Math 449: Numerical Applied Mathematics
Lecture 23

10/27/2017 Wenzhen

Today's topic: Interpolation

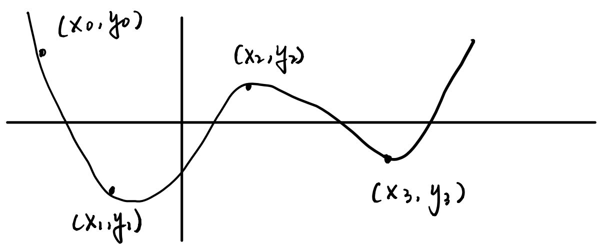
Let P_n be the set of all polynomials $p_n : \mathbb{R} \to \mathbb{R}$ of degree $\leq n$

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, \ldots, a_n \in \mathbb{R}$ are arbitrary coefficients.

Problem: Suppose $x_0, \ldots, x_n$ are distinct real and $y_0, \ldots, y_n$ be real #s.

Find $p_n \in P_n$ s.t. $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$



Why do we care?

1. missing data

   unknown function f

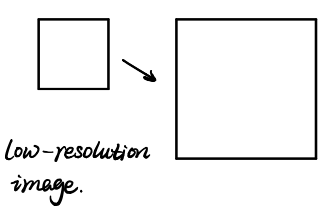
   take finite # of measurements. $y_i = f(x_i)$

   But I don't know f for other values of x

   Polynomial intep "fills in" missing values.

   $y = p_n(x)$ consistent w/ measurement. estimate of f.

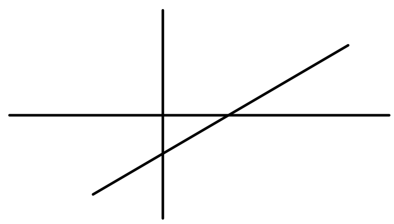
Ex. Image upscaling



Suppose we want to differentiate or integrate a function

Hard for arbitrary functions. Trivial for polynomials. Instead of integrating f, approximat f by a polynomial and integrate that instead

Not interpolation: best fit line.



interpolating polynomials go through all pts (x_i, y_i)

For our purposes, no difference between interpolation and "extrapolation".

Can we find such a p_n?

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

To interpolate, must satisfy

$$\left.\begin{array}{l} p_n(x_0) = y_0 \\ p_n(x_1) = y_1 \\ \quad\vdots \\ p_n(x_n) = y_n \end{array}\right\} n+1 \text{ linear equations}$$

$$a_0 + a_1 x + \cdots + a_n x^n = y_i$$

Instead of doing this, we represent p_n in a different way.

Monomials $\{1, x, x^2, \ldots, x^n\}$ are a basis for P_n.

$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ $\qquad a_0, a_1, \ldots, a_n$ are coeff. in the monomial basis.

Lagrange Interpolation.

Given distinct $x_0, x_1, \cdots, x_n$

Claim: For each $k = 0, \cdots, n$ we can find a polynomial $L_k \in P_n$ (Lagrange basis poly)

s.t. $\begin{cases} L_k(x_k) = 1 \\ L_k(x_i) = 0 & i \neq k \end{cases}$

Want to find $L_1 \in P_2$

Quadratic Vanishing @ x_0, x_2

$$q(x) = (x - x_0)(x - x_2)$$

$$q(x_1) = (x_1 - x_0)(x_1 - x_2) \neq 0$$ Assuming x_0, x_1, x_2 are distinct.

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \begin{cases} 0 & \text{at } x_0\ x_2 \\ 1 & \text{at } x_1 \end{cases}$$

Def.

Given distinct $x_0, x_1, \ldots, x_n$

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \qquad k = 0, 1, \ldots, n$$

$\{L_0, L_1, \ldots, L_n\}$ are the lagrange basis polynomials. for the pts $x_0, \ldots, x_n$.

Solution to interpolation prob.

Let $p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k$

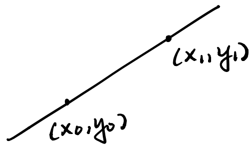
then $p_n(x_i) = \sum_{k=0}^{n} L_k(x_i)\, y_k = 0 + \cdots + 0 + L_i(x_i)\, y_i + 0 + \cdots + 0 = y_i$

for $i = 0, 1, \ldots, n$

instead of using monomial basis & solving for coefficients $a_0, \cdots, a_n$.

We use a different basis where the coefficients are just $y_0, \cdots, y_n$.

Ex. Linear Interpolation



$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$y = p_1(x)$ rearranges "point-slope" for a line.

$$y - y_0 = m(x - x_0)$$

$$= \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

Thm $\{L_0, \dots, L_n\}$ are a basis of P_n

<u>Proof</u> Suffices to show linearly independent.

Suppose
$$p_n = \sum_{k=0}^{n} L_k y_k = 0$$

Want to show that $y_0 = \cdots = y_n = 0$

if this is true, then $p_n(x_i) = 0$ for $i = 0, 1, \dots, n$.

$$= y_i$$

So all the $y_i = 0$. corrollary.