Examples on $V = \mathbb{R}^n$

- 2-norm $\quad \|v\|_2 = \sqrt{\vec{v} \cdot \vec{v}}$
- 1-norm $\quad \|v\|_1 = \|v_1\| + \cdots + |v_n|$
- p-norm $\quad (p \geq 1) \quad \|v\|_p = (|v_1|^p + \cdots + |v_p|^p)^{\frac{1}{p}}$
- ∞ - norm $\quad \|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_p|\}$

Properties of norms.

① when $n=1$, these are all identical to the abs value.

② $$e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{matrix}$$ ith col of identity matrix.

$\|e_i\| = 1$

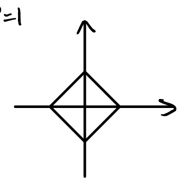
③ All are invariant under permutations of the entries.

e.g. $\|(v_1, v_2)\| = \|(v_2, v_1)\|$

If P is a permutation matrix, $\|Pv\| = \|v\|$

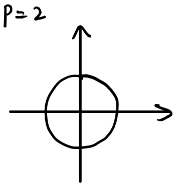
"Unit circles" in $\mathbb{R}^2$ $\quad \|v\| = 1$ for each of these norms.

$p=1$



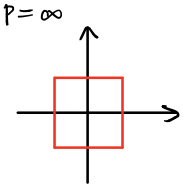
$|x| + |y| = 1$

$p=2$



$x^2 + y^2 = 1$

$p=\infty$



$\max\{|x|, |y|\} = 1$

Justification of "$p=\infty$"

If $v \neq 0$, and let $\tilde{v} = \frac{v}{\|v\|_\infty}$ so $\|\tilde{v}_i\| \leq 1$ and $\max_i \|\tilde{v}_i\| = 1$

$$\|\tilde{v}\|_p = \underbrace{(|\tilde{v}_1|^p + |\tilde{v}_2|^p + \cdots + |\tilde{v}_n|^p)}_{\text{at least } 1=1,\ \text{sum}}^{\frac{1}{p}}$$

$$1 \leq \|\tilde{v}\|_p \leq n^{\frac{1}{p}} \to 1 \quad \text{as } p \to \infty$$

Sandwitch lemma $\to$ $\lim_{p\to\infty} \|\tilde{v}\|_p = 1$ $\quad \|\tilde{v}\|_p = \left\| \frac{v}{\|v\|_\infty} \right\|_p = \frac{\|v\|_p}{\|v\|_\infty}$

So $\lim_{p\to\infty} \frac{\|v\|_p}{\|v\|_\infty} = 1$.

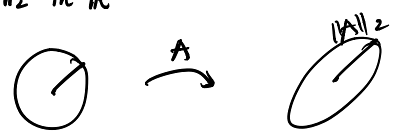
Matrix norms

Def. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the corresponding matrix norm on $\mathbb{R}^{n\times n}$ (aka. matrix norm) subordinate to the norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined as:

$$\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|.$$

$\|A\|$ is the maximum scaling factor when A is applied to $\vec{v}$.

Ex. $\|\cdot\|_2$ in $\mathbb{R}^2$



2000 500

Properties of matrix norms.

Thms 1. $\|I\| = 1$ for any $\|\cdot\|$

2. $\|Av\| \leq \|A\|\|v\|$

3. $\|AB\| \leq \|A\|\|B\|$

1. proof. $\|I\| = \max_{v\neq 0} \frac{\|Iv\|}{\|v\|} = \max_{v\neq 0} \frac{\|v\|}{\|v\|} = 1$

2. If $v = 0$ $\quad Av = 0$
$\|v\| = 0$ $\quad \|Av\| = 0$

Inequality is just $0 \leq 0$.

Otherwise $\frac{\|Av\|}{\|v\|} = \max_{u\neq 0} \frac{\|Au\|}{\|u\|} = \|A\|$

$\Rightarrow \|Av\| \leq \|A\|\|v\|$

3. $\|AB\| = \max_{v\neq 0} \frac{\|ABv\|}{\|v\|}$

by 2, $\leq \max_{v\neq 0} \frac{\|A\|\|Bv\|}{\|v\|}$

$= \|A\| \max_{v\neq 0} \frac{\|Bv\|}{\|v\|}$

$= \|A\|\|B\|$ $\boxtimes$

why are matrix norms actually norms on $\mathbb{R}^{n\times n}$?

1. $\|A\| \geq 0$ since $\frac{\|Av\|}{\|v\|} \geq 0 \quad \forall \vec{v} \neq 0.$

if $A=0$, then $\|A\|=0$, since $\|0\vec{v}\| = \|0\| = 0$, conversely, if $\|A\|=0$.

then we must have $\|Av\|=0$. $\forall \vec{v} \Rightarrow A\vec{v}=0 \ \forall v \Rightarrow A=0.$

2. $\|\lambda A\| = \max_{v\neq 0} \frac{\|\lambda A v\|}{\|v\|} = |\lambda| \max_{\lambda\neq 0} \frac{\|Av\|}{\|v\|} = |\lambda| \|A\|$

3. $\|A+B\| = \max_{v\neq 0} \frac{\|Av+Bv\|}{\|v\|} \leq \max_{v\neq 0} \frac{\|Av\|+\|Bv\|}{v} \leq \max_{u\neq 0} \frac{\|Au\|}{\|u\|} + \max_{u\neq 0} \frac{\|Bu\|}{\|u\|}$

Matrix p-norms.

Thm $\|A\|_\infty = \underbrace{\max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|}_{c.}$ max, absolute row sum.

Proof. Show $c \leq \|A_\infty\| \leq c$ (trick to use in Analysis).

For each $i=1,\dots,n$ and $v\in\mathbb{R}^n$

$|(Av)_i| =$