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Executive Summary

This paper investigates the model risk and return in the context of four different financial optimization models. The objective of this paper is to compare the performance of mean-variance optimization (MVO), robust mean-variance optimization, risk parity optimization (ERC), and market portfolio based on the respective optimal allocation of assets. By using the optimal weights estimated from the monthly return and variances in December 2004 till September 2008, the performance is tested out-of-sample in October 2008 and November 2008.

The MVO portfolio performs the best in October 2008 while the robust MVO portfolio with 90% confidence interval is the best in November 2008. On the other hand, the robust MVO portfolio with 90% confidence interval performs the worst in October 2008 and Market portfolio is the worst in November 2008. Both the ERC and Market portfolio generate negative portfolio return in November and October. Also, the robust MVO at the 95% confidence interval has a better performance in October 2008 than that at the 90% confidence interval, but this is not the case in November 2008.

Besides the portfolio return and variance, efficient frontiers are also drawn to compare the estimated MVO frontier, true MVO frontier, actual MVO frontier with the robust MVO ones. Estimated efficient frontier is the curve based on the optimization using the estimates we have used. It is normally more optimistic than the true and actual efficient frontier due to estimation errors. On the other hand, true efficient frontier uses true parameters, while actual one is computed by applying the portfolio weights from the estimated frontier to the true expected returns and covariance matrix. The actual portfolio performs even worse than the true one. In summary, the performance of the three efficient frontiers could be denoted symbolically as the following [1].

Estimated efficient frontier > True efficient frontier > Actual efficient frontier

However, it works the opposite in this project. For the MVO, true efficient frontier is above the actual efficient frontier, which is above the estimated efficient frontier. For both robust MVO with 90% and 95% confidence interval, actual efficient frontier is above the estimated efficient frontier. The performance can be denoted as following.

MVO: True efficient frontier > Actual efficient frontier > Estimated efficient frontier
Robust MVO: MVO: Actual efficient frontier > Estimated efficient frontier

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1. Introduction

In this project, the assets considered for investment are all the constituents of S&P 500. Stock symbols for the 20 assets are F, CAT, DIS, MCD, KO, PEP, WMT, C, WFC, JPM, AAPL, IBM, PFE, JNJ, XOM, MRO, ED, T, VZ, and NEM. Four distinct optimization results in different allocation of assets determined by different computational model and objective function.

The main parameters used for constructing four different optimization models are $\mu, Q, \lambda,$ and r_f. Sample mean μ is computed by using monthly (last trading day of each month) adjusted closing prices for each stock from 30-Dec-2004 to 30-Sep-2008. The covariance matrix Q is computed from the variance and covariance of return. Risk aversion coefficient λ represents the expected risk-return tradeoff and acts as a scaling factor in the reverse optimization estimate of excess returns, and is defined as the quantity that Idzorek suggests according to the Black-Litterman's model [2]. Also, the risk-free rate is calculated as the average of the 10-year treasury rate in 2005, 2006, 2007, and 2008 [3]. These main parameters used in the following models are estimated as following:

$$\bar{\mathrm{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} r_i - \bar{\mathrm{r}}$$

$$\lambda = \frac{\mathrm{E}[r_{mkt}] - r_f}{\sigma_{mkt}^2}$$

$$r_f = \frac{(0.0429 + 0.048 + 0.0463 + 0.0366)}{4} = 0.04345$$

$$\text{Monthly } r_f = \frac{0.04345}{12} = 0.00362$$

2. Market Portfolio

The market portfolio is a portfolio with the assets' weights determined by market capitalization. Market capitalization, a measure of the market value of a company, is estimated by multiplying the stock price each share by its total number of outstanding shares. In this paper, we retrieved all the historical market cap from companiesmarketcap.com [4], then we take the average of the market cap in 2005, 2006, 2007, and 2008 as the final input.

The asset's weights in the market portfolio, also termed as capitalization weights, equal to the proportion of each asset's total capital value to the total market capital value. Using the capitalization weights, the market portfolio has a negative realized return in both October and November in 2008, mainly because in October all 20 assets except ED have negative monthly returns and in November about half of the assets grow negatively. However, the market portfolio is completely diversified; thus the portfolio variance is small.

