Concentration inequalities in Statistical learning

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December 10, 2020

1 Background

Inequalities in statistics provide a means of bounding measures and quantities. They are usually used to specify bounds on quantities when these bounds are particularly difficult or intractable to compute. Inequalities play an important role in the algorithm of statistical learning and machine learning. They are involved in underpinning methods or approaches used in actual cases.

Here are several famous concentration inequalities involved in statistical learning.

- Markov's Inequality
- Chebychev's Inequality
- Bounded Differences Inequality

In this project, I will focus on the Bounded Differences Inequality.

1.1 Chebychev's Inequality

Theorem 1.1. *For some $a \in X$ where $X \subseteq R$, let f be a non-negative function such that $\{f(x) \geq b, forall x \geq a\}$, where $b \in Y$ where $Y \subseteq R$. Then the following ineuqality holds,*

$$P(x \geq a) \leq \frac{\mathbb{E}f(x)}{b} \tag{1}$$

1.2 Chernoff's bound

Suppose the function f is monotonically increasing. Thus, for every $x \geq a$, $f(x) \geq f(a)$. Substitute $b = f(a)$ in Chebychev's Inequality, and we have:

$$P(x \geq a) \leq \frac{\mathbb{E}f(x)}{f(a)} \tag{2}$$

Markov's Inequality says that

$$P(x \geq a) \leq \frac{\mathbb{E}x}{a} \tag{3}$$

It holds for $a > 0$ and nonnegetive x,

$$P(|x - \mathbb{E}(x)| \geq a) \leq \frac{\mathbb{E}\left\{|x - \mathbb{E}(x)|^2\right\}}{a^2} = \frac{\mathrm{Var}\{x\}}{a^2} \tag{4}$$

The Chernoff's bound is then,

$$P(x \geq a) \leq \frac{\mathbb{E}e^{sx}}{e^{sa}} \tag{5}$$

1.3 Chernoff technique

Theorem 1.2. *[Bar20] For $t > 0$:*

$$P(X - \mathbb{E}X \geq t) \leq \inf_{\lambda > 0} e^{-\lambda t} M_{X-\mu}(\lambda) \tag{6}$$

where $M_{X-\mu}(\lambda) = \mathbb{E}\exp(\lambda(X-\mu))($ for $\mu = \mathbb{E}X)$ is the moment-generating function of $X - \mu$.

By using Hoeffding's Inequality, we will have

Theorem 1.3. *[Bar20] For a random variable $X \in [a, b]$ with $\mathbb{E}X = \mu$ and $\lambda \in \mathbb{R}$,*

$$\ln M_{X-\mu}(\lambda) \leq \frac{\lambda^2(b-a)^2}{8} \tag{7}$$

1.4 Sub-Gaussian Random Variables

Definition 1.1. *[Bar20] X is sub-Gaussian with parameter σ^2 if, for all $\lambda \in \mathbb{R}$,*

$$\ln M_{X-\mu}(\lambda) \leq \frac{\lambda^2(\sigma)^2}{2} \tag{8}$$

Examples:

- $X \sim N(\mu, \sigma^2)$ is sub-Gaussian with parameter σ^2;
- $X \in [a, b]$ is sub-Gaussian with parameter $(b-a)^2/4$;
- X_i is independent, sub-Gaussian with parameters σ_i^2 implies $\sum_i X_i$ is sub-Gaussian with parameter $\sum_i \sigma_i^2$.

1.5 Pre-Gaussian Random Variables

Definition 1.2. *[Bar20] X is pre-Gaussian with parameters (σ^2, b) if, for all $|\lambda| < 1/b$,*

$$\log M_{X-\mu}(\lambda) \leq \frac{\lambda^2\sigma^2}{2} \tag{9}$$

Example:

- Sub-Gaussian X with parameter σ^2 is pre-Gaussian with parameters (σ^2, b) for all $b > 0$.

