Math 493: Mathematical Statistics
Lecture 05

09/11/2017

Today's topic: Independence

E, F are said to be independent if $P(E \cap F) = P(E) \cap P(F)$

If $P(E), P(F) \neq 0$, then E, F are independent if and only if $P(E|F) = P(E)$ $\quad P(F|E) = P(F)$

Events $E_1, \ldots, E_n$ are independent if $\underbrace{P(E_{i_1} \cap \cdots \cap E_{i_k})}_{\text{for every subcollection } E_{i_1}, \ldots, E_{i_k}} = P(E_{i_1}) \cdots P(E_{i_k})$

Example 4f (Binomial Distribution)

Flip a coin n times independently.

H : event i^{th} flip comes up head

1. Find prob. that at least 1 flip comes up head

$$P(H_1 \cup H_2 \cup \cdots \cup H_n) = 1 - \underbrace{P(H_1^c \cap H_2^c \cap \cdots \cap H_n^c)}_{P(H_1^c) \cdot \cdots \cdot P(H_n^c)}$$

$P(H_i) = p$ $\qquad P(H_1 \cup \cdots \cup H_n) = 1 - (1-p)^n$

$P(H_i^c) = 1 - p$

$0 < p < 1$ $\qquad \lim_{n \to \infty} P(H_1 \cup \cdots \cup H_n) = 1$

Find prob. of exactly k heads.

$$P(H_1 \cap H_2^c \cap \cdots \cap H_n^c) = p^k (1-p)^{n-k}$$

○ ○ ○ $\cdots$ ○

1 2 3 $\cdots$ n

We have $\binom{n}{k}$ configurations with exactly k heads

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Remarks

$P(\cdot | E)$ is a probability function

① $P(S|E) = 1$

② $E_1, E_2, \ldots$ mutually $\qquad$ then $P(\cup E_i | E) = \sum P(E_i | E)$

Conditional Independence

Events A_1, A_2 are said to be independent given event E if $P(A_1 \cap A_2 | E) = P(A_1|E)\, P(A_2|E)$

Example 5A.

E = a person chosen at random is accident prone.

A_1 = a police-holder has an accident during the 1^{st} yr.

A_2 = a police-holder has an accident during the 2^{nd} yr.

Assumptions:

$P(E) = 0.3$

$P(A_i | E) = 0.4$

$P(A_i | E^c) = 0.2$

A_1 and A_2 are independent given E

$$P(E_i|F) = \frac{P(F|E_i)\,P(E_i)}{P(F)}$$

Question: $P(A_2 | A_1)$

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$$

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_2 \cap A_1 \cap E) + P(A_2 \cap A_1 \cap E^c) \\ &= P(A_2 | (A_1 \cap E))\, P(A_1 \cap E) + P(A_2 | (A_1 \cap E^c))\, P(A_1 \cap E^c) \end{aligned}$$

By Bayes

$$P(A_2|A_1) = \underbrace{P(A_2|A_1 \cap E)}_{\text{claim: } P(A_2|E)} P(E|A_1) + \underbrace{P(A_2|A_1 \cap E^c)}_{P(A_2|E^c)} \underbrace{P(E^c|A_1)}_{1-P(E|A_1)}$$

Claim: if A_1, A_2 are independent given E then $P(A_2 | A_1 \cap E) = P(A_2 | E)$

$$P(A_1 \cap A_2 | E) = P(A_1|E)\, P(A_2|E)$$