

# THE CYCLE DESCENT STATISTIC ON PERMUTATIONS

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**ABSTRACT.** In this paper we study the cycle descent statistic on permutations. Several involutions on permutations and derangements are constructed. Moreover, we present a bijection between permutations and perfect matchings.

**Keywords:** Permutations; Cycle descents; Perfect matchings; Eulerian polynomials;

## 1. INTRODUCTION

Let  $\mathfrak{S}_n$  be the symmetric group of all permutations of  $[n]$ , where  $[n] = \{1, 2, \dots, n\}$ . We write an element  $\pi$  in  $\mathfrak{S}_n$  as  $\pi = \pi(1)\pi(2)\cdots\pi(n)$ . An *excedance* in  $\pi$  is an index  $i$  such that  $\pi(i) > i$  and a *fixed point* in  $\pi$  is an index  $i$  such that  $\pi(i) = i$ . A fixed-point-free permutation is called a *derangement*. Denote by  $\mathcal{D}_n$  the set of derangements of  $[n]$ . As usual, let  $\text{exc}(\pi)$ ,  $\text{fix}(\pi)$  and  $\text{cyc}(\pi)$  denote the number of excedances, fixed points and cycles in  $\pi$  respectively. For example, the permutation  $\pi = 3142765$  has the cycle decomposition  $(1342)(57)(6)$ , so  $\text{cyc}(\pi) = 3$ ,  $\text{exc}(\pi) = 3$  and  $\text{fix}(\pi) = 1$ .

The Eulerian polynomials  $A_n(x)$  are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} \quad \text{for } n \geq 1,$$

and have been extensively investigated. Foata and Schützenberger [5] introduced a  $q$ -analog of the Eulerian polynomials defined by

$$A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)}.$$

Brenti [2, 3] further studied  $q$ -Eulerian polynomials and established the link with  $q$ -symmetric functions arising from plethysm. Brenti [3, Proposition 7.3] obtained the exponential generating function for  $A_n(x; q)$ :

$$1 + \sum_{n \geq 1} A_n(x; q) \frac{z^n}{n!} = \left( \frac{1 - x}{e^{z(x-1)} - x} \right)^q.$$

In particular, Brenti [3, Corollary 7.4] derived the following identity:

$$\sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} (-1)^{\text{cyc}(\pi)} = -(x - 1)^{n-1}. \quad (1)$$

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From then on, there is a large literature devoted to various generalizations and refinements of the joint distribution of excedances and cycles (see [1, 7, 13] for instance). For example, Ksavrelof and Zeng [7] provided bijective proofs for (1) and the following formula:

$$\sum_{\pi \in \mathcal{D}_n} x^{\text{exc}(\pi)} (-1)^{\text{cyc}(\pi)} = -x - x^2 - \dots - x^{n-1}.$$

In particular, their bijection leads to a refinement of the above identity:

$$\sum_{\pi \in \mathcal{D}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cyc}(\pi)} = -x^{n-i},$$

where  $\mathcal{D}_{n,i}$  is the set of derangements  $\pi$  of  $[n]$  such that  $\pi(n) = i$ .

For any  $\pi \in \mathfrak{S}_n$ , we define a *standard cycle decomposition* of  $\pi \in \mathfrak{S}_n$  by requiring that each cycle is written with its smallest element first, and the cycles are written in increasing order of their smallest element. A permutation is said to be cyclic if there is only one cycle in its cycle decomposition. Let  $(c_1, c_2, \dots, c_i)$  be a cycle in standard cycle decomposition of  $\pi$ . We say that  $c_j$  is a *cycle descent* if  $c_j > c_{j+1}$ , where  $1 < j < i$ . Denote by  $CDES(\pi)$  the set of cycle descents of  $\pi$  and let  $\text{cdes}(\pi) = |CDES(\pi)|$  be the number of cycle descents of  $\pi$ . For example, for  $\pi = (1342)(57)(6)$ , we have  $CDES(\pi) = \{4\}$  and  $\text{cdes}(\pi) = 1$ . For  $\pi \in \mathfrak{S}_n$ , it is clear that  $\text{exc}(\pi) + \text{cyc}(\pi) + \text{cdes}(\pi) = n$ . Thus

$$A_n(x; q) = q^n \sum_{\pi \in \mathfrak{S}_n} \left(\frac{x}{q}\right)^{\text{exc}(\pi)} \left(\frac{1}{q}\right)^{\text{cdes}(\pi)}.$$

Let  $\mathfrak{S}_{n,i}$  be the set of permutations  $\pi \in \mathfrak{S}_n$  with  $\pi(i) = 1$ . Let  $\pi^{-1}$  be the inverse of  $\pi$ . Hence  $\pi^{-1}(1) = i$  for  $\pi \in \mathfrak{S}_{n,i}$ . Recently, we observed the following formulas for  $n \geq 2$ :

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_{n,i}} (-1)^{\text{cdes}(\pi)} t^{\pi^{-1}(1)} &= \begin{cases} 2^{n-2}t & \text{if } i = 1, \\ 0 & \text{if } i = 2, \dots, n-1, \\ 2^{n-2}t^n & \text{if } i = n, \end{cases} \\ \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{cdes}(\pi)} t^{\pi^{-1}(1)} &= 2^{n-2}(t + t^n), \\ \sum_{\pi \in \mathcal{D}_n} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} t^{\pi^{-1}(1)} &= \sum_{i=2}^n (-1)^{n-i} x^{i-1} t^i, \\ \sum_{\pi \in \mathcal{D}_n} (-1)^{\text{cdes}(\pi)} &= \frac{1}{2}[1 - (-1)^{n-1}]. \end{aligned}$$

The above formulas can be easily proved by taking  $x = 1$  in Theorem 1. Motivated by the these formulas, we shall study the cycle descent statistic of permutations. In the next section, we present the main results of this paper and collect some notation and definitions that will be needed in the rest of the paper.

## 2. DEFINITIONS AND MAIN RESULTS

Consider the following enumerative polynomials

$$P_{n,i}(x, y, q, t) = \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\text{exc}(\pi)} y^{\text{cdes}(\pi)} q^{\text{fix}(\pi)} t^{\pi^{-1}(1)}.$$

It is remarkable that the polynomials  $P_{n,i}(x, -1, 1, t)$  and  $P_{n,i}(x, -1, 0, t)$  have simple closed formulas. We state them as the first main result of this paper.

**Theorem 1.** *For  $n \geq 2$ , we have*

$$P_{n,i}(x, -1, 1, t) = \begin{cases} t(1+x)^{n-2} & \text{if } i = 1, \\ 0 & \text{if } i = 2, \dots, n-1, \\ t^n x(1+x)^{n-2} & \text{if } i = n, \end{cases} \quad (2)$$

and

$$P_{n,i}(x, -1, 0, t) = (-1)^{n-i} x^{i-1} t^i. \quad (3)$$

The first purpose of this paper is to provide bijective proofs for the explicit formulas (2) and (3).

A signed permutation  $(\pi, \phi)$  of  $[n]$  is a permutation  $\pi \in \mathfrak{S}_n$  together with a map  $\phi : [n] \mapsto \{+1, -1\}$  and we call  $\phi(i)$  the sign of  $i$ . For simplicity, we indicate the sign of  $\pi(i)$  by writing  $\pi(i)^+$  or  $\pi(i)^-$ . Let  $(\pi, \phi)$  be a signed permutation. Let  $NEG(\pi, \phi)$  be the set of numbers  $\pi(i)$  with the sign  $-1$ , i.e.

$$NEG(\pi, \phi) = \{\pi(i) \mid \phi(\pi(i)) = -1\},$$

and let  $neg(\pi, \phi) = |NEG(\pi, \phi)|$ .

**Definition 2.** *A negative cycle-descent permutation  $(\pi, \phi)$  of  $[n]$  is a signed permutation  $(\pi, \phi)$  such that  $NEG(\pi, \phi) \subseteq CDES(\pi)$ .*

We define

$$b_n(y, q) = \sum_{i=1}^n P_{n,i}(1, y, q, 1) = \sum_{\pi \in \mathfrak{S}_n} y^{\text{cdes}(\pi)} q^{\text{fix}(\pi)}.$$

It is easy to verify that  $b_n(2, 1)$  is the number of negative cycle-descent permutations of  $[n]$  since

$$b_n(2, 1) = \sum_{\pi \in \mathfrak{S}_n} 2^{\text{cdes}(\pi)}$$

and  $b_n(2, 0)$  is the number of negative cycle-descent derangements of  $[n]$  since

$$b_n(2, 0) = \sum_{\pi \in \mathfrak{S}_n} 2^{\text{cdes}(\pi)} 0^{\text{fix}(\pi)} = \sum_{\pi \in \mathcal{D}_n} 2^{\text{cdes}(\pi)}.$$

Now we present the second main result of this paper.