1.6 Martingales

Definition 1.3. *[Bar20] A sequence Y_n of random variables adapted to a filtration $\mathcal{F}_n$ is a martingale if, for all n,*

$$\begin{gathered} \mathbb{E}\,|Y_n| < \infty \\ \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] = Y_n \end{gathered} \tag{10}$$

Note:

- $\mathcal{F}_n$ is a filtration means these σ-fields are nested: $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.
- Y_n is adapted to $\mathcal{F}_n$ means that each Y_n is measurable with respect to $\mathcal{F}_n$.

1.7 Martingale Difference Sequences

Definition 1.4. *[Bar20] A sequence D_n of random variables adapted to a filtration $\mathcal{F}_n$ is a martingale Difference sequence if, for all n,*

$$\begin{gathered} \mathbb{E}\,|D_n| < \infty \\ \mathbb{E}\left[D_{n+1} \mid \mathcal{F}_n\right] = 0 \end{gathered} \tag{11}$$

2 Results

2.1 Pre-Gaussian Random Variables

Theorem 2.1. *[Bar20] For X pre-Gaussian with parameters (σ^2, b),*

$$P(X \geq \mu + t) \leq \begin{cases} \exp\left(-\frac{t^2}{2\sigma^2}\right) & \text{if } 0 \leq t \leq \sigma^2/b \\ \exp\left(-\frac{t}{2b}\right) & \text{if } t > \sigma^2/b \end{cases} \tag{12}$$

Because $t^2/\sigma^2 > t^2/(\sigma^2 + bt)$ and $t^2/bt > t^2/(\sigma^2 + bt)$, we have

$$P(X \geq \mu + t) \leq \exp\left(-\frac{t^2}{2(\sigma^2 + bt)}\right) \tag{13}$$

Proof. Assume $\mu = 0$. For $0 \leq \lambda < 1/b$,

$$\begin{aligned} P(X \geq t) &\leq \exp(-\lambda t)\mathbb{E}\exp(\lambda X) \\ &\leq \exp\left(-\lambda t + \frac{\lambda^2\sigma^2}{2}\right) \end{aligned} \tag{14}$$

Without the constraint $[0, 1/b)$ on λ, the minimum occurs at $\lambda * = t/\sigma^2$. Thus if

$$t/\sigma^2 < 1/b \Longleftrightarrow t < \sigma^2/b \tag{15}$$

we have

$$P(X \geq t) \leq \exp\left(-\lambda^* t + \lambda^{*2}\sigma^2/2\right) = \exp\left(-t^2/\left(2\sigma^2\right)\right) \tag{16}$$

The function $f : t \to -\lambda t + \frac{\lambda^2\sigma^2}{2}$ is monotonically decreasing in $[0, \lambda *]$, and obviously also in $[0, 1/b] \subset [0, \lambda *]$. If t is larger, the minimum will occur at $\lambda = 1/b$. Therefore, substituting the λ gives

$$P(X \geq t) \leq \exp\left(-t/b + \sigma^2/\left(2b^2\right)\right) \leq \exp(-t/(2b)) \tag{17}$$

where the second inequality follows from $t \geq \sigma^2/b$. □

For independent X_i, pre-Gaussian with parameters (σ_i^2, b_i), the sum $X = X_1 + \cdots + X_n$ is pre-Gaussian with parameters $\sum_i \sigma_i^2, max_i b_i$.

Actually, for $\mathbb{E}X_i = 0$, we have

$$\begin{aligned} M_X(\lambda) &= \prod_i \mathbb{E}\exp\left(\lambda X_i\right) \\ &\leq \prod_i \exp\left(\lambda^2\sigma_i^2/2\right) = \exp\left(\lambda^2 \sum_i \sigma_i^2/2\right) \end{aligned} \tag{18}$$

When $|\lambda| < 1/b_i$, for all i, the ineuqality holds.

