Math 449: Numerical Applied Mathematics
Lecture 31

11/15/2017 Wenzhen

Today's topic: Chebyshev Polynomial

Today: Finish Chebyshev Polynomial

Chebyshev Polynomials, on $[-1,1]$, $T_n = \cos(n \cos^{-1} x)$

Satisfy recurrence, $T_0(x) = 1$, $T_1(x) = x$,

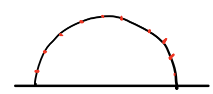
$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

$\|f\|_\infty = \max\limits_{x\in[a,b]} |f(x)|$ $\qquad \|T_n\|_\infty = 1$, $n = 0, 1, 2, \dots$

We want to choose $\cancel{\pi_n}$ interp pts so that $\|\pi_{n+1}\|_\infty$ is small

Error of Lagrange interpolation satisfies $\|f - p_n\|_\infty \leq \dfrac{M_{n+1}}{(n+1)!}\|\pi_{n+1}\|_\infty$ want $\to 0$ as $n\to\infty$.

idea: on $[-1,1]$, take $x_0, x_1, \dots, x_n$ to be roots of T_{n+1}



Thm let $f: [-1,1] \to \mathbb{R}$ and p_n be the interpolating poly. corresponding to chebyshev nodes.

then $\|\pi_{n+1}\|_\infty \leq 2^{-n}$

$\Rightarrow \|f - p_n\|_\infty \leq \dfrac{M_{n+1}}{2^n (n+1)!}$

Proof. $\pi_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$ $\leftarrow$ a monic polynomial. $T_{n+1}(x) = x^{n+1} + \dots$

it has same roots as $T_{n+1}(x) = 2^n x^{n+1} + \dots$

$= 2^n \pi_{n+1}(x)$

$\|\pi_{n+1}(x)\|_\infty = 2^{-n} \cancelto{1}{\|T_{n+1}\|_\infty}$ $\qquad \boxtimes$

Rmk 1 For $[a,b]$, map $[-1,1]$ to $[a,b]$ $\qquad$ scaling the interval

affine map.

Take image of Chebyshev nodes in $[a,b]$.

Rmk 2. Guarantee Convergence $\|f - p_n\|_\infty \to 0$ as $n \to \infty$.

as long as f is "nice" (e.g. C^1)

This is the foundation for Clenshaw – Curtis quadrature / Fejer quadrature.

Take Chebyshev nodes instead of equally-spaced like Newton Cotes.

Recall $v \in \mathbb{R}^n$, $p \geq 1$ $\quad \|v\|_p = (|v_1|^p + |v_2|^p + \cdots + |v_n|^p)^{1/p}$

$p=2$ is special, $\because$ comes from inner product $\|v\|_2 = (v \cdot v)^{1/2}$

Def. The p-norm of $f \in C[a,b]$ is

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

More generally, if $w: (a,b) \to \mathbb{R}$ is a positive, cont's, integrate "weight function"

weighted function: $w:(a,b) \to \mathbb{R}$,

weighted p-norm: $\|f\|_p = \left(\int_a^b w(x) |f(x)|^p dx \right)^{1/p}$

Lebegue

Space of function s.t. $|f|_p$ is integrable on (a,b) is called $\mathcal{L}^p(a,b)$

$w \cdot |f|_p$ is integrable $L_w^p(a,b)$ "weighted L^p space"

L^2 inner product on (a,b)

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

weighted L^2: $\langle f, g \rangle_w = \int_a^b w(x) f(x) g(x) dx$

$$\langle f, f \rangle = \int_a^b |f(x)|^2 dx = \|f\|_2^2$$

$$\Rightarrow \|f\|_2 = (\langle f, f \rangle)^{1/2}$$

Inner Product Space.

A real inner product on a real vector space V is a function $V \times V \to \mathbb{R}$
Satisfying some condition $\quad (f,g) \mapsto \langle f, g \rangle$

1. Symmetric $\quad \langle f, g \rangle = \langle g, f \rangle$

2. Positive-definite $\quad \langle f, f \rangle \geq 0 \quad = 0$ iff $f = \vec{0} \in V$

3. bilinear
$$\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$$
$$\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$$
$$\langle \lambda f, g \rangle = \lambda \langle f, g \rangle = \langle f, \lambda g \rangle \quad \forall \lambda \in \mathbb{R},\ f, g, h \in V$$

Every linear product $\langle \cdot,\cdot \rangle$ defines a norm on V

$$\|f\| = \langle f, f \rangle^{1/2}$$

triangle inequality

Cauchy – Schwatz Inequality.

$$|\langle f, g \rangle| \leq \|f\| \, \|g\|$$

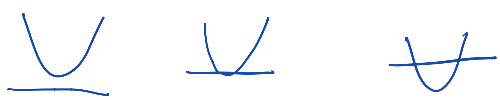
Proof. For any $\lambda \in \mathbb{R}$,

$$\lambda^2 \langle f, f \rangle + 2\lambda \langle f, g \rangle + \langle g, g \rangle$$

$$0 \leq \|\lambda f + g\|^2 = \langle \lambda f + g, \lambda f + g \rangle = \lambda^2 \|f\|^2 + 2\lambda \langle f, g \rangle + \|g\|^2$$

RHS is a quadratic polynomial in λ, that is always positive

$ax^2 + bx + c$ must have either a double real root. $\Rightarrow b^2 - 4ac \leq 0$.



$a = \langle f, f \rangle$ $b = 2\langle f, g \rangle$ $c = \langle g, g \rangle$

$$(\cancel{2}\langle f, g \rangle)^2 \leq 4 \langle f, f \rangle \langle g, g \rangle$$

$$|\langle f, g \rangle| \leq \|f\| \, \|g\|$$