# Math 449: Numerical Methods
# Homework 7
# Wenzhen Zhu

Part I: Theory

Problem 1. Prove that Simpson's rule is exact for degree $\leqslant 3$ polynomials.

Simpson's rule $\int_a^b f(x)\,dx = (b-a)\left(\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right)$

$f(x) = x^3$ $\int_a^b x^3\,dx = (b-a)\left(\frac{1}{6}a^3 + \frac{2}{3}\left(\frac{a+b}{2}\right)^3 + \frac{1}{6}b^3\right.$

$= (b-a)\left(\frac{1}{6}a^3 + \frac{1}{12}(a^3+3a^2b+3ab^2+b^3) + \frac{1}{6}b^3\right)$

$= (b-a)\left(\frac{1}{6}a^3 + \frac{1}{12}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{1}{4}b^3\right)$

$= \frac{b-a}{4}(a^3 + a^2b + ab^2 + b^3)$

$= \frac{1}{4}(b-a)(a+b)^3$

$= \frac{1}{4}(b^4 - a^4)$

$= \int_a^b x^3\,dx$

**Problem 2.**

**a.** Find a function $f$ on the interval $[-1,1]$ where

$$|E_1(f)| = \frac{(b-a)^3}{12}M_2 > 0,$$

i.e., where the trapezoid rule attains the maximum error allowed by Theorem 7.1, but does not integrate $f$ exactly.

**b.** Find a function $f$ on the interval $[-1,1]$ where

$$|E_2(f)| = \frac{(b-a)^5}{2880}M_4 > 0,$$

i.e., where Simpson's rule attains the maximum error allowed by Theorem 7.2, but does not integrate $f$ exactly.

$f(x) = x^2$ $\int_{-1}^1 x^2\,dx = \frac{2}{3}$

By trapzoid rule: $(b-a)\left(\frac{f(a)+f(b)}{2}\right)$

$2\left(\frac{1^2+(-1)^2}{2}\right) = 2$

$E_1(f) = \frac{(1-(-1))^3}{12}\,2 = \frac{4}{3}$ ①

$E_1(f) = \left|2 - \frac{2}{3}\right| = \frac{4}{3}$ ②

① = ②, so within maximum error

b. $f(x) = x^4$ $\int_{-1}^1 x^4\,dx = \frac{2}{5}$

By simpson's rule.

$\frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

$= \frac{1}{3}(1 + 4\cdot 0 + 1)$

$= \frac{2}{3}$

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$\text{Error} = \left|\frac{2}{3} - \frac{2}{5}\right| = \frac{4}{15}$ ① $M_4 = 24$

$|E_2(f)| = \frac{2^5}{2880}\cdot 24 = \frac{4}{15}$ ②

① = ②

**Problem 3** (Süli–Mayers, Exercise 7.3). A quadrature formula on the interval $[-1,1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \le 1$:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_0 f(-\alpha) + w_1 f(\alpha).$$

The formula is required to be exact whenever $f$ is a polynomial of degree 1.

- **a.** Show that $w_0 = w_1 = 1$, independent of the value of $\alpha$.
- **b.** Show also that there is one particular value of $\alpha$ for which the formula is exact for all polynomials of degree 2. Find this $\alpha$, and
- **c.** show that, for this value, the formula is also exact for all polynomials of degree 3.

a. By the statement, the formula is required to be exact whenever $f$ is a polynomial of degree 1.

$f(x) = 1 \Rightarrow \quad \int_{-1}^{1} f(x)\,dx = \omega_0 + \omega_1 = 2$

$f(x) = x \Rightarrow \quad \int_{-1}^{1} f(x)\,dx = -\alpha\omega_0 + \alpha\omega_1 = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0$

$$\Rightarrow \alpha(\omega_1 - \omega_0) = 0 \Rightarrow \left.\begin{array}{l}\omega_1 - \omega_0 = 0\\ \omega_1 + \omega_0 = 2\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_0 = 1\\ \omega_1 = 1\end{array}\right.$$

b. $f(x) = x^2$

$\int_{-1}^{1} x^2 = \omega_0\alpha^2 + \omega_1\alpha^2 = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

$\alpha^2(\omega_0 + \omega_1) = \alpha^2 \cdot 2 = \frac{2}{3}$

$\alpha^2 = \frac{1}{3}$

$\alpha = \pm\frac{1}{\sqrt{3}} \quad \because \alpha \in (0,1] \quad \therefore \alpha = \frac{1}{\sqrt{3}}$

c. $f(x) = x^3$

$\int_{-1}^{1} x^3 = \frac{1}{4}x^4\Big|_{-1}^{1} = 0$

$\int_{-1}^{1} x^3 = -\omega_0 \cdot \alpha^3 + \omega_1\alpha^3 = -\alpha^3 + \alpha^3 = 0$