Math 449: Numerical Methods
Homework 3
Wenzhen Zhu

Part 1: Theory.

P1. Prove the following statement about the order of growth of sequences, expressed using O notation.

(a) If f is a polynomial of degree d, then $f(n) = O(n^d)$

Let $f(n)$ have a general polynomial form:

$$f(n) = t_0 + t_1 n + t_2 \cdot n^2 \cdots + t_d n^d$$
$$\leqslant |t_0 + t_1 n + t_2 \cdot n^2 \cdots + t_d n^d|$$
$$\lesssim |t_0 n^d + t_1 n^d + t_2 \cdot n^d + \cdots + t_d n^d|$$
$$\leqslant n^d (|t_0 + t_1 + \cdots + t_d|)$$
$$= c \cdot n^d \quad \text{where } c \text{ is a constant.}$$
$$= O(n^d)$$

(b) $(\forall d \in N)\ (n^d = O(e^n))$

$$e_n = \sum_{k=0}^{\infty} \frac{n^k}{k!} = 1 + n + \frac{n^2}{2} + \frac{n^3}{6} + \frac{n^4}{24} + \cdots$$

$$e_n \geqslant \frac{n^d}{d!}$$

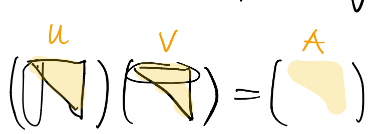
$$d!\, e_n \geqslant n^d$$
$$c \cdot e_n \geqslant n^d$$
$$\therefore O(e^n) = n^d$$

P2. Show product of 2 $n \times n$ upper triangular matrices is again an upper triangular matrix.

Hint: Split $\sum_{k=1}^{n} u_{ik} v_{kj}$ into $\sum_{k=1}^{j} u_{ik} v_{kj} + \sum_{k=j+1}^{n} u_{ik} v_{kj}$

Pf. By defn of upper triangular matrices, suppose U, V are both upper triangular matrix,

we have $u_{ij} = v_{ij} = 0 \quad (i > j)$



Let $A = UV$

$$a_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj} = \overset{0}{\cancel{\sum_{k=1}^{j} u_{ik} v_{kj}}} + \overset{0}{\cancel{\sum_{k=j+1}^{n} u_{ik} v_{kj}}}$$

when $i > j > k$, $u_{ik} = 0$. $\quad k > j \Rightarrow v_{kj} = 0$

$\therefore a_{ij} = 0$ when $i > j$. which also satisfies upper triangular defn. ∎

P.3.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

Show $\det(U) = u_{11} u_{22} \cdots u_{nn}$.

Pf by induction :

① base case ($n=1$) :

$U = (u_{11})$ $\det(U) = u_{11}$ which is true obviously.

② indution step :

Let U' be a u.t.m with $n+1$ by $n+1$, U be utm with $n \times n$.

$$\det(U') = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} & u_{1,n+1} \\ 0 & u_{22} & \cdots & u_{2n} & u_{2,n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & u_{nn} & u_{n,n+1} \\ 0 & \cdots & 0 & 0 & u_{n+1,n+1} \end{vmatrix} = u_{n+1,n+1} \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{vmatrix}$$

$$\det(U') = u_{n+1,n+1} \cdot \det(U) = (u_{11} u_{22} \cdots u_{nn})\, u_{n+1,n+1}$$

$\boxtimes$