A2

March 19, 2019

Zhu, Zhaoyang
1002321445

Q1

a).

$L(c, \mathbf{x}|\boldsymbol{\theta}, \pi) = \prod_{c=0}^{9}(\pi_c \prod_{d=1}^{784} \theta_{cd}^{x_d}(1-\theta_{cd})^{1-x_d})$

$logL = l_{mle} = \sum_{c=0}^{9} log(\pi_c) + \sum_{c=0}^{9}\sum_{d=1}^{784}(x_d)log(\theta_{cd}) + (1-x_d)log(1-\theta_{cd})$

For fixed value of c = k:

$\frac{dl}{d\theta_{kd}} = \sum_{c=0}^{9}\sum_{d=1}^{784}(x_d)\frac{I(c=k)}{\theta_{cd}} - (1-x_d)\frac{I(c=k)}{1-\theta_{cd}}$

$= \sum_{d=1}^{784}\sum_{c=0}^{9} I(x_d=1)\frac{I(c=k)}{\theta_{cd}} - (1-I(x_d=1))\frac{I(c=k)}{1-\theta_{cd}}$

For fixed value of d = j:

$\implies \frac{dl}{d\theta_{kj}} = \frac{\sum_{d=1}^{784} I(x_d=1)I(c=k)}{\theta_{kj}} - \frac{\sum_{d=1}^{784}(1-I(x_d=1))I(c=k)}{1-\theta_{kj}} = 0$

$\implies \frac{1}{\hat{\theta}_{kj}} = \frac{\sum_{d=1}^{784}(1-I(x_d=1))I(c=k)+\sum_{d=1}^{784} I(x_d=1)I(c=k)}{\sum_{d=1}^{784} I(x_d=1)I(c=k)}$

$\implies \hat{\theta}_{kj} = \frac{\sum_{d=1}^{784} I(x_d=1)I(c=k)}{\sum_{d=1}^{784} I(c=k)}$

$\implies \hat{\theta}_{kj} = \frac{\sum_{d=1}^{784} I(x_d=1 \wedge c=k)}{\sum_{d=1}^{784} I(c=k)}$

b).

$\pi(\boldsymbol{\theta}) = Beta(2,2) = constant * (\theta_{cd}(1-\theta_{cd}))$

$L(c, \mathbf{x}|\boldsymbol{\theta}, \pi) = \prod_{c=0}^{9}(\pi_c \prod_{d=1}^{784} \theta_{cd}^{x_d}(1-\theta_{cd})^{1-x_d})$

$\implies \pi(\theta, \pi|\mathbf{x}, c) \propto \pi(\boldsymbol{\theta})L(c, \mathbf{x}|\boldsymbol{\theta}, \pi)$

$l_{map} = log(\pi(\theta, \pi|c, \mathbf{x})) = log(\pi(\boldsymbol{\theta})) + l_{mle}$

$$\implies l_{map} = log(\theta_{cd}) + log(1-\theta_{cd}) + l_{mle}$$

For fixed value of c = k and d = j:

$$\implies \frac{dl_{map}}{d\theta_{kj}} = \frac{1}{\theta_{kj}} - \frac{1}{1-\theta_{kj}} + \frac{\sum_{d=1}^{784} I(x_d=1)I(c=k)}{\theta_{kj}} - \frac{\sum_{d=1}^{784}(1-I(x_d=1))I(c=k)}{1-\theta_{kj}} = 0$$

$$\implies \frac{1}{\hat{\theta}_{kj}} = \frac{2+\sum_{d=1}^{784}(1-I(x_d=1))I(c=k)+\sum_{d=1}^{784} I(x_d=1)I(c=k)}{1+\sum_{d=1}^{784} I(x_d=1)I(c=k)}$$

$$\implies \hat{\theta}_{kj} = \frac{1+\sum_{d=1}^{784} I(x_d=1 \wedge c=k)}{2+\sum_{d=1}^{784} I(c=k)}$$

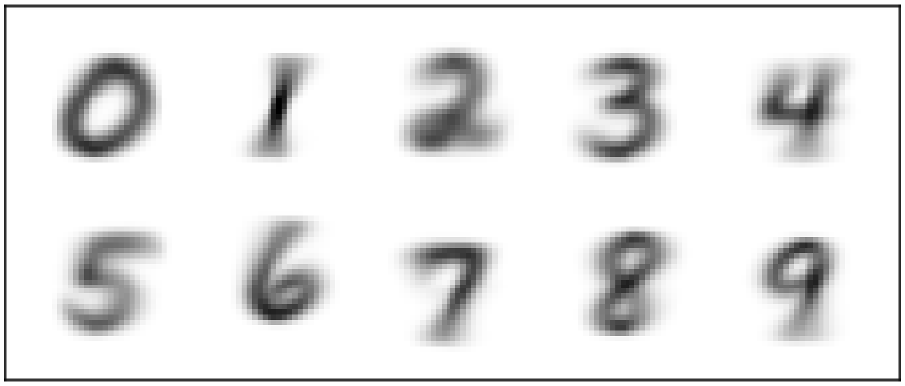
c).

```
In [1]: from __future__ import absolute_import
        from __future__ import print_function
        from future.standard_library import install_aliases
        install_aliases()

        import numpy as np
        import os
        import gzip
        import struct
        import array
        from scipy.stats import multivariate_normal
        from autograd import grad
        from scipy.misc import logsumexp as lse
        import matplotlib.pyplot as plt
        import matplotlib.image
        from urllib.request import urlretrieve
        %matplotlib inline

        def download(url, filename):
            if not os.path.exists('data'):
                os.makedirs('data')
            out_file = os.path.join('data', filename)
            if not os.path.isfile(out_file):
                urlretrieve(url, out_file)


        def mnist():
            base_url = 'http://yann.lecun.com/exdb/mnist/'

            def parse_labels(filename):
                with gzip.open(filename, 'rb') as fh:
                    magic, num_data = struct.unpack(">II", fh.read(8))
                    return np.array(array.array("B", fh.read()), dtype=np.uint8)

            def parse_images(filename):
```

```
        with gzip.open(filename, 'rb') as fh:
            magic, num_data, rows, cols = struct.unpack(">IIII", fh.read(16))
            return np.array(array.array("B", fh.read()),
                            dtype=np.uint8).reshape(num_data, rows, cols)

    for filename in ['train-images-idx3-ubyte.gz',
                     'train-labels-idx1-ubyte.gz',
                     't10k-images-idx3-ubyte.gz',
                     't10k-labels-idx1-ubyte.gz']:
        download(base_url + filename, filename)

    train_images = parse_images('data/train-images-idx3-ubyte.gz')
    train_labels = parse_labels('data/train-labels-idx1-ubyte.gz')
    test_images  = parse_images('data/t10k-images-idx3-ubyte.gz')
    test_labels  = parse_labels('data/t10k-labels-idx1-ubyte.gz')

    return train_images, train_labels, test_images, test_labels


def load_mnist():
    partial_flatten = lambda x : np.reshape(x, (x.shape[0], np.prod(x.shape[1:])))
    one_hot = lambda x, k: np.array(x[:,None] == np.arange(k)[None, :], dtype=int)
    train_images, train_labels, test_images, test_labels = mnist()
    train_images = partial_flatten(train_images) / 255.0
    test_images  = partial_flatten(test_images)  / 255.0
    train_labels = one_hot(train_labels, 10)
    test_labels = one_hot(test_labels, 10)
    N_data = train_images.shape[0]

    return N_data, train_images, train_labels, test_images, test_labels


def plot_images(images, ax, ims_per_row=5, padding=5, digit_dimensions=(28, 28),
                cmap=matplotlib.cm.binary, vmin=None, vmax=None):
    """Images should be a (N_images x pixels) matrix."""
    N_images = images.shape[0]
    N_rows = np.int32(np.ceil(float(N_images) / ims_per_row))
    pad_value = np.min(images.ravel())
    concat_images = np.full(((digit_dimensions[0] + padding) * N_rows + padding,
                             (digit_dimensions[1] +
                              padding) * ims_per_row + padding), pad_value)
    for i in range(N_images):
        cur_image = np.reshape(images[i, :], digit_dimensions)
        row_ix = i // ims_per_row
        col_ix = i % ims_per_row
        row_start = padding + (padding + digit_dimensions[0]) * row_ix
        col_start = padding + (padding + digit_dimensions[1]) * col_ix
        concat_images[row_start: row_start + digit_dimensions[0],
                             col_start: col_start + digit_dimensions[1]] = cur_image
    cax = ax.matshow(concat_images, cmap=cmap, vmin=vmin, vmax=vmax)
    plt.xticks(np.array([]))
    plt.yticks(np.array([]))
    return cax


def save_images(images, filename, **kwargs):
    fig = plt.figure(1)
    fig.clf()
    ax = fig.add_subplot(111)
    plot_images(images, ax, **kwargs)
    fig.patch.set_visible(False)
    ax.patch.set_visible(False)
    plt.savefig(filename)
```

```
In [2]: # binarilize our data into 1 and 0
        def binarilize(x):
            n, d = x.shape
            for i in range(n):
                for j in range(d):
                    if x[i][j] >= 0.5:
                        x[i][j] = 1
                    else:
                        x[i][j] = 0
            return x

        #Q1 d
        # the input x is a vector of pixels
        # the input theta is a c by d matrix
        # calculate the log likelihood for single training image
        def naive_loglik(x, pi_c, theta, label):
            pi = np.array([pi_c]*10)
            result = 0
            for c in range(len(theta)):
                if label[c] != 1:
                    result += np.log(pi[c])
                    for d in range(len(x)):
                        result += x[d] * np.log(theta[c][d]) + \
                        np.log(1 - theta[c][d]) * (1 - x[d])
            return -result

        # get average log likelihood
        def get_ave_loglik(x, y, theta, pi_c):
            total = 0
            for n in range(len(x)):
                total += naive_loglik(x[n], pi_c, theta, y[n])
            return total/len(x)


        # the input x is a matrix n by d
        # the input y is a matrix n by c
        # fit using MAP estimate
        def map_fit(x, y):
            theta = np.zeros((10, 784))
            for c in range(10):
                for d in range(784):
                    theta[c][d] = (1 + np.sum(np.where((x[:, d] == 1) & (y[:, c] == 1),
                         1, 0)))/ (2 + np.sum(y[:, c]))

            return theta

        num, x_train, y_train, x_test, y_test = load_mnist()
        x_train = binarilize(x_train)
        x_test = binarilize(x_test)
        theta = map_fit(x_train, y_train)
        fig = plt.figure(1)
        fig.clf()
        ax = fig.add_subplot(111)
        plot_images(theta, ax)
```

```
Out[2]: <matplotlib.image.AxesImage at 0x14ed7928278>
```



d).

