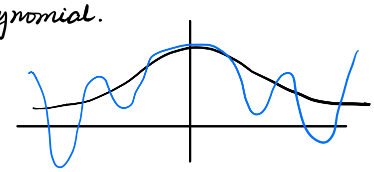
Math 449: Numerical Applied Mathematics
Lecture 30

11/13/2017 Wenzhen

Today's topic: Interpolation Error, Punge's Phenomenon, and Chebysheb Polynomial

Today: Interpolation Error, Runge's phenomenon, and Chebyshev polynomial.

e.g. $f(x) = \frac{1}{1+x^2}$ oscillation get worse as $n \to \infty$.



Recall, for $\vec{v} \in \mathbb{R}^n$, $\|\vec{v}\|_\infty = \max_{i=1,\dots,n} |v_i|$

Let $C[a,b]$ denote the space of continuous functions $f: [a,b] \to \mathbb{R}$

$\|f\|_\infty := \max_{x \in [a,b]} |f(x)|$ $\quad$ $\|\cdot\|_\infty$ is a norm on the vector space $C[a,b]$

If $f \in C[a,b]$, p is a polynomial on $[a,b]$ approximating f, we can measure the error

$\|f-p\|_\infty$ (largest absolute error for $x \in [a,b]$)

Runge's phenomenon

$\|f-p_n\|_\infty \not\to 0$ as $n \to \infty$ in fact, may $\to \infty$!

Thm (Weirestrass)

Let $f \in C[a,b]$, for all $\varepsilon > 0$, $\exists$ a polynomial s.t. $\|f-p\|_\infty \leq \varepsilon$.

i.e., $|f(x)-p(x)| \leq \varepsilon \quad \forall x$.

We don't know how to find this p.

e.g. using interpolation.

Goal: make $\|f-p_n\|_\infty$ as small as possible.

Recall: $f(x)-p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$

$\pi_{n+1}(x) = (x-x_0)\cdot \dots \cdot (x-x_n)$

we have control over

$|f(x)-p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \pi_{n+1}(x)$ where $M_{n+1} := \max_{x \in [a,b]} |f^{(n+1)}(x)| = \|f^{(n+1)}(x)\|$

$\Rightarrow \|f-p_n\|_\infty \leq \frac{M_{n+1}}{(n+1)!} \|\pi_{n+1}(x)\|_\infty$

We can choose $x_0, \dots, x_n$. Let's do this to try to make $\|\pi_{n+1}\|_\infty$ small.

Chebyshev polynomials T_n, $n = 0, 1, \dots \infty$

T_n is a degree n polynomial 以前 Cheby 拼成 Tseby

Idea: Find monic polynomials

$\swarrow$

leading term's coeff $= 1$

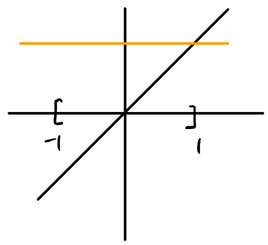
with the smallest possible $\|\cdot\|_\infty$ on $[-1, 1]$

$n=0$ 1

$n=1$ x

$n=2$ $x^2 - \frac{1}{2}$

$n=3$ $x^3 - \frac{3}{4}x$



T_n : normalize these degrees-n polynomials, so $\|T_n\|_\infty = 1$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

General Formula

$$T_n(x) = \cos(n \cdot \cos^{-1}(x)) \quad \text{for } n = 0, 1, 2, \dots$$

Why are these polynomials?

$n=0$ $T_0(x) = \cos(0) = 1$

$n=1$ $T_1(x) = \cos(\cos^{-1}(x)) = x$

Trig identity.

$$\cos[(n+1)\theta] + \cos[(n-1)\theta] = 2\cos\theta \cos n\theta$$

Proof. Recall $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i(n+1)\theta} + e^{i(n-1)\theta} = e^{in\theta}(e^{i\theta} + e^{-i\theta})$$

$$e^{i\theta} + e^{-i\theta} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$$
$$= 2\cos\theta$$

$$e^{i(n+1)\theta} + e^{i(n-1)\theta} = e^{in\theta} \cdot 2\cos\theta$$

Take real parts of both sides

$$\cos[(n+1)\theta] + \cos[(n-1)\theta] = \cos n\theta \cdot 2\cos\theta$$

Apply identity with $\theta = \cos^{-1}(x)$

$$\cos((n+1)\cos^{-1}(x)) = T_{n+1}(x) \qquad \begin{cases} \cos(\cos^{-1}(x)) = x \\ \cos(n\cos^{-1}(x)) = T_n x \end{cases}$$

$$\cos((n-1)\cos^{-1}(x)) = T_{n-1}(x)$$

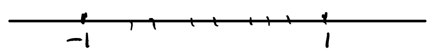
$$T_{n+1}(x) + T_{n-1}(x) = 2x\,T_n(x) \Rightarrow T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

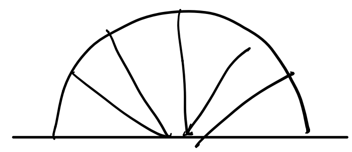
T_n can be defined by this recurrence w/ base case $T_0(x) = 1$ $T_1(x) = x$.

Ex. $T_2(x) = 2x T_1(x) - T_0(x) = 2x^2 - 1$

Since $-1 \leq \cos\theta \leq 1 \Rightarrow -1 \leq T_n(x) \leq 1$ minimize "overshoot" error.

Defn. The Chebyshev nodes are the roots of T_n $x_j = \cos \frac{2j-1}{2n} \pi$ $j = 1, \dots, n$





Idea for degree $= n$ Lagrange interpolation, take the interpolation pts. to be roots of $T_{n+1}(x)$