**Theorem 3.** *For  $n \geq 1$ , we have*

$$b_{n+1}(y, 1) = b_n(y, 1) + \sum_{i=1}^n b_i(y, 1) \binom{n}{i-1} (y-1)^{n-i} \quad (4)$$

with the initial condition  $b_1(y, 1) = 1$  and

$$b_{n+1}(y, 0) = \sum_{i=0}^{n-1} \binom{n}{i} [b_{i+1}(y, 0) + b_i(y, 0)] (y-1)^{n-i-1} \quad (5)$$

with initial conditions  $b_0(y, 0) = 1, b_1(y, 0) = 0$ .

By taking  $y = 2$  in the identity (4), we obtain Klazar's recurrence for  $w_{12}(n)$  (see [6, Eq. (39)] for details), which can be written as follows:

$$b_{n+1}(2, 1) = b_n(2, 1) + \sum_{i=1}^n b_{n+1-i}(2, 1) \binom{n}{i}. \quad (6)$$

It is interesting that the number  $b_n(2, 1)$  is also the common cardinality of the following sets:

- (i) The set of drawings of rooted plane trees with  $n + 1$  vertices (see [6]);
- (ii) The set of Klazar trees with  $n + 1$  vertices (see [4]);
- (iii) The set of perfect matchings on the set  $[2n]$  in which no even number is matched to a larger odd number (see [4]).
- (iv) The set of ordered partitions of  $[n]$  all of whose left-to-right minima occur at odd locations (see [10]).

In this paper, we shall focus on perfect matchings. Let  $\mathbb{P}_A = A \times \{0, 1\}$ , where  $A = \{i_1, \dots, i_k\}$  is a finite set of positive integers with  $i_1 < i_2 < \dots < i_k$ . When  $A = [n]$ , we write  $\mathbb{P}_A$  as  $\mathbb{P}_n$ . A perfect matching is a partition of  $\mathbb{P}_A$  into 2-element subsets or matches. For any match  $\{(i, x), (j, y)\}$  in a perfect matching, we say that  $(i, x)$  is the partner of  $(j, y)$ . For convenience, we represent a perfect matching as a dot diagram with vertices arranged in two rows.

**Example 4.** We give a dot diagram of a perfect matching  $M$  of  $\mathbb{P}_8$  as follows:

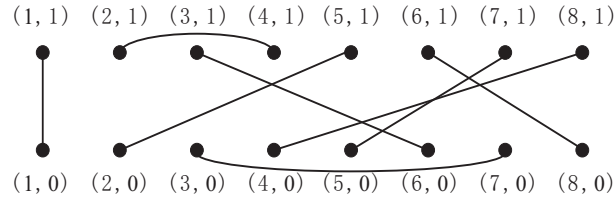


Fig.1. A perfect matching  $M$  of  $\mathbb{P}_8$

Thus, for any perfect matching  $M$  of  $\mathbb{P}_A$ , we say that  $\mathbb{P}_A$  is the vertex set of  $M$  and every match is an edge of  $M$ . We use  $V(M)$  and  $E(M)$  to denote vertices set and edges set in  $M$  respectively. Moreover, an edge is called an *arc* if it joins two dots in the same row; otherwise, this edge is called a *line*. For any line  $\{(i, 0), (j, 1)\}$ , it is said to be a *upline* if  $i < j$ , a *downline* if  $i > j$  and a *vertical line* if  $i = j$ . For any perfect matching  $M$ , let  $\text{arc}(M)$ ,  $\text{down}(M)$  and  $\text{ver}(M)$  be the numbers of arc, down lines and vertical lines in  $M$  respectively.

**Example 5.** In the perfect matching of Example 4, the edge  $\{(1, 1), (1, 0)\}$  is a vertical line, the edges  $\{(3, 1), (6, 0)\}$  and  $\{(6, 1), (8, 0)\}$  are two downlines, the edges  $\{(5, 1), (2, 0)\}$ ,  $\{(7, 1), (5, 0)\}$  and  $\{(8, 1), (4, 0)\}$  are three uplines, the edges  $\{(2, 1), (4, 1)\}$  and  $\{(3, 0), (7, 0)\}$  are two arcs; finally,  $\text{arc}(M) = 2$ ,  $\text{down}(M) = 2$ ,  $\text{ver}(M) = 1$ .

**Definition 6.** A perfect matching  $M$  of  $\mathbb{P}_n$  is a *Callan perfect matching* if  $M$  with no uplines.

**Example 7.** We give a dot diagram of a Callan perfect matching  $M$  of  $\mathbb{P}_8$  as follows:

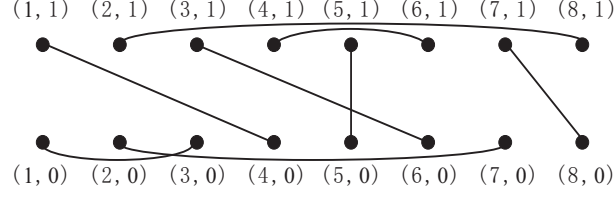


Fig.2. A Callan perfect matching  $M$  of  $\mathbb{P}_8$

Let  $m_n$  be the number of Callan perfect matchings of  $\mathbb{P}_n$ . Callan [4] proved that  $m_n$  satisfies the recurrence (6). So the number of negative cycle descent permutations of  $[n]$  equals to the number of Callan perfect matching of  $\mathbb{P}_n$ .

Let  $M$  be a perfect matching of  $\mathbb{P}_n$ . We say that  $M'$  is a *sub-perfect matching* of  $M$  if  $M'$  is a perfect matching such that  $V(M') \subseteq V(M)$  and  $E(M') \subseteq E(M)$ . For any  $V \subseteq [n]$ , if there is a sub-perfect matching  $M'$  of  $M$  with  $V(M') = V \times \{0, 1\}$ , then  $M'$  is said to be the sub-perfect matching induced by  $V$  and is denoted by  $M[V]$ .

Denote by  $\mathcal{G}(M)$  a graph which is obtained from  $M$  by identifying two vertices  $(i, 0)$  and  $(i, 1)$  as a new vertex  $i$  for any  $i \in [n]$ . It is easy to see that the graph  $\mathcal{G}(M)$  is the union of some disjoint cycles. For a cycle  $C$  in  $\mathcal{G}(M)$ , suppose  $C$  has the vertices set  $V$ . Note that there is a sub-perfect matching of  $M$  induced by  $V$ . We say that  $M[V]$  is a connected component of  $M$ . Let  $\text{com}(M)$  be the number of connected components in a perfect matching  $M$ . If a perfect matching  $M$  has exactly one connected component, i.e.,  $\text{com}(M) = 1$ , then we say that  $M$  is a connected perfect matching.

**Example 8.** For the perfect matching  $M$  of Example 7, we draw the graph  $\mathcal{G}(M)$  as follows:

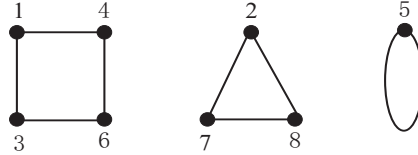


Fig.3. A graph  $\mathcal{G}(M)$ .

So we have  $\text{com}(M) = 3$ .

We can now conclude the third main result of this paper from the discussion above.

**Theorem 9.** There is a bijection  $\Gamma_n$  between the set of negative cycle descent permutations of  $[n]$  and the set of Callan perfect matchings of  $\mathbb{P}_n$ . Moreover, for any negative cycle-descent permutation  $(\pi, \phi)$  of  $[n]$ , we have

$$\text{com}(\Gamma_n(\pi, \phi)) = \text{cyc}(\pi), \quad \text{ver}(\Gamma_n(\pi, \phi)) = \text{fix}(\pi),$$

and

$$\text{down}(\Gamma_n(\pi, \phi)) = \begin{cases} \text{neg}(\pi, \phi) & \text{if } (1, 1) \text{ and its partner are in the same row,} \\ \text{neg}(\pi, \phi) + 1 & \text{otherwise.} \end{cases}$$

Let  $\Gamma|_{\mathcal{D}_n}$  denote the restriction of  $\Gamma_n$  on the set of negative cycle-descent derangements of  $[n]$ . So the following corollary is immediate.

