Math 449: Numerical Methods
Lecture 03

Sep 6th, 2017

Today's topic: Fixed point iteration

$g : [a,b] \to [a,b]$

$\xi \in [a,b]$ is a fixed pt if $g(\xi) = \xi$

Fixed pt iteration:

pick $x_0 \in [a,b]$ For $k = 0, 1, 2, \dots$ $x_{k+1} = g(x_k)$

Def. g is a contraction if $|g(x) - g(y)| \le L|x-y|$ for all $x, y \in [a,b]$, where $0 < L < 1$

Contraction mapping theorem:

A contraction $g : [a,b] \to [a,b]$ has a unique fixed pt $\xi \in [a,b]$, and $x_{k+1} = g(x_k)$ converges to ξ for every $x_0 \in [a,b]$.

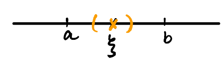
Lemma

If $|g'(x)| \le L$ $\forall$ $x \in (a,b)$, then $|g(x) - g(y)| \le L|x-y|$ $\forall$ $x, y \in [a,b]$

ξ is stable if $x_k \to \xi$ whenever x_k is sufficiently close to ξ.

Theorem Fixed Pt Stability Test

Suppose g has a fixed pt ξ and is continuously differentiable in some interval around ξ.



(a) If $|g'(\xi)| < 1$, then ξ is stable

(b) If $|g'(\xi)| > 1$, then ξ is unstable.

(c) If $|g'(\xi)| = 1$, insufficient info to determine stability.

Proof (a)

$\because$ g' is continuous, we can take a small interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$

s.t. $|g'(x)| \le L < 1$ on I_δ

Therefore, g is a contraction on I_δ, so $\forall$ $x_0 \in I_\delta$, $x_k \to \xi$ $\boxtimes$

Proof (b)

Similarly, if $|g'(\xi)| > 1$, then $\exists$ I_δ where $|g'(x)| \ge L > 1$

By mean value theorem MVT, $|x_{k+1} - \xi| = |g(x_k) - g(\xi)| = |g'(\eta_k)|\,|x_k - \xi| \ge L|x_k - \xi|$

so $x_0, x_1, x_2, \dots$ gets "pushed away" from ξ $\boxtimes$

Example Heron's Method for $\sqrt{y}$ $(y>0)$

$g(x) = \frac{1}{2}(x - \frac{y}{x})$ $\quad x \neq 0$

$g'(x) = \frac{1}{2}(1 - \frac{y}{x^2})$

If $\xi = \pm\sqrt{y}$, then $\xi^2 = y$, $\therefore g'(\xi) = \frac{1}{2}(1 - \frac{\xi^2}{\xi^2}) = 0 < 1$ $\quad \therefore \xi$ is stable.

Example Heron's method $(y=0)$

$g(x) = \frac{1}{2}x \Rightarrow g'(x) = \frac{1}{2} \Rightarrow g'(\xi) = \frac{1}{2}$

Convergence faster

Example

$g(x) = x - f(x) = x - x^2 + y$

$\xi = \pm\sqrt{y}$ are fixed pts

$g'(x) = 1 - 2x$

$g'(\sqrt{y}) = 1 - 2\sqrt{y}$

$g'(-\sqrt{y}) = 1 + 2\sqrt{y}$

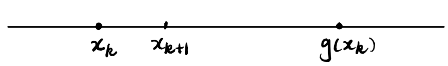
$-\sqrt{y}$ is always unstable for $y>0$

$+\sqrt{y}$ is stable when $0<y<1$ and unstable when $y>1$ e.g. $y=2$

not enough info when $y = 0, 1$.

Relaxation of fixed pt iteration methods

Want to slow down iteration to stabalize fixed pt (avoid overshooting)



Given "relaxation parameter λ, $x_{k+1} = (1-\lambda)x_k + \lambda g(x_k)$

Ex. $\lambda = 1$ $\quad x_{k+1} = g(x_k)$

$\lambda = \frac{1}{2}$ $\quad x_{k+1} = \frac{x_k + g(x_k)}{2}$ $\quad$ go halfway

$\lambda = 0$ $\quad x_{k+1} = x_k$

Usually $0<\lambda<1$ to slow down the fixed pt iteration

Can speed up by taking $\lambda>1$ called "over relaxation"

Def $g_\lambda(x) = (1-\lambda)x + \lambda g(x)$

prop Suppose $\lambda \neq 0$, then ξ is a fixed pt of g iff it's a fixed pt of g_λ

Proof $(\Rightarrow)$ If $g(\xi)=\xi$, then $g_\lambda(\xi) = (1-\lambda)\xi + \lambda g(\xi) = (1-\lambda)\xi + \lambda\xi = \xi$

$(\Leftarrow)$ If $g_\lambda(\xi)=\xi$, then $\xi = (1-\lambda)\xi + \lambda g(\xi)$

$\xi = \lambda g(\xi)$ ∎

Stability of relaxed fixed pt iteration

$$g_\lambda'(x) = (1-\lambda) + \lambda g'(x)$$

If ξ is a fixed pt, $g_\lambda'(\xi) = (1-\lambda) + \lambda g'(\xi)$

Idea: Stabalize ξ by taking λ approximately optimal. $g_\lambda'(\xi)=0$.

$$0 = (1-\lambda) + \lambda g'(\xi)$$

$$\lambda(1-g'(\xi)) = 1$$

$$\lambda = \frac{1}{1-g'(\xi)}$$

So assuming $g'(\xi)\neq 1$, we can always find λ so that ξ is a stable, fixed pt of g_λ.