Theorem 2.2. *[Bar20] For independent X_i, pre-Gaussian with parameters (σ_i^2, b_i), with mean μ_i,*

$$P\left(\frac{1}{n}\sum_{i=1}^n \left(X_i - \mu_i\right) \geq t\right) \leq \begin{cases} \exp\left(-nt^2/\left(2\sigma^2\right)\right) & \text{for } 0 \leq t \leq \sigma^2/b \\ \exp(-nt/(2b)) & \text{for } t > \sigma^2/b \end{cases} \tag{19}$$

where $\sigma^2 = \sum_i \sigma_i^2$ and $b = \max_i b_i$.

2.2 Bernstein's Inequality

Consider a random variable X with mean μ, variance σ^2, and bound $|X-\mu| \leq b$. Then X is pre-Gaussian with parameters $(2\sigma^2, 2b)$. Hence,

$$P(X \geq \mu+t) \leq \exp\left(-\frac{t^2}{4\left(\sigma^2+bt\right)}\right) \tag{20}$$

When we imporve the constants from $4\left(\sigma^2+bt\right)$ to $2(\sigma^2+bt/3)$, we get the Bernstein's Inequality.

Theorem 2.3. *[Kut02] Let $\xi_1, \ldots, \xi_m$ be independent random variable, with $|\xi_k - \mathbb{E}\xi_k| \leq b$ for all k. Let $X = \sum_{k=1}^m \xi_k$, and let $\sigma^2 = Var(X)$. Let $\mu = \mathbb{E}X$. Then we have*

$$P(X \geq \mu+t) \leq \exp\left(-\frac{t^2}{2(\sigma^2+bt/3)}\right) \tag{21}$$

2.3 Concentration Bounds for Martingales

Theorem 2.4. *[Bar20] Consider a martingale Difference sequence D_n which is adapted to a filtration $\mathcal{F}_n$. Assume it satisfies*

$$\text{for } |\lambda| \leq 1/b_n \text{ a.s., } \mathbb{E}\left[\exp\left(\lambda D_n\right) \mid \mathcal{F}_{n-1}\right] \leq \exp\left(\lambda^2 \sigma_n^2/2\right) \tag{22}$$

Then $\sum_{i=1}^n D_i$ is pre-Gaussian, with $(\sigma^2, b) = (\sum_{i=1}^n \sigma_i^2, max_i b_i)$.

$$P\left(\left|\sum_i D_i\right| \geq t\right) \leq \begin{cases} 2\exp\left(-t^2/\left(2\sigma^2\right)\right) & \text{if } 0 \leq t \leq \sigma^2/b \\ 2\exp(-t/(2b)) & \text{if } t > \sigma^2/b \end{cases} \tag{23}$$

Proof. Given $|\lambda| < 1/b_n$,

$$\begin{aligned} \mathbb{E}\exp\left(\lambda \sum_i D_i\right) &= \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{n-1} D_i\right) \mathbb{E}\left[\exp\left(\lambda D_n\right) \mid \mathcal{F}_{n-1}\right]\right] \\ &\leq \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{n-1} D_i\right)\right] \exp\left(\lambda^2 \sigma_n^2/2\right) \end{aligned} \tag{24}$$

Iterating shows that $\sum_i D_i$ is pre-Gaussian. □

Theorem 2.5. *[Bar20] Consider a martingale difference sequence D_i that a.s. falls in an interval of length B_i. Then*

$$P\left(\left|\sum_i D_i\right| \geq t\right) \leq 2\exp\left(-\frac{2t^2}{\sum_i B_i^2}\right) \tag{25}$$

2.4 Bounded Differences Inequality

Theorem 2.6. *[Sri] Suppose $f : \mathcal{X}^n \to \mathbb{R}$ satisfies the following bounded differences inequality: for all $x_1, \ldots, x_n, x_i' \in \mathcal{X}$,*

$$\left|f\left(x_1, \ldots, x_n\right) - f\left(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\right)\right| \leq B_i \tag{26}$$

