Math 449: Numerical Applied Mathematics
Lecture 20

10/20/2017 Wenzhen

Today's topic: Iterative Methods in Numerical Linear Algebra

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Today: Iterative methods in numerical linear algebra

Linear Systems $A\vec{x} = \vec{b}$

- direct methods
  terminate w/ exact answer after finitely many steps
- iterative methods
  start w/ guess $x^{(0)}$ iterate $x^{(0)}, x^{(1)}, x^{(2)}, \dots,$ get answer in limit.

Why not newton's method?

$$f(\vec{x}) = A\vec{x} - b$$

$$J_f(x) = A$$

$$x^{(k+1)} = x^{(k)} A^{-1}(Ax^{(k)} - b) \quad = A^{-1} b$$

Requires solving the oringinal linear system

Recall geometric series

if $|r| < 1$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \cdots$$

$$\approx 1 + r + r^2 + \cdots + r^k$$

Note:

$$(1-r)(1+r+r^2+\cdots+r^k) = 1 - \cancel{r} + \cancel{r} - \cancel{r^2} + \cancel{r^2} + \cdots \cancel{-} \cancel{r^k} + \cancel{r^k} - r^{k+1} = 1 - r^{k+1}$$

$$\to 1 \text{ as } k \to \infty$$

$$1 + r + \cdots + r^k = \frac{1}{1-r} - \frac{r^{k+1}}{1-r} \to \frac{1}{1-r} \quad \text{as } k \to \infty$$

Generalization to matrices

$$(I - M)^{-1} \approx I + M + M^2 + \cdots + M^k$$

$$(I-M)(I + M + M^2 + \cdots + M^k) = I - M^{k+1} \quad \text{when does this cvg} \to I$$

" " $M^k \to 0$ as $k \to \infty$.

Claim: If $\rho(M) < 1$, then $M^k \to 0$ as $k \to \infty$.

i.e. $\| M^k - 0 \| \to 0$ so $M^k \to 0$ in $\|\cdot\|$

(and in every other norm,

Equivalently Contraction map cvg $\to$ fixed pt. at 0)

Just proved

Prop: If $\rho(M) < 1$, then $I - M$ is invertible and

$$(I-M)^{-1} = I + M + M^2 + \cdots$$

To solve, $(I-M)x = b$

$$x = (I-M)^{-1} \cdot b$$

$$= (I + M + M^2 + M^3 + \cdots) b$$

Approx by $x^{(k)} = (I + M + M^2 + \cdots + M^k) b \to x$ as $k \to \infty$

Observe:

$$x^{(k+1)} = \left(I + M + M^2 + \cdots + M^k + M^{k+1}\right) b$$

$$= b + (M + M^2 + \cdots + M^{k+1}) b$$

$$= b + M(I + M + \cdots + M^k) b$$

$$= b + M x^{(k)}$$

So take iteration :

$$\begin{cases} x^{(0)} = b \\ x^{(k+1)} = b + M x^{(k)} \end{cases}$$

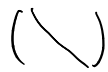
only uses matrix-vector multiplication $O(n^2)$

If we can get a good approx with $\ll n$ steps,

this does better than $O(n^2)$

Especially true for "sparse" matrices, where most entries $= 0$.

i.e., $O(n^2)$ nonzero entries

Ex. diagonal matrices 

tridiagonal matrix 

Prop Fixed pt ξ is a solution to $(I-M)\xi = b$

Proof. $\xi = b + M\xi$

$\xi - M\xi = b$

$(I-M)\xi = b$ $\boxtimes$

So if $\rho(M) < 1$, then $x^{(k)} \to \xi$ as $k \to \infty$.

To solve $A\vec{x} = \vec{b}$ split $A = I - M \Leftrightarrow M = I - A$ and if $\rho(M) < 1$, apply prev method.

What if we can't do this. i.e. what if $\rho(I-A) \geq 1$

Split $A = N - P$

N: nice, easy to solve

P: perturbation

iteration : $N x^{(k+1)} = b + P x^{(k)}$

Thm: If $\rho(N^{-1}P) < 1$, then $x^{(k)} \to \xi$ as $k \to \infty$

$\Leftrightarrow x^{(k+1)} = N^{-1}b + N^{-1}P x^{(k)}$

$(N-P)\vec{x} = \vec{b} \quad \Leftrightarrow \quad (I - N^{-1}P)\vec{x} = N^{-1}b$