Math 449: Numerical Applied Mathematics
Lecture 32

11/17/2017 Wenzhen

Today's topic: Orthogonal Polynomial

L^2 - inner product on $C[a,b]$

$\langle f, g \rangle = \int_a^b f(x) g(x)\,dx$ or if $w: [a,b] \to \mathbb{R}$ is a weighted function

$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\,dx$

$\|f\| = \langle f, f \rangle^{1/2}$ is a norm. e.g. for L^2, $\|f\| = \left(\int_a^b f(x)^2 dx\right)^{1/2}$

Cauchy-Schwatz

Corallary $\|f+g\| \leq \|f\| \|g\|$

Proof. $\|f+g\|^2 = \langle f+g, f+g \rangle = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2$

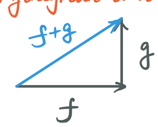
$\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$

$= (\|f\| + \|g\|)^2$

2 key tools:

1. Orthogonality $f, g \in V$, are orthogonal if $\langle f, g \rangle = 0$

Pythegrean thm for inner products



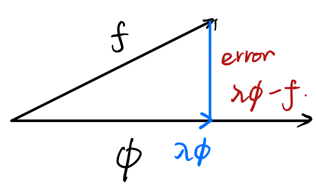
$\Rightarrow \|f+g\|^2 = \|f\|^2 + \cancel{2\langle f, g \rangle}^{\,0} + \|g\|^2 = \|f\|^2 + \|g\|^2$

minimizes the error $\|\lambda\phi - f\|$

best approx.

2. Projection

Given $\phi \in V$, $\phi \neq 0$, projection of f along ϕ is the vector $\lambda\phi$ that



minimizes the error $\underbrace{\|\lambda\phi - f\|}_{\text{best approx.}}$

$\|\lambda\phi - f\|^2 = \lambda^2 \|\phi\|^2 - 2\lambda \langle f, \phi \rangle + \|f\|^2$ is minimized when

$0 = \frac{d}{d\lambda} = 2\lambda\|\phi\|^2 - 2\langle f, \phi \rangle \Rightarrow \lambda = \frac{\langle f, \phi \rangle}{\|\phi\|^2} = \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle}$

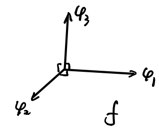
$\text{proj}_\phi f = \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \phi$

Error $\underset{}{\overset{\lambda\phi}{}}\lambda\phi - f$ is orthogonal to ϕ, since $\langle \lambda\phi - f, \phi \rangle = \lambda\langle \phi, \phi \rangle - \langle f, \phi \rangle$

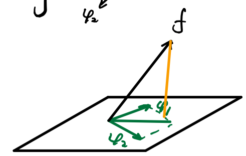
$= \langle f, \phi \rangle - \langle f, \phi \rangle$

$= 0$

Suppose $\exists$ orthogonal system, $\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ where $\langle \varphi_i, \varphi_j \rangle = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases}$



Proj of f onto the space spanned by $\{\varphi_0, \dots, \varphi_n\}$ is



$$\text{proj}_{\varphi_0} f + \cdots + \text{proj}_{\varphi_n} f = \frac{\langle f, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle} \varphi_0 + \cdots + \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} \varphi_n$$

idea: If we construct an orthogonal system of polynomials, then we can easily find the best approx. to f in $\|\cdot\|_2$ by projecting.

i.e. find polynomials s.t.

$$\int_a^b w(x) \varphi_i(x) \varphi_j(x)\, dx = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases}$$

work on $[-1, 1]$ can always do change of variables $[-1,1] \to [a,b]$

$w \equiv 1$, (ordinary L^2 inner product).

Legendre Polynomials

Start with $\varphi_0(x) = 1$

Constructed so that we add a degree n polynomial φ_n orthogonal to $\varphi_0, \dots, \varphi_{n-1}$

For degree 1, look at x.

$$\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x\, dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = 0 \Rightarrow \begin{cases} \varphi_1(x) = x \\ \varphi_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \end{cases}$$

奇函数

$$\langle \varphi_2, \varphi_0 \rangle = \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx = 0 \Rightarrow \varphi_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

Legendre polynomials are normalized so that $\varphi_n(1) = 1$

Recursive definition

$\varphi_0(x) = 1$, $\varphi_1(x) = x$, $(n+1)\varphi_{n+1}(x) = (2n+1)x\varphi_n(x) - n\varphi_{n-1}(x)$

$\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ is an orthogonal basis for P_n.

$\Rightarrow$ Best approx. of f by $p \in P_n$, minimizing $\|f - p\|_2$ is given by projecting f onto $\text{span}\{\varphi_0, \dots, \varphi_n\} = P_n$

Chebyshev Polynomial

$T_n(x) = \cos(n \cdot \cos^{-1}(x))$ are orthogonal on $[-1, 1]$ with $w(x) = \frac{1}{\sqrt{1 - x^2}}$

with respect to weighted ℓ^2-inner product.

$$\langle T_m, T_n \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos(m \cos^{-1}(x)) \cos(n \cos^{-1}(x))\, dx$$

Substitute $\theta = \cos^{-1}(x)$ $x = \cos\theta$

$$dx = -\sin\theta\, d\theta = -\sqrt{\sin^2\theta}\, d\theta = -\sqrt{1 - \cos^2\theta}\, d\theta = -\sqrt{1 - x^2}\, d\theta$$

$$\langle T_m, T_n \rangle = + \int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = + \frac{1}{2}\left[\int_0^{\pi} \cos(m+n)\theta\, d\theta + \int_0^{\pi} \cos(m-n)\theta\, d\theta \right]$$

$$= \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \end{cases}$$

if $m = n$ $\cos(2m)\theta = 0$

$m \neq n$ $\cos 0 = 1$