Math 449: Numerical Methods
Lecture 00
Aug 28th, 2017

Today's topic: Introduction

Notion of "solution" to a problem

Answer 1: Prove that a solution exists.

Answer 2: A symbolic solution

$$ax^2 + bx + c = 0$$

when $a \neq 0$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad$ quadratic formular

higher order polynomial

for quintic and above, there is no formula.

Answer 3: A numerical solution

An algorithm that approximates the exact solution to any desired level of accuracy.

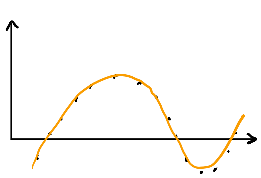
ex. bi-section algorithm

Numerical Analysis is concerned w/:

- developing such algorithms
- proving properties of these algorithms.
  - consistency correctness
  - convergence
  - stability
  - rate of convergence.
- implementing these algorithms.

Problems where numerical analysis is useful:

- root finding
- optimization: finding max / min of a given function.
- numerical linear algebra: Solving linear equations, finding eigenvalues and eigenvectors
- interpolation functions: given values of a function at some points, approximate value at any point.



- numerical quadrature: approximate the integral of a function.
- ODEs / PDEs: dig deeply Euler's methods

Numerical Root Finding

Solve $f(x) = 0$

when you cannot solve symbolically.

Start with a motivating example.

Heron's square root algorithm (2000 yrs old)

want to approximate $\sqrt{y}$

i.e. find a root of $f(x) = x^2 - y$

Heron's Algorithm:
1. Start with a guess x_0
2. For $k = 0, 1, 2, 3, \ldots$

$f'(x) = 2x$

Let $x_{k+1} = \frac{1}{2}(x_k + \frac{y}{x_k})$

Ex. Approx $\sqrt{2}$ $x_0 = 1$

$x_0 = 1$

$x_1 = \frac{1}{2}(x_0 + \frac{2}{x_0}) = \frac{1}{2}(1+2) = 1.5$

$x_2 = \frac{1}{2}(x_1 + \frac{2}{x_1}) = \frac{1}{2}(\frac{3}{2} + \frac{2}{\frac{3}{2}}) = \frac{1}{2} \cdot \frac{7}{3} = \frac{7}{6} = 1.466$

Consistency?

if $x_k \to \xi$, is it true that $\xi^2 = y$?

Yes. $x_{k+1} = \frac{1}{2}(x_k + \frac{y}{x_k})$

Take $\lim_{k\to\infty}$ on both sides:

$$\xi = \frac{1}{2}(\xi + \frac{y}{\xi})$$

$$2\xi = \xi + \frac{y}{\xi}$$

$$2\xi^2 = \xi^2 + y$$

$$\xi^2 = y$$

Sources of Eumerical Error.

Limitation, computers have finite space and time.

1. We can't run the algorithm ∞ times. (more often, problems is more fundamental)
2. Rounding error : we can only store a finite representation of a real number.
   So rounding is needed.

We can't do exact arithmetics in real numbers.