$$p(c|\mathbf{x},\boldsymbol{\theta},\pi)p(\mathbf{x},\boldsymbol{\theta},\pi) = p(\mathbf{x},\boldsymbol{\theta},\pi,c)$$

$$\implies p(c|\mathbf{x},\boldsymbol{\theta},\pi) = \frac{p(\mathbf{x},\boldsymbol{\theta},\pi,c)}{p(\mathbf{x},\boldsymbol{\theta},\pi)} = \frac{p(\mathbf{x},c|\boldsymbol{\theta},\pi)}{p(\mathbf{x}|\boldsymbol{\theta},\pi)} = \frac{p(\mathbf{x},c|\boldsymbol{\theta},\pi)}{\sum_{c=0}^{9} p(\mathbf{x},c|\boldsymbol{\theta},\pi)}$$

$$\implies p(c|\mathbf{x},\boldsymbol{\theta},\pi) = \frac{\pi_c \prod_{d=1}^{784} \theta_{cd}^{x_d}(1-\theta_{cd})^{1-x_d}}{\sum_{c=0}^{9} \pi_c \prod_{d=1}^{784} \theta_{cd}^{x_d}(1-\theta_{cd})^{1-x_d}}$$

$$\implies \quad log(p(c|\mathbf{x},\boldsymbol{\theta},\pi)) \quad = \quad log(\pi_c) + \sum_{d=1}^{784} x_d log(\theta_{cd}) + (1-x_d)log(1-\theta_{cd}) - \sum_{c=0}^{9}(log(\pi_c) + \sum_{d=1}^{784} x_d log(\theta_{cd}) + (1-x_d)log(1-\theta_{cd}))$$

$$\implies log(p(c|\mathbf{x},\boldsymbol{\theta},\pi)) = -\sum_{i=0,i\neq c}^{9}(log(\pi_i) + \sum_{d=1}^{784}(x_d)log(\theta_{id}) + log(1-\theta_{id})(1-x_d))$$

e).

```
In [3]: # 1E
        # input pi is a pre defined probability.
        # input x is a vector
        def naive_predict(x, theta, pi_c):
            probability = []
            pi = [pi_c]*10
            for c in range(len(theta)):
                product = 1
                for d in range(len(x)):
                    product = product * theta[c][d]**x[d]*(1 - theta[c][d])**(1 - x[d])
                probability.append(product * pi[c])
            return np.argmax(probability)

        # ave log likelihood on part of the data
        train_ave_loglik = get_ave_loglik(x_train[0:10000], y_train[0:10000], theta, 0.1)
        test_ave_loglik = get_ave_loglik(x_test, y_test, theta, 0.1)

        # predict on the test set use part of the test data
        prediction_test = []
        for n in range(len(x_test)):
            prediction_test.append(naive_predict(x_test[n], theta, 0.1))
        prediction_train = []
        for n in range(len(x_train)):
            prediction_train.append(naive_predict(x_train[n], theta, 0.1))
        accuracy_test = 0
        for i in range(len(prediction_test)):
            if prediction_test[i] == np.argmax(y_test[i]):
                accuracy_test += 1
        accuracy_test = accuracy_test/len(prediction_test)
        accuracy_train = 0
        for i in range(len(prediction_train)):
            if prediction_train[i] == np.argmax(y_train[i]):
                accuracy_train += 1
        accuracy_train = accuracy_train/len(prediction_train)
        print("The test accuracy is %s, The train accuracy is %s."%\
              (accuracy_test, accuracy_train))
        print("The average log likelihood for test set is %s, for training set is %s."%\
              (np.round(test_ave_loglik, 2), np.round(train_ave_loglik, 2)))
```

```
The test accuracy is 0.8426, The train accuracy is 0.8358833333333333.
The average log likelihood for test set is 2433.1, for training set is 2433.49.
```

Q2

a).

True

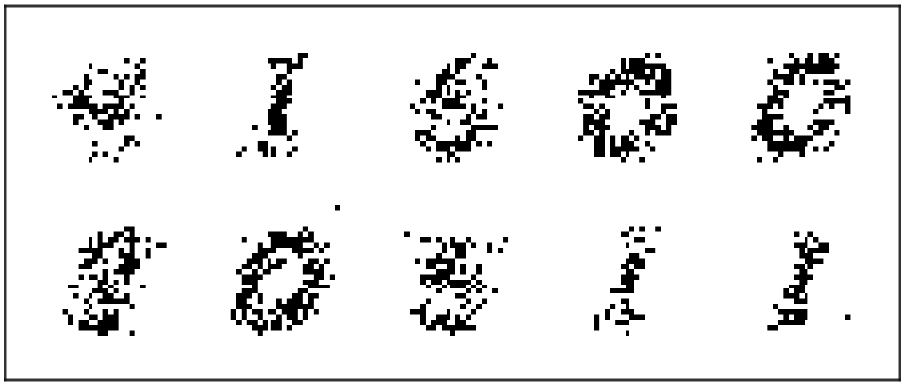
b).

False

c).

```
In [4]: #sample from c
        random_c = np.random.randint(0, 9, 10)
        random_picture = np.zeros((10, 784))
        for i in range(10):
            for j in range(784):
                random_picture[i][j] = np.random.binomial(1, theta[random_c[i]][j])

        fig = plt.figure(2)
        fig.clf()
        ax = fig.add_subplot(111)
        plot_images(random_picture, ax)
        print("The random images are %s."%(random_c))
```

```
The random images are [4 1 5 0 0 8 0 5 1 1].
```



d).

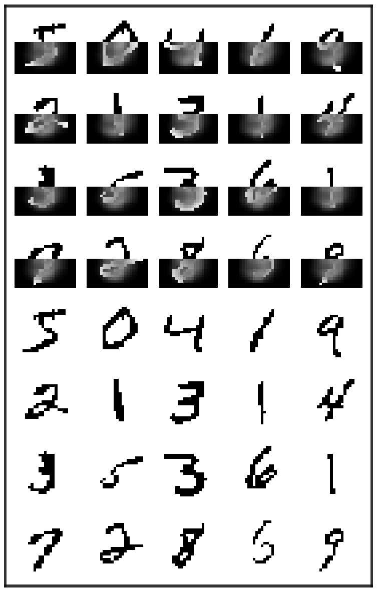
$$p(x_{i\in bot}|\mathbf{x}_{top},\boldsymbol{\theta},\pi) = \sum_{c=0}^{9} p(x_{i\in bot},c|\mathbf{x}_{top},\boldsymbol{\theta}_c,\pi_c)$$

$$= \sum_{c=0}^{9} \frac{p(x_{i\in bot},c,\mathbf{x}_{top}|\boldsymbol{\theta}_c,\pi_c)}{p(\mathbf{x}_{top}|\boldsymbol{\theta}_c,\pi_c)}$$

$$= \sum_{c=0}^{9} \frac{\prod_{d=1}^{382} \theta_{cd}^{x_d} (1-\theta_{cd})^{1-x_d} \theta_{ci}^{x_i} (1-\theta_{ci})^{1-x_i} \pi_c}{\prod_{d=1}^{382} \theta_{cd}^{x_d} (1-\theta_{cd})^{1-x_d}}$$

$$\implies p(x_{i \in bot} | \mathbf{x}_{top}, \boldsymbol{\theta}, \pi) = \sum_{c=0}^{9} \theta_{ci}^{x_i} (1-\theta_{ci})^{1-x_i} \pi_c$$

```
In [5]: def bot_part(x, theta, pi_c):
            bot_image = np.zeros(392)
            bot = x[392:]
            for d in range(392):
                result = 0
                for c in range(10):
                    result += theta[c][d + 392]**bot[d] * \
                    (1 - theta[c][d + 392])**(1 - bot[d]) * pi_c
                bot_image[d] = result
            return np.concatenate((x[:392], bot_image))


        new_image = x_train.copy()
        new_image = new_image[:20]
        for i in range(len(new_image)):
            new_image[i] = bot_part(new_image[i], theta, 0.1)


        fig = plt.figure(3)
        fig.clf()
        ax = fig.add_subplot(111)
        plot_images(np.concatenate((new_image, x_train[0:20]), axis = 0), ax)

Out[5]: <matplotlib.image.AxesImage at 0x14ed7b5e780>
```



Q3

a).

w is a 10 by 784 matrix = 7840 parameters

b).

Let $w = [w_1, w_2, … , w_c]^T $

$$\nabla_{w_c} log(p(c|\mathbf{xw})) = x - \frac{xe^{w_c^T \mathbf{x}}}{\sum_{c'=0}^{9} e^{w_{c'}^T \mathbf{x}}}$$

c).

```
In [6]: # the input w is a matrix
        def logistic_likelihood(x, index, w):
            bot = 0
            item = []
            for c in range(len(w)):
                item.append(np.dot(w[c].T, x))
            b = max(item)
            for c in range(len(w)):
                bot += np.exp(item[c] - b)
            bot = b + np.log(bot)
            return item[index] - bot


        # the input x is a vector
        # the inmput w is the matrix of w
        # conduct gradient ascent for one training image
        def single_gradient(x, c, w, iteration, lr, cr=0.001):
            this_w = w
            this_likelihood = logistic_likelihood(x, c, w)
            diff = 1000
            like = []
            while (iteration > 0) & (diff > cr):
                bot = 0
                next_w = this_w
                for i in range(10):
                    bot += np.exp(np.dot(w[i].T, x))
                next_w[c] = this_w[c] + lr * \
                (x - x * np.exp((np.dot(this_w[c].T, x))) / bot)
                this_w = next_w
                old_likelihood = this_likelihood
                this_likelihood = logistic_likelihood(x, c, this_w)
                diff = np.abs(old_likelihood - this_likelihood)
                iteration -= 1
                like.append(old_likelihood)
            return this_w, like[-1]

        # gradient ascent
        def gradient_ascent(x, y, stepsize, iterations):
            w = np.zeros((10, 784))
            likelihood = []
            for n in range(len(x)):
                index = np.argmax(y[n])
                result = single_gradient(x[n], index, w, iterations, stepsize)
                w = result[0]
                likelihood.append(result[1])
            return w, likelihood
```

d).

