Math 449: Numerical Applied Mathematics
Lecture 15

10/04/2017 Wenzhen

Today's topic: p-norm

Def. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the corresponding matrix norm on $\mathbb{R}^{n\times n}$ (aka. matrix norm) subordinate to the norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined as:

$$\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|.$$

ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -7 & -8 & -9 \end{pmatrix}$ $\quad \|A\|_\infty = |7| + |-8| + |-9| = 24$

$\vec{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad \|v\|_\infty = 1$

$A\vec{v} = \begin{pmatrix} -4 \\ -7 \\ 24 \end{pmatrix} \Rightarrow \|A\vec{v}\|_\infty = 24$

<u>Thm</u> $\quad \|A\|_\infty = \overbrace{\max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|}^{C.}$ $\quad$ max, absolute row sum.

Proof $\quad \|A\|_1 \leq C$ and $C \leq \|A\|_1$

$$\|Av\|_1 = \sum_{i=1}^{n} |Av_i| = \sum_{i=1}^{n} \left|\sum_{i=1}^{n} a_{ij} v_j\right| \leq C \sum_{j=1}^{n} |v_j| = C\|v_j\| = C\|v\|_1 \Rightarrow \|A\|_1 \leq C$$

On the other hand, if max occurs in column m, take $v = e_m$ $\quad \|v\|_1 = 1$

$\Rightarrow Av$ is just the m^{th} column of A.

$$\|Av\|_1 = \sum_{i=1}^{n} |a_{im}| = C \leq \|A\|_1$$

matrix 2-norm

Def. Given $A \in \mathbb{R}^{n\times n}$, let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A^TA = (A^TA)^T$

$\sigma_i = \lambda_i^{1/2} \quad (i = 1, \dots, n)$

are called the singular value of A.

$\lambda_i \in \mathbb{R}$ because A^TA is symmetric (spectral theorem)

why $\lambda \geqslant 0$?

Suppose λ is an eigenvalue of A^TA with eigenvectors v

$$A^TA\,v = \lambda v$$

$$v^TA^TAv = \lambda v^Tv$$

$$(Av)^T(Av) = \lambda v^Tv$$

$$Av \cdot Av = \lambda v \cdot v$$

$$\|Av\|_2^2 = \lambda \|v\|_2^2 \quad \Rightarrow \lambda = \frac{\|Av\|_2^2}{\|v\|_2^2} \geqslant 0$$

$$\sigma = \frac{\|Av\|_2}{\|v\|_2} \geqslant 0$$

A^TA is symmetric positive semi-definite. possibly $Av=0$ for $v \neq 0$.

<u>Thm</u> $\|A\|_2 = \sigma_n$, the largest singular values of A

Proof. $\|A\|_2 = \max\limits_{\|v\|_2=1} \|Av\|_2$

Suppose v maximizes $\|Av\|_2$ subject to $\|v\|_2 = 1$

Then it also maximizes $\|Av\|_2^2$ subject to $\|v\|_2^2 = 1$

$f(u) = \|Au\|_2^2 \qquad g(u) = \|u\|_2^2$

$\quad = Au \cdot Au \qquad\quad = u \cdot u$

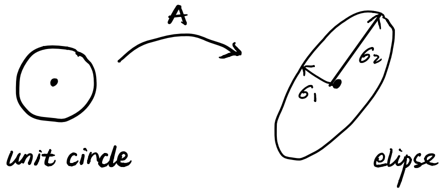
Lagrange Multiplier theorem:

$$\nabla f(v) = \lambda \nabla g(v)$$

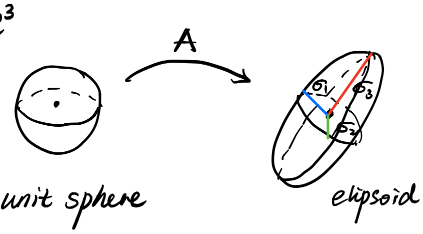
$$2A^TAv = 2\lambda v$$

if $\vec{v}$ maximizes, v is an eigenvector of A^TA with eigenvalue λ

Visual in $\mathbb{R}^2$



in $\mathbb{R}^3$



$\sigma_1 < \sigma_2 < \sigma_3$

σ_i = radii of elipsoid along principal axes.

Def. Let $\|\cdot\|$ be a norm. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix.
The condition number of A is defined by

$$\kappa(A) = \|A\|\,\|A^{-1}\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|} \quad \max_{v\neq 0} \frac{\|v\|}{\|Av\|}$$

Note $\kappa(A) = \kappa(A^{-1})$

Ex. In $\mathbb{R}^2$, $\kappa_2(A) = \dfrac{\sigma_2}{\sigma_1}$ In $\mathbb{R}^n$, $\kappa_2(A) = \dfrac{\sigma_n}{\sigma_1}$