Stat155
Game Theory
Lecture 13: General-Sum Games

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Outline for today

- Two-player general-sum games
  - Definitions: payoff matrices, dominant strategies, safety strategies, Nash equilibrium.
  - Example: Cheetahs and gazelles
- Multiplayer general-sum games
  - Nash equilibrium
  - Example: Polluting factories

General-sum games

Notation

- A two-person general-sum game is specified by two payoff matrices, $A, B \in \mathbb{R}^{m \times n}$.
- Simultaneously, Player I chooses $i \in \{1, \ldots, m\}$ and the Player II chooses $j \in \{1, \ldots, n\}$.
- Player I receives payoff a_{ij}.
- Player II receives payoff b_{ij}.

General-sum games

Dominated pure strategies

A pure strategy e_i for Player I is *dominated* by $e_{i'}$ in payoff matrix A if, for all $j \in \{1, \ldots, n\}$,

$$a_{ij} \leq a_{i'j}.$$

Similarly, a pure strategy e_j for Player II is *dominated* by $e_{j'}$ in payoff matrix B if, for all $i \in \{1, \ldots, m\}$,

$$b_{ij} \leq b_{ij'}.$$

General-sum games

Safety strategies

- A *safety strategy* for Player I is an $x_* \in \Delta_m$ that satisfies

$$\min_{y \in \Delta_n} x_*^\top A y = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^\top A y.$$

- x_* maximizes the worst case expected gain for Player I.
- Similarly, a *safety strategy* for Player II is a $y_* \in \Delta_n$ that satisfies

$$\min_{x \in \Delta_m} x^\top B y_* = \max_{y \in \Delta_n} \min_{x \in \Delta_m} x^\top B y.$$

- y_* maximizes the worst case expected gain for Player II.

General-sum games

Nash equilibria

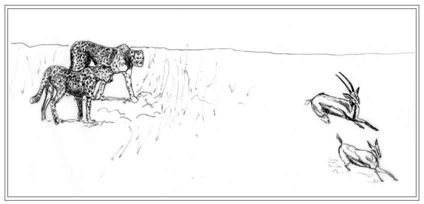
A pair $(x_*, y_*) \in \Delta_m \times \Delta_n$ is a *Nash equilibrium* for payoff matrices $A, B \in \mathbb{R}^{m \times n}$ if

$$\max_{x \in \Delta_m} x^\top A y_* = x_*^\top A y_*,$$

$$\max_{y \in \Delta_n} x_*^\top B y = x_*^\top B y_*.$$

- If Player I plays x_* and Player II plays y_*, neither player has an incentive to unilaterally deviate.
- x_* is a *best response* to y_*, y_* is a best response to x_*.
- In general-sum games, there might be many Nash equilibria, with different payoff vectors.

Example: Cheetahs and Gazelles



(Karlin and Peres, 2016)

Payoff matrices

	large	small
large	$(\ell/2, \ell/2)$	(ℓ, s)
small	(s, ℓ)	$(s/2, s/2)$

$(s \leq \ell)$.

Example: Cheetahs and Gazelles

Payoff matrices

	large	small
large	$(\ell/2, \ell/2)$	(ℓ, s)
small	(s, ℓ)	$(s/2, s/2)$

$(s \leq \ell)$.

- Dominant strategy?
- For $\ell \geq 2s$, large is a dominant strategy.
- Suppose $\ell \leq 2s$.
- Pure Nash equilibria? (large, small), (small, large).
- But who gets the large gazelle?

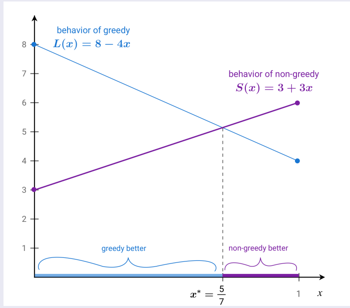
Example: Cheetahs and Gazelles

- Mixed Nash equilibrium?
- If Cheetah I plays $\Pr(\text{large}) = x$, Cheetah II's payoffs are:

$$\begin{aligned} \text{large:} \qquad & L(x) = \frac{\ell}{2}x + \ell(1-x), \\ \text{small:} \qquad & S(x) = sx + \frac{s}{2}(1-x). \end{aligned}$$

- Equilibrium is when these are equal: $x^* = (2\ell - s)/(\ell + s)$.