Compare with naive bayes, our accuracy on test set is higher. At the same time, if we spend sometime tuning the hyperparameter stepsize, we can increase our accuracy. Meanwhile, the average log likelihood is much lower than naive bayes classifier.

```
In [18]: def logistic_predict(x, w):
             probability = []
             bot = 0
             for c in range(len(w)):
                 bot += np.exp(np.dot(w[c].T, x))
             for c in range(len(w)):
                 probability.append(np.exp(np.dot(w[c].T, x)) / bot)
             return np.argmax(probability)

         w, logistic_like = gradient_ascent(x_train, y_train, 0.00006, 100)
         ave_logistic_like_train = np.sum(logistic_like) / len(x_train)

         # average log likelihood on test set
         ave_logistic_like_test = []
         for n in range(len(x_test)):
             c = np.argmax(y_test[n])
             ave_logistic_like_test = logistic_likelihood(x_test[n], c, w)
         ave_logistic_like_test = np.sum(ave_logistic_like_test) / len(x_test)

         # accuracy on test set
         l_prediction = []
         for n in range(len(x_test)):
             l_prediction.append(logistic_predict(x_test[n], w))
         l_accuracy = 0
         for i in range(len(l_prediction)):
             if l_prediction[i] == np.argmax(y_test[i]):
                 l_accuracy += 1
         l_accuracy = l_accuracy/len(l_prediction)
```

```
    # accuracy on training set
    lt_prediction = []
    for n in range(len(x_train)):
        lt_prediction.append(logistic_predict(x_train[n], w))
    lt_accuracy = 0
    for i in range(len(lt_prediction)):
        if lt_prediction[i] == np.argmax(y_train[i]):
            lt_accuracy += 1
    lt_accuracy = lt_accuracy/len(lt_prediction)
    print("The average likelihood on training set is %s, on test set is %s."%\
          (np.round(ave_logistic_like_train, 2), ave_logistic_like_test))
    print("The accuracy on test set is %s, on training set is %s."%\
          (np.round(l_accuracy, 2), np.round(lt_accuracy, 2)))
```

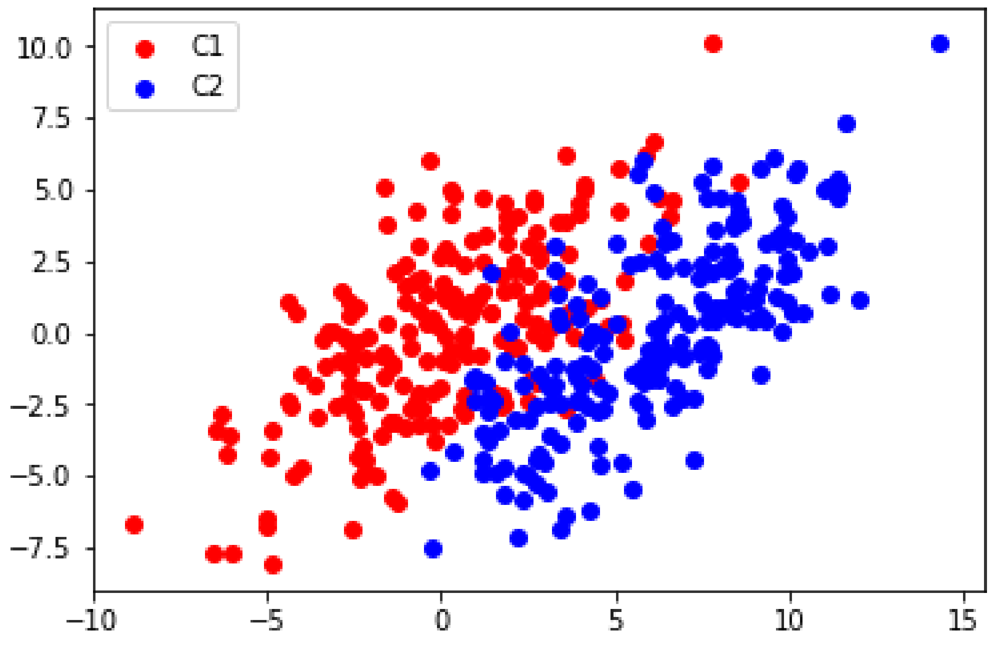
```
The average likelihood on training set is -0.62, on test set is -2.1299229047144762e-14.
The accuracy on test set is 0.86, on training set is 0.86.
```

Q4

a).

```
In [8]: mu1 = np.array([0.1, 0.1])
        mu2 = np.array([6, 0.1])
        sigma = np.array([10, 7, 7, 10]).reshape(2, 2)
        c1 = np.random.multivariate_normal(mu1, sigma, 200)
        c2 = np.random.multivariate_normal(mu2, sigma, 200)
        plt.figure()
        plt.scatter(c1[:, 0], c1[:, 1], c='red', label='C1')
        plt.scatter(c2[:, 0], c2[:, 1], c='blue', label='C2')
        plt.legend()
```

```
Out[8]: <matplotlib.legend.Legend at 0x14ed7ea0f28>
```



b).

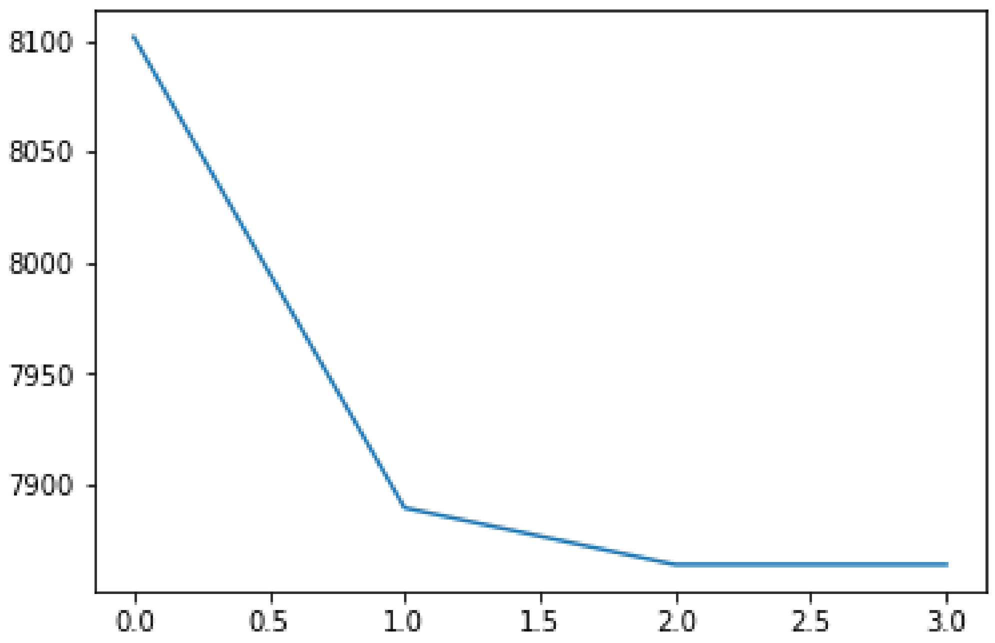
```
In [9]: def cost(mu, x, r):
            c = 0
            for n in range(0, len(x)):
                for k in range(0, len(mu)):
                    c += r[n][k]*(np.linalg.norm(x[n] - mu[k]))**2
            return c


        def km_e_step(x, mu, r):
            for n in range(0, len(x)):
                distance = []
                for k in range(0, len(mu)):
                    distance.append(np.linalg.norm(x[n] - mu[k])**2)
                closet = np.argmin(distance)
                for k in range(0, len(mu)):
                    if k == closet:
                        r[n][k] = 1
                    else:
                        r[n][k] = 0
            return r


        def km_m_step(x, r, mu):
```

