Math 449: Numerical Methods
Homework 2
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Problem 1.

For Heron's method

$$e_k = x_k - \sqrt{y} \qquad e_{k+1} = \frac{e_k^2}{2x_k}$$

(a) Show if $x_k \to \sqrt{y}$ and $y > 0$, method cvg quadratically.

let $\varepsilon_k = e_k$, $e_{k+1} = \frac{e_k^2}{2x_k} \Rightarrow \frac{\varepsilon_{k+1}}{e_k^2} = \frac{1}{2x_k} = \frac{1}{2\sqrt{y}}$ ①

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^2} = \lim_{k\to\infty} \frac{1}{2\sqrt{y}} \text{ by ①}$$

$$= \mu > 0$$

$\therefore$ cvg quadratically.

(b) If $y = 0$, show that $x_k \to 0$ at least linearly but not quadratically.

$$\lim_{k\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \lim_{k\to 0} \frac{\frac{1}{2}|x_k|}{|x_k|} = \frac{1}{2}. = \mu \quad \Rightarrow \text{linearly}$$

$q = 2$, $\lim_{k\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k^2} = \lim_{k\to\infty} \frac{1}{2\sqrt{y}} = \infty \Rightarrow$ slower than 2, which is quadratically.

Problem 2.

Simple iterative method $x_k = 2x_k - y \cdot x_k^2$ for $y \neq 0$.

which is used to compute reciprocal $\frac{1}{y}$ without any division operations.

(a) Show $0/1$ are the only fixed pts of $g(x) = 2x - yx^2$

$$g(\xi) = 2\xi - y \cdot \xi^2 = \xi$$

$$\xi - y \cdot \xi^2 = \xi(1 - y \cdot \xi) = 0 \Rightarrow \xi_1 = 0 \text{ or } \xi_2 = \frac{1}{y}.$$

(b) Determine whether each fixed pt is stable / not

$g'(x) = 2 - 2y \cdot x$ by part (a) fixed pt $\xi_1 = 0$, $\xi_2 = \frac{1}{y}$

By fixed pt stability test,

$|g'(0)| = 2 > 1$, then $\xi_1 = 0$ is unstable

$|g'(\frac{1}{y})| = 0 < 1$, then $\xi_2 = \frac{1}{y}$ is stable.

(c). Iteration newton's method. particular choice of f. which has $\frac{1}{y}$ as a root.

$f(x) = \frac{1}{x} - y.$

Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k + \frac{1/x_k - y}{1/x_k^2} = x_k + x_k^2\left(\frac{1}{x_k} - y\right) = \boxed{2x_k - yx_k^2}$$

which agrees with the problem.