APS 1022

Financial Engineering II

Option Value Project

by Group 7

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Executive Summary

Options are financial contracts that provide investors the rights to buy or sell an underlying asset at a predefined price after a specified period of time. There exist numerous methods to price options. Common approaches include the Monte Carlo Simulation Method, the Lattice Method, the Finite Difference Method, the Closed Form for analytical model method, etc. In this report, the team will specifically focus on the Monte Carlo Simulation Method Lattice Method and use these two methods to price different types of options.

The types of options that are explored to compare the two pricing methods above are Asian Options, Exotic Options like lookback and floating lookback option and American put options.

Monte Carlo Method(MC) is based on the risk neutral valuation and the risk neutrality. Firstly, a large number of random scenarios for the price of the underlying asset will be simulated. Then, the associated payoff for each scenario will be computed. The price of the option is calculated by taking the discounted average payoff of all the simulated random paths. The sample size of the Monte Carlo simulation is set to be 10,000. The sunykatuib has a unit time of 1 week. The maturity of the option is two months. The current price of the underlying asset is $100 and the volatility of the asset is 25%. The risk-free rate used in this implementation is 2%.

Compared to Monte Carlo simulation, the Binomial Lattice(BL) approach requires less effort and computation, in which it can build the binomial tree(up to 256 paths) for the stock price and then calculate the payoffs for each stock. The value of option is the one that is at time 0 by discounting the forward payoffs.

After the implementation of both methods, we find that option prices derived from Monte Carlo Simulation and Binomial Lattice methods do not differ significantly. Both Monte Carlo simulation and Lattice are reliable option pricing models.

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1. Introduction

Options are a type of financial derivative product that allows investors of the product to buy or sell a security at a chosen price. Options are divided into a call or put option. A call option allows the holder to buy a given asset at a stated strike price within a certain timeframe. In contrast, a put option allows the holder to sell a given asset at a stated strike price within a certain timeframe.

The objective of this project is to price the following three styles of option with the Monte Carlo Simulation and Binomial Lattice approach.

1. Asian Option: an option type where the payoff depends on the average price of an underlying asset over a specific time frame stated in the contract.
2. Exotic Option: an option type that differs from traditional option (i.e. European, Asian) in the payoff structures and strike prices. In this project, we will be focusing on the Lookback option and Floating Lookback option [1].
3. American Option: a special option type that allows the holder to exercise the option rights at any time before and during the day of expiration. Since the contract allows the holder to exercise the rights at any time point during the life of the contract, American option carries an added premium cost [2].

The parameters that will be used to implement the two approaches are listed in *Table 1*.

Table 1. *Parameters*

Parameters	Value
Risk-free Rate (r)	2%
Current Price (S_0)	\$100
Strike Price (K)	\$105
Volatility(σ)	25%
Maturity Time (T)	2 months ≃ 8 weeks ≃ ⅙ year
Simulation Period (dT)	1 week ≃ 1/48 year
Number of Scenarios (numPaths)	10,000
Number of Steps (numSteps)	8

2. Option Payoff Structure

The option payoff for Asian and Exotic options derived from Monte Carlo Simulation and Binomial Lattice method are demonstrated in the following section.

1) Asian Option
   For the Asian option, payoff is determined by the arithmetic average of the underlying stock price over the time period. For the Asian call, the payoff of each path C_i is the difference between the average price over the path and strike price \$105. If the average stock price in any path falls below strike price \$105, the payoff of this path turns out to be zero. The Asian put works as the opposite.

Asian call payoff in i^{th}path = max (average price ini^{th}path - strike price \$105, 0)

Asian put payoff in i^{th}path = max (strike price \$105 - average price ini^{th}path, 0)

2) Lookback Option
   For the Lookback option, payoff is determined by the maximum or minimum underlying stock price throughout the life of the option. The payoff of Lookback call C_i is the difference between maximum value in each path and the strike price \$105, while the payoff of Lookback put C_i is the difference between strike price and minimum value in each path. If the difference is smaller than or equal to zero, the payoff of this path is zero.