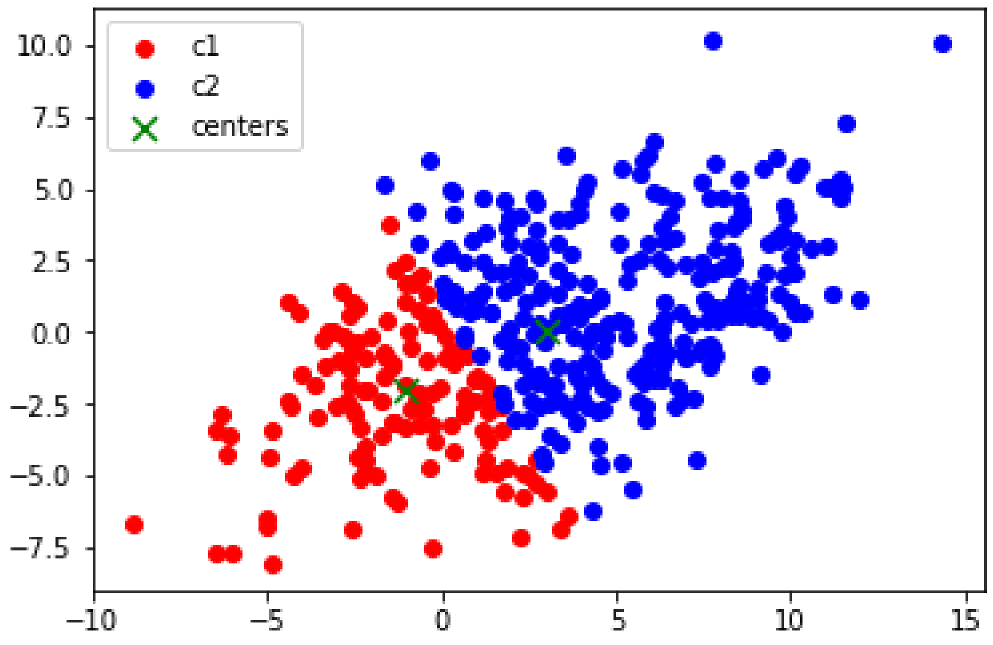
```
    top = np.zeros(mu.shape[1])
    bot = 0
    for k in range(0, len(mu)):
        for n in range(0, len(x)):
            top += r[n][k] * x[n]
            bot += r[n][k]
        mu[k] = top/bot
    return mu


def k_mean(k, x, mu, iterations, cr):
    r_init = np.zeros(len(x)*k).reshape(len(x), k)
    mu_init = mu
    diff = 100
    iteration = 1
    cost_list = []
    while (iteration > 0) & (diff > cr):
        if iteration == 1:
            iteration = iterations
            new_r = km_e_step(x, mu_init, r_init)
            new_mu = km_m_step(x, r_init, mu_init)
            this_cost = cost(new_mu, x, new_r)
            previous_cost = this_cost
            diff = previous_cost
        else:
            new_r = km_e_step(x, new_mu, new_r)
            new_mu = km_m_step(x, new_r, new_mu)
            previous_cost = this_cost
            this_cost = cost(new_mu, x, new_r)
            diff = previous_cost - this_cost
        cost_list.append(this_cost)
        iteration -= 1
    return cost_list, new_r, new_mu


# c_1 is the class_one predicted
# c_2 is the class_two predicted
# t_1 is the true class_one
# t_2 is the true class_two
def misclassification_rate(c_1, c_2, t_1, t_2):
    misclassification = 0
    for i in c_1:
        if i in t_2:
            misclassification += 1
    for i in c_2:
        if i in t_1:
            misclassification += 1
    misclassification_rate = misclassification/(len(c_1) + len(c_2))
```

```python
    return misclassification_rate

# assign class to EM algorithm or K-mean
def assign_class(x, result, em = False):
    if em:
        responsibility = result[1]
        log_lik = result[0]
        plt.figure()
        plt.plot(log_lik)
        em_class_one = np.array([])
        em_class_two = np.array([])
        # assign points to the class with larger responsibility
        for i in range(0, len(x)):
            if responsibility[i][0] >= responsibility[i][1]:
                em_class_one = np.concatenate((em_class_one, x[i]))
            else:
                em_class_two = np.concatenate((em_class_two, x[i]))

        em_class_one = em_class_one.reshape(len(em_class_one)//2, 2)
        em_class_two = em_class_two.reshape(len(em_class_two)//2, 2)
        plt.figure()
        plt.scatter(em_class_one[:, 0], em_class_one[:, 1], c="red", label="c1")
        plt.scatter(em_class_two[:, 0], em_class_two[:, 1], c="blue", label="c2")
        plt.legend()
        return em_class_one, em_class_two
    else:
        plt.figure()
        total_cost = result[0]
        plt.plot(total_cost)
        plt.figure()
        r = result[1]
        new_mu = result[2]
        class_1 = np.array([])
        class_2 = np.array([])
        for i in range(0, r.shape[0]):
            if r[i][0] == 1:
                class_1 = np.concatenate((class_1, x[i]))
            else:
                class_2 = np.concatenate((class_2, x[i]))
        class_1 = class_1.reshape(len(class_1)//2, 2)
        class_2 = class_2.reshape(len(class_2)//2, 2)
        plt.scatter(class_1[:, 0], class_1[:, 1], c="red", label="c1")
        plt.scatter(class_2[:, 0], class_2[:, 1], c="blue", label="c2")
        plt.scatter(new_mu[:, 0], new_mu[:, 1], c="green", marker='x',
                    label="centers", s=80)
        plt.legend()
        return class_1, class_2
```

```
init_mu1 = np.array([0, 0])
init_mu2 = np.array([1, 1])
x = np.concatenate((c1, c2))
mu = np.concatenate((init_mu1, init_mu2)).reshape(2, 2)
result = k_mean(2, x, mu, 20, 0.0001)
class_1, class_2 = assign_class(x, result)
k_mean_mr = misclassification_rate(class_1, class_2, c1, c2)
print("k mean misclassification rate is %s"%(k_mean_mr))
```

```
k mean misclassification rate is 0.31
```





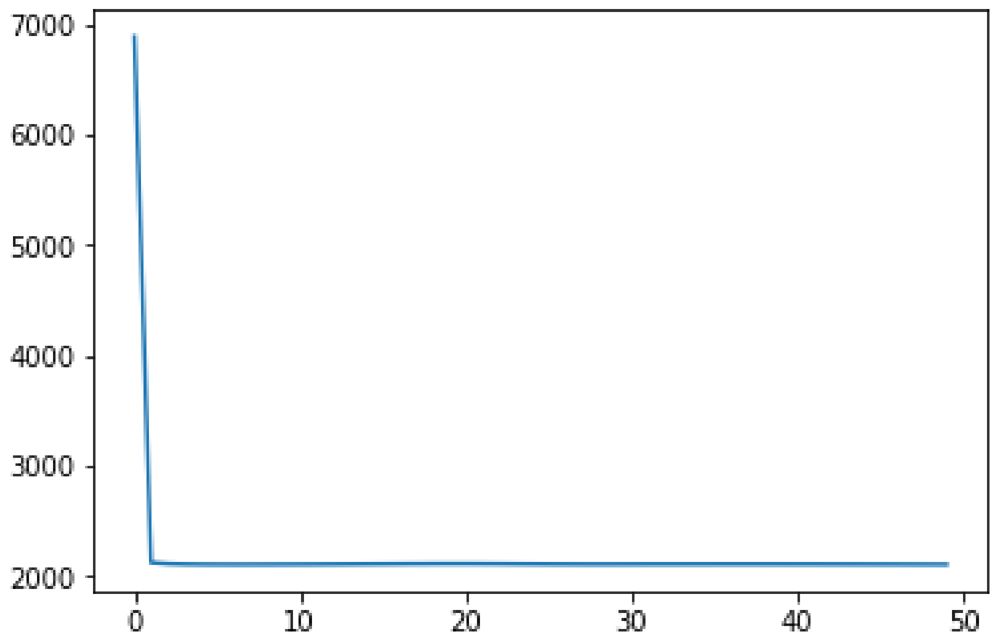
c).

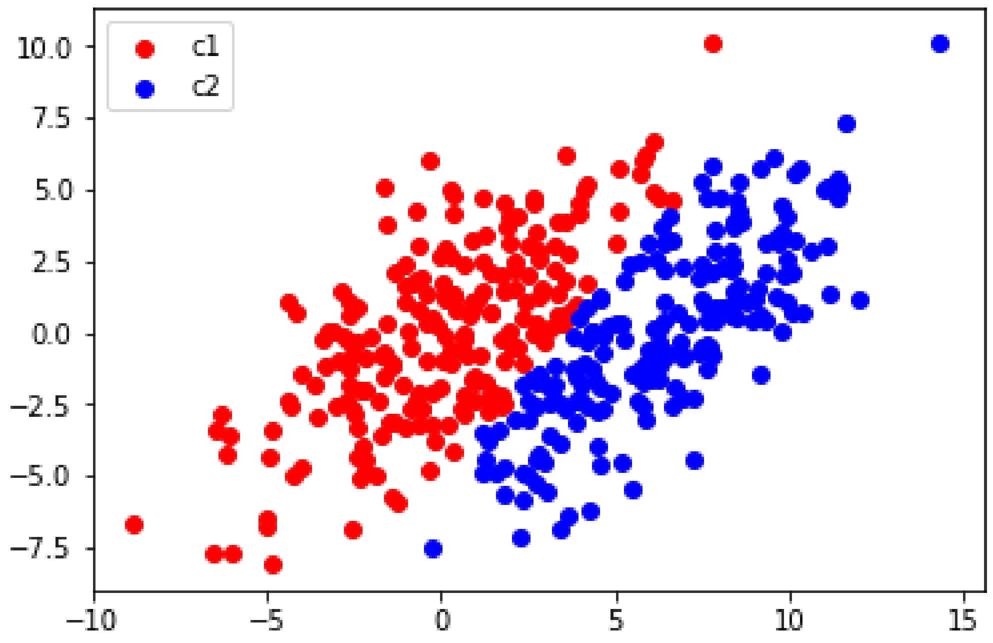
```
In [10]: def normal_density(x, mu, sigma):
             return multivariate_normal.pdf(x, mu, sigma)


         # the input sigma is a list
         def log_likelihood(x, pi, mu, sigma):
             result = 0
             for i in range(0, len(x)):
                 loss = 0
                 for k in range(0, len(mu)):
                     loss += pi[k] * normal_density(x[i], mu[k], sigma[k])
                 result += np.log(loss)
             return result


         def em_e_step(x, mu, sigma, pi, bot):
             top = pi * normal_density(x, mu, sigma)
             return top/bot

         # the input gamma = Gamma[:,k]
         def em_m_step(x, gamma):
             n_k = np.sum(gamma)
             mu_k = 0
```

```
    sigma_k = 0
    for n in range(0, len(gamma)):
        mu_k += gamma[n] * x[n]
    mu_k = mu_k / n_k
    for n in range(0, len(gamma)):
        ys = (x[n] - mu_k).reshape(2,1)
        sigma_k += gamma[n] * np.dot(ys, ys.T)
    sigma_k = sigma_k / n_k
    pi_k = n_k/ len(x)
    return mu_k, sigma_k, pi_k



# the input sigma is a list, cr is a small number
# the input mu is a list
# the input pi is list

def em_algo(x, mu, sigma, pi, iteration, cr):
    diff = 100
    gamma = np.zeros((len(x), len(pi)))
    loglik = []
    new_mu = mu
    new_sigma = sigma
    new_pi = pi
    old_loglik = log_likelihood(x, new_pi, new_mu, new_sigma)
    while (iteration > 0) & (diff > cr):
        # E-Step
        for n in range(0, len(x)):
            bot = 0
            for k in range(0, len(pi)):
                bot += pi[k] * normal_density(x[n], new_mu[k],
                            new_sigma[k])

            for k in range(0, len(pi)):
                gamma[n][k] = em_e_step(x[n], new_mu[k],
                        new_sigma[k], new_pi[k], bot)
        new_mu = []
        new_pi = []
        new_sigma = []
        # M-setp
        for k in range(0, len(pi)):
            result_k = em_m_step(x, gamma[:, k])
            new_mu.append(result_k[0])
            new_pi.append(result_k[2])
            new_sigma.append(result_k[1])

        this_loglik = log_likelihood(x, new_pi, new_mu, new_sigma)
        # calculate the log-likelihood and check convergence
```

```
            diff = np.abs(old_loglik - this_loglik)
            loglik.append(-old_loglik)
            iteration -= 1
            old_loglik = this_loglik
        return (loglik, gamma)

    combined_mu = [init_mu1, init_mu2]
    covariance = [np.array([[1,0], [0, 1]]), np.array([[1,0], [0, 1]])]
    pi = [0.5, 0.5]
    em_result = em_algo(x, combined_mu, covariance, pi, 50, 0.01)
    em_class_one, em_class_two = assign_class(x, em_result, em=True)
    em_mr = misclassification_rate(em_class_one, em_class_two, c1, c2)
    print("K mean misclassification rate is %s, EM misclassification rate is %s"\
          %(k_mean_mr, em_mr))
```

```
K mean misclassification rate is 0.31, EM misclassification rate is 0.1025
```



