ELEC6229: Receding Horizon Method

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***Abstract*— This report summarizes the implementation of receding horizon method and discusses the paramater choosing for control and prediction horizons.**

I. Introduction of system model

In this coursework, we are required to manage a super apple market on Mars using Receding Horizon Control (RHC), and we are demanded to minimize the cost within a 52 weeks period. The specifications of the problem is given by:

- The total time period is 52 weeks.
- The re-order number **r** is fixed to be the choice in coursework 1 where $r = 1$.
- The penalty for short stock (demand excess stock) is 20 coins per week and the warehouse cost is 5 coins per unit per week (pupw). At the end of the period, we will be charged 10 coins per unit apple to return them to earth.
- The order number **y** is flexible and to be determined by RHC.

For the convenience of analysis. For the range $1 \leq k \leq 52, k \in N^+$, define the stock at the beginning of week k is z_k^- and at the end of the week k is z_k^+. The demand x_k and the order number y_k. The system model then becomes:

$$\begin{aligned} z_k^- &= z_{k-1}^+ + y_{k-1} && (1) \\ z_k^+ &= \max(z_k^- - x_k, 0) && (2) \end{aligned}$$

The weekly consumption of super apple on Mars is a random variable (RV) with following distribution:

TABLE I

Apple consumption PDF

Demand (x)	0	1	2	3	4	5	6
probability (p)	0.04	0.08	0.28	0.4	0.16	0.02	0.02

The weekly consumption is generated using rejection method, which principle has been introduced in the former coursework.

Following is the organization of remaining sections: in section II, the initial analysis of the reasonable choice of y will be discussed before simulation. Section III promotes the optimal choice of y from the perspective of RHC. Section IV discussed parameters adjustments of RHC. Section V conclude the work and evaluate the performance of RHC.

II. Analysis before simulation

According to Table. I, the expectation of weekly consumption is $E(p_D(x)) = \sum_{x=0}^{6} p_D(x) \cdot x = 2.7$. Then, from the perspective of probability theory, the optimal solution is that at the beginning of each week, the stock is 2.7 units. Since the choice of order number y need to be an integer. Then, the suboptimal choice is maintaining the stock at 3 at the beginning of each week.

According to the system model which described in section I, we can only place an order when $r \leq 1$, then the choice of y for each week k should be:

$$y_k = \begin{cases} 0 & \text{for} \quad r_k > 1 \\ 2 & \text{for} \quad r_k = 1 \\ 3 & \text{for} \quad r_k = 0 \end{cases} \tag{3}$$

It should be mentioned that we place an order on the Friday of each week, then the order number y_k for the week k is aiming to maintain the stock for the week $k+1$.

III. Receding Horizon Method implementation

The basic idea of receding horizon method is to predict N steps ahead of the system at time k to minimize a cost function $J(y)$, which is a function of $Y = [y_k, y_{k+1}, \ldots, y_{k+N-1}]$ and $X = [x_k, x_{k+1}, \ldots, x_{k+N-1}]$. And assign the first input y_k as the input to the system. For a specific sequence of Y, the cost function for the system specified in section I is:

$$\begin{aligned} J_N(X,Y) &= \sum_{i=k}^{k+N-1} 5 \cdot \max\left(\left(z_i^+ - x_i\right), 0\right) \\ &+ 20 \cdot \delta\left(\max\left(\left(z_i^+ - x_i + 0.1\right), 0\right)\right) \end{aligned} \tag{4}$$

Here 0.1 is added to guarantee that the the algorithm won't get penalty when the demand is equals the stock. Simultaneous Eq.1, Eq.2 and Eq.4, we can get the totall cost within the window for a specific set pair X, Y before we reach the end of the period:

$$\begin{aligned} J_N(X,Y) = &\sum_{i=k}^{k+N-1} 5 \cdot \max\left(\left(z_{i-1}^+ + y_{i-1} - x_i\right), 0\right) \\ &+ 20 \cdot \delta\left(\max\left(\left(z_{i-1}^+ + y_{i-1} - x_i + 0.1\right), 0\right)\right) \end{aligned} \tag{5}$$

For each possible sequence of Y, we need to traverse the demand space. According to Table. I, we have 7^N possible demand sequences where N is the window size. Then the weighted cost for Y is defined by:

$$J_N(Y) = \sum_{X \in \mathbb{X}} J_N(X,Y) \cdot p(X) \tag{6}$$

Where $\mathbb{X}$ is the demand space. The probability $p(X)$ is used to weight the cost according to itself probability, and it can be calculated from Table. I. For example, when $N = 3$, the probability for $Y = [3,3,3]$ is $p(Y) = 0.4^3 = 0.0064$.

