Math 493: Mathematical Statistics
Lecture 02
Sep 1st, 2017

events

$$(\underset{\underset{\text{outcomes}}{\uparrow}}{X},\ \overset{\overset{\text{events}}{\downarrow}}{\mathcal{F}},\ \underset{\underset{\text{prob. measure}}{\uparrow}}{P})$$

$E_1, E_2 \in \mathcal{F}$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - \underbrace{P(E_1 \cap E_2)}_{P(E_1 E_2)}$$

Exclusion - inclusion principle

Let $E_1, \dots, E_n \in \mathcal{F}$,

$$P(E_1 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cap \cdots \cap E_{i_r}) \qquad \text{pair-wise intersection}$$

check $n=3$:

$$P(E_1 \cup E_2 \cup E_3) = \overbrace{P(E_1) + P(E_2) + P(E_3)}^{r=1} \overbrace{- P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_2)}^{r=2} \overbrace{+ P(E_1 \cap E_2 \cap E_3)}^{r=3}$$

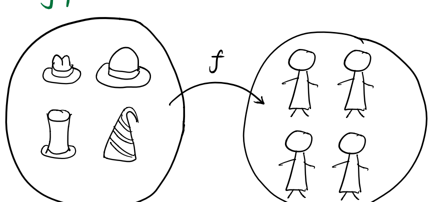
Proof

$$P(E_1 \cup (E_2 \cup E_3)) = P(E_1) + \underbrace{P(E_2 \cup E_3)}_{P(E_2) + P(E_3) - P(E_2 \cap E_3)} - P(E_1 \cap (E_2 \cup E_3))$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_2 \cap E_3) - P((E_1 \cap E_2) \cup (E_2 \cap E_3))$$

$$(P(E_1 \cap E_2) + P(E_2 \cap E_3) - P(E_1 \cap E_2 \cap E_3)$$

Example 5m (matching problem)



$\mathcal{H} = \{h_1, \dots, h_N\}$ $\qquad$ $P = \{p_1, \dots, p_N\}$

X = set of all bijective maps $f: \mathcal{H} \to P$

$\underbrace{\# X}_{\text{number of elements of } X} = N!$

$\mathcal{F}$ = all subset of X $\qquad$ $E \in \mathcal{F}$, $P(E) = \dfrac{\# E}{\# X}$

E_i = event that person p_i selects hat h_i $= \{ f \in X : f(h_i) = p_i \}$

Want: $P(E_1^c \cap E_2^c \cap \cdots \cap E_N^c)$

$$P(E_{i_1} \cap \cdots \cap E_{i_r}) = \frac{(N-r)!}{N!}$$

$$E_1^c \cap \cdots \cap E_N^c = (E_1 \cup \cdots \cup E_N)^c$$

$$P((E_1 \cup \cdots \cup E_N)^c) = 1 - P(E_1 \cup \cdots \cup E_N) \qquad \frac{(N-r)!}{N!}$$

$$P(E_1^c \cap \cdots \cap E_N^c) = 1 - \sum_{r=1}^{N} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cap \cdots \cap E_{i_r})$$

$$= \sum_{r=0}^{N} (-1)^r \frac{N!}{(N-r)!\,r!} \frac{(N-r)!}{N!} \qquad \binom{N}{r} = \frac{N!}{(N-r)!\,r!}$$

$$= \sum_{r=0}^{N} (-1)^r \frac{1}{r!}$$

$5! = \frac{1}{120}$
$4! = 24$
$3! = 6$

$$= \underbrace{1 - 1}_{0} + \underbrace{\frac{1}{2} - \frac{1}{6}}_{\frac{1}{3}} + \underbrace{\frac{1}{24} - \frac{1}{120}}$$

$$\lim_{r \to \infty} \sum_{r=0}^{N} (-1)^r \frac{1}{r!} = \frac{1}{e} = 0.368 \approx 40\%$$

Taylor Series of e^x