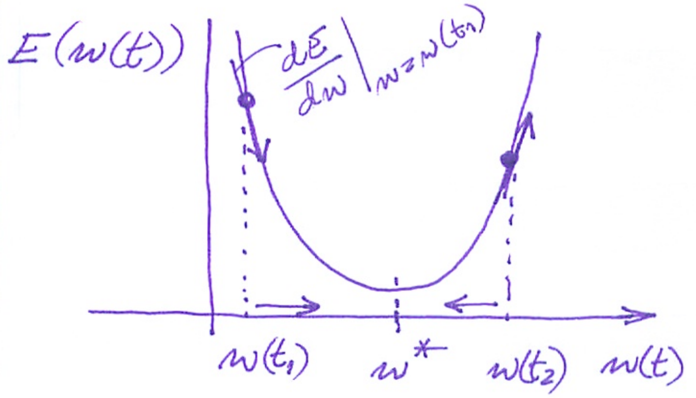
Construct a simple Lyapunov function (for 1-d case)



$x \rightarrow O \xrightarrow{w} O\ y$

$y = w(t)x$ ← state variable

Define $E(w(t)) =$

$= \frac{1}{2}(D-y)^2 = \frac{1}{2}(D-wx)$

Assume $\exists w^*$: minimum location (equilibrium point)

Observe: $E \geq 0$, $E = 0$ iff $w = w^*$

E is scalar (function of weight(s))

E is continuously differentiable wrt. w

Condition for $E'_t \leq 0$? (When is E a L-function?)

$$\frac{dE}{dt} = \frac{dE}{dw}\frac{dw}{dt} \approx \frac{dE}{dw} \cdot \underbrace{\frac{\Delta w}{\Delta t}}_{=1} \approx \underbrace{\frac{dE}{dw}}_{\text{given}} \cdot \underbrace{\Delta w}$$

← this is the learning rule, not yet stated

$$\frac{dE}{dw} = (D - wx)(-x) = -(D - \underbrace{wx}_{y})x$$

Now, propose the learning rule to be

$$\Delta w \propto -\frac{dE}{dw} = -\alpha \frac{dE}{dw}$$

learn rate, $\alpha > 0$ const.

as suggested by the figure: update w negatively proportional to the gradient

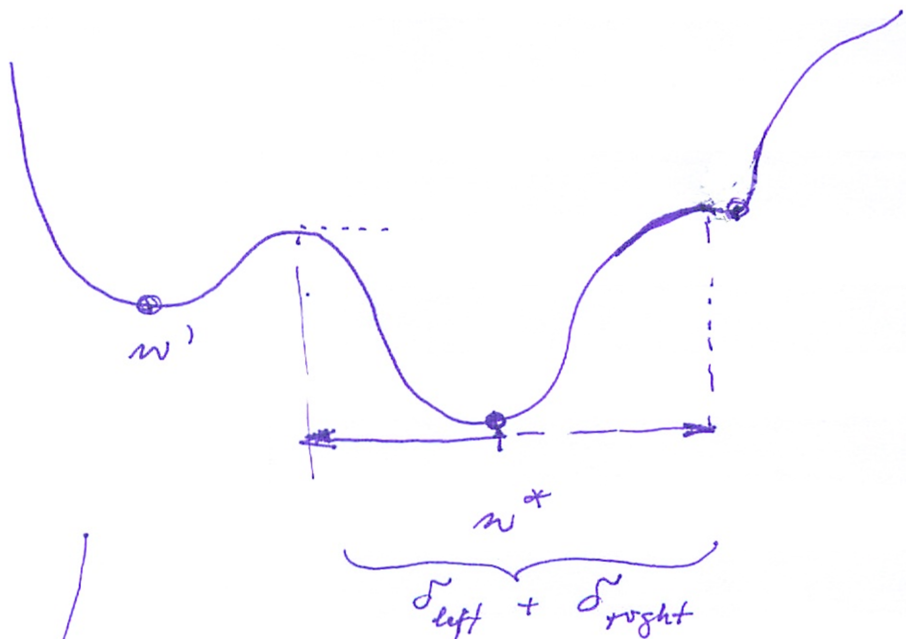
Then

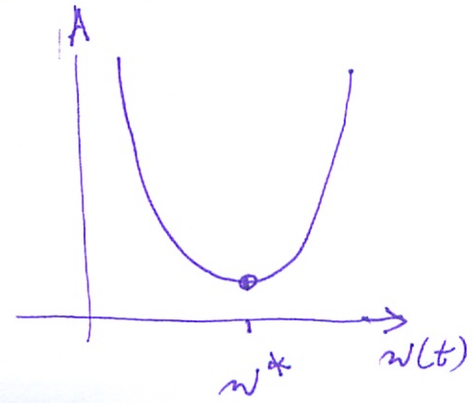
$$\frac{dE}{dt} = -(D-y)x \cdot \alpha(D-y)x = -\alpha(D-y)^2 x^2 \leq 0$$

Construct a simple Lyapunov function, cont'd

I.e., if we train with the above learning rule, namely, the weight update is done in the direction of the negative gradient of E, then E is a Lyapunov function for w^* in $|w - w^*| < \delta$ ($\delta > 0$, ~~small~~), which guarantees convergence of the learning (or stabilization) if w started within the δ neighborhood of w^*.

The neighborhood size δ depends on the locations of other possible equilibrium points





$\delta = \infty$