Then

$$P(|f(X) - \mathbb{E}f(X)| \geq t) \leq 2\exp\left(-\frac{2t^2}{\sum_i B_i^2}\right) \tag{27}$$

Proof. Use the Chernoff's bound (5), we have

$$P(X-\mathbb{E}[X]\geq t)\leq e^{-st}e^{\mathbb{E}[X-\mathbb{E}[Z]]} \tag{28}$$

Now let,

$$V_i=\mathbb{E}\left[X\mid x_1,\ldots,x_i\right]-\mathbb{E}\left[X\mid x_1,\ldots,x_{i-1}\right],\forall i=1,..,n \tag{29}$$

Then $V=\sum_{i=1}^n V_i=X-\mathbb{E}[X]$.
Therefore,

$$P(Z-\mathbb{E}[X]\geq t)\leq e^{-st}\mathbb{E}\left[e^{\sum_{i=1}^n sV_i}\right]=e^{-st}\prod_{i=1}^n\mathbb{E}\left[e^{sV_i}\right] \tag{30}$$

Let V_i be bounded by the interval $[L_i,U_i]$. Because $|X-X_i'|\leq B_i$ (26), it follows that $|V_i|\leq B_i$. Thus, we have $|U_i-L_i|\leq B_i$.

Because

$$\mathbb{E}\left[e^{sV_i}\right]\leq e^{\frac{s^2(U_i-L_i)^2}{8}}\leq e^{\frac{s^2B_i^2}{8}} \tag{31}$$

We then have

$$P(X-\mathbb{E}[X]\geq t)\leq e^{-ts}\prod_{i=1}^n e^{\frac{s^2B_i^2}{8}}=e^{s^2\sum_{i=1}^n\frac{B_i^2}{8}-st} \tag{32}$$

Then we can minimize the bound respect to s and have

$$2s\sum_{i=1}^n\frac{B_i^2}{8}-t=0\Rightarrow s=\frac{4t}{\sum_{i=1}^n B_i^2} \tag{33}$$

Then the bound is

$$P(X-\mathbb{E}[X]\geq t)\leq e^{\left(\frac{4t}{\sum_{i=1}^{nt}B_i^2}\right)^2\sum_{i=1}^n\frac{B_i^2}{8}-\left(\frac{4t^2}{\sum_{i=1}^n B_i^2}\right)}\Rightarrow P(X-\mathbb{E}[X]\geq t)\leq e^{-\frac{2t^2}{\sum_{i=1}^n B_i^2}} \tag{34}$$

It follows the Bounded Differences ineuqality. □

Example 1: Rademacher Averages [Bar20]
For a set $A\subset\mathbb{R}^n$, Consider

$$Z=\sup_{a\in A}\langle\epsilon,a\rangle \tag{35}$$

where $\epsilon=(\epsilon_1,\ldots,\epsilon_n)$ is a sequence of i.i.d uniform $\{\pm 1\}$ random variables. Define the Rademacher complexity of A as $R(A)=\mathbb{E}Z/n$. (a measure of the size of A). The bounded difference approach implies that Z is concentrated around $R(A)$:

Corollary 2.6.1. *Z is sub-Gaussian with parameter $4\sum_i\sup_{a\in A}a_i^2$.*

Proof. Rewrite $Z=f(\epsilon)=f(\epsilon_1,\ldots,\epsilon_n)$. Note that a change of ϵ_i will results in a change in Z of no more than $B_i=\sup_{a\in A}2\,|a_i|$. Then the result follows. □

Example 2: Exmpirical Process [Bar20]
For a class F of functions $f:\mathcal{X}\to[0,1]$, suppose that $X_1,\ldots,X_n$ are i.i.d on $\mathcal{X}$. Consider that

$$Z=\sup_{f\in F}\left|\mathbb{E}f(X)-\frac{1}{n}\sum_{i=1}^n f\left(X_i\right)\right|=:\|\underbrace{P-P_n}_{\text{emp proc}}\|_F \tag{36}$$