Table 1. Market Portfolio Summary

	Return	Variance	Standard deviation	Sharpe ratio
Market	0.00712715	0.00090414	0.03006884	0.11660954
Market in Oct	-0.11672675	0.00090414	0.03006884	-4.00240177
Market in Nov	-0.03998382	0.00090414	0.03006884	-1.45016071

After the market portfolio is given, risk aversion coefficient λ used in the mean-variance optimization and robust mean-variance optimization is defined as the quantity that Idzorek suggests according to the Black-Litterman model:

$$\lambda = \frac{\mathrm{E}[r_{mkt}] - r_f}{\sigma_{mkt}^2} = 3.878$$

where $\mathrm{E}[r_{mkt}]$ is the market portfolio expected return, r_f is the risk-free rate, σ_{mkt}^2 is the market portfolio variance.

3. Mean-variance optimization

Mean-variance analysis, or modern portfolio theory (MPT), is the process of weighing risk against expected return so as to maximize the expected return for a given level of risk. In the MPT theory, it is assumed that investors are risk averse, in which case investors tend to choose the portfolio with less risk. They choose the portfolio with higher expected return only if they can accept more risk [5]. However, the tradeoff between expected return and risk is determined by risk aversion characteristic, which is different for all investors. Also, the return used in optimization is assumed to be normally distributed, and the transaction costs or taxes incurred during trading is ignored.

For the mean-variance optimization, the objective of the portfolio optimization process is to find the optimal allocation of assets that maximizes the return after deducting the risk with risk aversion parameter. In this case, short selling is allowed, and the allocation of assets should be summed up to 1. Each stock is limited to a maximum of 200% long or short position. The MVO model can be expressed as:

$$\max_{x \in R^n} \mu^T x - \lambda x^T Q x$$
$$s.t. 1_n^T x = 1$$
$$-2 \leq x_i \leq 2 \ \forall i$$

where x is the optimal weights, μ is the estimated return, Q is the covariance matrix, λ is the risk aversion value.

Using the optimal weights, the portfolio return, variance, standard deviation, and Sharpe ratio are generated as the following. It is obvious that the performance of the MVO portfolio is the best during November 2008 with the highest realized return 25% and Sharpe ratio 2.5.

Table 2 MVO Portfolio Summary

	Return	Variance	Standard deviation	Sharpe ratio
MVO	0.08102368	0.00990579	0.09952784	0.77770042
MVO in Oct	0.18579978	0.00990579	0.09952784	1.83043196
MVO in Nov	0.25014593	0.00990579	0.09952784	2.47694605

4. Robust mean-variance optimization

Robust mean-variance optimization is similar to the MVO in the way of dealing with optimization problem, except that it incorporates uncertainty into the optimization model. Since mean-variance portfolio relies heavily on parameters mean and variance, the optimal allocation of the assets is extremely sensitive to the changes in the mean of asset's return. Thus, the estimation error on the expected return can generate a huge negative impact on portfolio optimization process. Robust MVO, in this case, improves stability by creating an uncertainty set around the estimated returns and optimize around it. Box uncertainty set is defined as following:

$$\mu^{true} \in \mathbb{R}^n : |\mu_i^{true} - \mu_i| \leq \delta_i, i = 1, \ldots, n$$

where μ^{true} is the vector of 'true' expected return and δ_i is the maximum 'distance' between μ_i and μ_i^{true}.

The distance δ_i is defined by the estimation error of μ_i and is set proportionally to the standard errors:

$$\delta_i = \varepsilon_1 (\frac{\sigma_i}{\sqrt{T}})$$

where ε_1 is a sizing parameter, T is the number of observations, σ_i is the variance diagonal matrix.

For the robust mean-variance optimization, the objective is to find the optimal weights that maximizes the return after deducting the risk with risk aversion parameter. The uncertainty is added as a penalty on the target returns as $\delta_i x$. In this case, short selling is allowed, and the allocation of assets should be summed up to 1. Each stock is limited to a maximum of 200% long or short position. The robust MVO model can be expressed as:

$$\max_{x \in R^n} \mu^T x - \lambda x^T Q x$$
$$s.t.\ \mu^T x - \delta^T |x| \geq R$$
$$1_n^T x = 1$$
$$-2 \leq x_i \leq 2\ \forall i$$

where x is the optimal weights, μ is the estimated return, Q is the covariance matrix, λ is the risk aversion value, δ_i is the distance, R is the target return.