Lookback call payoff in i^{th}path = max (highest price ini^{th}path - strike price \$105, 0)

Lookback put payoff in i^{th}path = max (strike price \$105 - lowest price ini^{th}path, 0)

3) Floating Lookback
   For the Floating Lookback option, the strike price is set to be the optimal value (minimum or maximum value) achieved by the underlying stock price over the option's lifespan. The payoff of Floating Lookback call C_i is the difference between the final price at maturity time and the minimum value in each path, while the payoff of Floating Lookback put C_i is the difference between the maximum value and the final price at maturity time in each path. The payoff of the path would become zero if the difference is smaller than or equal to zero.

Floating Lookback call payoff in i^{th}path = max ((final price - lowest price) ini^{th}path, 0)

Floating Lookback put payoff in i^{th}path = max ((highest price - final price) ini^{th}path, 0)

Finally, the option price is calculated by converting the expected payoff at the maturity time to the present value just as the formula shown below.

$$C = e^{-r*T} * \left(\sum_{i=1}^{256} C_i * P_i\right)$$

where C_i stands for the payoff in i^{th} path and

P_i is the probability of i^{th} path occuring

A summary of the payoff structure for Asian options and exotic options is shown in *Table 2*.

Table 2. *Asian and Exotic Option Payoff Structure*

Option	Payoff
Asian Call	$Max\{\overline{S} - K,\ 0\}$
Asian Put	$Max\{K - \overline{S},\ 0\}$
Lookback Call	$Max\{S_{max} - K,\ 0\}$
Lookback Put	$Max\{K - S_{min},\ 0\}$
Floating Lookback Call	$Max\{S_T - S_{min},\ 0\}$
Floating Lookback Put	$Max\{S_{max} - S_T,\ 0\}$

where

$\overline{S}$ = Average stock price over the period of T
S_{max} = Highest stock price over the period of T
S_{min} = Lowest stock price over the period of T
S_T = Stock price at maturity time T

3. Monte Carlo Simulation

Monte Carlo simulation is a mathematical model used to estimate the probability of different outcomes of an uncertain event which can be used to price options. The method is based on repeated computation and random sampling by Monte Carlo using Geometric Brownian Motion (GBM) with a constant risk neutral drift of *r* and volatility σ. The Geometric Random Walk equation, discretized GBM, is given by:

$$S_{t+1} = S_t e^{\left(r - \frac{\sigma^2}{2}\right)dT + \epsilon_t},\ \epsilon_t \sim N(0, 1)\ t \in \{0, 1 ... T\}$$ ***Eqn (1)***

3.1 Methodologies

The algorithms for different types of options will be discussed in the following sections.

3.1.1 Asian and Exotic Options

The general algorithm of estimating the price for Asian and exotic options are as follow:

1. For each path $i = 1,..., numPath$
   a) For each time step $j = 1,..., numSteps$ of the path i
      i. Simulate the stock price S_t using ***Eqn(1)***
2. Compute $\overline{S}$, S_{max} , S_{min} , S_T.
3. Compute the average option price $\overline{p} = e^{rT} \times \frac{1}{numPath} \sum p_i$ and standard deviation of option price $std(p) = \sqrt{Var(Option\ Price)}$ $where\ var(p) = \frac{\sum_{i=1}^{numPath} (p_i - \overline{p})^2}{numPath}$

3.1.2 American Put Option

Due to the flexible exercise time, an added premium cost is introduced in American put option. The formula of the premium is given by:

$$E[S_T|S_t] = Ke^{-r*(T-t)}N(-d_2) - S_tN(-d_1) \qquad \textit{Eqn(2)}\ [3]$$

where

$$d_1 = \frac{log(\frac{S_t}{K})+(r+\frac{\sigma^2}{2})(T=t)}{\sigma\sqrt{(T-t)}} \qquad \textit{Eqn (3)}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} \qquad \textit{Eqn (4)}$$

The algorithm of Monte Carlo American put option is as follow:

1. For each path $i = 1,..., numPath$
   a. For each time step $j = 1,..., numSteps$ of the path i and while the optimal stock stopping (exercise) time τ_i for path i has not been found
      i. Simulate the stock price S_t using ***Eqn(1)***
      ii. Compute d_1 and d_2 using ***Eqn(3)(4)***
      iii. Compute the expected added premium $E[S_T|S_t]$ using ***Eqn(2)***
      iv. Compute payoff p_i.
         1. If $p_i < E[S_T|S_t]$, the exercise boundary has not been crossed, therefore proceed to the next time step
         2. otherwise, the current time t will be the optimal stopping time τ for this path and proceed to the next path
   b. If no optimal stopping time is found at expiration T, then $\tau_i = T$.