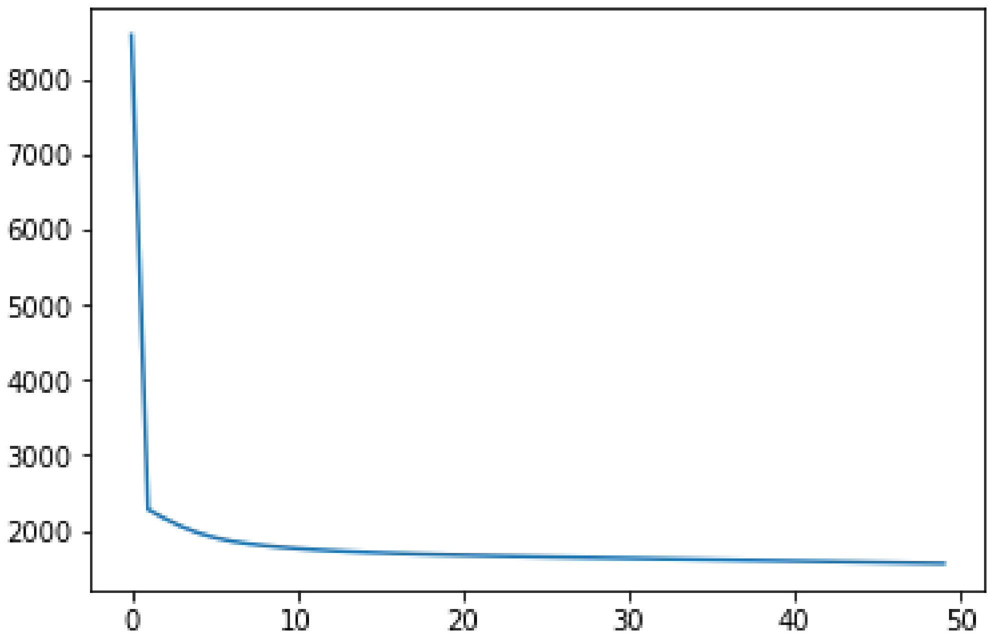


d).

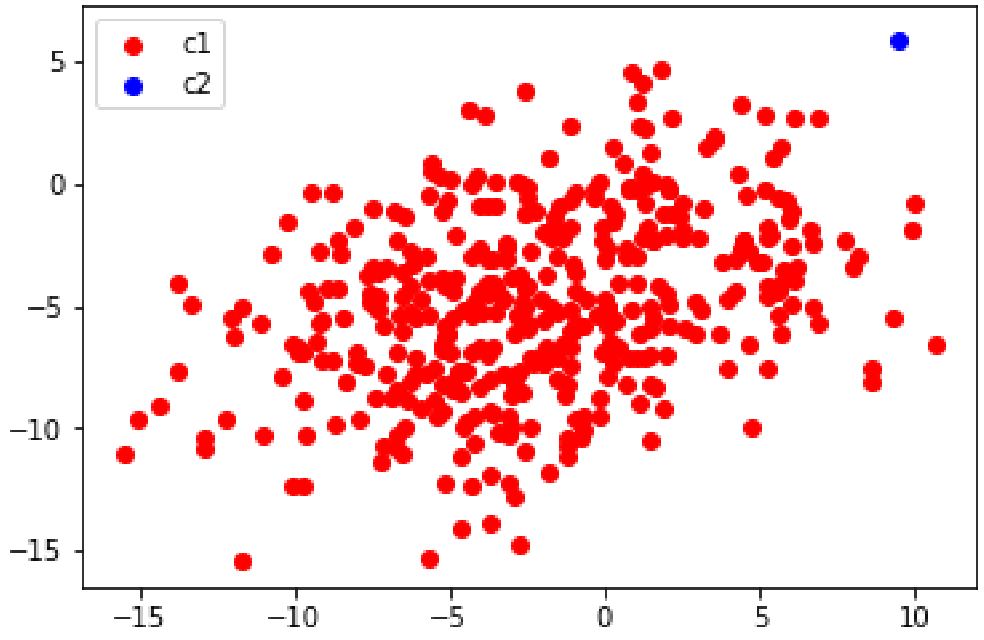
From the above graphs we can see that, k-mean converges faster thanEM algorithm, but EM algorithm performs much better for this data.

```
In [13]: #4 d).
         # From the above graphs we can see that, k-mean converges faster than
         # EM algorithm, but EM algorithm performs much better for this data.
         # Try data with different starting point

         realization_k = []
         realization_e = []
         init_mu1 = np.array([-5, -5])
         init_mu2 = np.array([10, 5])
         em_mu = [init_mu1, init_mu2]
         k_mu = np.concatenate((init_mu1, init_mu2)).reshape(2,2)
         k_result = k_mean(2, x, k_mu, 20, 0.000001)
         em_result = em_algo(x, em_mu, covariance, pi, 50, 0.1)
         em_class_one, em_class_two = assign_class(x, em_result, em=True)
         k_class_one, k_class_two = assign_class(x, k_result)
         realization_k.append(misclassification_rate(k_class_one,\
                                                     k_class_two, c1, c2))
         realization_e.append(misclassification_rate(em_class_one,\
                                                     em_class_two, c1, c2))
         # new data
         mu = np.random.random(2)
```

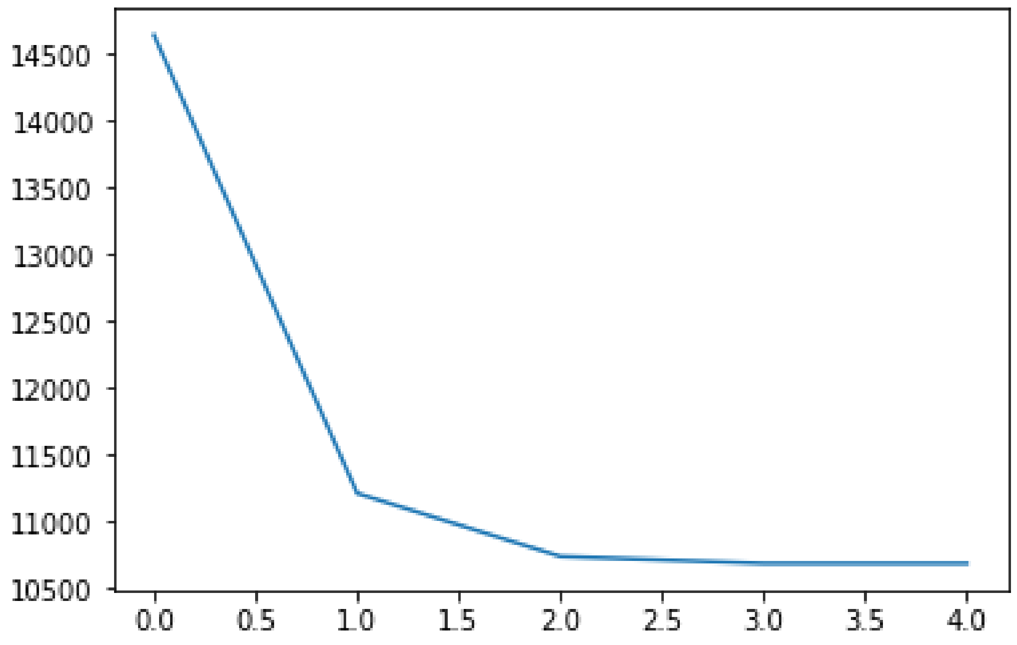
```
c1 = np.random.multivariate_normal(mu, sigma, 200)
c1 = np.random.multivariate_normal(mu, sigma, 200)
x = np.concatenate((c1, c2))
k_result = k_mean(2, x, k_mu, 20, 0.000001)
em_result = em_algo(x, em_mu, covariance, pi, 50, 0.01)
em_class_one, em_class_two = assign_class(x, em_result, em=True)
k_class_one, k_class_two = assign_class(x, k_result)
realization_k.append(misclassification_rate(k_class_one,\
                                            k_class_two, c1, c2))
realization_e.append(misclassification_rate(em_class_one,\
                                            em_class_two, c1, c2))

# new data + some random noise [5, 0)
for i in range(len(c1)):
    c1[i] = c1[i] + 5 * np.random.random_sample(2) - 5
for i in range(len(c2)):
    c2[i] = c2[i] + 5 * np.random.random_sample(2) - 5
x = np.concatenate((c1, c2))

k_result = k_mean(2, x, k_mu, 20, 0.000001)
em_result = em_algo(x, em_mu, covariance, pi, 50, 0.01)
em_class_one, em_class_two = assign_class(x, em_result, em=True)
k_class_one, k_class_two = assign_class(x, k_result)
realization_k.append(misclassification_rate(k_class_one,\
                                            k_class_two, c1, c2))
realization_e.append(misclassification_rate(em_class_one,\
                                            em_class_two, c1, c2))

# new data + some random noise [5, 0)
for i in range(len(c1)):
    c1[i] = c1[i] + 2 * np.random.random_sample(2) - 2
for i in range(len(c2)):
    c2[i] = c2[i] + 2 * np.random.random_sample(2) - 2
x = np.concatenate((c1, c2))

k_result = k_mean(2, x, k_mu, 20, 0.000001)
em_result = em_algo(x, em_mu, covariance, pi, 50, 0.01)
em_class_one, em_class_two = assign_class(x, em_result, em=True)
k_class_one, k_class_two = assign_class(x, k_result)
realization_k.append(misclassification_rate(k_class_one,\
                                            k_class_two, c1, c2))
realization_e.append(misclassification_rate(em_class_one,\
                                            em_class_two, c1, c2))
```

