Math 449: Numerical Applied Mathematics
Lecture 12

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Today's topic: LU Decomposition

Theorem

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{matrix}} & \begin{matrix} \cdots & a_{1n} \\ & \vdots \\ \cdots & a_{kn} \end{matrix} \\ \begin{matrix} \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & a_{nn} \end{matrix} \end{pmatrix}$$

$A^{(k)}$: k^{th} leading principle submatrix of A.

Theorem If $A \in \mathbb{R}^{n\times n}$, and if $A^{(k)} \in \mathbb{R}^{k\times k}$ is nonsingular for $k=1,\dots,n-1$. then we can decompose $A = LU$. LU decomp exists

Ex. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad A^{(1)} = (0)$ singular. does not have an LU decomp, even though $\det(A) = -1 \neq 0$

Counter Ex.

$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \det(A) = -1 \neq 0$

A doesn't have LU decomp

but after swapping rows, we have.

$A\vec{x} = \vec{b}$ has unique solution $\vec{x}$

Swap the rows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{U}$$

Swapping rows = pivoting

Q: What matrix P swaps the rows of A?

Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

check

$$PA = \underset{P}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} = \underset{A \text{ after swap rows.}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

P is called permutation matrix. A permutation matrix P is such that PA consists of A with its rows permuted.

In particular, taking $A = I$ shows us that P is just the result of permuting the corresponding rows of I.

In general, we want to permute the rows of an $n \times n$ matrix, take P to be the corresponding permutation of the $n \times n$ identity matrix.

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\qquad PA = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$R3 \to R1$
$R2 \to R3$
$R1 \to R2$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \to 3^{rd} \text{ row of } I \\ \to 2^{nd} \text{ row} \\ \to 1^{st} \end{array}$$

Interesting fact about permutation matrix.

1. The product of 2 permutation matrix is also a permutation.
2. Every permutation matrix is orthogonal.
$$PP^T = I = P^T P$$
So P is invertible w/ $P^{-1} = P^T$
$$\begin{pmatrix} — & e_i^T & — \end{pmatrix} \begin{pmatrix} | \\ e_i \\ | \end{pmatrix} = \begin{pmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{pmatrix}$$
$$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th row}.$$
3. Every permutation can be written as a product of row swaps.

Theorem

Every square matrix A can be decomposed as
$$\underset{\text{perm}}{P}A = \underset{\substack{\text{unit} \\ \text{lower tri}}}{L}\ \underset{\text{upper tri}}{U}$$

Remark this decomposition is generally not unique!

Ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underset{L}{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}}$

$(P = I)$

If $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\quad PA = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \underset{L}{\begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{pmatrix}}$

Proof. Induction on n

base case: $n=1$

every 1×1 matrix satisfies this condition $\quad A=(a)\ P=(1)$

$\quad PA=(a)=(1)(a)$

(vacuously) $\quad (a)=(1)(a)$

induction step: $n\to n+1$

Suppose true, for matrices of size $n\times n$, show also true for $(n+1)\times(n+1)$

look at 1^{st} col of A, suppose $a_{r1}=\alpha$ has the largest absolute value in 1^{st} col.

swap row 1 and row r

$$\begin{pmatrix} a_{11} & \\ \vdots & \\ \alpha & \\ \vdots & \\ a_{n+1,1} & \end{pmatrix}$$

$P^{(1r)}$ is permutation, matrix swapping rows 1 and r

$$P^{(1r)}A=\left(\begin{array}{c|c} \alpha & \text{row vec} \\ \vdots & \\ a_{11} & n\times n \text{ matrix} \\ \vdots & \\ a_{n+1,1} & \end{array}\right)=\begin{pmatrix} \alpha & w^T \\ p & B \end{pmatrix}$$

Claim $=\begin{pmatrix} 1 & 0^T \\ m & I \end{pmatrix}\begin{pmatrix} \alpha & v^T \\ 0 & C \end{pmatrix}$ for some $m, v\in\mathbb{R}^n$, $C\in\mathbb{R}^{n\times n}$

Take $v=w$, if $\alpha=0$, then the entire 1^{st} col is 0 $\Rightarrow p=0$.

So take $m=0$, $C=B$.

If $\alpha\neq 0$, take $m=\dfrac{p}{\alpha}$

$B=mv^T+C \Leftrightarrow C=B-mv^T$

By inductive assumption, exists P^*, L^*, U^* s.t. $P^*C=L^*U^*$

$C=(P^*)^T L^* U^*$

Altogether, $P^{(1r)}A=\begin{pmatrix} 1 & 0^T \\ m & I \end{pmatrix}\begin{pmatrix} \alpha & v^T \\ 0 & (P^*)^T L^* U^* \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ m & (P^*)^T L^* \end{pmatrix}\begin{pmatrix} \alpha & v^T \\ 0 & U^* \end{pmatrix}$

$$=\begin{pmatrix} 1 & 0^T \\ 0 & (P^*)^T \end{pmatrix}\begin{pmatrix} 1 & 0^T \\ P^*m & L^* \end{pmatrix}\begin{pmatrix} \alpha & v^T \\ 0 & U^* \end{pmatrix}$$

So take $P=\begin{pmatrix} 1 & 0^T \\ 0 & P^* \end{pmatrix}P^{1r}$ $\qquad PA=LU$ $\qquad\boxtimes$