Math 449: Numerical Applied Mathematics
Lecture 17

10/09/2017 Wenzhen

Today's topic: Solutions of Nonlinear Systems

Today: Solutions of nonlinear systems

Let $D \subset \mathbb{R}^n$ be a nonempty closed set

$f: D \longrightarrow \mathbb{R}^n$ a continuous function (i.e. a vector field)

Problem: Find $\xi \in D$ s.t. $f(\xi) = \vec{0} \in \mathbb{R}^n$

in components $\xi = (\xi_1, \xi_2, \dots, \xi_n)$

$f(\xi) = (f_1(\xi_1, \dots, \xi_n), \dots, f_n(\xi_1, \dots, \xi_n))$

$$f(\xi) = \vec{0} \iff \begin{cases} f_1(\xi_1, \dots, \xi_n) = 0 \\ \vdots \\ f_n(\xi_1, \dots, \xi_n) = 0 \end{cases}$$

Previous case: $n=1$, $D = [a,b] \subset \mathbb{R}$

What we did before: convert to a fixed problem

$g: D \longrightarrow D$

A fixed pt $\xi \in D$ s.t. $g(\xi) = \xi$

fixed pt iteration:

given $x^{(0)} \in D$ initial guess.

for $k = 0, 1, 2, \dots$

$x^{(k+1)} = g(x^{(k)})$

- stability of fixed pts

  does $x^{(k)} \to \xi$ as $k \to \infty$ if $x^{(0)}$ is "close enough" to ξ ?

- How to choose a "good" g, whose fixed pts, are roots of f ?

def. $g: D \to D$ is a contraction or D with respect to the norm $\|\cdot\|$

if $\|g(x) - g(y)\| \leq L\|x-y\| \quad \forall x, y \in D$ for some $L \in (0,1)$

Prop. A linear map $g(x) = Ax$ is a contraction in $\|\cdot\|$ iff $\|A\| < 1$

Proof. If $\|A\|<1$, then $\|Ax - Ay\| = \|A(x-y)\| \leq \|A\| \cdot \|x-y\|$

So take $L = \|A\|$

Conversely, if $\|A\| \geq 1$, then $\exists$ some $x \neq 0$, $\ni$ $\|Ax\| \geq \|A\|\|x\| \geq \|x\|$

$\|Ax - A\vec{0}\| \geq \|x - 0\|$

So distance between $x, 0$ doesn't contract. $\boxtimes$

Key fact about $\mathbb{R}^n$: completeness

$x^{(k)}$ is called a Cauchy Sequence with respect to $\|\cdot\|$

if $\|x^{(k)} - x^{(l)}\| \to 0$ as $k, l \to \infty$.

Every Cauchy sequence in $\mathbb{R}^n$ converges to some limit.

Since D is closed, if the sequence $x^{(k)} \in D$, then so is ξ.

more general : Contraction Mapping Theorem.

If $g: D \to D$ is a contraction in any norm $\|\cdot\|$, then g has a unique fixed pt $\xi \in D$, and the fixed pt iteration converges to ξ for any $x^{(0)} \in D$.

Another facts about $\mathbb{R}^n$:

All norms are equivalent.

If I have $\|\cdot\|$ and $|||\cdot|||$. are norms on $\mathbb{R}^n$, $\exists$ some constant $C>0$,

s.t. $\|\cdot\| \leq C \cdot |||\cdot|||$

For practical purposes: so if $|||x^{(k)} - \xi||| \to 0$,

then $\|x^{(k)} - \xi\| \leq C|||x^{(k)} - \xi||| \to 0$, too.

So if a sequence converges in one norm, then it must also cvg in every norm.

pf: 1. Show $x^{(k)}$ is Cauchy.

$$d(x^{(k)}, x^{(l)})$$

if $k < l$, then

$$\| x^{(k)} - x^{(l)} \| \leqslant \| x^{(k)} - x^{(k+1)} \| + \| x^{(k+1)} - x^{(k+2)} \| + \cdots + \| x^{(l-1)} - x^{(l)} \|$$

By the contraction property,

$$\begin{aligned}
\| x^{(k)} - x^{(k+1)} \| &= \| g(x^{(k-1)}) - g(x^{(k)}) \| \\
&\leqslant L \| x^{(k-1)} - x^{(k)} \| \\
&\leqslant L^2 \| x^{(k-2)} - x^{(k-1)} \| \\
&\vdots \\
&\leqslant L^k \| x^{(0)} - x^{(1)} \|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \| x^{(k)} - x^{(l)} \| &\leqslant \| x^{(0)} - x^{(1)} \| \left(L^k + L^{k+1} + L^{k+2} + \cdots \right) \\
&= \| x^{(0)} - x^{(1)} \| L^k \left(1 + L + L^2 + \cdots \right) \\
&= \| x^{(0)} - x^{(1)} \| \frac{L^k}{1-L} \to 0 \quad \text{as } k \to \infty
\end{aligned}$$

So $x^{(k)} \longrightarrow \xi \in D$ as $k \to \infty$

Since g is continuous,

$$x^{(k+1)} = g(x^k)$$

$$\xi = g(\xi)$$

So ξ is a fixed pt of g

Uniqueness: if ξ, η are both fixed pts, then

$$\| \xi - \eta \| = \| g(\xi) - g(\eta) \| \leqslant L \| \xi - \eta \| \Rightarrow \| \xi - \eta \| = 0 \Rightarrow \xi = \eta \quad \boxtimes$$