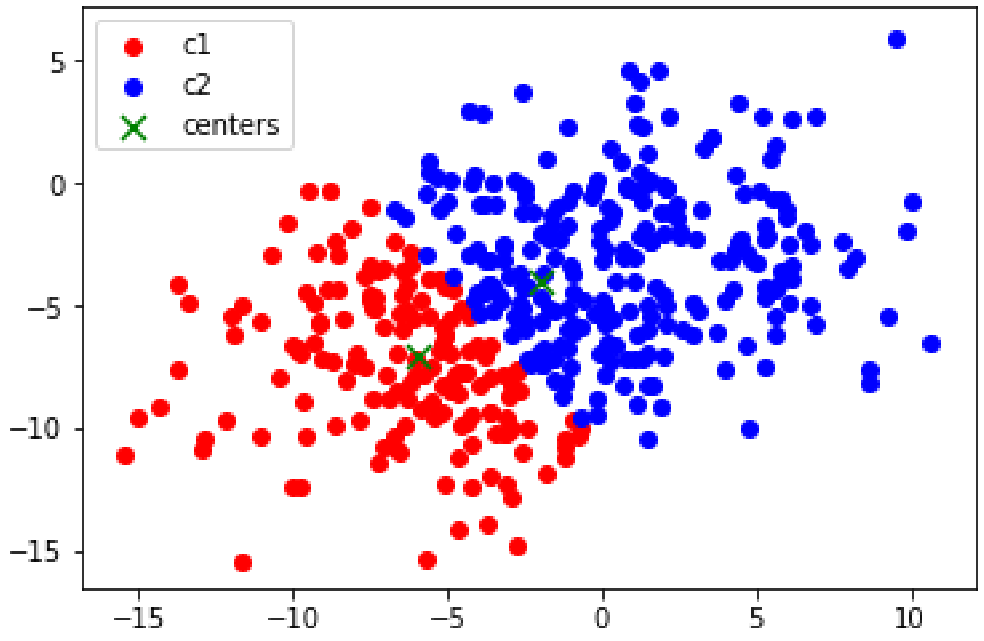


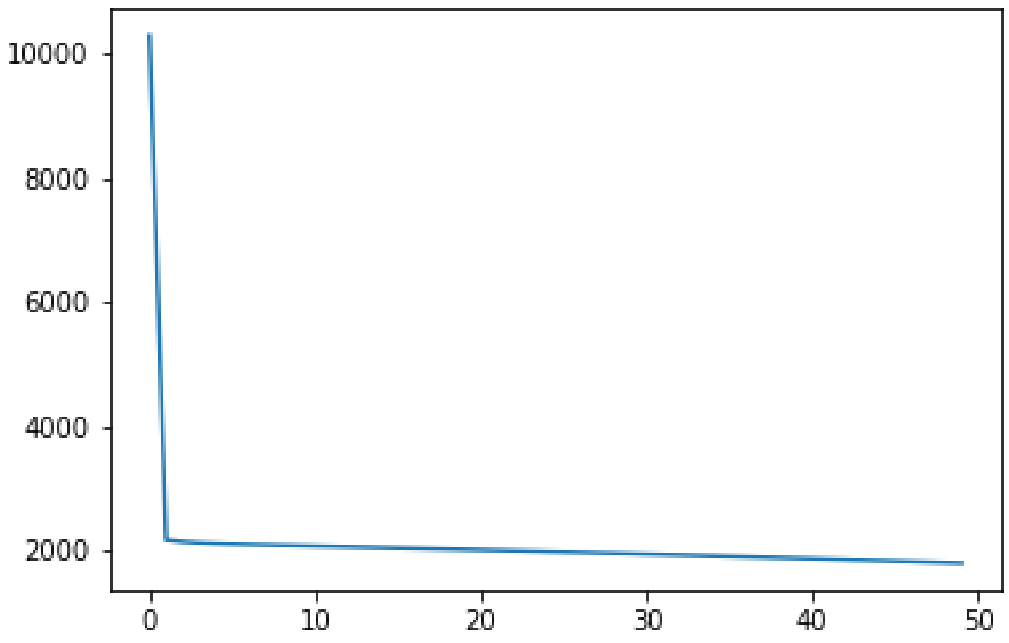
8000
7000
6000
5000
4000
3000
2000
0
10
20
30
40
50

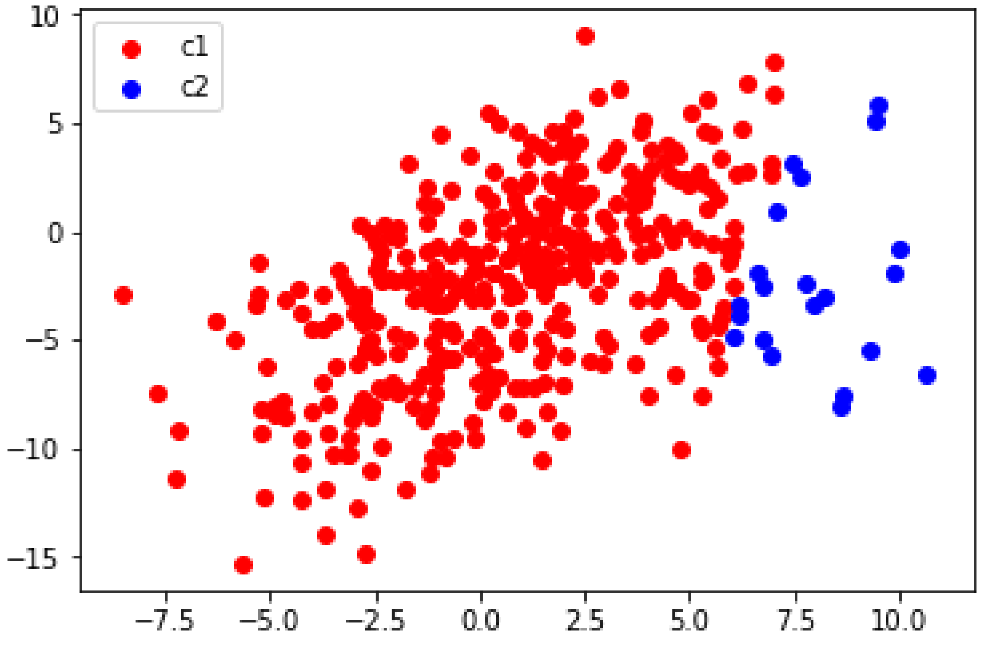


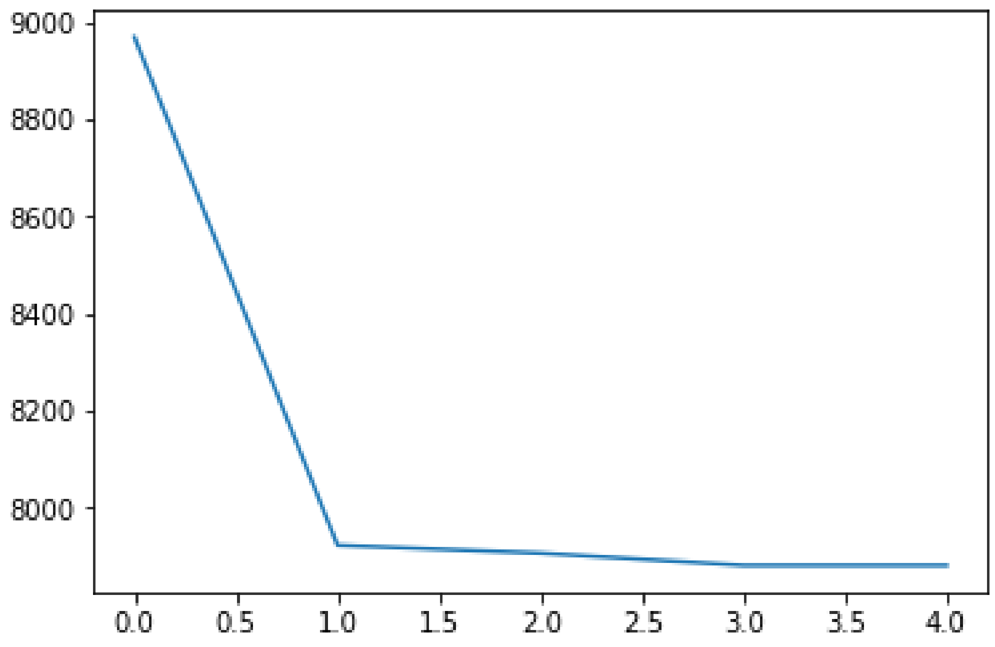
c1
c2
5
0
−5
−10
−15
−15
−10
−5
0
5
10


14500
14000
13500
13000
12500
12000
11500
11000
10500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