2. Compute the average optimal stopping time $\bar{\tau} = \frac{1}{numPath}\sum \tau_i$

3. Compute the average option price $\bar{p} = e^{rT} \times \frac{1}{numPath}\sum p_i$

3.2 Result

Illustration of the geometric random walks of Monte Carlo simulation for the asset price is shown in *Figure 1.* The result of option pricing with Monte Carlo simulation is shown in *Table 3.*

Figure 1. *Geometric Random Paths of Monte Carlo Simulation for Stock Price*

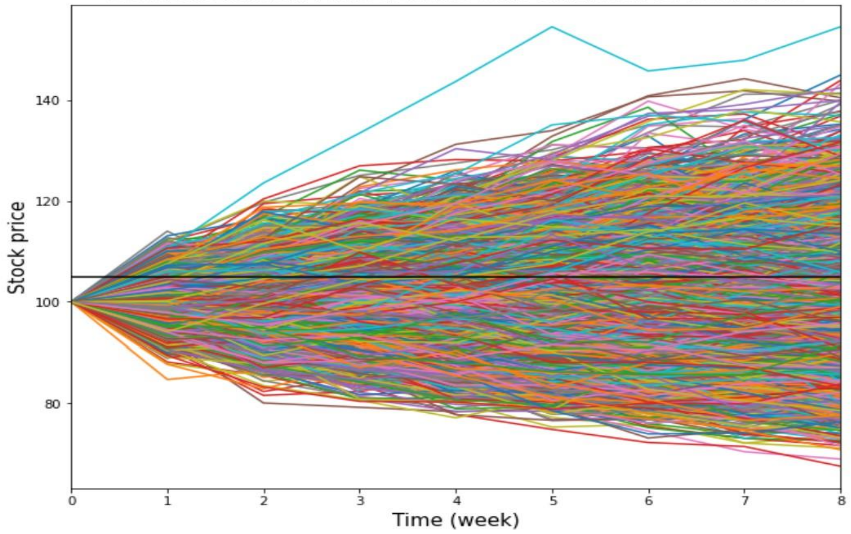

Table 3. *Monte Carlo Simulation Asian and Exotic Option Pricing Result*

Option	Estimated Price	95% Confidence Interval
Asian Call	$0.6800	[0.6421, 0.7180]
Asian Put	$5.5107	[5.4202, 5.6012]
Lookback Call	$3.3156	[3.2111, 3.4202]
Lookback Put	$10.9693	[10.8599, 11.0786]
Floating Lookback Call	$6.2692	[6.1409, 6.3975]
Floating Lookback Put	$6.2437	[6.1311, 6.3563]

Table 4. Monte Carlo Simulation American Put Option Pricing Result

Option	Estimated Price	95% Confidence Interval	Estimated Stoptime	95% Confidence Interval
American Put	$6.9298	[$6.8078., $ 7.0518]	6.2522 week	[6.2084, 6.2960]

4. Binomial Lattice

Binomial Lattice Model is another option valuation method that can be used to price an underlying asset by generating all the possible paths of price throughout the option contract period. There are two assumptions with the binomial option pricing model. Firstly, prices move in a series of discrete time steps and the time between steps is constant. Secondly, there are only two possible outcomes at each time step, moving up or moving down. Also, the sizes of up and down moves are constant throughout all steps. The size of up move factor is given by$e^{\sigma\sqrt{dT}}$, while the size of down move factor is given by $e^{-\sigma\sqrt{dT}}$.

4.1 Methodologies

The binomial lattice algorithms for different types of options will be discussed in the following sections.

