Math 449: Numerical Applied Mathematics
Lecture 26

11/03/2017 Wenzhen
Today's topic: Hermite Interpolation

Today: Hermite interpolation

Recall: Given $x_0, \dots, x_n$, distinct

$y_0, \dots, y_n$ } not distinct
$z_0, \dots, z_n$,

find $p_{2n+1} \in P_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i \qquad i = 0, 1, \dots, n$$
$$p'_{2n+1}(x_i) = z_i$$

Construct Hermite basis $\{H_0, \dots, H_n, K_0, \dots, K_n\}$ of P_{2n+1} s.t. $H_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & i \neq k \end{cases}$

$$H_k'(x_i) = 0 \qquad K_k(x_i) = 0 \qquad K_k'(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left[H_k(x) y_k + K_k(x) z_k \right]$$

Lagrange interpolation uses nodal polynomials

$$\pi_{n+1}(x) = (x - x_0) \cdots (x - x_n)$$

which has roots at $x_0, \dots, x_n$

Idea: If we want derivative to also vanish, need double roots $(x-x_0)^2 \cdots (x-x_n)^2$

Consider $[L_k(x)]^2 \qquad L_k \in P_n$

By construction, $[L_k(x_i)]^2 = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$

degree $2n$, has 1st property we want from H_k

$$\frac{d}{dx}[L_k(x)]^2 = 2 L_k(x) L'_k(x) \quad \text{at } x = x_i \qquad \frac{d}{dx}(L_k(x))^2 = \begin{cases} 2L_k'(x_i) & i = k \\ 0 & i \neq k \end{cases}$$

Take $$H_k(x) = \underbrace{[L_k(x_i)]^2}_{\text{degree } 2n} \underbrace{(1 - 2L_k'(x_k)(x - x_k))}_{\text{degree } = 1}$$

$$H_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_k'(x_i) = \begin{cases} 2L_k'(x_k) - 2L_k'(x_k) & i = k \\ 0 & i \neq k \end{cases}$$

$$K_k(x) = \underbrace{[L_k(x)]^2}_{\text{degree } 2n} \underbrace{(x - x_k)}_{\text{degree } 1} \qquad K_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$K_k'(x_i) = 2L_k(x_i)\,L_k'(x_i)\,(x-x_k) + [L_k(x_i)]^2 = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases}$$

Example $(n=1)$

$x_0 = 0,\ x_1 = 1$

$$L_0(x) = \frac{x-x_1}{x_0-x} = \frac{x-1}{0-1} = 1-x$$

$$L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-0}{1-0} = x$$

$$H_0(x) = [L_0(x)]^2(1-2L_0'(0)(x-0)) = (1-x)^2(1+2x) = 1-3x^2+2x^3$$

check $H_0(0)=1 \quad H_0(1)=0$

$$H_0'(x) = -6x+6x^2 \qquad H_0'(0)=0 \quad H_0'(1)=0$$

$$H_1(x) = [L_1(x)]^2(1-2L_1'(1)(x-1)) = x^2(1-2(x-1)) = x^2(3-2x) = -x^3+3x^2$$

$$H_1(0)=0 \quad H_1(1)=1$$

$$H_1'(x) = 6x-6x^2$$

$$H_1'(0) = 0$$

$$K_0(x) = [L_0(x)]^2(x-x_0) = (1-x)^2 x = x-2x^2+x^3$$

Given $f:[a,b]$ by P_{2n+1} $\quad P_{2n+1}(x_i) = f(x_i)$

$\quad P_{2n+1}'(x_i) = f'(x_i)$

Theorem For each $x\in[a,b]$, exists ξ:

$$f(x)-P_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

Proof. Fix $x\in[a,b]$ if $x=x_i$, both sides vanish

otherwise, define

$$\psi(t) = [f(t)-P_{2n+1}(t)] - \frac{f(x)-P_{2n+1}(x)}{[\pi_{n+1}(x)]^2}[\pi_{n+1}(t)]^2 \quad \text{vanishes at } x_0,\dots,x_n$$

ψ vanishes at $n+1$ pts.

By construction, ψ' also vanishes at the $n+1$ oringinal pts $x_0,\dots,x_n$

So intotal, ψ' vanishes at $2n+2$ pts.

Rolle $\Rightarrow$ ψ'' vanishes at $2n+1$ pts

$\vdots$

$\psi^{(n+2)}$ vanishes at a pt ξ.

$$\psi^{(2n+2)}(t) = f^{(2n+2)}(t) - \cancel{P_{2n+1}^{(2n+2)}}\overset{0}{} (t) - \frac{f(x) - P_{2n+1}}{[\lambda_{n+1}(x)]^2} (2n+2)!$$

$$(\lambda_{n+1}(x))^2 = (x-x_0)^2 \cdots (x-x_n)^2 = x^{2n+2} + \text{lower-order terms.}$$

$$0 = \psi^{(2n+2)}(\xi) = f^{(n+2)}(\xi) - \frac{f(x) - P_{2n+1}(x)}{(\lambda_{n+1}(x))^2} (2n+2)!$$

$\boxtimes$