**Corollary 10.**  $\Gamma|_{\mathcal{D}_n}$  is a bijection between the set of negative cycle-descent derangements of  $[n]$  and the set of Callan perfect matchings of  $\mathbb{P}_n$  which have no vertical lines.

The rest of this paper is organized as follows. In Section 3, we give the proof of the explicit formula (2) in Theorem 1. In Section 4, we give the proof of the explicit formula (3) in Theorem 1. In Section 5, we give a bijective proof of the recurrence relation (4). In Section 6, we give a bijective proof of the recurrence relation (5). In Section 7, we give a bijective proof of Theorem 9.

### 3. PROOF OF THE EXPLICIT FORMULA (2) IN THEOREM 1

Suppose that  $\pi = \pi(i_1) \dots \pi(i_k)$  is a permutation on the set  $\{i_1, \dots, i_k\}$  of positive integers with  $i_1 < i_2 < \dots < i_k$ . Throughout this paper, we always let

$$\text{red}(\pi) := \text{red}(\pi(i_1)) \dots \text{red}(\pi(i_k)) \in \mathfrak{S}_k,$$

where  $\text{red}$  is a map defined by  $\text{red}(i_j) = j$  for any  $j = 1, 2, \dots, k$ .

Let  $P_n(x, y, 1, 1) = \sum_{i=1}^n P_{n,i}(x, y, 1, 1)$ . We give a recurrence for  $P_{n,i}(x, y, 1, 1)$  in the following lemma.

**Lemma 11.** For any  $n \geq 2$ , we have

$$P_{n+1,i}(x, y, 1, 1) = \begin{cases} P_n(x, y, 1, 1) & \text{if } i = 1, \\ xP_{n-1}(x, y, 1, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1) + y \sum_{j=i}^n P_{n,j}(x, y, 1, 1) & \text{if } i = 2, \dots, n+1, \end{cases}$$

with initial conditions

$$P_{1,1}(x, y, 1, 1) = 1, P_{2,1}(x, y, 1, 1) = 1, P_{2,2}(x, y, 1, 1) = x.$$

*Proof.* For any  $\pi = \pi(1)\pi(2) \dots \pi(n+1) \in \mathfrak{S}_{n+1,1}$ , we have  $\pi(1) = 1$ . Let  $\tilde{\pi} = \pi(2) \dots \pi(n+1)$ . Then  $\tilde{\pi}$  is a permutation on the set  $\{2, 3, \dots, n\}$  and  $\text{red}(\tilde{\pi}) \in \mathfrak{S}_n$ . Obviously,

$$\text{exc}(\pi) = \text{exc}(\text{red}(\tilde{\pi})) \text{ and } \text{cdes}(\pi) = \text{cdes}(\text{red}(\tilde{\pi})).$$

So we have  $P_{n+1,1}(x, y, 1, 1) = P_n(x, y, 1, 1)$ .

For any  $i \geq 2$ , let  $\pi = \pi(1)\pi(2) \dots \pi(n+1) \in \mathfrak{S}_{n+1,i}$ . Let  $\sigma = (1, c_1, c_2, \dots, c_m)$  be the cycle in the standard cycle decomposition of  $\pi$  which contains the entry 1. So  $\pi$  can be split into the cycle  $\sigma$  and a permutation  $\tau$  on the set  $\{1, 2, \dots, n+1\} \setminus \{1, c_1, \dots, c_m\}$ , i.e.,  $\pi = \sigma \cdot \tau$ . Clearly,  $m \geq 1$ ,  $i \geq 2$  and  $c_m = i$  since  $\pi \in \mathfrak{S}_{n+1,i}$ . We distinguish two cases:

**Case 1.**  $m = 1$ .

Deleting the cycle  $(1, c_1) = (1, i)$  from the standard cycle decomposition of  $\pi$ , we obtain the permutation

$$\tau = \pi(2) \dots \pi(i-1)\pi(i+1) \dots \pi(n+1)$$

which is defined on the set  $\{1, 2, \dots, n+1\} \setminus \{1, i\}$ . Note that  $\text{red}(\tau) \in \mathfrak{S}_{n-1}$  and

$$\text{exc}(\pi) = \text{exc}(\text{red}(\tau)) + 1, \text{cdes}(\pi) = \text{cdes}(\text{red}(\tau)).$$

This provides the term

$$xP_{n-1}(x, y, 1, 1).$$

**Case 2.**  $m \geq 2$ .

Suppose that  $c_{m-1} = j$  for some  $2 \leq j \leq n+1$ . Deleting the number  $c_m = i$  from the standard cycle decomposition of  $\pi$ , we obtain a permutation

$$\tilde{\pi} = (1, c_1, \dots, c_{m-1}) \cdot \tau$$

which is defined on the set  $\{1, \dots, i-1, i+1, \dots, n+1\}$ . Hence  $\text{red}(\tilde{\pi}) \in \mathfrak{S}_n$ . Moreover, if  $c_{m-1} = j \leq i-1$ , then

$$\text{red}(\tilde{\pi}) \in \mathfrak{S}_{n,j}, \text{exc}(\pi) = \text{exc}(\text{red}(\tilde{\pi})) + 1, \text{cdes}(\pi) = \text{cdes}(\text{red}(\tilde{\pi})).$$

This provides the term

$$x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1).$$

If  $c_{m-1} = j \geq i+1$ , then

$$\text{red}(\tilde{\pi}) \in \mathfrak{S}_{n,j-1}, \text{exc}(\pi) = \text{exc}(\text{red}(\tilde{\pi})), \text{cdes}(\pi) = \text{cdes}(\text{red}(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^n P_{n,j}(x, y, 1, 1).$$

In conclusion, for any  $i \geq 2$  we have

$$P_{n+1,i}(x, y, 1, 1) = x P_{n-1}(x, y, 1, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1) + y \sum_{j=i}^n P_{n,j}(x, y, 1, 1).$$

□

### A proof of the identity (2):

Note that

$$\sum_{\pi \in \mathfrak{S}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} t^{\pi^{-1}(1)} = t^i \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} = t^i P_{n,i}(x, -1, 1, 1).$$

In order to prove the identity (2) in Theorem 1, it is sufficient to show that

$$P_{n,i}(x, -1, 1, 1) = \begin{cases} (1+x)^{n-2} & \text{if } i = 1, \\ 0 & \text{if } i = 2, \dots, n-1, \\ x(1+x)^{n-2}, & \text{if } i = n. \end{cases} \quad (7)$$

### (i) An induction proof of the explicit formula (7).

*Proof.* It is easy to verify that

$$P_{2,1}(x, -1, 1, 1) = 1, P_{2,2}(x, -1, 1, 1) = x.$$

Assume that the explicit formula (7) holds for any  $2 \leq k \leq n$ . By Lemma 11, we have

$$\begin{aligned} P_{n+1,1}(x, -1, 1, 1) &= P_n(x, -1, 1, 1) \\ &= P_{n,1}(x, -1, 1, 1) + P_{n,n}(x, -1, 1, 1) \\ &= (1+x)^{n-2} + x(1+x)^{n-2} = (1+x)^{n-1}, \end{aligned}$$

$$\begin{aligned}
P_{n+1,n+1}(x, -1, 1, 1) &= xP_{n-1}(x, -1, 1, 1) + x \sum_{j=2}^n P_{n,j}(x, -1, 1, 1) - \sum_{j=n+1}^n P_{n,j}(x, -1, 1, 1) \\
&= xP_{n-1,1}(x, -1, 1, 1) + xP_{n-1,n-1}(x, -1, 1, 1) + xP_{n,n}(x, -1, 1, 1) \\
&= x(1+x)^{n-2} + x^2(1+x)^{n-2} = x(1+x)^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
P_{n+1,i}(x, -1, 1, 1) &= xP_{n-1}(x, -1, 1, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, -1, 1, 1) - \sum_{j=i}^n P_{n,j}(x, -1, 1, 1) \\
&= xP_{n-1,1}(x, -1, 1, 1) + xP_{n-1,n-1}(x, -1, 1, 1) - P_{n,n}(x, -1, 1, 1) \\
&= x(1+x)^{n-2} - P_{n,n}(x, -1, 1, 1) = 0
\end{aligned}$$

for any  $2 \leq i \leq n$ .

## (ii) A bijection proof of the explicit formula (7).

Now we give a bijective proof of (7) by establishing an involution  $\psi_{n,i}$  on  $\mathfrak{S}_{n,i}$ .

