Back substitution

idea: Gaussian Elimination

1st step: reduce to an equivalent upper triangular system by row operations on column 1, col 2, ...

2nd step: solve upper triangular system by backsubstitution

Triangular System:

def. A matrix $L \in \mathbb{R}^{n \times n}$ is lower-triangular ( ) if $L_{ij} = 0$, when $i < j$

$U \in \mathbb{R}^{n \times n}$ is upper-triangular ( ) if $u_{ij} = 0$ when $i > j$

L is unit lower triangular if $l_{ii} = 1$

unit upper $u_{ii} = 1$

Back-substitution:

Suppose we have an $n \times n$ system. $U \in \mathbb{R}^{n \times n}$ is upper triangular and $u_{ii} \neq 0 \ \forall i$

$$U\vec{x} = \vec{b}$$

$$\begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ & u_{22} & \cdots & u_{n2} \\ & & u_{n-1,n-1} & u_{n\,n-1} \\ & & & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$u_{n-1,n-1}\, x_{n-1} + u_{n-1,n} \cdot x_n = b_{n-1}$$

$$x_{n-1} = \frac{1}{u_{n-1,n-1}} (b_{n-1} - u_{n-1,n}\, x_n)$$

i^{th} row: $u_{ii} x_i + \sum_{j=i+1}^{n} u_{ij} x_j = b_i$

$$\Rightarrow u_{ii} x_i = b_i - \sum_{j=i+1}^{n} u_{ij} \cancel{x_j}$$

$$x_i = \frac{1}{u_{ii}} \left(b_i - \sum_{j=i+1}^{n} u_{ij} \right)$$

Similarly, $L\vec{x}=\vec{b}$ solved by forward-sub, if $l_{ii}\neq 0$

$$\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & & 0 \\ l_{n1} & \cdots & & l_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$l_{11}x_1 = b_1 \qquad x_1 = \frac{b_1}{l_{11}}$$

$$\sum_{j=1}^{i-1}(l_{ij}x_j + l_{ii}x_i) = b_i$$

$$x_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}x_j\right)$$

Operation count for backsubstitution

1^{st} step: $x_n = \frac{1}{u_{nn}} b_n$ 1 division

2^{nd} step: $x_{n-1} = \frac{1}{u_{n-1,n-1}} b_{n-1} - u_{n-1,n}\cdot x_n$ 1 mult 1 subtraction 1 div.

$\vdots$

i^{th} step: 1 div $(i-1)$ mult $(i-1)$ sub

mult count: $\sum_{i=1}^{n}(i-1) = 0+1+\cdots+(n-1) = \frac{n(n-1)}{2}$

sub count $\frac{n(n-1)}{2}$

division n.

total: $\frac{n(n-1)}{2}\cdot 2+n = n^2-n+n = n^2$