This is called a uniform law of large numbers, if Z converges to 0. It shows that Z is concentrated about $\mathbb{E}Z$:

Corollary 2.6.2. *Z is sub-Gaussian with parameter $1/n$.*

Proof. Rewrite $Z=g(X_1,\ldots,X_n)$. Note that a change of X_i will results in a change in Z of no more than $B_n=1/n$. Then the result follows. □

2.5 Concentration

Recall the bounded differences inequality without absolute values.

$$P(f(X) - \mathbb{E}f(X) \geq t) \leq \exp\left(-\frac{2t^2}{\sum_i B_i^2}\right) \tag{37}$$

when we apply $-t$ to the inequality, we have

$$P(f(X) - \mathbb{E}f(X) \leq -t) \leq \exp\left(-\frac{2t^2}{\sum_i B_i^2}\right) \tag{38}$$

This two inequalities show that $f(X)$ is concentrated around $\mathbb{E}f(X)$ within the radius of $t \approx \sqrt{n}$. Here is an example of the application by the inequality.

- Let $\mathcal{G}_{n,p}$ be a random graph over n vertices where each edge is included in the graph independently with probability p. Note that we have m random variables, one indicator variable for each edge being included. Note that the chromatic number of the graph is a function with bounded difference. [Maj17]

3 Open questions and research directions

3.1 Open questions

In 2000, Talagrand [Tal95] proposed more inequalities in dertermining bounds. Generally he extends the Bounded Differences Inequality to further applicable fields. For example, he extends it towards the form of integration. Besides, he also discusses the optimization (or sharpening) on the bounds based on different objects. Can we develop further specific inequalities for more narrowed objects? In other words, for some specific application and problems, can we simplify the Bounded Differences Inequality for more precise using? Also, can we use the original Bounded Differences Inequality to proof other concept?

For example the "Hamming Distance": [Maj17]

Let $x, x' \in \Omega := \Omega_1 \times \ldots \Omega_n$. We define

$$d_H\left(x, x'\right) := \left|\left\{i : x_i \neq x_i'\right\}\right| \tag{39}$$

and define the set $A_k := \{x : x \in \Omega, d_H(x, A) \leq k\}$

Can we prove that

$$P[X \in A] \cdot P\left[d_H(X, A) \geqslant t\right] \leqslant \exp\left(-t^2/2n\right) \tag{40}$$

3.2 Doob construction

A Doob martingale is a construction of a stochastic process to approximates a given random variable. It has the properties of the martingale with respect to the given filtration.

The Doob construction can be used to help prove the Bounded Differences Inequality by following brief steps. [Bar20]

$$\begin{aligned} Y_i &= \mathbb{E}\left[f(X) \mid X_1^i\right] \\ D_i &= Y_i - Y_{i-1} \\ f(X) - \mathbb{E} f(X) &= \sum_{i=1}^n D_i \end{aligned} \tag{41}$$

Then

$$\begin{aligned} D_i = Y_i - Y_{i-1} &= \mathbb{E}\left[f(X) \mid X_1^i\right] - \mathbb{E}\left[f(X) \mid X_1^{i-1}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f(X) \mid X_1^i\right] - f(X) \mid X_1^{i-1}\right] \end{aligned} \tag{42}$$

Thus D_i is a random variable that falls in an interval of length no more than B_i.

Similarly, can we use the Doob construction in proof of extended bounded differences inequalities? For example, the inequalities proposed by Talagrand in 2020.