For any  $\pi \in \mathfrak{S}_n$ , suppose that  $\pi = C_1 \dots C_k$  is the standard cycle decomposition of  $\pi$ . Let  $\hat{\pi} = a_1 a_2 \dots a_n$  be the permutation obtained from  $\pi$  by erasing the parentheses in its standard cycle decomposition. If  $\hat{\pi} \neq 12 \dots n$ , then  $\hat{\pi}$  has some *top-descent*, i.e., the number  $a_i$  satisfying  $a_i > a_{i+1}$ , and let  $q_\pi$  be the *last top-descent* which appears in the sequence  $\hat{\pi}$ . For example, let us consider the permutation  $\pi = 1472365$  in  $\mathfrak{S}_7$ . Its standard cycle decomposition is  $(1)(24)(375)(6)$  and  $\hat{\pi} = 1243756$ . Then  $\hat{\pi}$  has exactly two top-descents 4 and 7, and  $q_\pi = 7$ .

We define a map  $\Phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$  as follows:

For any  $\pi \in \mathfrak{S}_n$ , if  $q_\pi$  is the last element of a cycle  $C_i$  for some  $i$ , then let  $\Phi(\pi)$  be the permutation obtained from  $\pi$  by erasing the right and left parentheses “)” “(” after the number  $q_\pi$  in the standard cycle decomposition of  $\pi$ ; otherwise, let  $\Phi(\pi)$  be the permutation obtained from  $\pi$  by inserting a right parentheses “)” and a left parentheses “(” after the number  $q_\pi$  in the standard cycle decomposition of  $\pi$ . For example, if  $\pi = (1)(24)(375)(6)$ , then  $\hat{\pi} = 1243756$  and  $q_\pi = 7$ , and so  $\Phi(\pi) = (1)(24)(37)(5)(6)$ . If  $\sigma = (1)(24)(37)(5)(6)$ , then  $\Phi(\sigma) = (1)(24)(375)(6)$ . Clearly, we have

$$\hat{\pi} = \widehat{\Phi(\pi)}, \quad q_\pi = q_{\Phi(\pi)}, \quad \text{exc}(\pi) = \text{exc}(\Phi(\pi)), \quad \text{cdes}(\pi) - \text{cdes}(\Phi(\pi)) = \pm 1.$$

Denote by  $\Omega_{n,1}$  the set of the permutations  $\pi \in \mathfrak{S}_{n,1}$  such that  $\hat{\pi} = 123, \dots, n$ . It is easy to verify that the weight of  $\Omega_{n,1}$  is  $(1+x)^{n-2}$ . For any permutation  $\pi \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}$ , we have  $\Phi(\pi) \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}$ . So for any  $\pi \in \mathfrak{S}_{n,1}$ , let

$$\psi_{n,1}(\pi) = \begin{cases} \Phi(\pi) & \text{if } \pi \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}, \\ \pi & \text{if } \pi \in \Omega_{n,1}. \end{cases}$$



Hence,

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_{n,1}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} &= \sum_{\pi \in \Omega_{n,1}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} \\ &= \sum_{\pi \in \Omega_{n,1}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} = \sum_{\pi \in \Omega_{n,1}} x^{\text{exc}(\pi)} = (1+x)^{n-2}. \end{aligned}$$

For example, we list all  $\pi \in \mathfrak{S}_{4,1}$  and  $\psi_{4,1}(\pi)$  in Table 1.

$\pi \in \mathfrak{S}_{4,1}$	$x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)}$	$\hat{\pi}$	$q_\pi$	$\psi_{4,1}(\pi)$
(1)(2)(3)(4)	1	1234		(1)(2)(3)(4)
(1)(23)(4)	$x$	1234		(1)(23)(4)
(1)(2)(34)	$x$	1234		(1)(2)(34)
(1)(234)	$x^2$	1234		(1)(234)
(1)(24)(3)	$x$	1243	4	(1)(243)
(1)(243)	$-x$	1243	4	(1)(24)(3)

Table.1. Involution  $\psi_{4,1}$

For  $2 \leq i \leq n$ , denote by  $\mathcal{A}_{n,i}$  the set of permutations  $\pi \in \mathfrak{S}_{n,i}$  such that the numbers  $q_\pi$  and 1 isn't in the same cycle in the standard cycle decomposition of  $\pi$ . Let  $\mathcal{A}_{n,i}^* = \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}$  for short. For any  $\pi \in \mathcal{A}_{n,i}^*$ , let  $\pi = C_1 \dots C_k$  be the standard cycle decomposition of  $\pi$ , suppose the length of the cycle  $C_1$  is  $l+1$  and

$$\hat{\pi} = 1, \dots, i, a_1, \dots, a_{n-l-1}.$$

Then we have

$$a_1 < a_2 < \dots < a_{n-l-1}$$

since  $q_\pi$  is an element in the cycle  $C_1$ .

Now suppose that  $C_1 = (1, c_{11}, \dots, c_{1l})$ . Let  $Q$  be the set of indices  $j \in \{1, 2, \dots, l\}$  such that  $c_{1j}$  is not the largest number in the set  $\{1, 2, \dots, n\} \setminus \{c_{1,j+1}, \dots, c_{1l}\}$ , i.e.,

$$Q = \{j \mid 1 \leq j \leq l, c_{1j} < \max\{1, 2, \dots, n\} \setminus \{c_{1,j+1}, \dots, c_{1l}\}\}.$$

Let  $\Omega_{n,i}$  be the set of permutations  $\pi \in \mathcal{A}_{n,i}^*$  such that  $Q = \emptyset$ . For any  $i = 2, \dots, n-1$ , we have  $l \in Q$  since  $i < n$ , and so  $\Omega_{n,i} = \emptyset$ . Moreover,  $\pi \in \Omega_{n,n}$  if and only if

$$\hat{\pi} = 1, k, \dots, n, 2, 3, \dots, k-1$$

for some  $k > 1$ .

We define a map  $\Psi$  from  $\mathcal{A}_{n,i}^* \setminus \Omega_{n,i}$  to itself as follows:

For any  $\pi \in \mathcal{A}_{n,i}^* \setminus \Omega_{n,i}$ , we have  $Q \neq \emptyset$ . Let  $m = m_\pi = \min Q$ . Then we have

$$c_{11} < \dots < c_{1,m-1} \text{ and } c_{1,m-1} > c_{1m}.$$

Furthermore, we distinguish the following two cases:

**Case 1.**  $2 \leq m \leq l$ .

Let

$$\Psi(\pi) = (1, c_{1m}, \dots, c_{1l}) \cdot C_2 \dots C_k \cdot (c_{11}, \dots, c_{1,m-1}).$$

Then  $\Psi(\pi)$  has at least two cycles,  $m_{\Phi(\pi)} = 1$  since  $c_{1m} < c_{1,m-1}$ , and so  $\Psi(\pi) \in \mathcal{A}_{n,i}^* \setminus \Omega_{n,i}$ . Moreover, we have  $\text{exc}(\pi) = \text{exc}(\Psi(\pi))$  and  $\text{cdes}(\pi) = \text{cdes}(\Psi(\pi)) + 1$

**Case 2.**  $m = 1$  and there are at least two cycles in the standard cycle decomposition of  $\pi$ .

Suppose that  $C_1 = (1, c_{11}, \dots, c_{1l})$  and  $C_k = (c_{k1}, \dots, c_{ks})$  are the first cycle and the last cycle in the standard cycle decomposition of  $\pi$  respectively, where  $s$  is the length of the cycle  $C_k$ . Let

$$\Psi(\pi) = (1, c_{k1}, \dots, c_{ks}, c_{11}, \dots, c_{1l}) \cdot C_2 \dots C_{k-1}.$$

Then

$$m_{\Psi(\pi)} = s + 1 \geq 2,$$

and so  $\Psi(\pi) \in \mathcal{A}_{n,i}^* \setminus \Omega_{n,i}$ . Moreover, we have  $\text{exc}(\pi) = \text{exc}(\Psi(\pi))$  and  $\text{cdes}(\pi) = \text{cdes}(\Psi(\pi)) - 1$

When  $2 \leq i \leq n - 1$ , for any  $\pi \in \mathfrak{S}_{n,i}$ , let

$$\psi_{n,i}(\pi) = \begin{cases} \Phi(\pi) & \text{if } \pi \in \mathcal{A}_{n,i} \\ \Psi(\pi) & \text{if } \pi \in \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i} \end{cases}.$$

For example, we list all  $\pi \in \mathfrak{S}_{4,2}$  and  $\psi_{4,2}(\pi)$  in Table 2, and  $\pi \in \mathfrak{S}_{4,3}$  and  $\psi_{4,3}(\pi)$  in Table 3.