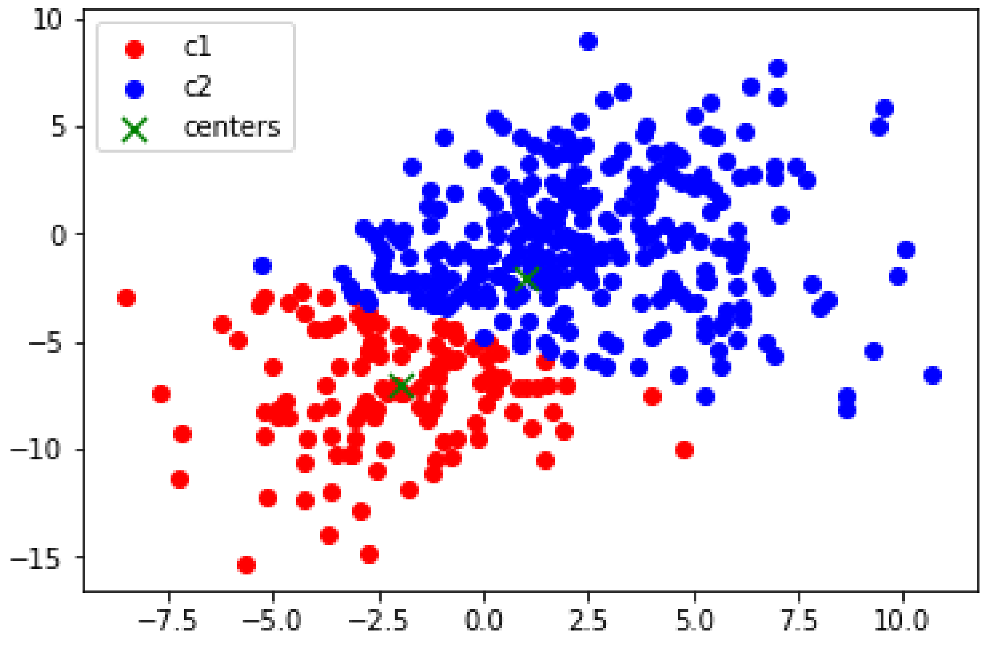

c1
c2
centers
5
0
−5
−10
−15
−15
−10
−5
0
5
10


10000
8000
6000
4000
2000
0
10
20
30
40
50

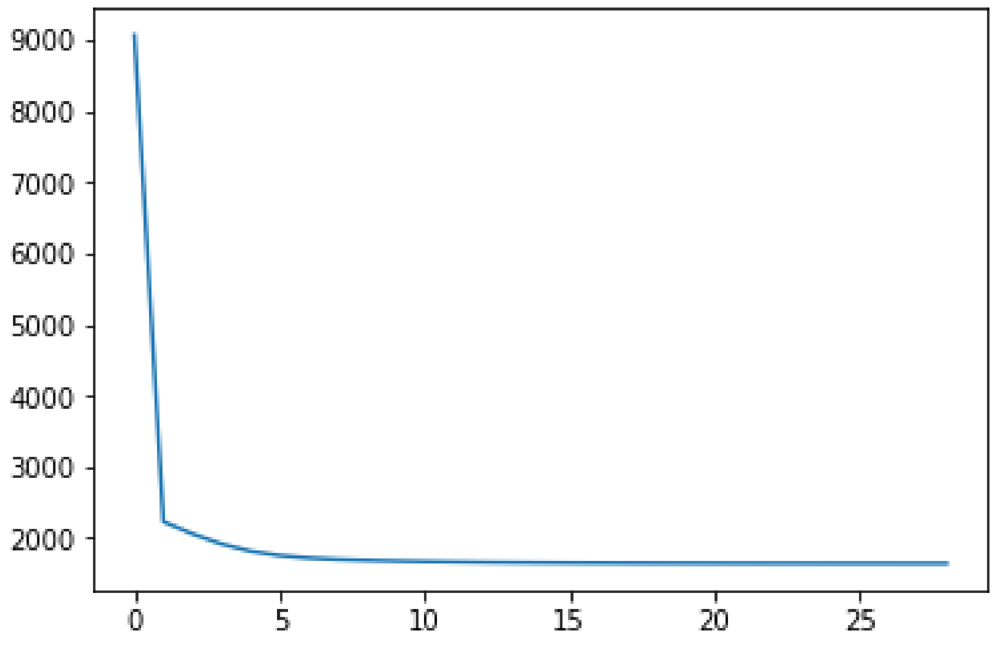

c1
c2
10
5
0
−5
−10
−15
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

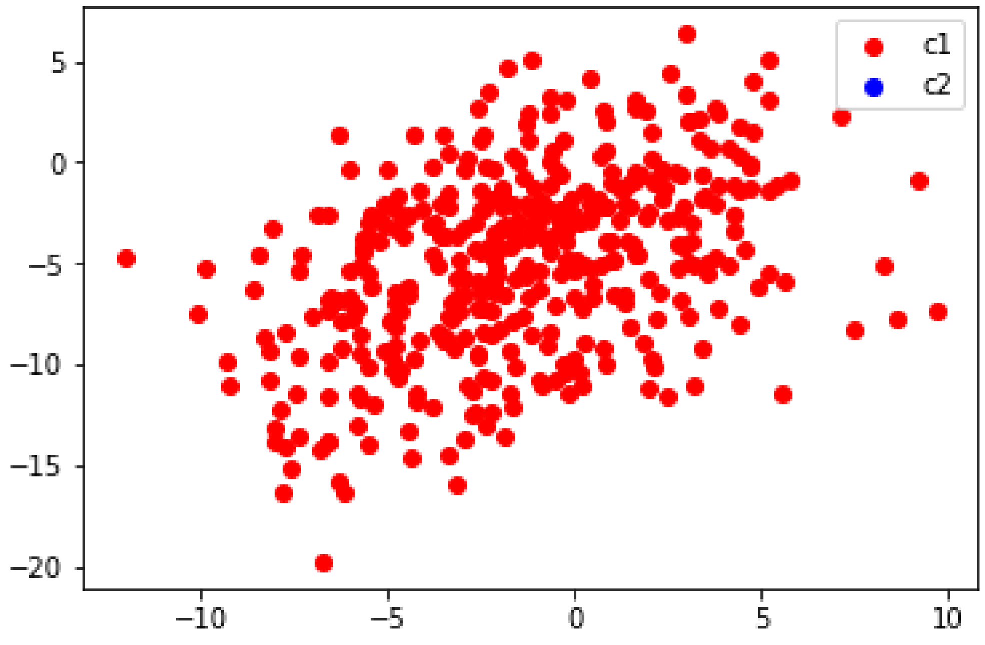


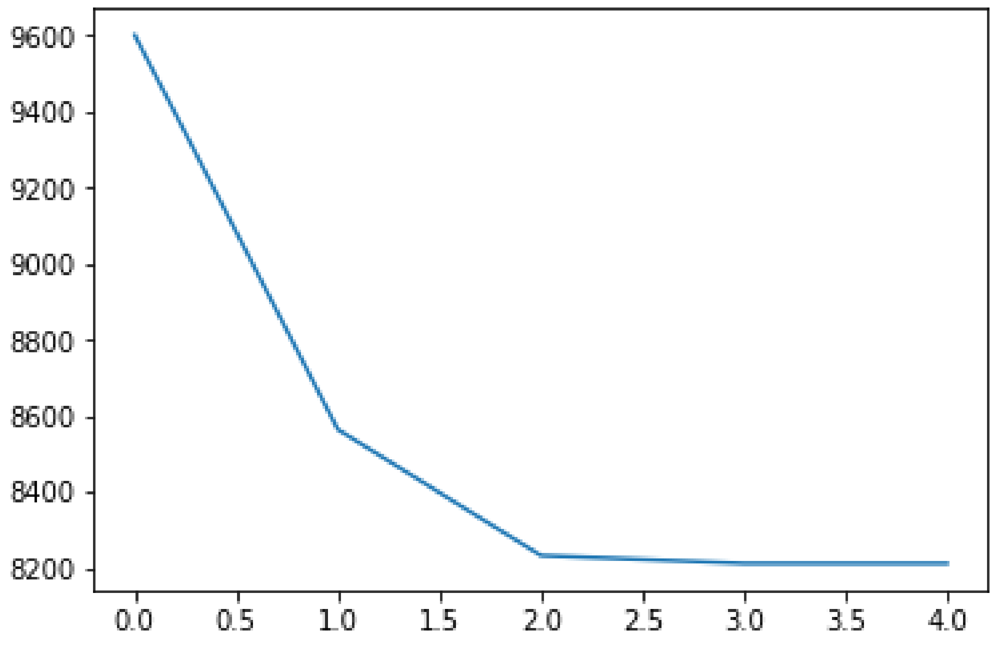
9000
8800
8600
8400
8200
8000
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

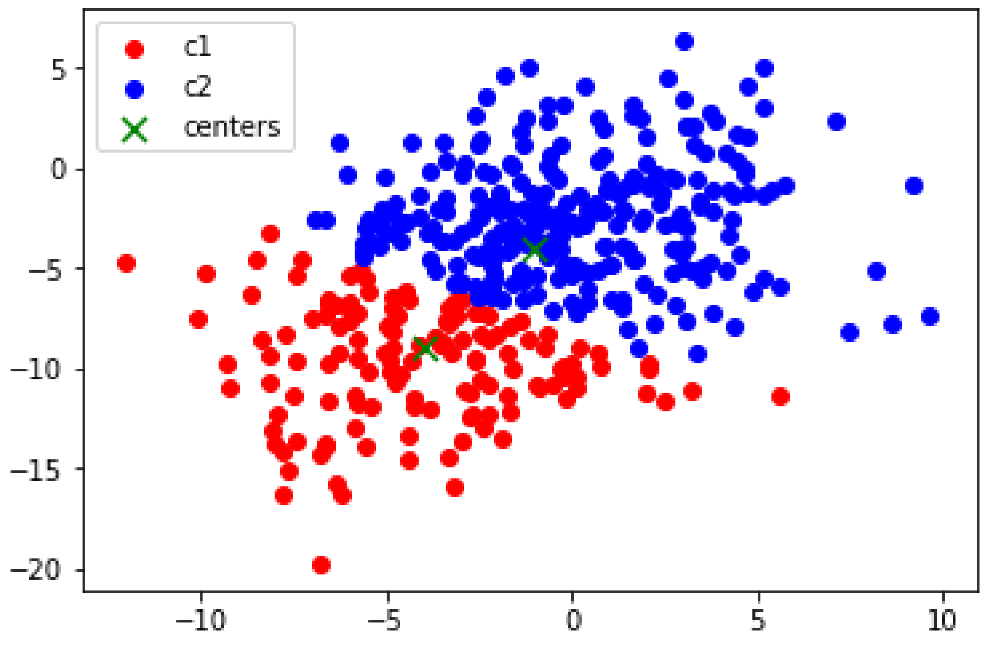


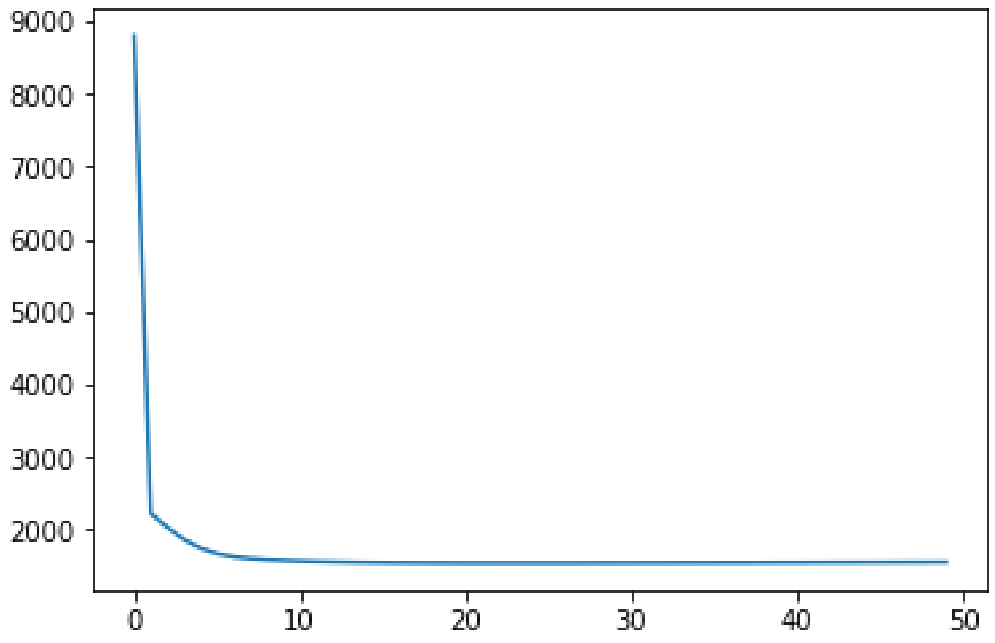
c1
c2
centers
10
5
0
−5
−10
−15
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