4.1.1 Asian Options and Exotic Options

There are several steps to calculate option price with binomial model:

1. Calculate the size of up and down move factors and the corresponding probability
   a. Simulate the up move factor as u = $e^{\sigma\sqrt{dT}}$ and the probability p = $\frac{e^{r*dT} - d}{u - d}$
   b. Simulate the down move factor as d = $e^{-\sigma\sqrt{dT}}$ and the probability q = 1 - p
2. Build binomial stock price tree
   a. Simulate Price = $P_0 u^m d^{8-m}$ where m denotes the number of up move factors in a path
3. Build binomial probability tree
   a. Simulate Probability = $p^m q^{8-m}$ where m denotes the number of up move factors in a path
4. Calculate option payoff and discount prices to the present value by risk-free rate

Calculate size of up and down move and the corresponding probability

According to the equations below, the size of the up move factor is 1.035 and the size of the down move factor is 0.966 at each time step. Also, the probability of an up move is 0.531 and the probability of down move is 0.469.

$$u = e^{\sigma\sqrt{dT}} = e^{0.25\sqrt{1/52}} = 1.037$$
$$d = e^{-\sigma\sqrt{dT}} = e^{-0.25\sqrt{1/52}} = 0.965$$
$$p = \frac{e^{r*dT} - d}{u - d} = \frac{e^{0.02\sqrt{1/52}} - 0.966}{1.035 - 0.966} = 0.497$$
$$q = 1 - p = 0.503$$

Build binomial tree using up and down move factors

Since there are 8 time steps in total and only 2 outcomes at each time step, the number of all possible paths is 2^8,which is 256 paths in total. By using itertool, we could generate all the combinations of up and down moves factors as shown in *Table 5*.

Table 5. *Binomial Tree with up and down move factors*

	0	1	2	3	4	5	6	7	8
0	100	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743
1	100	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	0.964559
2	100	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	0.964559	1.036743
3	100	1.036743	1.036743	1.036743	1.036743	1.036743	1.036743	0.964559	0.964559
4	100	1.036743	1.036743	1.036743	1.036743	1.036743	0.964559	1.036743	1.036743
...	...	...	...	...	...	...	...	...	...
251	100	0.964559	0.964559	0.964559	0.964559	0.964559	1.036743	0.964559	0.964559
252	100	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	1.036743	1.036743
253	100	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	1.036743	0.964559
254	100	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	1.036743
255	100	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559	0.964559

256 rows × 9 columns

Build binomial stock price tree

With the initial stock price $100 and the up and down move factors, we could get the entire binomial price tree by cumulatively multiplying the up and down move factors to the initial stock value. As shown in the *Table 6* below, column 0 denotes the initial price $100, while column 1 to column 8 represents all 256 possible prices at the corresponding time steps. The final stock price at the maturity time ranges from $75.78 to $131.96 as shown in the last column.

Table 6. *Binomial Stock Price Tree*

	0	1	2	3	4	5	6	7	8
0	100.0	103.674334	107.483674	111.432983	115.527403	119.772265	124.173097	128.735631	133.465807
1	100.0	103.674334	107.483674	111.432983	115.527403	119.772265	124.173097	128.735631	124.173097
2	100.0	103.674334	107.483674	111.432983	115.527403	119.772265	124.173097	119.772265	124.173097
3	100.0	103.674334	107.483674	111.432983	115.527403	119.772265	124.173097	119.772265	115.527403
4	100.0	103.674334	107.483674	111.432983	115.527403	119.772265	115.527403	119.772265	124.173097
...	...	...	...	...	...	...	...	...	...
251	100.0	96.455889	93.037385	89.740037	86.559550	83.491784	86.559550	83.491784	80.532742
252	100.0	96.455889	93.037385	89.740037	86.559550	83.491784	80.532742	83.491784	86.559550
253	100.0	96.455889	93.037385	89.740037	86.559550	83.491784	80.532742	83.491784	80.532742
254	100.0	96.455889	93.037385	89.740037	86.559550	83.491784	80.532742	77.678572	80.532742
255	100.0	96.455889	93.037385	89.740037	86.559550	83.491784	80.532742	77.678572	74.925557

256 rows × 9 columns

Build binomial probability tree

The up move factor has the probability of 0.497 and the down move factor has the probability of 0.503. As shown in the *Table 7* below, the probability of up and down move at each time step is listed. For example, the first path contains only up moves, so the probability is 0.497 at all time steps. The second path represents the path with 7 up moves and 1 down move, so the probability is 0.497 from time step 1 to time step 7 and probability is 0.503 at time step 8.