$\pi \in \mathfrak{S}_{4,2}$	$x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)}$	$\hat{\pi}$	$q_\pi$	$m_\pi$	$\psi_{4,2}(\pi)$
(12)(3)(4)	$x$	1234		1	(142)(3)
(142)(3)	$-x$	1423	4	2	(12)(3)(4)
(12)(34)	$x^2$	1234		1	(1342)
(1342)	$-x^2$	1342	4	3	(12)(34)
(1432)	$x$	1432	3	2	(132)(4)
(132)(4)	$-x$	1324	3	1	(1432)

Table.2. Involution  $\psi_{4,2}$

$\pi \in \mathfrak{S}_{4,3}$	$x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)}$	$\hat{\pi}$	$q_\pi$	$m_\pi$	$\psi_{4,3}(\pi)$
(13)(2)(4)	$x$	1324	3	1	(143)(2)
(143)(2)	$-x$	1432	3	2	(13)(2)(4)
(13)(24)	$x^2$	1324	3	1	(1243)
(1243)	$-x^2$	1243	4	3	(13)(24)
(1423)	$-x^2$	1423	4	2	(123)(4)
(123)(4)	$x^2$	1234		1	(1423)

Table.3. Involution  $\psi_{4,3}$

Hence,

$$\sum_{\pi \in \mathfrak{S}_{n,i}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} = \sum_{\pi \in \mathcal{A}_{n,i}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} = 0.$$

When  $i = n$ , it is not hard to see that the weight of  $\Omega_{n,n}$  is  $x(1+x)^{n-2}$ . For any  $\pi \in \mathfrak{S}_{n,n}$ , let

$$\psi_{n,n}(\pi) = \begin{cases} \Phi(\pi) & \text{if } \pi \in \mathcal{A}_{n,n} \\ \Psi(\pi) & \text{if } \pi \in \mathfrak{S}_{n,n} \setminus (\mathcal{A}_{n,n} \cup \Omega_{n,n}) \\ \pi & \text{if } \pi \in \Omega_{n,n} \end{cases}.$$

For example, we list all  $\pi \in \mathfrak{S}_{4,4}$  and  $\psi_{4,4}(\pi)$  in Table 4.

$\pi$	$x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)}$	$\hat{\pi}$	$q_\pi$	$m_\pi$	$\psi_{4,4}(\pi)$
(14)(2)(3)	$x$	1423	4		(14)(2)(3)
(14)(23)	$x^2$	1423	4		(14)(23)
(134)(2)	$x^2$	1342	4		(134)(2)
(1234)	$x^3$	1234			(1234)
(124)(3)	$x^2$	1243	4	1	(1324)
(1324)	$-x^2$	1324	3	2	(124)(3)

Table.4. Involution  $\psi_{4,4}$

Hence,

$$\begin{aligned}
& \sum_{\pi \in \mathfrak{S}_{n,n}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} \\
&= \sum_{\pi \in \mathcal{A}_{n,n}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,n} \setminus (\mathcal{A}_{n,n} \cup \Omega_{n,n})} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} + \sum_{\pi \in \Omega_{n,n}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} \\
&= \sum_{\pi \in \Omega_{n,n}} x^{\text{exc}(\pi)}(-1)^{\text{cdes}(\pi)} = x(1+x)^{n-2}.
\end{aligned}$$

□

#### 4. PROOF OF THE EXPLICIT FORMULA (3) IN THEOREM 1

Let  $P_n(x, y, 0, 1) = \sum_{i=1}^n P_{n,i}(x, y, 0, 1)$ . Similarly, we first give the recurrence for  $P_n(x, y, 0, 1)$ .

**Lemma 12.** *For any  $n \geq 2$  and  $2 \leq i \leq n+1$ , we have*

$$P_{n+1,i}(x, y, 0, 1) = xP_{n-1}(x, y, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1) + y \sum_{j=i}^n P_{n,j}(x, y, 0, 1).$$

*Proof.* For any  $\pi = \pi(1)\pi(2)\dots\pi(n+1) \in \mathcal{D}_{n+1,i}$ , let  $\sigma = (1, c_1, c_2, \dots, c_l)$  be the cycle in the standard cycle decomposition of  $\pi$  which contains the number 1. So  $\pi$  can be split into the cycle  $\sigma$  and a permutation  $\tau$  on the set  $\{1, 2, \dots, n+1\} \setminus \{1, c_1, \dots, c_l\}$ , i.e.,  $\pi = \sigma \cdot \tau$ . Clearly,  $l \geq 1$ ,  $i \geq 2$  and  $c_l = i$  since  $\pi \in \mathfrak{S}_{n+1,i}$ . We distinguish the following two cases:

**Case 1.**  $l = 1$ .

Deleting the cycle  $(1, c_1) = (1, i)$  from the standard cycle decomposition of  $\pi$ , we obtain the permutation

$$\tau = \pi(2)\dots\pi(i-1)\pi(i+1)\dots\pi(n+1)$$

which is defined on the set  $\{2, \dots, i-1, i+1, \dots, n+1\}$ . Note that  $\text{red}(\tau) \in \mathcal{D}_{n-1}$ ,

$$\text{exc}(\pi) = \text{exc}(\text{red}(\tau)) + 1 \text{ and } \text{cdes}(\pi) = \text{cdes}(\text{red}(\tau)).$$

This provides the term

$$xP_{n-1}(x, y, 0, 1).$$

**Case 2.**  $l \geq 2$ .

Suppose that  $c_{l-1} = j$  for some  $2 \leq j \leq n+1$ . Deleting the number  $c_l = i$  from the standard cycle decomposition of  $\pi$ , we obtain a permutation

$$\tilde{\pi} = (1, c_1, \dots, c_{l-1}) \cdot \tau$$

which is defined on the set  $\{1, \dots, i-1, i+1, \dots, n+1\}$ . Note that  $\text{red}(\tilde{\pi}) \in \mathcal{D}_n$ . Moreover, if  $c_{l-1} = j \leq i-1$ , then

$$\text{red}(\tilde{\pi}) \in \mathcal{D}_{n,j}, \text{exc}(\pi) = \text{exc}(\text{red}(\tilde{\pi})) + 1, \text{cdes}(\pi) = \text{cdes}(\text{red}(\tilde{\pi})).$$

This provides the term

$$x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1).$$

If  $c_{l-1} = j \geq i+1$ , then

$$\text{red}(\tilde{\pi}) \in \mathcal{D}_{n,j-1}, \text{exc}(\pi) = \text{exc}(\text{red}(\tilde{\pi})), \text{cdes}(\pi) = \text{cdes}(\text{red}(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^n P_{n,j}(x, y, 0, 1).$$

Thus, for any  $i \geq 2$  we have

$$P_{n+1,i}(x, y, 0, 1) = x P_{n-1}(x, y, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1) + y \sum_{j=i}^n P_{n,j}(x, y, 0, 1).$$

□

### A proof of (3) in Theorem 1:

*Proof.* Note that

$$\sum_{\pi \in \mathcal{D}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} t^{\pi^{-1}(1)} = t^i \sum_{\pi \in \mathcal{D}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} = t^i P_{n,i}(x, -1, 0, 1).$$

Therefore, it is sufficient to show that

$$P_{n,i}(x, -1, 0, 1) = (-1)^{n-i} x^{i-1} \quad (8)$$

for any  $n \geq 2$  and  $2 \leq i \leq n$ .

#### (i) An induction proof of (8):

It is easy to check that

$$P_{2,2}(x, -1, 0, 1) = x.$$

Assume that the formula holds for any  $2 \leq k \leq n$ . By Lemma 12, we have

$$\begin{aligned}
P_{n+1,i}(x, -1, 0, 1) &= xP_{n-1}(x, -1, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, -1, 0, 1) - \sum_{j=i}^n P_{n,j}(x, -1, 0, 1) \\
&= x \sum_{j=2}^{n-1} P_{n-1,j}(x, -1, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, -1, 0, 1) - \sum_{j=i}^n P_{n,j}(x, -1, 0, 1) \\
&= \sum_{j=2}^{n-1} (-1)^{n-1-j} x^j + \sum_{j=2}^{i-1} (-1)^{n-j} x^j - \sum_{j=i}^n (-1)^{n-j} x^{j-1} \\
&= (-1)^{n+1-i} x^{i-1}
\end{aligned}$$

for any  $2 \leq i \leq n$ .