Using the optimal weights, the portfolio return, variance, standard deviation, and Sharpe ratio are generated in the *Table 3*.

Table 3. Robust MVO Portfolio Summary

	Return	Variance	Standard deviation	Sharpe ratio
Robust MVO 90%	0.09766826	0.01259233	0.11221555	0.83809623
Robust MVO 95%	0.11818736	0.04762100	0.21822235	0.52499906
Robust MVO in Oct 90%	-0.20130022	0.01259233	0.11221555	-1.82613772
Robust MVO in Oct 95%	0.12937691	0.04762100	0.21822235	0.57627497
Robust MVO in Nov 90%	0.35766454	0.01259233	0.11221555	3.15503249
Robust MVO in Nov 95%	0.25440369	0.04762100	0.21822235	1.14920794

5. Risk Parity optimization with no short selling

The risk parity optimization seeks to allocate the assets based on a risk-weighted basis. Risk parity optimization, also referred to as equal risk contribution, attempts to diversify portfolio risk by ensuring each asset contribute same level of risk. Therefore, the objective of the risk parity optimization is the minimize the difference of risk between each asset and the constraint is no short selling. The model can be formulated as:

$$\min_{x \in R^n} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i (Qx)_i - x_j (Qx)_j)^2$$
$$s.t.\ 1_n^T x = 1$$
$$x \geq 0$$

where x is the optimal weights, Q is the covariance matrix.

Using the optimal weights, the portfolio return, variance, standard deviation, and Sharpe ratio are generated in the *Table 4*.

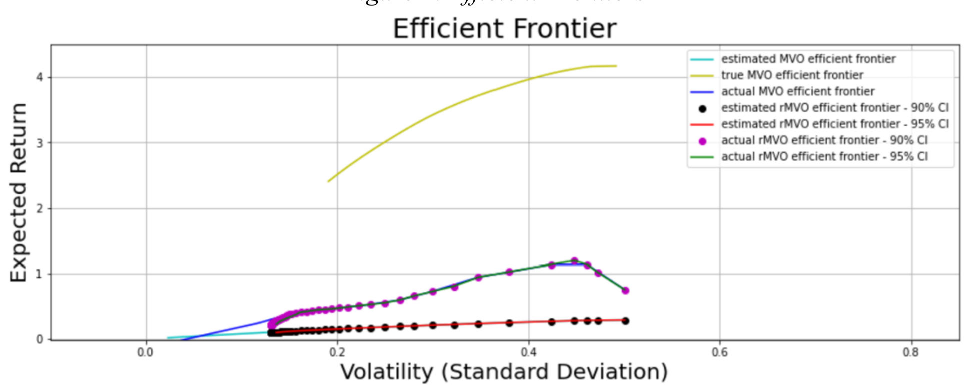
Table 4. ERC Portfolio Summary

	Return	Variance	Standard deviation	Sharpe ratio
ERC	0.00627249	0.00047175	0.02171988	0.12208440
ERC in Oct	-0.14619216	0.00047175	0.02171988	-6.89750458
ERC in Nov	-0.00856764	0.00047175	0.02171988	-0.56116654

6. Efficient Frontier

The efficient frontier represents a set of optimal portfolios that offer the lowest risk for a given level of minimum expected return or the highest expected return for a maximum defined level of risk. Theoretically, any portfolio that lies below or to the right of the efficient frontier is sub-optimal.