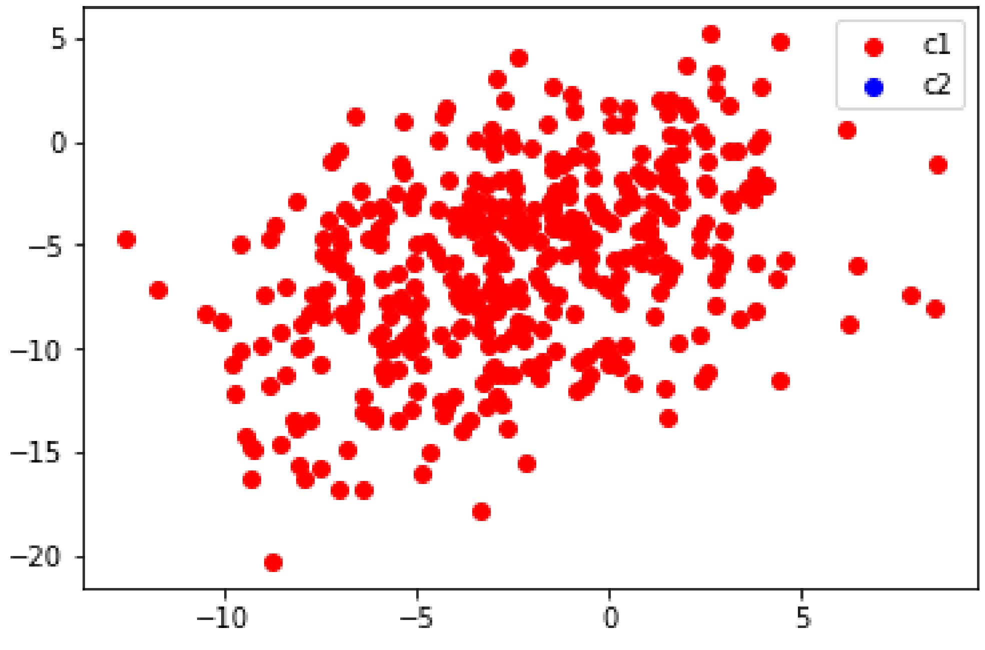

9000
8000
7000
6000
5000
4000
3000
2000
0
5
10
15
20
25

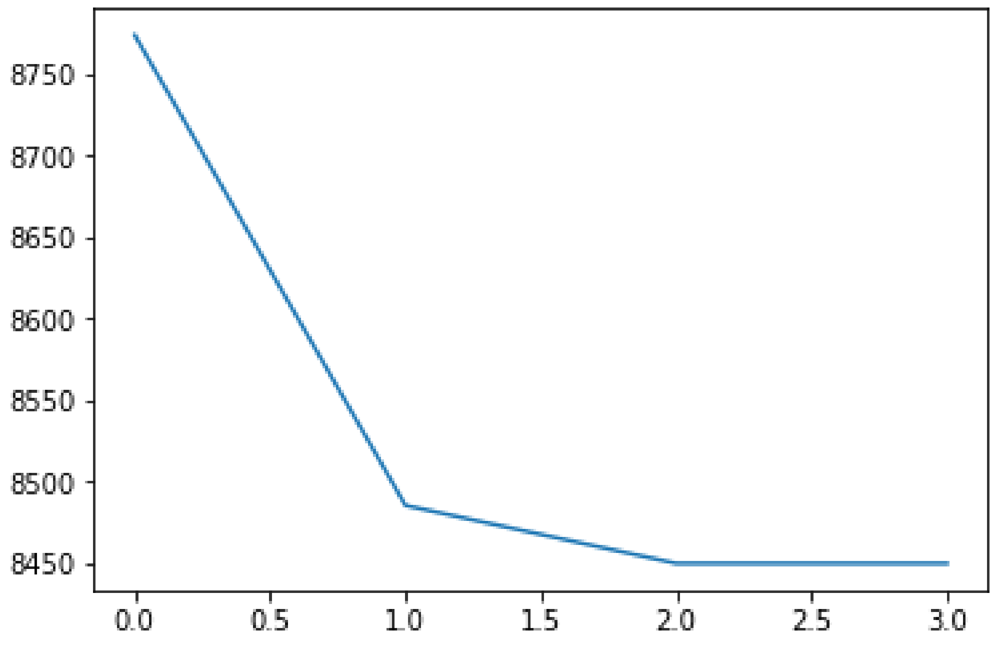

c1
c2
5
0
−5
−10
−15
−20
−10
−5
0
5
10

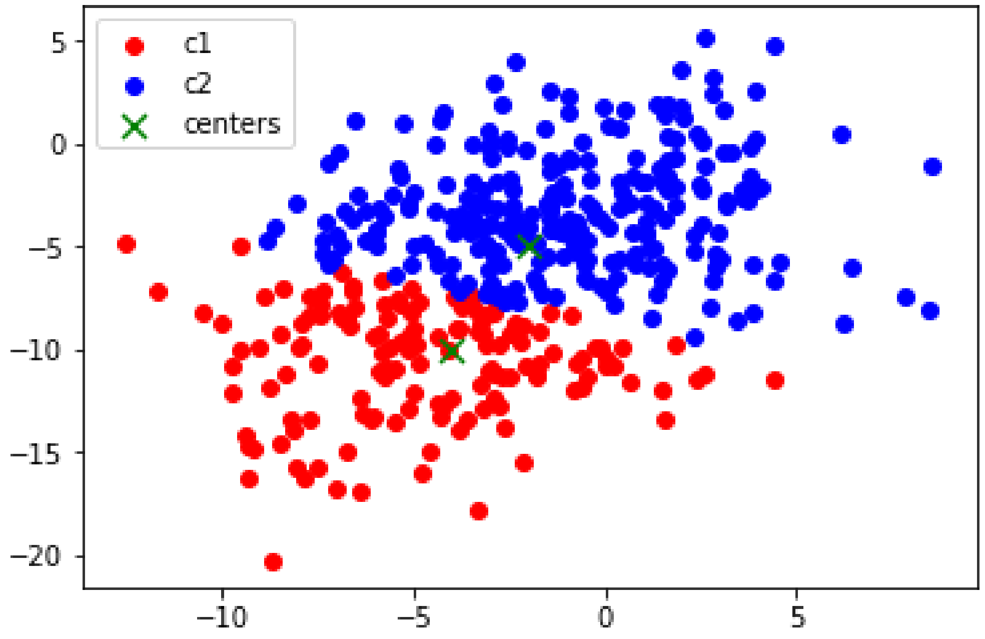

9600
9400
9200
9000
8800
8600
8400
8200
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0


c1
c2
centers
5
0
−5
−10
−15
−20
−10
−5
0
5
10


9000
8000
7000
6000
5000
4000
3000
2000
0
10
20
30
40
50

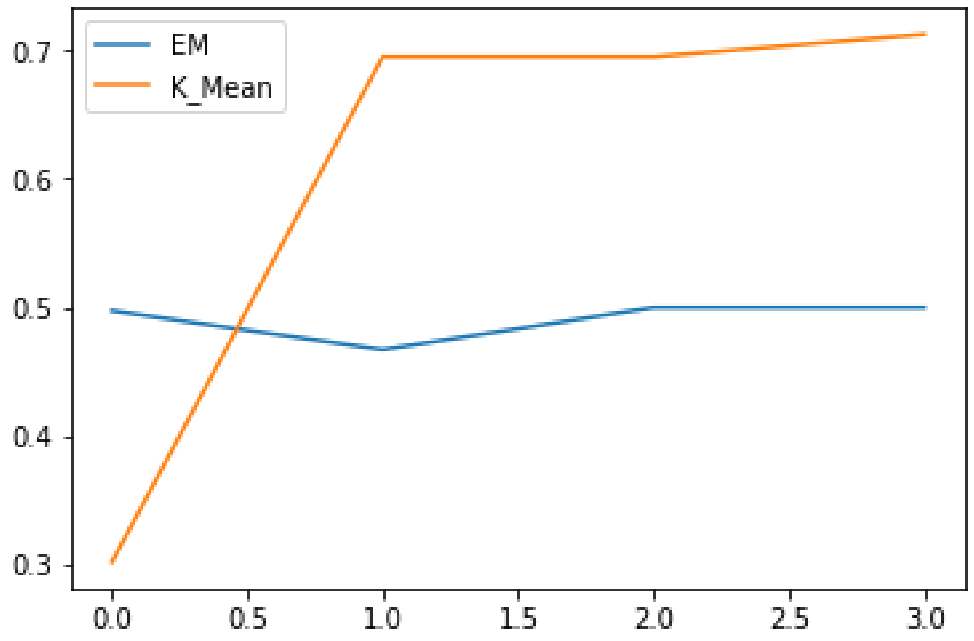

c1
c2
5
0
−5
−10
−15
−20
−10
−5
0
5





```
In [15]: plt.figure()
         plt.plot(realization_e, label="EM")
         plt.plot(realization_k, label="K_Mean")
         plt.legend()
```

```
Out[15]: <matplotlib.legend.Legend at 0x14ed7c50e10>
```



From our realization of different data, we can see that EM algorithm performs best when we know the data is from gaussian distribution. It may perform terrible in some senarios, but overall it performs better than K-mean algorithms.