(b) For $k = 0, 1, 2, \ldots$ and $c^k = 10^{k+1}$, calculate and compare the iterates of the quadratic penalty method with $\lambda^k = 0$ $\forall k$ and multipliers with $\lambda^0 = 0$

$$\mathcal{L}(x,\lambda) = \frac{1}{2}(x_1^2 - x_2^2) - 3x_2 + \lambda x_2 + \frac{c}{2}x_2^2$$

$$\frac{\partial L}{\partial x_1^k} = x_1^k = 0$$

$$\frac{\partial L}{\partial x_2^k} = -x_2^k - 3 + \lambda^k + c^k x_2^k = 0 \qquad x_2^k = \frac{3-\lambda^k}{c^k - 1}$$

① $x^k = \left(0, \frac{3-\lambda^k}{c^k-1}\right)$

$c^k = 10^{k+1}$

$k=0 \Rightarrow c^0 = 10^{0+1} = 10$ $\quad x^0 = (0, \frac{1}{3})$

$k=1$ $\quad c^1 = 100$ $\quad x^1 = (0, \frac{1}{33})$

$k=2$ $\quad c^2 = 1000$ $\quad x^2 = (0, \frac{1}{333})$

② multiplier method with $\lambda^0 = 0$ $\qquad$ let $h(x) = \frac{3-\lambda^k}{c^k - 1}$

$k=0 \Rightarrow c^0 = 10$, $\lambda^0 = 0$ $\quad x^0 = (0, \frac{1}{3})$

$k=1 \Rightarrow c^1 = 100$ $\quad \lambda^1 = \lambda^0 + c^0 h(x_1^0) = 0 + 10 \cdot \frac{1}{3} = \frac{10}{3}$

$$x_2^1 = \frac{3 - \frac{10}{3}}{100 - 1} = -\frac{1}{297}$$

$k=2 \Rightarrow c^2 = 1000$

$$\lambda^2 = \lambda^1 + c^1 h(x_2^1) = \frac{10}{3} + 100 \cdot \left(-\frac{1}{297}\right) = 3$$

$x_2^2 = 0$ $\quad \Rightarrow \vec{x}_2 = (0,0)$

(d) Suppose c is take to be constant in the method of multipliers. For what values of c would the augmented Lagrangian have a minimum and for what values of c would the method converge.

$$\lambda^{k+1} = \lambda^k + c^k h(x_2^k) = \lambda^k + c^k\left(\frac{3-\lambda^k}{c^k-1}\right) = \frac{3c^k - \lambda^k}{c^k - 1}$$

$$\lambda^{k+1} - \lambda^* = \frac{3c^k - \lambda^k}{c^k - 1} - 3 = \frac{3c^k - \lambda^k - 3c^k + 3}{c^k - 1} = \frac{3-\lambda^k}{c^k - 1}$$

so min exist when $c^k > 1$.

method converge when $\frac{1}{c^k - 1} < 1 \quad \Rightarrow c^k > 2$.