Today: Sources of numerical error
Convergence of Heron's method.

1. Can only iterate finitely many times

2. finite computer representation of real numbers.

Real numbers must be represented as a finite sequence of 0 and 1's

2 main way of doing this:

fixed points vs. floating pts.

Fixed points

$\pm b_1 b_2 b_3 \cdots b_n$

put a decimal pts in a fixed position

General: $\underbrace{\pm b_1 b_2 b_3 \cdots b_n \cdot 2^{-k}}_{\text{fairly narrow}}$ for some fixed k

Floating point: like scientific notation

6.02×10^{23}
6.63×10^{-34}

$\pm 1.\underbrace{b_1 b_2 \cdots b_n}_{\text{significant}} \cdot 2^{\overbrace{\pm e_1 e_2 \cdots e_n}^{\text{exponent}}}$ sign bits tell us positive / negative

In practice, computers implement IEEE standard.

For each arithmetic operation, a rounding error is made that is small relative to the order of magnitude.

There are a couple of situations

Caution: make sure that small errors don't become large.

Ex. subtraction of 2 large nearby numbers.

$$(2.0^{53}+1)-2.0^{53} = 0.0.$$

relatively small.

really weired example.

$$x_+ = \frac{-b+\sqrt{b^2-4ac}}{2a}$$ If a, c small relative to $b>0$, then $\sqrt{b^2-4ac} \approx b$.

$$x_- = \frac{-b-\sqrt{b^2-4ac}}{2a}$$ ← no problem

$$x_+ = \frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}} = \frac{\overset{2}{\overset{+4ac}{\cancel{b^2}-(\cancel{b^2}-4ac)}}}{2\cancel{a}(-b-\sqrt{b^2-4ac})} = \frac{2c}{-b-\sqrt{b^2-4ac}} = \frac{c}{a x_-} = \frac{c}{a}\cdot\frac{1}{x_-}$$

Return to Heron's Method. for $\sqrt{y}$

Prove it's converging to the correct number.

Start w/ rough guess x_0

For $k = 0, 1, 2, \ldots$; $\quad x_{k+1} = \frac{1}{2}\left(x_k + \frac{y_k}{x_k}\right)$

Last time, consistent, i.e. if $x_k \to \xi \neq 0$ then $\xi^2 = y$.

Does it converge? When it converge? How quickly does it converge?

Define error $e_k = x_k - \sqrt{y}$ how much x_k is away from the true answer.

so $x_k \to \sqrt{y} \iff e_k \to 0$ as $k \to \infty$

Special case 1: $y = 0$.

Suppose $x_0 \neq 0$.

$x_{k+1} = \frac{1}{2}(x_k + 0) = \frac{1}{2} x_k$ At every step of the algorithm.

$\Rightarrow x_k = 2^{-k} x_0 \to 0$ as $k \to \infty$ $\qquad \lim_{k\to\infty} 2^{-k}\cdot x_0 = 0$

Case 2: $y > 0$.

Quick observation: $\frac{1}{2}\left(\sqrt{y} + \frac{y}{\sqrt{y}}\right) = \sqrt{y}$ (written $\sqrt{y}$ as $\frac{1}{2}\left(\sqrt{y}+\frac{y}{\sqrt{y}}\right)$).

if I plug in $\sqrt{y}$ ~~as~~ an input, we get $\sqrt{y}$ as output

$\sqrt{y}$ is called a fixed point of the function $x_k \mapsto \frac{1}{2}\left(x_k + \frac{y}{x_k}\right)$

Given the error $e_k = x_k - \sqrt{y}$, what can we say about $e_{k+1} = x_{k+1} - \sqrt{y}$?

I can rewrite $e_{k+1} = x_{k+1} - \sqrt{y} = \frac{1}{2}\left(x_k + \frac{y}{x_k}\right) - \frac{1}{2}\left(\sqrt{y} + \frac{y}{\sqrt{y}}\right)$

$$= \frac{1}{2}(x_k - \sqrt{y}) + \frac{1}{2}\left(\frac{y}{x_k} - \frac{y}{\sqrt{y}}\right)$$

$$= \frac{1}{2} e_k + \frac{\overset{\sqrt{y}}{\cancel{y}}}{2}\left(\frac{\sqrt{y} - x_k}{x_k \cancel{\sqrt{y}}}\right) = \frac{1}{2}e_k - \frac{1}{2}\underbrace{(x_k - \sqrt{y})}_{e_k}\frac{\sqrt{y}}{x_k} = \frac{1}{2}e_k\left(1 - \frac{\sqrt{y}}{x_k}\right)$$

$$= \frac{1}{2}e_k\left(1-\frac{\sqrt{y}}{x_k}\right) = \frac{1}{2}e_k\left(\frac{x_k-\sqrt{y}}{x_k}\right) = \frac{1}{2}e_k^2\cdot\frac{1}{x_k}$$

Want to show $e_k \to 0$

Suppose $x_k \geq \sqrt{y}$, so $e_k \geq 0$

$\Rightarrow$ $e_{k+1} \geq 0$, too

$$0 \leq e_{k+1} = \frac{1}{2}e_k^2\cdot\frac{1}{x_k} \leq \frac{1}{2}e_k^2\cdot\frac{1}{\underbrace{\cancel{x_k-\sqrt{y}}}_{e_k}} = \frac{1}{2}e_k$$

and

Error is at least halved at each step.

If $x_0 > \sqrt{y}$, then $0 \leq e_k \leq 2^{-k}e_0 \to 0$ as $k\to\infty$

Squeeze theorem.

$\therefore$ $e_k \to 0$

Here we have shown that Heron's method convergence as long as $x_0 \geq \sqrt{y}$.

If $0 < x_0 < \sqrt{y}$., then $e_0 < 0$. $\Rightarrow e_0 > 0$.
the formula still holds.

So take $x_1 > \sqrt{y}$ "initial step"

$\Rightarrow$ $x_k \to \sqrt{y}$ whenever $x_0 > 0$.

Q. What happens if $x_0 < 0$.?

Can show that $x_k \to -\sqrt{y}$.

In summary.

show Heron's method converges to roots of $f(x) = x^2 - y$.

Root-finding more generally

How to find $f(x) = 0$. for f more generally.

First question: does f have a root?

For example: $f(x) = x^2+1$
$x \in \mathbb{R}$.