When the prediction window contains the final week, the cost function should be different. Besides, according to my implementation, when the prediction window trying to go beyond the 52-the week, the window size will be reduced to guarantee the 52-the week is the final predict week.

$$
\begin{aligned}
J_N(X,Y) &= \sum_{i=k}^{k+N-1} 5 \cdot \max\left(\left(z_{i-1}^{+} + y_{i-1} - x_i\right), 0\right) \\
&+ 20 \cdot \delta\left(\max\left(\left(z_{i-1}^{+} + y_{i-1} - x_i + 0.1\right), 0\right)\right) \\
&+ 5 \cdot \delta(i-52)\left(\max\left(\left(z_{i-1}^{+} + y_{i-1} - x_i + 0.1\right), 0\right)\right)
\end{aligned} \tag{7}
$$

From the concept of RHC, we need to determine the prediction horizon size and control horizon size, the window mentioned before is the former one. Here, for this specific task, the control horizon, which is the key parameter to decide the optimal input y, is chosen to be $N_C = 1$, since the whole process is a stochastic and non-Markov process. Then the optimal choice $\hat{y}_k$ for time k is got by finding the total cost starting with

$$
\hat{y}_k = \arg\min_{y_i} \sum_{y_k = y_i, Y \in \mathbb{Y}} J_N(Y) \quad y_i \in [0,6] \tag{8}
$$

The simulation results for 500 runs is shown in Fig. 1. The mean cost over the 52-week-period is 377.24, which is got by taking the average over 5000 runs, and the standard deviation is 57.63. Then according to the central limit theory, we are 99.5% sure that the total cost is within $[375.14, 379.34]$

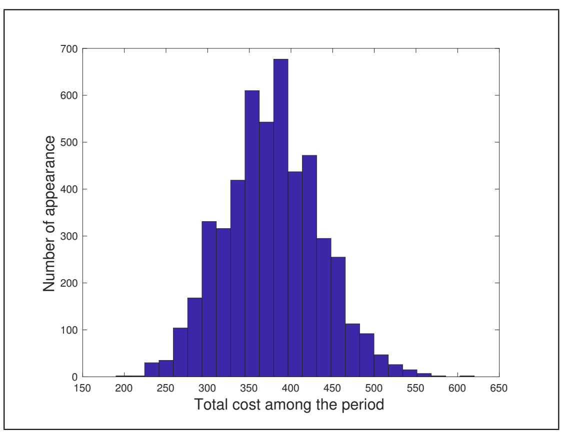

Fig. 1. Cost simulation results for 500 times based on RHC.

Table. II shows the relationship between weekly stock and order number for that week. This result is identical to my analysis before simulation: maintaining the stock $z_k^- = 3$ all the time. Here the order number is range from $[1,6]$, since when $x_k^+ \geq 2$ we cannot order an apple, but when $x_k^+ \leq 1$, we do need to order some apple since the expectation of apple consumption is 2.7.

Inspecting Table. II, we found that when $x_k^+ \geq 2$ the choice of $y_k = 1$. This is due to that r is fixed to be 1, which means when $x_k^+ \geq 2$, the first week of in the prediction window will not order any apple no matter what order number we choose. Then, when $N_C = 1$, the costs specified in Eq.8 are the same for each y_i, then, the algorithm will choose y_k to be the first element in the order space $[0,1,\ldots,6]$ as $\hat{y}$, which is 1.

TABLE II

Simulation results for order number and stock

Week k	0	1	2	3	4	5	6	7	8	...
x_k^+	0	1	0	0	1	0	3	1	2	...
y_k	3	2	3	3	2	3	0	2	1	...
Week k	43	44	45	46	47	48	49	50	51	52
x_k^+	0	0	0	0	1	1	0	1	0	0
y_k	3	3	3	3	2	2	3	2	3	End

IV. RHC parameters discussion

The key parameters for RHC method are prediction horizon size N_P and control horizon size N_C. The results got in last section is based on $N_P = 3$ and $N_C = 1$.

For this specific problem, I think the choice of N_P has limited influence on the results. Because the process is stochastic, the demand for each week is an RV that we can only know its distribution and have no idea of what the real demand will be, then we cannot predict the real stock for the future. All our cost calculation (Eq.6) are based on probability, which will always choose to maintain $z_k^- = 3$, which is only determined by y_k, z_{k-1}^+, and irrelevant to the future order number. Thus extend N_P will not improve the performance.

For the choice of N_C, it shares the same reason with the choice of N_P, since we have only the PDF of future demand, then, according to Eq.8, the algorithm tries to minimize the cost, but we only got the PDF of the demand, then, the optimal choice will be maintaining $z_k^- = 3$ when k is within the N_P window, and choose $\hat{y}_k = 3$ when k go beyond N_P.

V. Conclusion and evaluation

There are two way to implement Eq.6: Monte Carlo method or traverse all the possible Y with respect probability. My implementation used the second method. However, Eq.6 can also be achieved approximately by Monte Carlo method, where we can randomly generate N demand sequences $X_n \sim p(X)$. And the cost for each Y is got based on these X_n.

Due to the low efficiency of generating apple consumption RVs, the Monte Carlo method expects more processing time than my implementation, which is a disadvantage. However, as N_P increase, $p(Y)$ may cause floating overflow, I think this may be improved by using $\sum -\log(p(y_i))$ instead of $p(Y)$ as the weight.

Besides, from my perspective, $r = 1$ limit the overall performance of the algorithm, since such parameter is optimal with fixed $y = 3$, but here y is flexible. The best choice r is 2, which guarantee the minimum stock $x_k^- = 3$ when $x_{k-1}^+ \leq 2$. I also simulated when $r = 2$, where I have 99.95% confidence the cost is in the range $[348.02, 360.90]$.

In conclusion, the RHC method is implemented to the system, which markedly minimzed the cost ($E(J) = 377.24$) compared to the result in the coursework II ($E(J) = 413.17$). And the relevant issues are properly discussed.