Example: Cheetahs and Gazelles



(Karlin and Peres, 2016)

- Example: $\ell = 8$, $s = 6$.
- Equilibrium is when $L(x) = S(x)$: $x^* = (2\ell - s)/(\ell + s) = 5/7$.
- Think of x as the proportion of a population of cheetahs that would greedily pursue the large gazelle. For a randomly chosen pair of cheetahs, if $x > x^*$, $S(x) > L(x)$, and non-greedy cheetahs will do better. And vice versa. Evolution pushes the proportion to x^*. This is the *evolutionarily stable strategy*.

Comparing two-player general-sum and zero-sum games

Zero-sum games

1. A pair of safety strategies is a Nash equilibrium (minimax theorem)
2. Hence, there is always a Nash equilibrium.
3. If there are multiple Nash equilibria, they form a convex set, and the expected payoff is identical within that set.
   Thus, any two Nash equilibria give the same payoff.

General-sum games

1. A pair of safety strategies might be unstable.
   (Opponent aims to maximize their payoff, not minimize mine.)
2. There is always a Nash equilibrium (Nash's Theorem).
3. There can be multiple Nash equilibria, with different payoff vectors.

Comparing two-player general-sum and zero-sum games

Zero-sum games

4. If each player has an equalizing mixed strategy
   (that is, $x^\top A = v\mathbf{1}^\top$ and $Ay = v\mathbf{1}$),
   then this pair of strategies is a Nash equilibrium.
   (from the principle of indifference)

General-sum games

4. If each player has an equalizing mixed strategy
   for their opponent's payoff matrix
   (that is, $x^\top B = v_2\mathbf{1}^\top$ and $Ay = v_1\mathbf{1}$),
   then this pair of strategies is a Nash equilibrium.

Outline

- Two-player general-sum games
  - Definitions: payoff matrices, dominant strategies, safety strategies, Nash equilibrium.
  - Example: Cheetahs and gazelles
- Multiplayer general-sum games
  - Nash equilibrium
  - Example: Polluting factories

Multiplayer general-sum games

Notation

- A k-person general-sum game is specified by k utility functions, $u_j : S_1 \times S_2 \times \cdots \times S_k \to \mathbb{R}$.
- Player j can choose strategies $s_j \in S_j$.
- Simultaneously, each player chooses a strategy.
- Player j receives payoff $u_j(s_1, \ldots, s_k)$.

- $k = 2$: $u_1(i, j) = a_{ij}$, $u_2(i, j) = b_{ij}$.
- For $\mathbf{s} = (s_1, \ldots, s_k)$, let $\mathbf{s}_{-i}$ denote the strategies without the ith one:

$$\mathbf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k).$$

- And write $(s_i, \mathbf{s}_{-i})$ as the full vector.

Multiplayer general-sum games

Definition

A vector $(s_1^*, \ldots, s_k^*) \in S_1 \times \cdots \times S_k$ is a *pure Nash equilibrium* for utility functions $u_1, \ldots, u_k$ if, for each player $j \in \{1, \ldots, k\}$,

$$\max_{s_j \in S_j} u_j(s_j, \mathbf{s}_{-j}^*) = u_j(s_j^*, \mathbf{s}_{-j}^*).$$

- If the players play these s_j^*, nobody has an incentive to unilaterally deviate: each player's strategy is a best response to the other players' strategies.

Multiplayer general-sum games

Definition

A sequence $(x_1^*, \ldots, x_k^*) \in \Delta_{S_1} \times \cdots \times \Delta_{S_k}$ (called a *strategy profile*) is a *Nash equilibrium* for utility functions $u_1, \ldots, u_k$ if, for each player $j \in \{1, \ldots, k\}$,

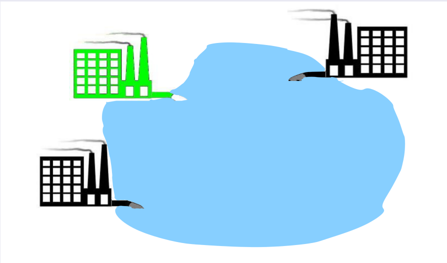
$$\max_{x_j \in \Delta_{S_j}} u_j(x_j, \mathbf{x}_{-j}^*) = u_j(x_j^*, \mathbf{x}_{-j}^*).$$

Here, we define

$$\begin{aligned} u_j(\mathbf{x}^*) &= \mathbb{E}_{s_1 \sim x_1, \ldots, s_k \sim x_k} u_j(s_1, \ldots, s_k) \\ &= \sum_{s_1 \in S_1, \ldots, s_k \in S_k} x_1(s_1) \cdots x_k(s_k) u_j(s_1, \ldots, s_k). \end{aligned}$$

- If the players play these mixed strategies x_j^*, nobody has an incentive to unilaterally deviate: each player's mixed strategy is a best response to the other players' mixed strategies.