**(ii) A bijective proof of (8):**

Next we give a bijective proof of the explicit formula (8) by establishing an involution  $\varphi_{n,i}$  on  $\mathcal{D}_{n,i}$ . Fix  $i \in \{2, \dots, n\}$ . By definition, the weight of each  $\pi \in \mathcal{D}_{n,i}$  is  $(-1)^{\text{cdes}(\pi)} x^{\text{exc}(\pi)}$ , hence the weight of the cyclic permutation

$$\sigma^i = (1, 2, \dots, i-1, n, n-1, \dots, i) \in \mathcal{D}_{n,i}$$

is  $(-1)^{n-i} x^{i-1}$ .

For any  $\pi \in \mathcal{D}_{n,i}$ , suppose that  $\pi = C_1 \dots C_k$  is the standard cycle decomposition of  $\pi$  and

$$C_k = (c_{k1}, \dots, c_{ks}).$$

**Case 1.**  $k = 1$  and  $C_k = (1, 2, \dots, i-1, n, n-1, \dots, i)$ .

Then let  $\varphi_{n,i}(\pi) = \pi$ .

**Case 2.**  $k \geq 2$  and  $\text{red}(C_k) = (1, 2, \dots, r-1, s, s-1, \dots, r)$  for some  $r = 2, 3, \dots, s$ .

Suppose that  $C_{k-1} = (c_{k-1,1}, c_{k-1,2}, \dots, c_{k-1,t})$  and  $c_{k,j}$  is the largest number in the set  $\{c_{k,1}, c_{k,2}, \dots, c_{k,l_k}\}$  for some  $j \in \{1, 2, \dots, l_k\}$ . If  $c_{k-1,2} < c_{k,j-1}$ , then let

$$\varphi_{n,i}(\pi) = C_1 \dots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, c_{k2}, \dots, c_{ks}, c_{k-1,2}, \dots, c_{k-1,t}),$$

and so we have

$$\text{exc}(\pi) = \text{exc}(\varphi_{n,i}(\pi)) \text{ and } \text{cdes}(\pi) = \text{cdes}(\varphi_{n,i}(\pi)) + 1;$$

if  $c_{k-1,2} > c_{k,j-1}$  then let

$$\varphi_{n,i}(\pi) = C_1 \dots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, \dots, c_{k,j-2}, c_{kj} \dots, c_{ks}, c_{k,j-1}, c_{k-1,2}, \dots, c_{k-1,t}),$$

and so we have

$$\text{exc}(\pi) = \text{exc}(\varphi_{n,i}(\pi)) \text{ and } \text{cdes}(\pi) = \text{cdes}(\varphi_{n,i}(\pi)) + 1.$$

**Case 3.**  $\text{red}(C_k) \neq (1, 2, \dots, r-1, s, s-1, \dots, r)$  for any  $r = 2, 3, \dots, s$ .

There exists a unique index  $\tilde{s}$  such that

$$\text{red}(c_{k1}, c_{k2}, \dots, c_{k\tilde{s}}) = 1, 2, \dots, r-1, \tilde{s}, \tilde{s}-1, \dots, r$$

for some  $r = 2, 3, \dots, \tilde{s}$  and

$$\text{red}(c_{k1}, c_{k2}, \dots, c_{k,\tilde{s}+1}) \neq 1, 2, \dots, \tilde{r}-1, \tilde{s}+1, \tilde{s}, \dots, \tilde{r}$$

for any  $\tilde{r} = 2, 3, \dots, \tilde{s} + 1$ . It is easy to check  $3 \leq \tilde{s} \leq s - 1$ . Moreover, suppose that  $c_{kj}$  is the largest number in the set  $\{c_{k1}, c_{k2}, \dots, c_{k\tilde{s}}\}$ . Then we have

$$c_{k,\tilde{s}+1} < c_{k,j-1} \text{ or } c_{k,\tilde{s}+1} > c_{k\tilde{s}}.$$

If  $c_{k,\tilde{s}+1} < c_{k,j-1}$  then

$$\varphi_{n,i}(\pi) = C_1 \dots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \dots, c_{ks}) \cdot (c_{k2}, \dots, c_{k\tilde{s}}),$$

we have

$$\text{exc}(\pi) = \text{exc}(\varphi_{n,i}(\pi)) \text{ and } \text{cdes}(\pi) = \text{cdes}(\varphi_{n,i}(\pi)) - 1;$$

if  $c_{k,\tilde{s}+1} > c_{k\tilde{s}}$  then let

$$\varphi_{n,i}(\pi) = C_1 \dots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \dots, c_{ks}) \cdot (c_{k2}, \dots, c_{k,j-1}, c_{k,\tilde{s}}, c_{kj}, \dots, c_{k,\tilde{s}-1}),$$

we have

$$\text{exc}(\pi) = \text{exc}(\varphi_{n,i}(\pi)) \text{ and } \text{cdes}(\pi) = \text{cdes}(\varphi_{n,i}(\pi)) - 1.$$

For the case with  $n = 4$ , we list all  $\pi$  and  $\varphi_{n,i}(\pi)$  in Table. 5.

$\pi \in \mathcal{D}_{4,2}$	$\varphi_{4,2}(\pi)$	$\pi \in \mathcal{D}_{4,3}$	$\varphi_{4,4}(\pi)$	$\pi \in \mathcal{D}_{4,4}$	$\varphi_{4,4}(\pi)$
(12)(34)	(1342)	(13)(24)	(1423)	(14)(23)	(1324)
(1342)	(12)(34)	(1423)	(13)(24)	(1324)	(14)(23)
(1432)	(1432)	(1423)	(1423)	(1234)	(1234)

Table. 5. Involutions  $\varphi_{n,i}(\pi)$  for  $n = 4$

Hence,

$$\sum_{\pi \in \mathcal{D}_{n,i}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} = (-1)^{n-i} x^{i-1} + \sum_{\pi \in \mathcal{D}_{n,i} \setminus \{\sigma^i\}} x^{\text{exc}(\pi)} (-1)^{\text{cdes}(\pi)} = (-1)^{n-i} x^{i-1}.$$

□

## 5. PROOF OF THE RECURRENCE RELATION (4)

Suppose that  $y$  is a positive integer. Let  $\mathfrak{S}_n(y)$  denote the set of pairs  $[\pi, \phi]$  such that  $\pi \in \mathfrak{S}_n$  and  $\phi$  is a map from the set  $CDES(\pi)$  to the set  $\{0, 1, \dots, y-1\}$ . So,  $b_n(y, 1) = |\mathfrak{S}_n(y)|$ .

For any  $[\pi, \phi] \in \mathfrak{S}_{n+1}(y)$ , we distinguish the following two cases:

**Case 1.**  $\pi(1) = 1$ .

Let  $\tau = \pi(2) \dots \pi(n+1)$ . Then  $\tau$  is a permutation defined on the set  $\{2, 3, \dots, n+1\}$  and

$$\text{red}(\tau) \in \mathfrak{S}_n.$$

Define a map  $\phi' : [n] \mapsto \{+1, -1\}$  by letting  $\phi'(i) = \phi(\text{red}^{-1}(i))$  for  $i = 1, 2, \dots, n$ . Then

$$[\text{red}(\tau), \phi'] \in \mathfrak{S}_n(y),$$

and so this provides the term  $b_n(y, 1)$ .

**Case 2.**  $\pi(1) \neq 1$ .

Let  $\sigma = (1, c_1, c_2, \dots, c_l)$  be the cycle in the standard cycle decomposition of  $\pi$  which contains the number 1. So,  $\pi$  is split into the cycle  $\sigma$  and a permutation  $\tau$  on the set  $\{1, 2, \dots, n+1\} \setminus \{1, c_1, \dots, c_l\}$ , i.e.,  $\pi = \sigma \cdot \tau$ . Clearly,  $l \geq 1$  since  $\pi(1) \neq 1$ .