Table 7. *Binomial Probability Tree*

	0	1	2	3	4	5	6	7
0	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753
1	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.503247
2	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.503247	0.496753
3	0.496753	0.496753	0.496753	0.496753	0.496753	0.496753	0.503247	0.503247
4	0.496753	0.496753	0.496753	0.496753	0.496753	0.503247	0.496753	0.496753
...	...	...	...	...	...	...	...	...
251	0.503247	0.503247	0.503247	0.503247	0.503247	0.496753	0.503247	0.503247
252	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.496753	0.496753
253	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.496753	0.503247
254	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.496753
255	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247	0.503247

256 rows × 8 columns

Afterwards, cumulative probability at the maturity time should be obtained to get the probability of the occurring payoffs in 256 paths. According to the formula below, the first path with all up moves has the probability of 0.0037 occurring and the last path with all down moves has the probability of 0.0041 occurring.

Pi = $p^{m}q^{8-m}$ where m denotes the number of up move factors in a path

Table 8. *Binomial Cumulative Probability Tree*

	0	1	2	3	4	5	6	7
0	0.496881	0.246890	0.122675	0.060955	0.030287	0.015049	0.007478	0.003715
1	0.496881	0.246890	0.122675	0.060955	0.030287	0.015049	0.007478	0.003762
2	0.496881	0.246890	0.122675	0.060955	0.030287	0.015049	0.007572	0.003762
3	0.496881	0.246890	0.122675	0.060955	0.030287	0.015049	0.007572	0.003809
4	0.496881	0.246890	0.122675	0.060955	0.030287	0.015238	0.007572	0.003762
...	...	...	...	...	...	...	...	...
251	0.503119	0.253129	0.127354	0.064074	0.032237	0.016018	0.008059	0.004055
252	0.503119	0.253129	0.127354	0.064074	0.032237	0.016219	0.008059	0.004004
253	0.503119	0.253129	0.127354	0.064074	0.032237	0.016219	0.008059	0.004055
254	0.503119	0.253129	0.127354	0.064074	0.032237	0.016219	0.008160	0.004055
255	0.503119	0.253129	0.127354	0.064074	0.032237	0.016219	0.008160	0.004106

256 rows × 8 columns

Calculate option payoff and discount prices to the present value by risk-free rate
Calculation of the payoff and discounted option price is the same as section 2.

4.1.2 American Put Option

Evaluating American put option using the binomial lattice is straightforward. Similar to section 4.1.1, the procedure of the calculation is shown below.

1. Calculate the size of up and down move factors and the corresponding probability
   a. Simulate the up move factor as u = $e^{\sigma\sqrt{dT}}$ and the probability p = $\frac{e^{r*dT}-d}{u-d}$
   b. Simulate the down move factor as d = $e^{-\sigma\sqrt{dT}}$ and the probability q = 1 - p
2. Build the recombing early payoff tree.
   a. Simulate early payoff = max($K - S_t$,0) for each time unit at each node
3. Backwards Pricing.
   a. for each t in range(8,0),
      i. calculate the value at each node by discounting using the risk-free rate.
      ii. compare with the early payoff.
      iii. take the largest one
      iv. use the largest value to reiterate the loop above
   b. The resulted option value is the value at t = 0

Build binomial tree using up and down move factors
This is the same as the value in section 4.1.1. Applying the same metrics indicated above, i.e. *u, d, p, q, S0, K,* the potential stock price at each period will be calculated.