Example: Polluting Factories



(Karlin and Peres, 2016)

If firm III purifies, the cost matrix (cost = − payoff) is

firm I \ firm II	purify	pollute
purify	(1,1,1)	(1,0,1)
pollute	(0,1,1)	(3,3,4)

If firm III pollutes, then it is

firm I \ firm II	purify	pollute
purify	(1,1,0)	(4,3,3)
pollute	(3,4,3)	(3,3,3)

(Karlin and Peres, 2016)

- Pure equilibria?
  (purify, purify, pollute), (purify, pollute, purify), (pollute, purify, purify).

  "Tragedy of the commons": (pollute, pollute, pollute)

Example: Polluting Factories

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- Set $\mathbf{x}_i = (p_i, 1 - p_i)$, that is, $\Pr(\text{Player } i \text{ plays purify}) = p_i$.
- For $0 < p_i < 1$, we have a Nash equilibrium iff

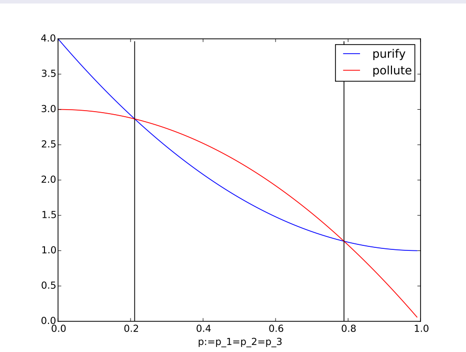
$$u_i(\text{purify}, \mathbf{x}_{-i}) = u_i(\text{pollute}, \mathbf{x}_{-i}).$$

- Solving shows there are two symmetric mixed Nash equilibria:

$$p_1 = p_2 = p_3 = \frac{3 \pm \sqrt{3}}{6}.$$

Example: Polluting Factories

- $p_i = \Pr(i \text{ plays purify})$.
- Plot: **cost** for $p_1 = p_2 = p_3$.
- Blue curve: $-u_i(\text{purify}, \mathbf{x}_{-i})$ $= p^2 + 2p(1-p) + 4(1-p)^2$
  Red curve: $-u_i(\text{pollute}, \mathbf{x}_{-i})$ $= 6p(1-p) + 3(1-p)^2$.



- Imagine that we draw random factories from a population with proportion p that pollute.
- What if p is a little less than $(3+\sqrt{3})/6 \approx 0.79$?
  purify has lower cost.
- What if p is a little more than $(3+\sqrt{3})/6 \approx 0.79$?
  pollute has lower cost.
- What about near $p = (3-\sqrt{3})/6 \approx 0.21$?
  Not an attractor!
- What about near $p = 0$?
  pollute has lower cost.

Example: Polluting Factories

If firm III purifies, the cost matrix (cost = − payoff) is

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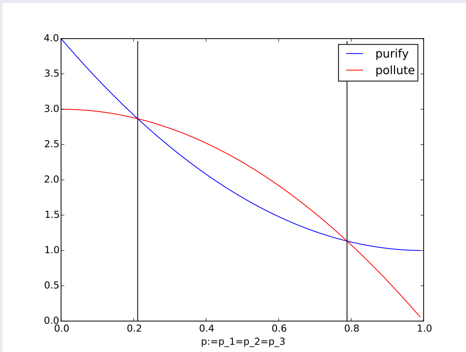

Nash equilibria:

- (p, p, p) with
  - $p = (3+\sqrt{3})/6 \approx 0.79$.
  - $p = (3-\sqrt{3})/6 \approx 0.21$.
  - $p = 0$.
- (purify, purify, pollute),
  (purify, pollute, purify),
  (pollute, purify, purify).

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