Math 449: Numerical Applied Mathematics
Lecture 10

09/22/2017

Today's topic: LU Decomposition

Fast Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad 2\times 2 + 2\times 2 = 8$$

Strassen 1969

Cost of Gaussian Elimination

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{12}\cdot a_{21}}{a_{11}} \end{pmatrix}}_{U}$$

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} \end{pmatrix}}$$

LU decomposition of A

unit - lower triangular matrix.

$$L = \begin{pmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{pmatrix} \qquad L^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{pmatrix}$$

$$U = L^{-1}A$$
$$A = LU$$

more generally, $n\times n$ matrix

$U = L_{(N)} \cdots L_{(1)} A$ where each $L_{(i)}$ climates 1 entry below diagonal

$$L_{(i)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

need to eliminate $N = \frac{n(n-1)}{2} = (n-1) + (n-2) + \cdots + 1$

Eliminate each term w/ row operation

$$A = (L_N \cdots L_1)^{-1} \cdot U = L\cdot U \qquad A = \underbrace{(L_{(N)} \cdots L_{(1)})^{-1}}_{L} U$$

Note: LU decomposition only works if we never encounter a 0 on the diagonal.

Example. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

Let $u_{11} = 0$ $u_{12} = 1$ but then $l_{21}\cdot u_{11} = 0 \neq 1$

Why no solution for $l_{21}, u_{11}, u_{12}, u_{22}$?

$0 = u_{11} + 0 \Rightarrow u_{11} = 0$

$1 = u_{12} + 0 \Rightarrow u_{12} = 1$

Once we have $A = LU$, it's easy to solve $A\vec{x} = \vec{b}$

$$L\underbrace{U\vec{x}}_{\vec{y}} = b$$

Step 1: Solve $L\vec{y} = \vec{b}$ by substitution

Step 2: solve $U\vec{x} = \vec{y}$ by back-substitution

$\boxed{O(n^2)}$

Reminder: not every matrix has an LU decomp.

$$Ax = LUx = Ly = b$$

Use Gaussian elimination method reduces

$$A\vec{x}=\vec{b}$$
$$\Downarrow$$
$$Ux = y = L^{-1}b = b'$$

It's valueble to keep track of L_1 because if we want to solve a prob, with a different right-hand side

$$A\vec{x}=\vec{b}_1 \qquad A\vec{x}=\vec{c}_1 \qquad A\vec{x}=d$$

Don't have to "redo" the reduction of A to upper triangular form.

Now we've reduced solving $A\vec{x}=\vec{b}$ to LU decomp of A

+ solve 2 triangular systems.

Cost of LU decomp of A?

$$L\,U = A$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & \ddots & \\ \cdots & \cdots & & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots \\ & u_{22} & u_{23} & \\ & & u_{33} & \\ & & & \ddots \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & & & \end{pmatrix}}_{A}$$

Row 1: $u_{1j} = a_{1j}$ for $j = 1, \dots, n$

Row 2: $l_{21} u_{11} = a_{21} \Rightarrow l_{21} = \frac{1}{u_{11}} a_{21}$

$$l_{21} u_{1j} + u_{2j} = a_{2j} \Rightarrow u_{2j} = a_{2j} - l_{21} u_{1j} \quad \text{for } j = 2 \dots n$$
$$= a_{2j} - \frac{1}{u_{11}} a_{21} u_{1j}$$

Row 3: $l_{31} \cdot u_{11} = a_{31} \Rightarrow l_{31} = \frac{1}{u_{11}} a_{31}$

$$l_{31} \cdot u_{12} + l_{32} u_{22} = a_{32} \Rightarrow l_{32} = \frac{1}{u_{22}} (a_{32} - l_{31} u_{12})$$

$$l_{31} u_{1j} + l_{32} \cdot u_{2j} + u_{3j} = a_{3j} \Rightarrow u_{3j} = a_{3j} - l_{31} u_{1j} - l_{32} u_{2j}$$

Row i: $l_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}\right)$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

To compute $A = LU$

for $i = 1, \cdots, n$:

  for $j = 1, \cdots, i-1$

    compute l_{ij}

  for $j = 1, \cdots, n$

    compute u_{ij}.

Need to make sure $u_{jj} \neq 0$.