Math 449: Numerical Methods
Lecture 04

Sep 8th, 2017

Today's topic: Relaxation Method

Last time:

$g: [a,b] \longrightarrow [a,b]$ $\quad\quad$ $\xi = g(\xi)$ is stable if $|g'(\xi)| < 1$

unstable if $|g'(\xi)| > 1$

Relaxation:

relaxation parameter

instead of $x_{k+1} = g(x_k)$ take $x_{k+1} = (1-\lambda)x_k + \lambda g(x_k) = g_\lambda(x_k)$ $\quad$ $\lambda \in (0,1) \longrightarrow$ "slow down"

Avoid over-shooting $\quad$ $\lambda > 1 \longrightarrow$ "accelerate"

If $g'(\xi) \neq 1$, then $\lambda = \dfrac{1}{1-g'(\xi)} \Rightarrow g_\lambda'(\xi) = 0$ $\quad$ shrinking parameter

so ξ is a stable fixed pt of g_λ

want to solve $f(x) = 0$

"Naïve Choice" $g(x) = x - f(x)$

Fixed pt ξ of g are roots of f.

$$g_\lambda(x) = (1-\lambda)x + \lambda(x - f(x)) = x - \lambda f(x)$$

This is the relaxation method for root finding.

$g'(x) = 1 - f'(x)$
$g'(\xi) = 1 - f'(\xi)$

$g'(\xi) = 1 \Leftrightarrow f'(\xi) = 0$

If $f'(\xi) \neq 0$, optimal choice is $\lambda = \dfrac{1}{1-g'(\xi)} = \dfrac{1}{f'(\xi)}$

$$\Rightarrow g_\lambda(x) = x - \frac{f(x)}{f'(\xi)}$$

Problem: We don't know ξ yet.

Generalized Relaxation Method: $x_{k+1} = x_k - \lambda(x_k) f(x_k)$

Obvious Choice $\lambda(x) = \dfrac{1}{f'(x)}$ $\quad$ if f continuously differentiable, $\lim_{x \to \xi} \lambda(x) = \dfrac{1}{f'(\xi)}$ $\quad$ so we don't have to know $f'(\xi)$

Newton's Method: $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ $\quad$ Assume $f'(x_k) \neq 0 \;\; \forall k$

Example: $f(x) = x^2 - y$

$f'(x) = 2x$

Newton's Method
$$x_{k+1} = x_k - \frac{(x^2 - y)}{2x_k}$$
$$= x_k - \frac{x_k^{\cancel{2}}}{2\cancel{x_k}} + \frac{y}{2x_k}$$
$$= \frac{1}{2}(x_k + y) \longleftarrow \text{Heron's Method}$$

Interpretation of Newton's Method.

Given a "guess" x_k

Taylor's Theorem (linearization about x_k)

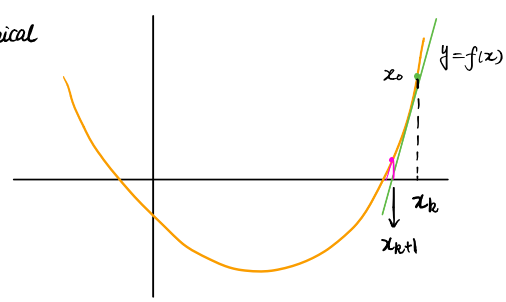
$$f(x_k + \underset{\text{nearby}}{\underset{\uparrow}{\Delta x_k}}) \approx f(x_k) + f'(x_k)\Delta x_k \quad = 0$$

Want to "move" by Δx_k to get closer to a root of f.

Solve for Δx_k so that $f(x_k) + f'(x_k)\Delta x_k = 0$

$$\Rightarrow \quad \Delta x_k = -\frac{f(x_k)}{f'(x_k)}$$

Graphical



linearization with slope $f'(x_0)$

x_{k+1} = root of linearization to f at x_k

= "tangential line" to f at $(x_k, f(x_k))$

Rate of Convergence

Def. $x_k \to \xi$ converges at least linearly if $\exists$ a sequence $\underset{\text{error}}{\varepsilon_k} \to 0$ s.t.

$$|x_k - \xi| \leq \varepsilon_k$$

and $\dfrac{\varepsilon_{k+1}}{\varepsilon_k} \to \mu \in (0,1)$ $\quad$ μ: error shrinking factor

$x_k \to \xi$ with order at least q $(q>1)$

if $\dfrac{\varepsilon_{k+1}}{\varepsilon_k^q} \to \mu > 0$ $\quad$ (not necessarily <1)

drop "at least" when $\varepsilon_k = |x_k - \xi|$ is actual error

Linear convergence

For large enough k, $\dfrac{\varepsilon_{k+1}}{\varepsilon_k} \approx \mu$

$\varepsilon_{k+1} \approx \mu \varepsilon_k$ $\quad$ μ is "shrinking" factor for error.

order q convergence:

For large k, $\varepsilon_{k+1} \approx \mu \varepsilon_k^q = \underbrace{(\mu \varepsilon_k^{q-1})}_{\text{shrinking factor}} \varepsilon_k$

Don't need $\mu<1$, for this case, because $\varepsilon_k \to 0 \Rightarrow (\mu \varepsilon_k^{q-1}) \to 0$

So $\mu \varepsilon_k^{q-1} < 1$ for large enough k.

Example. $x_k = 2^{-k} \longrightarrow 0$ $\quad$ $\lim_{k\to\infty} 2^{-k} = 0 = \xi$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

let $\varepsilon_k = |x_k - \xi| = 2^{-k}$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{2^{-k-1}}{2^{-k}} = \frac{1}{2} \to \frac{1}{2} = \mu$$

So $x_k \longrightarrow 0$ linearly.

Another Example

$x^k = 4^{-k}$ $\quad$ $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

$\varepsilon_k = |x_k - \xi| = 4^{-k}$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{4^{-k-1}}{4^{-k}} = \frac{1}{4} = \mu.$$

Also linear convergence, just w/ smaller μ.

Example $x_k = 2^{-2^k} \longrightarrow 0 = \xi$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{128}, \ldots$

Shrinking accelerates at each step.

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{2^{-2^{k+1}}}{2^{-2^k}} = 2^{-2^k - 2^{k+1}} = 2^{2^k(1-2)} = 2^{2k} \to 0 \notin (0,1).$$

Super convergence!!!

$$\frac{\varepsilon_{k+1}}{\varepsilon_k^2} = \frac{2^{-2^{k+1}}}{(2^{-2^k})^2} = \frac{2^{-2^{k+1}}}{2^{-2^{k+1}}} = 1 \to \mu$$ order 2 quadratic convergence.