Math 449: Numerical Applied Mathematics
Lecture 13

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Today's topic: Pivoting & Rounding Error & Begin norms

Today: Pivoting & rounding error Begin norms.

Pivoting:

last time, we proved a thm: Every $A \in \mathbb{R}^{n \times n}$ can be decomposed as $PA = LU$

idea: at each step, find the entry in the col of interest w/ largest absolute value.

Interchanging rows to put that entry on the diagonal.

$$A = \begin{pmatrix} a_{11} & \\ \vdots & \\ \alpha & \\ \vdots & \end{pmatrix}$$ $\alpha = a_{r1}$ has largest abs val in col 1.

interchange rows l and r.

$$P^{(lr)} A = \begin{pmatrix} \alpha & w^T \\ p & B \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0^T \\ m & I \end{pmatrix} \begin{pmatrix} \alpha & v^T \\ 0 & C \end{pmatrix}$$

Ex. $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}}_{U}$$

Using pivoted LU algo 3 has the largest abs val in col 1, so take.

$$PA = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 2/3 \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ requires pivoting.

why pivot?

Ex. $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$ where $\varepsilon > 0$ is very small.

$$= \begin{pmatrix} 1 & 0 \\ 1/\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 1 \\ 0 & \frac{\varepsilon - 1}{\varepsilon} \end{pmatrix}$$ $1 - \frac{1}{\varepsilon}$ may cause 1 being throw if ε is too small.

ex. if $\varepsilon = 10^{-16}$ $\quad 1 - \frac{1}{\varepsilon} = 1 - 10^{16} \approx -10^{16}$

So $1 - \frac{1}{\varepsilon}$ can be rounded to $-\frac{1}{\varepsilon}$ if ε is really small.

Decomposition depends on

$$\frac{1}{\varepsilon} + \left(1 - \frac{1}{\varepsilon}\right) = 1$$

$$\frac{1}{\varepsilon} + \left(-\frac{1}{\varepsilon}\right) = 0.$$

with pivoting $\quad PA = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}$

$$= \overset{L}{\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}} \overset{U}{\begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix}}$$

if ε is very small $1 - \varepsilon \approx 1$

if ε is very small,

$\varepsilon + (1-\varepsilon)$ rounded to $\varepsilon + 1$ bounded by

Basic Idea.

At each step of the algo $\alpha \neq 0$, $\Rightarrow$ $m = \frac{1}{\alpha} \cdot P$

$\because$ $|\alpha| >$ every entry of P, $\quad m$ only has entry w/ abs val ≤ 1.

Additional computational cost of pivoting.

Row interchange themselves $\Rightarrow$ 0 cost. essentially.

dominan cost : cost of computing contains ~~entry~~ entry with max abs val at each step.

1^{st} step: costs n.

2^{nd} step: $\quad (n-1)$

3^{rd} $\quad (n-2)$

⋮

n^{th} ~~step~~ row $\quad 1$

the total $\underline{(n-1) + (n-2) + \cdots + 1}$

~~total~~: $1+2+\cdots+n = O(n^2)$ decomp. costs $\frac{2}{3}n^3 + O(n^2)$.

How to talk about approx error for matrices and vectors?

e.g. $A\vec{x} = \vec{b}$

does a small perturbation in $\vec{b}$ a ~~small~~ large perturbation in X

How do we measure the size of vec/ matrix.

In $\mathbb{R}$, we measure error using ~~abs~~ abs val.

e.g. $|x_k - \xi|$

Generalize abs val to vectors & matrices.

Key properties of $|\cdot|$ in $\mathbb{R}$.

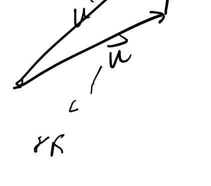
1. $|v| \geq 0 \ \forall v \in \mathbb{R}$, with $|v| = 0 \Leftrightarrow v = 0$. positive definite.

2. $|\lambda v| = |\lambda||v|$ ~~and v∈R~~. for every $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}$.

3. $|u+v| \leq |u|+|v|$ triangle inequality.

Def. A norm on a vector space V is a real-valued function $v \longrightarrow \|v\|$, Satisfying

1. $\|v\| \geq 0 \quad \forall v \in V$. and $\|v\| = 0 \Leftrightarrow v = 0$
2. $\|\lambda v\| = \|\lambda\| \|v\| \quad \lambda \in \mathbb{R}$ and $v \in V$.
3. $\|u+v\| \leq \|u\|+\|v\| \quad \forall u, v \in V$.



Basic idea.

~~At~~ each step at the algo

$\alpha \neq 0 \to m = \frac{1}{\alpha}$

norms.

- A gallery of norms in $\mathbb{R}^n$

Def. Euclidean ~~nor~~ (or 2-norm) on $\mathbb{R}^n$ is $v=(v_1, \dots, v_n)$

$$\|v\|_2 = \sqrt{v \cdot v} = (v_1 \cdot \dots v_n)^{1/2}$$

Def. 1 norm (aka taxi-cab norm). on $\mathbb{R}^n$ is.

$$\|v\|_1 = \|v_1\| + \|v_2\| \cdots + \|v_n\|$$

Def. For $p \geq 1$, the p-norm on $\mathbb{R}^n$ is

$$\|v\|_p = \left(|v_1|^p + |v_2|^p + \dots + |v_n|^p\right)^{1/p}.$$

Def ∞-norm. max norm on $\mathbb{R}^n$ is

$$\|v\|_\infty = \max\{|v_1|, \dots, |v_n|\} \qquad p=\infty.$$