Note that there is a unique index  $k \geq 1$  which satisfies  $c_{k-1} < c_k$  and  $c_k > c_{k+1} > \dots > c_l$ . For the sequence  $c_k \dots c_l$ , if  $\phi(c_i) = 0$  for some  $k \leq i \leq l-1$  then let  $k'$  be the largest index in  $\{k, k+1, \dots, l-1\}$  such that  $\phi(c_{k'}) = 0$ ; otherwise,  $k' = k-1$ . Let

$$\sigma' = (1, c_1, \dots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then  $\pi'$  is a permutation defined on the set  $[n+1] \setminus B$ , where

$$B = \{c_{k'+1}, \dots, c_l\},$$

and

$$\text{red}(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map  $\phi' : [n+1-|B|] \mapsto \{+1, -1\}$  by letting

$$\phi'(i) = \phi(\text{red}^{-1}(i))$$

for any  $1 \leq i \leq n+1-|B|$ . Then

$$[\text{red}(\pi'), \phi'] \in \mathfrak{S}_{n+1-|B|}(y).$$

Note that  $1 \leq |B| \leq n$  and  $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$ . For any  $k \leq i \leq l-1$ , let  $\theta(c_i) = \phi(c_i)$ . Then  $\theta$  is a map from the set  $\{c_{k'+1}, \dots, c_{l-1}\}$  to  $\{1, 2, \dots, y-1\}$ . So there are  $\binom{n}{|B|}$  ways to form the set  $B$  and  $(y-1)^{|B|-1}$  ways to form the map  $\theta$ . This provides the term

$$\sum_{i=1}^n b_{n+1-i}(y, 1) \binom{n}{i} (y-1)^{i-1}$$

Hence we derive the recurrence relation

$$\begin{aligned} b_{n+1}(y, 1) &= b_n(y, 1) + \sum_{i=1}^n b_{n+1-i}(y, 1) \binom{n}{i} (y-1)^{i-1} \\ &= b_n(y, 1) + \sum_{i=1}^n b_i(y, 1) \binom{n}{i-1} (y-1)^{n-i}. \end{aligned}$$

## 6. PROOF OF THE RECURRENCE RELATION (5)

Clearly, we have  $b_0(y, 0) = 1$  and  $b_1(y, 0) = 0$ . Suppose that  $y$  is a positive integer. Let  $\mathcal{D}_n(y)$  denote the set of pairs  $[\pi, \phi]$  such that  $\pi \in \mathcal{D}_n$  and  $\phi$  is a map from the set  $CDES(\pi)$  to the set  $\{0, 1, \dots, y-1\}$ . Hence  $b_n(y, 0) = |\mathcal{D}_n(y)|$ .

For any  $[\pi, \phi] \in \mathcal{D}_{n+1}(y)$ , let  $\sigma = (1, c_1, c_2, \dots, c_l)$  be the cycle in the standard cycle decomposition of  $\pi$  which contains the number 1. So,  $\pi$  is split into the cycle  $\sigma$  and a permutation  $\tau$  on the set  $[n+1] \setminus \{1, c_1, \dots, c_l\}$ , i.e.,  $\pi = \sigma \cdot \tau$ . Clearly,  $l \geq 1$  since  $\pi(1) \neq 1$ .

Note that there is a unique index  $k \geq 1$  which satisfies  $c_{k-1} < c_k$  and  $c_k > c_{k+1} > \dots > c_l$ . For the sequence  $c_k \dots c_l$ , if  $\phi(c_i) = 0$  for some  $k \leq i \leq l-1$  then let  $k'$  be the largest index in  $\{k, k+1, \dots, l-1\}$  such that  $\phi(c_{k'}) = 0$ ; otherwise,  $k' = k-1$ .

We distinguish the following two cases:

**Case 1.**  $k' = 0$

Let

$$B = \{c_1, \dots, c_l\}.$$

Note that  $\tau$  is a permutation defined on the set  $[n+1] \setminus \{1, c_1, \dots, c_l\}$  and

$$\text{red}(\tau) \in \mathfrak{S}_{n-|B|}.$$

Define a map  $\phi' : [n-|B|] \mapsto \{+1, -1\}$  by letting

$$\phi'(i) = \phi(\text{red}^{-1}(i))$$

for any  $1 \leq i \leq n-|B|$ . Then

$$[\text{red}(\tau), \phi'] \in \mathfrak{S}_{n-|B|}(y)$$

and there are  $b_{n-|B|}(y, 0)$  ways to form the pairs  $[\text{red}(\tau), \phi']$ .

Note that  $1 \leq |B| \leq n$  and  $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$ . For any  $k \leq i \leq l-1$ , let  $\theta(c_i) = \phi(c_i)$ . Then  $\theta$  is a map from the set  $\{c_k, \dots, c_{l-1}\}$  to  $\{1, 2, \dots, y-1\}$ . So there are  $\binom{n}{|B|}$  ways to form the set  $B$  and  $(y-1)^{|B|-1}$  ways to form the mapping  $\theta$ .

This provides the term

$$\sum_{i=1}^n b_{n-i}(y, 0) \binom{n}{i} (y-1)^{i-1}.$$

**Case 2.**  $k' \geq 1$

Let

$$\sigma' = (1, c_1, \dots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then  $\pi'$  is a permutation defined on the set  $[n+1] \setminus B$ , where

$$B = \{c_{k'+1}, \dots, c_l\},$$

and

$$\text{red}(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map  $\phi' : [n+1-|B|] \mapsto \{+1, -1\}$  by letting

$$\phi'(i) = \phi(\text{red}^{-1}(i))$$

for any  $1 \leq i \leq n+1-|B|$ . Then

$$[\text{red}(\pi'), \phi'] \in \mathcal{D}_{n+1-|B|}(y)$$

and there are  $b_{n+1-|B|}(y, 0)$  ways to form the pairs  $(\text{red}(\pi'), \phi')$ .

Note that  $1 \leq |B| \leq n-1$  and  $B \setminus \{c_l\} \subseteq CDES(\pi)$ . For any  $k \leq i \leq l-1$ , let  $\theta(c_i) = \phi(c_i)$ . Then  $\theta$  is a map from the set  $\{c_{k'+1}, \dots, c_{l-1}\}$  to  $\{1, 2, \dots, y-1\}$ . So there are  $\binom{n}{|B|}$  ways to form the set  $B$  and  $(y-1)^{|B|-1}$  ways to form the map  $\theta$ .

This provides the term

$$\sum_{i=1}^{n-1} b_{n+1-i}(y, 0) \binom{n}{i} (y-1)^{i-1}.$$

Hence, we have

$$b_{n+1}(y, 0) = \sum_{i=1}^n \binom{n}{i} b_{n-i}(y, 0) (y-1)^{i-1} + \sum_{i=1}^{n-1} \binom{n}{i} b_{n+1-i}(y, 0) (y-1)^{i-1}.$$



## 7. PROOF OF THEOREM 9

**Lemma 13.** *There is a bijection  $\Theta_n$  from the set of cyclic negative cycle-descent permutations of  $[n]$  to the set of connected Callan perfect matchings of  $\mathbb{P}_n$ .*

*Proof.* Let  $(\pi, \phi)$  be a cyclic negative cycle-descent permutations of  $[n]$ . Then there is exactly one cycle  $C$  in the standard cycle decomposition of  $\pi$ . Suppose

$$C = (c_1, c_2, \dots, c_n)$$

where  $c_1 = 1$ . Erase the parentheses, draw a bar after each element  $c_i$  which has sign  $+1$ , and add a bar before  $c_1$ . Regard the numbers between two consecutive bars as “blocks”. So, we decompose  $(\pi, \phi)$  into a sequence of blocks

$$B_1, B_2, \dots, B_k.$$

Suppose that the  $i$ -th block  $B_i$  contains  $t_i$  number  $b_{i1}, \dots, b_{it_i}$  with  $b_{i1} > \dots > b_{it_i}$ . We construct a perfect matchings  $M$  as follows:

- Step 1. For every block  $B_i$ , we connect the vertex  $(b_{i,j}, 0)$  to the vertex  $(b_{i,j+1}, 1)$  as a downline of  $M$  for any  $1 \leq j \leq t_i - 1$ .
- Step 2. For any odd integer  $i \in \{1, 2, \dots, k-1\}$ , we connect the vertex  $(b_{i,t_i}, 0)$  to the vertex  $(b_{i+1,t_{i+1}}, 0)$  as an arc of  $M$ . For any even integer  $i \in \{1, 2, \dots, k-1\}$ , we connect the vertex  $(b_{i,1}, 1)$  to the vertex  $(b_{i+1,1}, 1)$  as an arc of  $M$ .
- Step 3. If  $k$  is odd, we connect the vertex  $(b_{1,1}, 1) = (1, 1)$  to the vertex  $(b_{k,t_k}, 0)$  as a downline of  $M$ ; otherwise, connect the vertex  $(b_{1,1}, 1) = (1, 1)$  to the vertex  $(b_{k,1}, 1)$  as an arc of  $M$ .