Figure 1. Efficient Frontiers



In *Figure 1*, seven efficient frontiers are plotted. They are:

1) Estimated MVO efficient frontier: computed using the estimated expected return μ
2) True MVO efficient frontier computed using the true October 2008 expected return
3) Actual MVO efficient frontier computed by applying the estimated portfolio weight to the true expected returns and covariance.
4) Estimated robust MVO efficient frontier with 90% confidence interval
5) Estimated robust MVO efficient frontier with 95% confidence interval
6) Actual robust MVO efficient frontier with 90% confidence interval
7) Actual robust MVO efficient frontier with 95% confidence interval

6.1 MVO Efficient Frontiers

In the plot, it is obvious that the true MVO efficient frontier generates significantly higher expected return than the estimated efficient frontier. The reason behind this is that the MVO model we implemented allows short selling and during Oct. 2008, nearly all the assets of this portfolio drop harshly, therefore yielding significant return if we take short positions on these stocks. The perturbations in our estimates of the expected return result in loss of possible high return for our portfolio.

6.2 Robust MVO Efficient Frontiers

The estimated efficient frontier with 90% confidence interval is similar to the one with 95% confidence interval. The estimated Robust MVO efficient frontiers are similar to but slightly worse than the estimated MVO efficient frontier. This is reasonable because the robust MVO model with box uncertainty set gives very conservative solution (corner point), trying to minimize the model sensitivity to return (μ) estimation error. In our case, the true efficient frontier is higher than the estimated efficient frontiers, therefore it is reasonable that the actual efficient frontiers for both robust MVO and MVO lie in between the true and estimated efficient frontiers.

7. Result

7.1 October 2008

Comparing the results of all the portfolios for October 2008, the equal risk contribution (ERC) model with no short selling performed the worst in terms of Sharpe's ratio. This is because the estimated covariance and the expected return we computed are irrelevant as the majority of the assets dropped drastically in October 2008. Additionally, no short selling is allowed with the ERC portfolio. When the price of all the assets in the portfolio drop, taking a long position in any of the assets will result in a negative return. The same reason applies to the market capitalization portfolio. All other portfolios that allow short selling performs relatively better than ERC and market portfolio because these portfolios take short positions in some assets that drop significantly in October 2008, thus hedging some of the risks of the long positions' asset.

In terms of the returns, the same reasons applied. The ERC and market portfolio performed worst since the price of all the assets dropped, and the portfolio is no longer 'diversified'.

Table 5. Realized Return of October 2008

	Return	Variance	Standard deviation	Sharpe ratio
MVO	0.18579978	0.00990579	0.09952784	1.83043196
Robust MVO 90%	-0.20130022	0.01259233	0.11221555	-1.82613772
Robust MVO 95%	0.12937691	0.04762100	0.21822235	0.57627497
ERC	-0.14619216	0.00047175	0.02171988	-6.89750458
Market Cap	-0.11672675	0.00090414	0.03006884	-4.00240177

7.2 November 2008

Comparing the results of all the portfolios for November 2008, the ERC portfolio and the market portfolio still performed the worst in terms of Sharpe's Ratio. Unlike October, the true returns of November are more diversified with some assets yielding positive returns and some with negative returns. Sharpe's ratio of ERC portfolio and the market portfolio in November is better than Sharpe's Ratio in October, even though they are still negative. The negative Sharpe's ratio is possibly due to significant deviation between the estimated parameters (i.e., covariance) and the true parameters. Similarly, portfolios with short selling perform better in November both in terms of Sharpe's ratio and return, possibly because the portfolios take short positions in assets that are still dropping and take long positions in assets that are rebounding from the low price.

Table 6. Realized Return of November 2008

	Return	Variance	Standard deviation	Sharpe ratio
MVO	0.25014593	0.00990579	0.09952784	2.47694605
Robust MVO 90%	0.35766454	0.01259233	0.11221555	3.15503249
Robust MVO 95%	0.25440369	0.04762100	0.21822235	1.14920794
ERC	-0.00856764	0.00047175	0.02171988	-0.56116654
Market Cap	-0.03998382	0.00090414	0.03006884	-1.45016071

7.3 Conclusion

In general, enhanced portfolio optimization models outperform the simple mean-variance model in many perspectives. Both robust MVO and risk parity models try to diminish the impact of parameter estimation errors. With robust MVO, we tend to get more conservative solutions comparing to other strategies. With the risk parity model, the portfolio is more diversified from a risk perspective. The reason for the MVO portfolio to outperform other portfolios in this analysis may be due to the unexpected crisis that happened in 2008.

Reference

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