Math 449: Numerical Applied Mathematics
Lecture 25

11/01/2017 Wenzhen

Today's topic: Interpolation Error

Today: Interpolation error (Hermite Interpolation)

Given a function $f: [a,b] \longrightarrow \mathbb{R}$ and distinct pts $x_0, x_1, \dots, x_n \in [a,b]$ approx. f by the interpolating polynomial $p_n(x_i) = f(x_i)$

Thm If f is C^{n+1} on $[a,b]$, then for $\forall x \in [a,b]$, $\exists\ \xi \in [a,b]$ s.t.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$$

where $\pi_{n+1}(x) = (x-x_0)(x-x_1)\cdot \cdots \cdot (x-x_n)$ is the "nodal polynomial" vanishing at $x_0, \dots, x_n$

Key tool:

Rolle's theorem If $f: [a,b] \to \mathbb{R}$ is C^1 and $f(a) = f(b)$, then $f'(\xi) = 0$ $\xi \in (a,b)$

Proof. of main theorem

Def. Given a fixed x, if $x = x_i$ for $i = 0, \dots, n$, then $f(x) - p_n(x) = 0$ and $\pi_{n+1}(x) = 0$.

otherwise define a function $\varphi(t) = [f(t) - p_n(t)] - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t)$

Can see that $\varphi(x_0) = \cdots = \varphi(x_n) = 0$ and $\varphi(x) = 0$

So φ vanishes at $n+2$ pts.

Rolle $\Rightarrow$ φ' vanishes at n pts

φ'' $n-1$ pts

$\varphi^{(n+1)}$ a pt ξ

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p_n^{(n+1)}(\xi)$$

$\pi_{n+1}(t) = t^{n+1}$ + lower-order-terms

$\pi_{n+1}^{(n+1)} = (n+1)!$

$$0 = f^{(n+1)}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n+1)!$$

$$\frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n+1)! = f^{(n+1)}(\xi)$$

$$\Rightarrow f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$$

Cor If $M_{n+1} = \max_{\xi \in [a,b]} \left| f^{(n+1)}(\xi) \right|$ then $\left| f(x) - p_n(x) \right| \le \frac{M_{n+1}}{(n+1)!} \left| \pi_{n+1}(x) \right|$

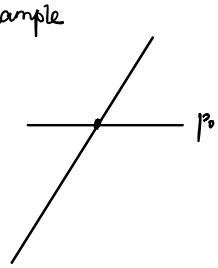
$\forall x \in [a,b]$

What might cause p_n to be a bad approximation of f?

- $f^{(n+1)}$ could be a very large at some points, so M_{n+1} would be large
  "wiggly function"

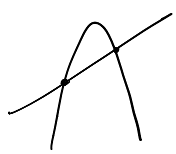
Concrete Example

$n=0$



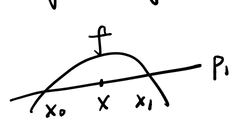
$f^{(n+1)} = f'$ is large.

$n=1$



$f^{(n+1)} = f''$ is large.

$\pi_{n+1}(x) = (x-x_0) \cdots (x-x_n)$ large (e.g. interpolation points are too far apart



far from x_0 and x_1

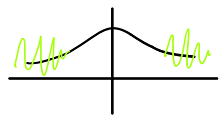
so $|\pi_2(x)| = |x - x_0|\,|x - x_1|$

more subtle example.

In fact, some cases p_n may fail to verge to f as $n \to \infty$

or may eaven get worse.

$f(x) = \dfrac{1}{1+x^2}$



Another type of polynomial interpolation : Hermite Interpolation.

idea: At $x_0, \dots, x_n$ specify function value and 1^{st} derivative.

$2(n+1)$ equation $\Rightarrow$ $2n+1$

polynomial of degree $2n+1$ has this # of coeffs.

Find $P_{2n+1} \in \mathcal{P}_{2n+1}$ s.t. $P_{(2n+1)}(x_i) = 1$ s.t. $P_{2n+1}(x_i) = y_i$ P

$P'_{2n+1}(x_i) = z_i$

idea construct a basis.

$\{H_0, H_1, \dots, H_n, K_0, \dots, K_n\}$

$H_k(x_i) = \begin{cases} 1 & i = k \\ 0, & i \neq k \end{cases}$ $\qquad H'_k(x_i) = 0.$

~~K_k~~

$K_k(x_i) = 0 \qquad K'_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$

$$P_{2n+1}(x) = \sum_{k=0}^{\infty} \left(H_k(x)\,\varphi(x) + K_k(x)(z_k) \right).$$