Math 449: Numerical Methods
Lecture 05
09/11/2017

Today's topic: Rates of convergence & Newton's Methods

Example:

$2^{-k} \to 0$ linearly error is halved. $\mu = \frac{1}{2}$

$4^{-k} \to 0$ also linearly $\mu = \frac{1}{4}$

$2^{2^{-k}} \to 0$ with order $= 2$ quadratically

In general, $2^{-q^k} \longrightarrow 0$ with order $= q$ $\forall\, q > 1$

Example

$x_k = 2^{-k^2} \longrightarrow 0$

$\varepsilon_k = |x_k - \xi| = x_k$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{2^{-(k+1)^2}}{2^{-k^2}} = \frac{2^{\cancel{-k^2}-2k-1}}{2^{\cancel{-k^2}}} = 2^{-k-1} \to 0 \notin (0,1) \quad \text{faster than linear}$$

For any $q > 1$,

$$\frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \frac{2^{-k^2-2k-1}}{2^{-qk^2}} = \frac{2^{-2k-1}}{2^{-(q-1)k^2}} \to \infty$$

Slower than order q for any $q > 1$.

Ex. $x_k = \frac{1}{k} \to 0$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{k}{k+1} \to 1 \notin (0,1)$$

Slower than linear (sublinear)

Newton's Method

To find $f(\xi) = 0$, take x_0 $\quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

Assume that $f'(x_k) \neq 0$ $\forall k$

Suppose $f(\xi) = 0$, $f'(\xi) \neq 0$

Fixed pt iteration: $g(x) = x - \frac{f(x)}{f'(x)}$

To check if ξ is a stable fixed pt: $\quad g(\xi) = \xi - \frac{0}{\text{nonzero}} = \xi$

To check the stabality, look at $g'(\xi)$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$\therefore g'(\xi) = \frac{\overset{0}{\cancel{f(\xi)}}\, f''(\xi)}{f'(\xi)^2} = 0$$

$\therefore$ ξ is a stable fixed pt.

$x_k \longrightarrow \xi$ for x_0 "close enough" to ξ. How fast ?

Taylor's theorem with mean-value remainder.

real value $-$ kth order taylor expansion

1st order $\quad f(\xi) = f(x) + f'(y)(\xi - x) \quad$ for some $y \in (x, \xi)$

intermediate

2nd order $\quad f(\xi) = f(x) + f'(y)(\xi - x) + \frac{1}{2}f''(y)(\xi - x)^2$

$\vdots$

kth order (general case) $\quad f(\xi) = f(x) + f'(y)(\xi - x) + \cdots + \frac{1}{(k-1)!}f^{k-1}(y)(\xi - x)^{k-1} + \frac{1}{k!}f^{k}(y)(\xi - x)^k$

If f is twice continuously differentiable in a neighborhood of ξ, with $f(\xi)=0$, $f'(\xi)\neq 0$, then $x_k \to \xi$ with order 2 for x close enough to ξ.

Proof

Taylor expand $f(\xi)$ around x_k

$$0 = f(\xi) = f(x_k) + f'(x_k)(\xi - x_k) + \frac{1}{2}f''(x_k)(\xi - x_k)^2$$

$$-f(x_k) = f'(x_k)(\xi - x_k) + \frac{1}{2}f''(x_k)(\xi - x_k)^2$$

$$-\frac{f(x_k)}{f'(x_k)} = (\xi - x_k) + \frac{f''(x_k)(\xi - x_k)^2}{2f'(x_k)}$$

$$\underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} - \xi = \frac{f''(x_k)(\xi - x_k)^2}{2f'(x_k)}$$

$$|x_{k+1} - \xi| = \left|\frac{f''(x_k)}{2f'(x_k)}\right| |x_k - \xi|^2$$

$$\frac{|x_{k+1} - \xi|}{|x_k - \xi|^2} = \left|\frac{f''(x_k)}{2f'(x_k)}\right|$$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k^2} \longrightarrow \left|\frac{f''(x_k)}{2f'(x_k)}\right| = \mu \qquad \boxtimes$$

Recall Heron's Method is Newton's Method for $f(x) = x^2 - y$

Q: Why is Heron's Method so fast for $y>0$ but not $y=0$?

A. $f'(x) = 2x \Rightarrow f'(\pm\sqrt{y}) = 2\sqrt{y}$ if $y>0$, $f'(\pm\sqrt{y}) > 0$ so convergence is quadratic.

If $y=0$, then $f'(0)=0$, so theorem doesn't apply. convergence is linear (HW2)

More general result.

with order ≥ 2

If ξ is a fixed pt of g, and $g'(\xi)=0$, then fixed pt iteration $x_{k+1} = g(x_k)$ cvgs at least quadratically

Also assume, g is twice continouslly differentiable near neighborhood of ξ.

Proof Taylor expand around ξ

$$g(x_k) = g(\xi) + \cancel{g'(\xi)(x_k-\xi)}^{\,0} + \frac{1}{2}g''(\xi)(x_k-\xi)$$

$$x_{k+1} = \xi + \frac{1}{2}g''(\xi)(x_k-\xi)$$

$$\frac{|x_{k+1}-\xi|}{|x_k-\xi|^2} = \frac{1}{2}|g''(\xi)| \to \frac{1}{2}|g''(\xi)| \quad \boxtimes.$$

More general thm

If g is q times continuously differentiable near ξ,

$$g'(\xi) = \cdots = g^{(q-1)}(\xi) = 0 \quad \text{and} \quad g^{q}(\xi) \neq 0$$

then $x_{k+1} \to \xi$ with order of q.

Proof. Take Taloy Expansion out to order of q

$$x_{k+1} = g(x_k) = \underbrace{g(\xi)}_{\xi''} + \underbrace{0 + \cdots + 0}_{\text{Taylor around } \xi} + \frac{1}{q!}g^{(q)}(\eta_k)(x_k-\xi)^q$$

$$x_{k+1} - \xi'' = \frac{1}{q!}g^{(q)}(\eta_k)(x_k-\xi)^q$$

$$\frac{\xi_{k+1}}{\xi_k^q} = \frac{1}{q!}\left|g^{(q)}(\eta_k)\right|$$

$$= \underbrace{\frac{1}{q!}\left|g^{(q)}(\xi)\right|}_{>0} = \mu$$

$\boxtimes$