Math 449: Numerical Applied Mathematics
Lecture 11

09/25/2017 Wenzhen

Today's topic: LU Decomposition

Today: LU decomposition (continued)

Last time: Gaussian elimination is equivalent to decomposing $A = LU$.

L: unit

Advantage: Storing L allows us to solve any other $A\vec{x} = \vec{c}$ without having to redo the elimination process for A.

Reduce solving $A\vec{x} = \vec{b}$ to:

LU decompose A.

Solve triangular system by substitution $O(n^2)$ ops.

LU decomp algorithm.

For $i = 1, \dots, n$.

$$l_{ij} = \frac{1}{u_{jj}}\left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}\right] \quad (j = 1, \dots, i-1)$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \qquad (j = i, \dots, n)$$

Note: Requires $u_{jj} \neq 0$ $j = 1, \dots, n$ cost of LU decomposition.

l_{ij}: 1 div, $j-1$ multiplications, $j-1$ subtractions

i^{th} row of L: $i-1$ divisions $\frac{(i-1)(i-2)}{2}$ multi $\frac{(i-1)(i-2)}{2}$ sub

i^{th} row of L $i-1 + (i-1)(i-2) = (i-1)^2$ ops.

all n rows of L:

$$\sum_{i=1}^{n} (i-1)^2 = 0^2 + 1^2 + 2^2 + \cdots + (n-1)^2$$

$$= \frac{n(n-1)(n-2)}{6} \quad \text{ops.}$$

To compute u_{ij} : $i-1$ multiplications, $i-1$ subtraction

j^{th} column of U: $\frac{j(j-1)}{2}$ mult $\frac{j(j-1)}{2}$ sub

j^{th} column of U: $j(j-1)$ ops

All of U: $\sum_{j=1}^{n} j(j-1) = \sum_{j=1}^{n} (j^2 - j)$

$$= \sum_{j=1}^{n} j^2 - \sum_{i=1}^{n} j$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1-3)}{6}$$

All of U

$$\frac{2n(n+1)(n-1)}{6}$$

Total cost of LU decomposition:

$$\frac{n(n-1)}{6}\left[(2n-1)+(2n+2)\right]$$

$$\frac{n(n-1)(4n+1)}{6} \text{ ops.} \qquad \frac{2}{3}n^3 + O(n^2)$$

So, if A has an LU decomposition, then solving $A\vec{x} = \vec{b}$ takes $\frac{2}{3}n^3 + O(n^2)$ ops.

Cost of computing A^{-1}

Cheaper to solve $A\vec{x}=\vec{b}$, then to compute $\vec{x}=A^{-1}\vec{b}$

1st column of A^{-1} tells in the solution of

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{e}_1$$

i^{th} column A^{-1} is the solution of $A\vec{x} = \vec{e}_i$ $\quad \vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i^{th}$ entry.

Also i^{th} col of I

Computing A^{-1} requires solving $A\vec{x} = e_i$ $\quad i=1, \dots, n$.

Find LU decomposition of A

$$\frac{2}{3}n^3 + O(n^2)$$

Solve n different triangular systems by substitution.

This cost $O(n^3)$

$A^{-1}\vec{b}$ takes $O(n^2)$

When does LU decomp fail?

e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Thm

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{matrix}}^{\,A^{(k)}} & \begin{matrix} \cdots & a_{1n} \\ & \vdots \\ \cdots & a_{kn} \end{matrix} \\ \begin{matrix} \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & a_{nn} \end{matrix} \end{pmatrix}$$

k^{th} leading principle submatrix of A.

Theorem If $A \in \mathbb{R}^{n \times n}$, and if $A^{(k)} \in \mathbb{R}^{k \times k}$ is nonsingular for $k=1, \dots, n-1$.

then we can decompose $A = LU$.

Ex. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad A^{(1)} = (0)$ singular.

After k rows of LU decomposition algorithm.

$$\begin{pmatrix} 1 & & \\ l_{21} & \ddots & \\ l_{k1} & \cdots & 1 \\ & & \end{pmatrix} \begin{pmatrix} u_{11} & \cdots & u_{1k} & u_{1n} \\ & \ddots & \vdots & \\ & & u_{kk} & u_{kn} \\ & & & \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ & & \vdots & & \\ a_{k1} & \cdots & a_{kk} & & a_{kn} \\ & & & & \end{pmatrix}$$

$$A^{(k)} = L^{(k)} U^{(k)}$$

$A^{(k)}$ nonsingular $\Leftrightarrow$ $\det(A^{(k)}) \neq 0$

$$\det A^{(k)} = (\det(L^{(k)}) \det(U^{(k)})) = u_{11} u_{22} \cdots u_{kk}$$

So if $\det A^{(k)} \neq 0$, then $u_{jj} \neq 0$ for $j = 1, \dots, k$

If this is true for $k = 1, \dots, n-1$. then we need to solve $A\vec{x} = \vec{b}$

then to compute $\vec{x} = A^{-1}\vec{b}$

$\det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = -1 \neq 0$ so we should be able to solve $A\vec{x} = \vec{b}$

Observe that if we switch rows 1 and 2.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \longmapsto \begin{array}{c} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} R2 \\ R1 \end{array}$$ this does have a LU decomposition.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ this procedure is called pivoting.