Math 449: Numerical Methods
Lecture 02

Sep 1st, 2017

Today: Finding roots via fixed point iteration

How do we know if a function has real roots?

Theorem:

Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is continuous and $f(a) \cdot f(b) \leq 0$, then f has a root in $[a,b]$

proof:

If $f(a)f(b) = 0$, then either a or b is a root.

If $f(a)f(b) < 0$, then one is negative and the other is positive, f is continuous, by intermediate value

Why must $\sqrt{y}$ exist for $y \geq 0$?

For $y = 0$, $\sqrt{0} = 0$

For $y < 0$, $f(x) = x^2 - y$

$f(0) < 0$ $\quad f(b) > 0$ for b suff large

Heron's Method

$$x_{k+1} = g(x_k) \qquad \text{s.t. } g(x) = \frac{1}{2}\left(x + \frac{y}{x}\right)$$

If g is continuous and $x_k \longrightarrow \xi$ as $k \to \infty$,

$$\lim_{k \to \infty} x_{k+1} = \lim_{k \to \infty} g(x_k)$$

$$\xi = g(\xi)$$

ξ is called a fixed point of g when $g(\xi) = \xi$

As seen last time, fixed pt of g are precise roots of $f(x) = x^2 - y$

Approach:

1. Given problem of finding a root of f
2. Construct a function g whose fixed points are roots of f.
3. Apply fixed-point iteration for $k = 0, 1, 2, \ldots$ $\quad x_{k+1} = g(x_k)$

Easy to come up with g whose fixed pts are roots of f.

Ex. $g(x) = x - f(x)$ so $g(\xi) = \xi - f(\xi) = \xi \Leftrightarrow f(\xi) = 0$.

This is usually not a good choice.

Ex. $f(x) = x^2 - 2$ $g(x) = x - x^2 + 2$

check : $g(\sqrt[4]{2}) = \sqrt[4]{2} - 2 + 2$

$$\left.\begin{aligned} x_0 &= 1 \\ x_1 &= g(1) = 2 \\ x_2 &= g(2) = 2 - 4 + 2 = 0 \\ x_3 &= g(0) = 2 \\ x_4 &= g(2) = 0 \\ &\vdots \end{aligned}\right\} \quad 0, 2, 0, 2 \quad \text{oscillating, doesn't converge.}$$

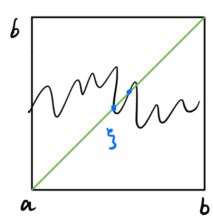
Motivates two questions :

1. When does a function g have a fixed point ? (How many ?)

2. Are these fixed points stable (iteration converges) ?

Theorem Brouwer's fixed pt theorem

If $g : [a,b] \to [a,b]$ is continuous, then $\exists\ \xi \in [a,b]$ not necessarily unique.



$f(x) = x - g(x)$

$f(a) = a - g(a) \leqslant 0$
$f(b) = b - g(b) \geqslant 0$

by intermediate value theorem (IVM) $f(\xi) = 0$ for some $\xi \in [a,b]$ s.t. $\xi - g(\xi) = 0$

so ξ is a fixed pt of g. ∎

Remark. It's estimated that $[a,b]$ is a closed interval
Otherwise conclusion needs not hold [HW]

Ex. $g(x) = x^3$ on $[-1, 1]$ fixed pts $[-1, 0, 1]$ $g(x) = x$. $\forall$ pts fixed.

Def. $f: [a,b] \to [a,b]$ is a contraction if $\exists$ a constant $0<L<1$ s.t.

$$|g(x) - g(y)| \le L|x-y| \quad \forall x, y \in [a,b]$$

Distance between x, y shrinks (or contracts) by a factor of at least L when g is applied.

Fact. Every contraction is continuous. $\lim_{x \to y} g(x) = g(y)$

Theorem (Contraction Mapping Theorem)

Every contraction $g: [a,b] \to [a,b]$ has a unique fixed pt $\xi \in [a,b]$

Futhermore, the iteration $x_{k+1} = g(x_k)$ converges to ξ for any starting guess $x_0 \in [a,b]$

Proof

Existence: g is continuous so ξ exists by Brouwer

Uniqueness: Suppose $\xi, \eta \in [a,b]$ are both fixed pts

$$|\xi - \eta| = |g(\xi) - g(\eta)| \le L|\xi - \eta| \quad 0<L<1$$

only true if $|\xi - \eta| = 0$ $\xi = \eta$

Convergence: Let $x_0 \in [a,b]$ ξ is fixed pt

$$|x_{k+1} - \xi| = |g(x_k) - g(\xi)| \le L|x_k - \xi|$$

Everytime I iterate, distance to ξ shrinks by at least a factor of L.

By induction,

$$0 \le |x_k - \xi| \le L^k |x_0 - \xi| \to 0$$

By squeeze thm, $|x_k - \xi| = 0$ $\boxtimes$

Lemma

If $|g'(x)| \le L$ $\forall x \in [a,b]$ then $|g(x) - g(y)| \le L|x-y|$ $\forall x, y \in [a,b]$

Proof by MVT

$g(x) - g(y) \le g'(\eta)(x-y)$ for some η between x, y.

$|g(x) - g(y)| \le |g'(\eta)||x-y| \le L|x-y|$ $\boxtimes$

By Heron's: $g(x) = \frac{1}{2}(x + \frac{y}{x})$

$g'(x) = \frac{1}{2}(1 - \frac{y}{x^2})$ on $[\sqrt{y}, b)$ $|g'(x)| \le \frac{1}{2}(1 - \frac{y}{b^2})$ error shrinks by at least $\frac{1}{2}$.