Build the recombing early payoff tree
Early exercise is an essential factor that influences the American Option value. The potential early payoff as well as the value of the final (8^{th}week) period of the put option are stored in the dictionary below, in which they are computed according to the formula $max(K - S_t, 0)$ in each nodes.

```
{0: [5.0],
 1: [1.3257, 8.5441],
 2: [0, 5.0, 11.9626],
 3: [0, 1.3257, 8.5441, 15.26],
 4: [0, 0, 5.0, 11.9626, 18.4404],
 5: [0, 0, 1.3257, 8.5441, 15.26, 21.5082],
 6: [0, 0, 0, 5.0, 11.9626, 18.4404, 24.4673],
 7: [0, 0, 0, 1.3257, 8.5441, 15.26, 21.5082, 27.3214],
 8: [0, 0, 0, 0, 5.0, 11.9626, 18.4404, 24.4673, 30.0744]}
```

The key name [0,1…...8] denotes which period (or week) the option is in, while the elements in each line denote the potential early exercise value at each node at that period.

Backwards Pricing
The value of the last period, which is shown in the last line of the dictionary above, is used to derive the discounted option value at period 7 without considering early exercise of the American put option. The equation is shown as the following in which i stands for the number of upward trends.

$$(p * S_{iu\,(8-i)d} + q * S_{(i+1)\,u\,(7-i)d})/R$$

Then we get the discounted value at period 7 as follows:

Table 9. *The Early Payoff, Discounted Value, And the final value of each node at week 7*

	earlypayoff	discounted	max
0	0.0000	0.000000	0.000000
1	0.0000	0.000000	0.000000
2	0.0000	0.000000	0.000000
3	1.3257	2.515185	2.515185
4	8.5441	8.500363	8.544100
5	15.2600	15.216190	15.260000
6	21.5082	21.464472	21.508200
7	27.3214	27.277686	27.321400

Compare the discounted value with the early payoff value at period 7 (shown in the second last line in the dictionary), treat the max of those two as the current option value for each node and iterate the above procedure for 7 weeks.

Table 10. *Pricing Result I*

```
value at each nodes in period 7 is
 0      0.000000
1      0.000000
2      1.265231
3      5.546905
4     11.962600
5     18.440400
6     24.467300
Name: max, dtype: float64
value at each nodes in period 6 is
 0      0.000000
1      0.636458
2      3.418544
3      8.771926
4     15.260000
5     21.508200
Name: max, dtype: float64
```

Table 11. *Pricing Result II*

```
value at each nodes in period 5 is
 0      0.320162
1      2.035685
2      6.110070
3     12.032013
4     18.440400
Name: max, dtype: float64
value at each nodes in period 4 is
 0      1.183000
1      4.084404
2      9.086481
3     15.260000
Name: max, dtype: float64
value at each nodes in period 3 is
 0      2.642021
1      6.598932
2     12.188205
Name: max, dtype: float64
value at each nodes in period 2 is
 0     4.631393
1     9.407794
Name: max, dtype: float64
```

4.2 Result

The following table is a summary of the option prices for Asian option, Lookback option, Floating Lookback option, and American Put using binomial lattice.

Table 12. *Binomial Lattice Option Pricing Result*

Option	**Price**
Asian Call	$0.6925
Asian Put	$5.5095
Lookback Call	$3.575
Lookback Put	$11.2485
Floating Lookback Call	$6.5979
Floating Lookback Put	$6.5478
American Put	$7.0322

5. Conclusion

Using two different methods to estimate the price of different kinds of the options, we have stored the results in the following table.

Table 13: *Result Comparison between MC and Lattice*

Option	Monte Carlo	Binomial Lattice	Difference
Asian Call	$0.68	$0.69	-$0.01
Asian Put	$5.51	$5.51	$0
Lookback Call	$3.32	$3.58	-$0.26
Lookback Put	$10.97	$11.25	-$0.28
Floating Lookback Call	$6.27	$6.60	-$0.33
Floating Lookback Put	$6.24	$6.55	-$0.31
American Put	$6.93	$7.03	-$0.10

By observing the table above, the estimated price for the Asian/Lookback Call is always smaller than that for Put, which is due to the fact the strike price K is greater than the initial price S_0 by the put call parity.

Comparing the price of the same option for MC and Lattice, we can see that estimation from both approaches result in very similar value, indicating that both MC and Lattice are reliable methods for pricing options.

6. Reference

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