Math 449: Numerical Applied Mathematics
Lecture 16

10/06/2017 Wenzhen

Today's topic: p-norm

Proof $\vec{b} = A\vec{x} \Rightarrow \|b\| = \|A\vec{x}\| \leq \|A\|\,\|x\|$

$$A\,\delta x = \delta b \Rightarrow \delta x = A^{-1}\delta b$$

$$\Rightarrow \|\delta x\| \leq \|A^{-1}\|\,\|\delta b\|$$

multiply $\|b\|\,\|\delta x\| \leq \underbrace{\|A\|\,\|A^{-1}\|}_{K(A)}\,\|x\|\,\|\delta b\|$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq K(A)\frac{\|\delta b\|}{\|b\|}$$

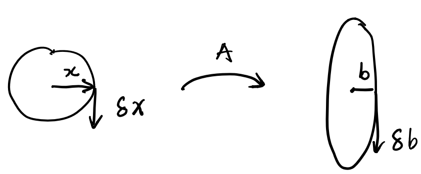
Cor $\dfrac{\|\delta b\|}{\|b\|} \leq K(A)\dfrac{\|\delta x\|}{\|x\|}$

Proof. $\begin{cases} Ax = b \\ A(x+\delta x) = b + \delta b \end{cases} \Leftrightarrow \begin{cases} A^{-1}b = x \\ A^{-1}(b+\delta b) = x + \delta x \end{cases}$

So by the theorem

$$\frac{\|\delta b\|}{\|b\|} \leq \underbrace{K(A^{-1})}_{K(A)}\frac{\|\delta x\|}{\|x\|}$$

$\boxtimes$



If A has a "really big" condition number, it's called "ill-conditioned"

Pivoting doesn't help. in the $\|\cdot\|_p$ norm, if P is a permutation matrix

$$K_p(PA) = K_p(A)$$

p-norm is invariant under permutations

$$\|Pv\|_p = \|v\|_p \Rightarrow \frac{\|PAv\|_p}{\|v\|_p} = \frac{\|Av\|_p}{\|v\|_p}$$

So $\|PA\|_p = \|A\|_p$

Similarly $\|(PA)^{-1}\|_p = \|A^{-1}\|_p$

Preconditioning.

idea: Replace $A\vec{x} = \vec{b}$ by $BA\vec{x} = B\vec{b}$

where B approximates A^{-1} Diagonal preconditioner: $B = \text{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{nn}}\right)$

Ex. $\begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 1 \end{matrix}$

$x_1 + x_2 = 1$ $\quad x_1 = 0$

$0x_1 + \varepsilon x_2 = \varepsilon$ $\quad x_2 = 1$

Diagonal preconditioner

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix}$

So replace $\begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\downarrow R_2 \times \frac{1}{\varepsilon}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2/\varepsilon \end{pmatrix}$

Notoriously ill-conditioned example Hilbert Matrix.

$$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Ex $(n=3)$ Entries on diagonal gets small really fast.

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \xrightarrow[\substack{R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{1}{3}R_1}]{\text{Gaussian Elimination}} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/12 \\ 0 & 1/12 & 4/45 \end{pmatrix} \xrightarrow[R_3 - R_2]{} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/12 \\ 0 & 0 & 1/180 \end{pmatrix}$$