Math 449: Numerical Methods
Homework 5
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Part I : Theory

Problem 1. Prove the following norm inequalities

a. $\forall\ v \in \mathbb{R}^n$, $\|v\|_2^2 \leq \|v\|_1 \|v\|_\infty$

Proof. by def. we have

$$\|\vec{v}\|_2^2 = v_1^2 + v_2^2 + \cdots + v_n^2$$

$$\|\vec{v}\|_1 = \sum_{i=1}^{n} |v_i|$$

$$\|\vec{v}\|_\infty = \max_{i=1\ldots n} |v_i|$$

$$|v_i|^2 \leqslant |v_i| \cdot \max_{i=1}^{n} |v_i|$$

$$|v_i| < \max_{i=1}^{n} |v_i|$$

$$\|v\|_2^2 \leqslant \|v\|_1 \cdot \|v\|_\infty$$

(b) For any norm $\|\cdot\|$ on $\mathbb{R}^n$, if λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$, then $\|A\| \geqslant |\lambda|$ in the induced norm.

Let $\tilde{v}$ be an eigenvector with λ being the corresponding eigenvalue.

We have $A\tilde{v} = \lambda \tilde{v}$

$$\frac{\|A\tilde{v}\|}{\|\lambda \tilde{v}\|} = 1$$

So $\|A\tilde{v}\| = |\lambda|\ \|\tilde{v}\|$

$$\Rightarrow\ |\lambda| = \frac{\|A\tilde{v}\|}{\|\tilde{v}\|} \leqslant \max_{v \neq 0} \frac{\|A\tilde{v}\|}{\|\tilde{v}\|} = \|A\|$$

Problem 2.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x_1, x_2) = \begin{pmatrix} x_1^3 - 3x_1x_2^2 - 1 \\ 3x_1^2x_2 - x_2^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Show $f(x_1, x_2) = \vec{0}$ has 3 solutions : $(1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$$\begin{cases} x_1^3 - 3x_1x_2^2 - 1 = 0 & ① \\ 3x_1^2x_2 - x_2^3 = 0 & ② \end{cases}$$

$x_2(3x_1^2 - x_2^2) = 0 \Rightarrow x_2 = 0$ or $3x_1^2 = x_2^2$

(i) back substitute $x_2 = 0$ in ①

$x_1^3 - 1 = 0 \Rightarrow x_1 = 1$.

Thus, one solution is $(1,0)$

(ii) back substitute $3x_1^2 = x_2^2$ in ①

$$x_1^3 - 3x_1 \cdot (3x_1^2) - 1 = 0$$
$$-8x_1^3 = 1$$
$$x_1^3 = -\frac{1}{8}$$
$$\therefore x_1 = -\frac{1}{2}$$

by $3x_1^2 = x_2^2$, $\quad x_2^2 = \frac{3}{4} \Rightarrow x_2 = \pm\frac{\sqrt{3}}{2}$

Thus, the other solutions are $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$