Math 449: Numerical Applied Mathematics
Lecture 07

09/15/2017

Today's topic: Intro to Numerical Linear Algebra

Intro to Numerical Linear Algebra

1. 1 – variable :

linear systems are trivial

$ax = b \quad (a \neq 0)$ has solution $x = \frac{b}{a} = a^{-1} b$

division appears in 1-D newton's method. $x_{k+1} = x_k - \frac{f(x)}{f'(x)}$

Focus : algorithms that are fast / efficient for very large systems ($n \times n$ matrices with n large)

Ex. Google's Page Rank eigenvector of matrix of size (# of webpages) × (# of webpages).

Motivations.

$A\vec{x} = \vec{b}$

$A \in \mathbb{R}^n \times \mathbb{R}^n \qquad \underbrace{\vec{x}}_{\text{unknown}}, \overbrace{\vec{b}}^{\text{known}} \in \mathbb{R}^n$

A is nonsingular.

an inverse.

unique soln $\vec{x} = A^{-1}\vec{b}$

Ex. $\begin{cases} \alpha x_1 + \beta x_2 = b_1 \\ \gamma x_2 + \delta x_2 = b_2 \end{cases} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

if $\det(A) = \alpha \cdot \delta - \beta \cdot \gamma \neq 0$ then $A^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$

Soln: $\vec{x} = A^{-1}\vec{b} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$= \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta b_2 - \beta b_1 \\ -\gamma b_1 + \alpha b_2 \end{pmatrix}$$

Check $Ax = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta b_2 - \beta b_1 \\ -\gamma b_1 + \alpha b_2 \end{pmatrix}$

$$= \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} (\alpha\delta - \beta\gamma) b_1 + 0 \cdot b_2 \\ 0 b_1 + (\alpha\delta - \beta\gamma) b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Cramer's Rule (works for $n \times n$ matrix) from $2\times2 \to n \times n$

Soln to $A\vec{x} = \vec{b}$ is $x_i = \dfrac{\det(A_i)}{\det(A)}$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

where A_i is the matrix obtained by replacing the i^{th} column of A by $\vec{b}$ $(i = 1, \dots, n)$

Ex. $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $A_1 = \begin{pmatrix} b_1 & \beta \\ b_2 & \delta \end{pmatrix}$ $A_2 = \begin{pmatrix} \alpha & b_1 \\ \gamma & b_2 \end{pmatrix}$

$\det(A) = \alpha\delta - \beta\gamma$

$\det(A_1) = \begin{vmatrix} b_1 & \beta \\ b_2 & \delta \end{vmatrix} = \delta b_1 - \beta b_2$ $\det(A_2) = \begin{vmatrix} \alpha & b_1 \\ \gamma & b_2 \end{vmatrix} = \alpha b_2 - \gamma b_1$

$x_1 = \dfrac{\delta b_1 - \beta b_2}{\alpha\delta - \beta\gamma}$ $x_2 = \dfrac{\alpha b_2 - \gamma b_1}{\alpha\delta - \beta\gamma}$

Expansion by rows / columns.

$\begin{pmatrix} \cdot & \cdot & \\ & & \\ & & \end{pmatrix}$ $\det A = \sum_{j=1}^{n} (-1)^{j-1} a_{1j} \det(A_{1j})$

A_{1j} : $(n-1) \times (n-1)$ matrix excluding row 1 and col j.

To compute $\det(n \times n)$, compute det of n different $(n-1)\times(n-1)$ matrices.

		# terms
$n=1$:	$\det(a_{11}) = a_{11}$	1
$n=2$:	det(	2
$n=3$:		6
$n=4$		24
$n=5$		120

in general, $\det(A)$ has $n!$ terms.

O- notation

describing the comparative growth rates of different sequences.

How many steps does it take to solve an $n \times n$ system ?

How quickly does this grow with n?

direction methods

Method that returns the answer after a finite # of steps.

Def. let (x_n) and (y_n) be sequence of real #.

" big O" $\quad x = O(y_n)$ if $|x_n| \leq c|y_n|$ for some $c > 0$, $N \in \mathbb{N}$

whenever $n \geq N$ for some $N \in \mathbb{N}$

Ex. $x_n = \sin(n) = O(1)$

$\sin(n) \in [-1, 1]$

Ex. $x_n = O(1)$

here $y_n = (1, 1, 1, \dots)$

$|x_n| \leq c$ for large enough n

$\Rightarrow$ x_n is a bounded sequence.

If $n \geq 1$, then $1 + \frac{1}{n} \geq 2$

Ex. $2n+1 = O(n)$

if $n \geq 1$, then $2n+1 \leq 2n + n = 3n$

So take $N = 1$, $c = 3$.

$n + 1000{,}000 = O(n)$.

General idea

look at dominant term that grows fastest as $n \to \infty$

ignore constant factors and lower-order terms.