It is easy to check that  $M$  is connected and has no uplines. So,  $M$  is a connected Callan perfect matching. Define  $\Theta_n$  as a map from the set of cyclic negative cycle descent permutations of  $[n]$  to the set of connected Callan perfect matchings of  $\mathbb{P}_n$  by letting  $\Theta_n(\pi, \phi) = M$ . Let  $(\pi, \phi)$  and  $(\pi', \phi')$  be two different cyclic negative cycle-descent permutations of  $[n]$ . Then the sequence of blocks of  $(\pi, \phi)$  and  $(\pi', \phi')$  are different. This implies  $\Theta_n(\pi, \phi) \neq \Theta_n(\pi', \phi')$ , and so the map  $\Theta_n$  is a bijection.

Conversely, let  $M$  be a connected Callan perfect matching of  $\mathbb{P}_n$ . Delete the edge incident with the vertex  $(1, 1)$  from  $M$ , identify two vertices  $(i, 0)$  and  $(i, 1)$  in  $M$  as a new vertex  $i$  for each  $i = 1, 2, \dots, n$ , denote by  $\mathcal{G}^*(M)$  the graph obtained from  $M$ . Then the graph  $\mathcal{G}^*(M)$  is a path on the vertex set  $[n]$  and can be written as

$$a_1 a_2 \dots a_n$$

where  $a_1 = 1$  and the set  $\{a_1 a_2, a_3 a_4, \dots, a_{n-1} a_n\}$  is the edge set of  $\mathcal{G}^*(M)$ . Draw a bar after each number  $a_i$  which satisfies either (1)  $i = n$  or (2) there is an arc of  $M$  in

$$\{(a_i, 0), (a_{i+1}, 0)\}, \{(a_i, 1), (a_{i+1}, 1)\},$$

and add a bar before  $a_1$ . Regard the numbers between two consecutive bars as “blocks”. So, we obtain a sequence of blocks

$$B'_1, B'_2, \dots, B'_k.$$

We construct a cyclic negative cycle-descent permutations  $(\pi, \phi)$  of  $[n]$  as follows:

- Step 1'. For each block  $B'_i$ , we write the numbers in  $B'_i$  in decreasing order, denote by  $\tau_i$  the obtained sequence, and let

$$\pi = (\tau_1, \tau_2, \dots, \tau_k).$$

- Step 2'. For any number  $j \in [n]$ , suppose  $j$  is in a block  $B'_i$  for some  $1 \leq i \leq k$ . If  $j$  is the smallest number in  $B'_i$ , then let the sign of  $j$  be  $+1$ ; otherwise, let the sign of  $j$  be  $-1$ . In fact, this defines a mapping  $\phi$  from  $[n]$  to  $\{+1, -1\}$ .

Then  $(\pi, \phi)$  is a cyclic negative cycle-descent permutations of  $[n]$ .  $\square$

**Example 14.** Let us consider a cyclic negative cycle-descent permutation

$$(1^+ 6^- 4^- 3^+ 2^+ 8^- 7^- 5^+)$$

of [8]. We erase the parentheses, draw a bar after each element which has sign  $+1$ , and add a bar before 1. Thus we obtain

$$|1|643|2|875|$$

and the sequence of blocks

$$B_1 = 1, B_2 = 643, B_3 = 2, B_4 = 875.$$

By Steps 1,2, and 3 in the proof of Lemma 13, we construct the following dot diagram.

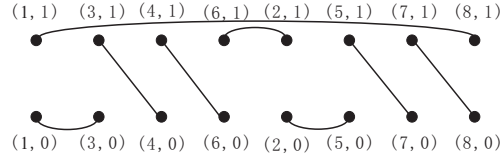


Fig.4. A dot diagram constructed by Step 1,2, and 3 in the proof of Lemma 13

Finally, we obtain a connect Callan perfect matching  $M$  corresponding with  $(1^+ 6^- 4^- 3^+ 2^+ 8^- 7^- 5^+)$  as follows:

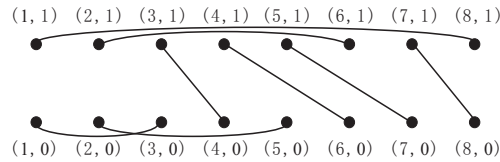


Fig.5. A connect Callan perfect matching  $M$  corresponding with  $(1^+ 6^- 4^- 3^+ 2^+ 8^- 7^- 5^+)$

Conversely, let us consider the connect perfect matching  $M$  in Fig.5. After deleting the edge  $\{(1,1), (8,1)\}$ , we can obtain the graph

$$\mathcal{G}^*(M) = 13462578$$

which has the edge set  $\{13, 34, 46, 62, 25, 57, 78\}$ . Note that there is 3 arcs

$$\{(1,0), (3,0)\}, \{(6,1), (2,1)\}, \{(2,0), (5,0)\}$$

in  $M$ . So, we draw bars after the numbers 1,6,2,8, and add a bar before 1. Thus we obtain

$$|1|346|2|578|$$

and the sequence of blocks

$$B'_1 = 1, B'_2 = 346, B'_3 = 2, B'_4 = 578.$$

By Steps 1' and 2' in the proof of Lemma 13, we construct a cyclic negative cycle-descent permutation

$$(1^+6^-4^-3^+2^+8^-7^-5^+)$$

of [8].

### A bijection proof of Theorem 9

*Proof.* Let  $(\pi, \phi)$  be a negative cycle-descent permutations of  $[n]$ . Suppose that  $\pi = C_1 \dots C_k$  is the standard cycle decomposition of  $\pi$  and

$$C_i = (c_{i1}, \dots, c_{i,l_i})$$

for each  $i = 1, 2, \dots, k$ . Then

$$\text{red}(C_i) \in \mathfrak{S}_{l_i},$$

Define a map  $\phi^i : [l_i] \mapsto \{+1, -1\}$  by letting

$$\phi^i(j) = \phi(\text{red}^{-1}(j)),$$

i.e., the sign of  $\text{red}(c_{ij})$  is the same as that of  $c_{ij}$ . Then

$$(\text{red}(C_i), \phi^i)$$

is a cyclic negative cycle-descent permutations of  $[l_i]$ . By Lemma 13, we have  $\Theta_{l_i}(\text{red}(C_i), \phi^i)$  is a connected Callan perfect matching. For any  $1 \leq j \leq l_i$ , we replace the labels  $(j, 0)$  and  $(j, 1)$  of vertices in  $\Theta_{l_i}(\text{red}(C_i), \phi^i)$  with  $(\text{red}^{-1}(j), 0)$  and  $(\text{red}^{-1}(j), 1)$  respectively and denote by  $M^i$  the perfect matching obtained from  $\Theta_{l_i}(\text{red}(C_i), \phi^i)$ . At last, let

$$M = M^1 \cup M^2 \cup \dots \cup M^k$$

where the notation  $M \cup M'$  denotes the union of two perfect matchings  $M$  and  $M'$  such that the vertex set of  $M \cup M'$  is  $V(M) \cup V(M')$  and the edge set of  $M \cup M'$  is  $E(M) \cup E(M')$ . So  $M$  is a Callan perfect matchings of  $\mathbb{P}_n$ . Define  $\Gamma_n$  as a map from the set of negative cycle descent permutations of  $[n]$  to the set of Callan perfect matchings of  $\mathbb{P}_n$  by letting  $\Gamma_n(\pi, \phi) = M$ . Note that  $\Gamma_n$  is injective, and so it is a bijection.

By the definition of  $\Gamma_n$ , it is easy to see that

$$\text{com}(\Gamma_n(\pi, \phi)) = \text{cyc}(\pi) \text{ and } \text{ver}(\Gamma_n(\pi, \phi)) = \text{fix}(\pi).$$

If the vertices  $(1, 1)$  and its partner are in the same row, then  $\text{down}(\Gamma_n(\pi, \phi)) = \text{neg}(\pi, \phi)$ ; otherwise,  $\text{down}(\Gamma_n(\pi, \phi)) = \text{neg}(\pi, \phi) + 1$ .  $\square$

**Example 15.** Let us consider a negative cycle-descent permutation

$$(1^+6^-3^+4^+)(2^+8^-7^+)(5^+)$$

of [8]. We draw the perfect matchings  $M^1$ ,  $M^2$  and  $M^3$  corresponding with the cycles  $C_1$ ,  $C_2$  and  $C_3$  respectively as follows:

Cycles	$C_1$	$C_2$	$C_3$
	$(1^+6^-3^+4^+)$	$(2^+8^-7^+)$	$(5^+)$
Perfect matchings	$M^1$	$M^2$	$M^3$

Finally, we obtain a Callan perfect matching  $M = M^1 \cup M^2 \cup M^3$  which is exactly that in Example 7.

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