Math 449: Numerical Applied Mathematics
Lecture 22

10/30/2017 Wenzhen

Today's topic: Interpolation Error

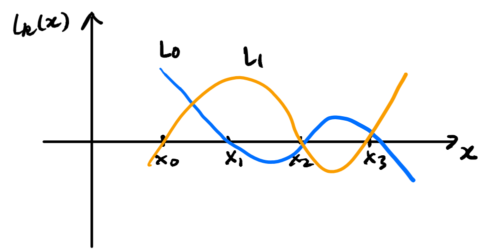
Today: Interpolation & Interpolation Error

Given distinct points $x_0, \dots, x_n \in \mathbb{R}$ and not necessarily distinct $y_0, \dots, y_n \in \mathbb{R}$

fit a polynomial $p_n \in P_n$ where $p_n(x_i) = y_i$ for $i = 0, 1, \dots, n$.

Define Lagrange basis polynomials

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$



degree-n polynomial with

$$L_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & i \neq k \end{cases}$$

Interpolating polynomial

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

$\{L_k\}_{k=0}^{n}$ is a basis for P_n, and p_n is the unique polynomial in P_n interpolating given pts $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Function Approximation

Given a function $f : [a, b] \to \mathbb{R}$ and distinct interpolation pts.

$x_0, x_1, \dots, x_n \in [a, b]$

Take $y_k = f(x_k)$ $\quad k = 0, 1, \dots, n$

interpolating polynomial $\quad p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$

agrees with f at $x_0, x_1, \dots, x_n$.

How well does p_n approx. f?

Can we find an upper bound on the error $|f(x) - p_n(x)|$?

Different way to approx. f by a degree-n polynomial : Taylor polynomial.

If f is n times differentiable, pick one pt x_0.

$$t_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Lagrange polynomial doesn't require any differentiality of f to define.

Taylor's theorem with remainder says that if f is C^{n+1}, then we have the error expression

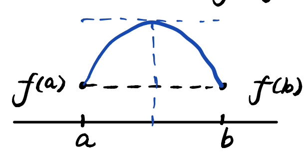
$$f(x) - t_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Our goal is to get a similar expression for $f(x) - p_n(x)$

Key Tool: Rolle's Theorem

If $f : [a,b] \to \mathbb{R}$ is C^1 (continuously differentiable once) and if $f(a) = f(b)$

then $f'(\xi) = 0$ at some $\xi \in (a,b)$



Proof. Since f is continuous on a closed interval

it attains a max and a min in $[a,b]$

if either a min / max is attained at an interior pt $\xi \in (a,b)$

then $f'(\xi) = 0$

Otherwise, both max and min attained at end pts, but values at endpts are equal (max = min) $\Rightarrow$ constant function.

$\Rightarrow f'(\xi) = 0$ for any $\xi \in (a,b)$ $\boxtimes$

Rmk This is a special case of the MVT and in fact is used to prove MVT and Taylor's theorem.

Theorem

If f is C^{n+1} on $[a,b]$, $x_0, \dots, x_n \in [a,b]$ distinct

p_n the interpolating polynomial.

Let $\pi_{n+1}(x) = (x-x_0)\cdots(x-x_n) \in P_{n+1}$ be nodal polynomial then $f(x)-p_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)$ for some ξ.

then for all $x \in [a,b]$, $\exists\ \xi \in (a,b)$:

$$f(x)-p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)$$

ξ may be different for different x.

Rmk. This looks like Taylor's Thm, where $\pi_{n+1}(x)$ plays the role of $(x-x_0)^{n+1}$

Start with special case $n=0$.

$n=0$ Take $x_0 \in [a,b]$, $p_0(x) = f(x_0)$

If $x = x_0$, then $f(x) - p_0(x) = 0$

$x \neq x_0$ fix x_1 and define

$$\varphi(t) = [f(t) - p_0(t)] - \frac{[f(x)-p_0(x)]}{\pi_1(x)}\pi_1(t)$$

If $t = x$, $\varphi(x) = [f(x) - p_0(x)] - \dfrac{f(x)-p_0(x)}{\pi_1(x)}\pi_1(x) = 0$

if $t = x_0$,

Rolle's Theorem

Proof. If $x = x_i$, for some $i = 0, \dots, n$, then both sides are 0.

otherwise, fix x and define $\varphi(t) = (f(t) - p_n(t)) - \frac{(f(x) - p_n(x)}{}$

Let $\varphi(t) = [f(t) - p_n(t)] - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t)$

As above, $\varphi(x_0) = \cdots = \varphi(x_n) = 0$ and $\varphi(x) = 0$

φ vanishes at $n+1$ distinct pts. Apply Rolle's Thm to construct pairs.