Revisit "Secant" Method

idea: $f'(x_k) = \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

defn: $x_{k+1} = x_k - \dfrac{f(x_k)}{\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}$

$$= x_k - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

Using Taylor's thm, we can show that

$$\frac{\varepsilon_{k+1}}{\varepsilon_k \varepsilon_{k-1}} \longrightarrow \frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right| = \mu$$

which is not quite quadratic.

Suppose, $\varepsilon_0, \varepsilon_1 \approx \dfrac{\delta}{\mu}$ for some constant δ.

$\varepsilon_{k+1} \approx \mu\, \varepsilon_k \varepsilon_{k-1}$ when k is large

So we have:

$$\varepsilon_2 \approx \mu\, \varepsilon_1 \varepsilon_0 \approx \mu \frac{\delta}{\mu}\frac{\delta}{\mu} = \frac{\delta^2}{\mu}$$

$$\varepsilon_3 \approx \mu\, \varepsilon_2 \varepsilon_1 \approx \mu \frac{\delta^2}{\mu}\frac{\delta}{\mu} = \frac{\delta^3}{\mu}$$

$$\varepsilon_4 \approx \mu\, \varepsilon_3 \varepsilon_2 \approx \mu \frac{\delta^3}{\mu}\frac{\delta^2}{\mu} = \frac{\delta^5}{\mu}$$

$$\varepsilon_5 \approx \qquad \frac{\delta^8}{\mu}$$

$$\varepsilon_6 \approx \qquad \frac{\delta^{13}}{\mu}$$

Fibonacci numbers: $F_{k+1} = F_k + F_{k-1}$ $\quad F_k = \phi^k \quad \phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$

So $\varepsilon_k \approx \dfrac{\delta^{F_k}}{\mu}$ as $k \to \infty$

$$\lim_{k\to\infty} \varepsilon_k = \frac{\delta^{\phi_k}}{\mu}$$

Recall $\quad c^{q^k} \longrightarrow 0$ with order q, so $x_k \to \xi$ with order ϕ

Steffensen's Method

$$f'(x_k) = \frac{f(x_k+h_k) - f(x_k)}{h_k} \qquad \text{where } \lim_{k\to\infty} h_k = 0$$

take $h_k = f(x_k)$

$$f'(x_k) = \frac{f(x_k+h_k) - f(x_k)}{h_k}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{\dfrac{f(x_k+f(x_k)) - f(x_k)}{f(x_k)}} = x_k - \frac{f(x_k)^2}{f(x_k+f(x_k)) - f(x_k)}$$

$$g(x) = x - \frac{f(x)^2}{f(x+f(x)) - f(x)} \qquad x_k \to \xi \text{ at least quadratically.}$$

Efficiency

Function evaluation (generally most expensive part at each step)