A Exercises

A.1 Exercise 1: Bernstein's Inequality

Prove the Bernstein's Inequality

If $\mathbb{P}(|X_i| \leq c) = 1$ and $\mathbb{E}(X_i) = \mu$, then for any $\epsilon > 0$,

$$\mathbb{P}\left(\left|\bar{X}_n - \mu\right| > \epsilon\right) \leq 2\exp\left\{-\frac{n\epsilon^2}{2\sigma^2 + 2c\epsilon/3}\right\} \tag{43}$$

Solution:

Lemma A.1. *[JW08] Suppose thath $|X| \leq c$ and $\mathbb{E}(X) = 0$, for any $t > 0$,*

$$\mathbb{E}(e^{tX}) \leq \exp\{t^2\sigma^2(\frac{e^{tc} - 1 - tc}{(tc)^2})\} \tag{44}$$

where $\sigma^2 = Var(X)$

Proof. Let $F = \sum_{r=2}^{\infty} \frac{t^{r-2}\mathbb{E}(X^r)}{r!\sigma^2}$. Then,

$$\mathbb{E}\left(e^{tX}\right) = \mathbb{E}\left(1 + tx + \sum_{r=2}^{\infty} \frac{t^r X^r}{r!}\right) = 1 + t^2\sigma^2 F \leq e^{t^2\sigma^2 F} \tag{45}$$

For $r \geq 2$, $\mathbb{E}\left(X^r\right) = \mathbb{E}\left(X^{r-2}X^2\right) \leq c^{r-2}\sigma^2$.

Therefore, we have

$$F \leq \sum_{r=2}^{\infty} \frac{t^{r-2}c^{r-2}\sigma^2}{r!\sigma^2} = \frac{1}{(tc)^2}\sum_{i=2}^{\infty} \frac{(tc)^r}{r!} = \frac{e^{tc} - 1 - tc}{(tc)^2} \tag{46}$$

Hence,

$$\mathbb{E}\left(e^{tX}\right) \leq \exp\left\{t^2\sigma^2 \frac{e^{tc} - 1 - tc}{(tc)^2}\right\} \tag{47}$$

□

From Lemma A.1, we can assume that $\mu = 0$ for simplicity. Then

$$\mathbb{E}\left(e^{tX_i}\right) \leq \exp\left\{t^2\sigma_i^2 \frac{e^{tc} - 1 - tc}{(tc)^2}\right\} \tag{48}$$

where $\sigma^2 = \mathbb{E}(X_i^2)$.

Then we have

$$\begin{aligned} \mathbb{P}\left(\bar{X}_n > \epsilon\right) &= \mathbb{P}\left(\sum_{i=1}^{n} X_i > n\epsilon\right) = \mathbb{P}\left(e^{t\sum_{i=1}^{n} X_i} > e^{tn\epsilon}\right) \\ &\leq e^{-tn\epsilon}\mathbb{E}\left(e^{t\sum_{i=1}^{n} X_i}\right) = e^{-tn\epsilon}\prod_{i=1}^{n}\mathbb{E}\left(e^{tX_i}\right) \\ &\leq e^{-tn\epsilon}\exp\left\{nt^2\sigma^2 \frac{e^{tc} - 1 - tc}{(tc)^2}\right\} \end{aligned} \tag{49}$$

Take $t = (1/c)\log(1 + \epsilon c/\sigma^2)$ we will have"

$$\mathbb{P}\left(\bar{X}_n > \epsilon\right) \leq \exp\left\{-\frac{n\sigma^2}{c^2} h\left(\frac{c\epsilon}{\sigma^2}\right)\right\} \tag{50}$$

where $h(u) = (1 + u)\log(1 + u) - u$.

By noting that $h(u) \geq u^2/(2 + 2u/3)$ for $u \geq 0$, we just show the Bernstein's Inequality.

A.2 Exercise 2

Prove that for any $z > 0$, if $m \geq 3(z_3)\ln(z+3)$, then $\frac{m}{\ln m} > z$. [Kut02]

Solution:

Firstly, we note that

$$\frac{d}{dm}\frac{m}{\ln m} = \frac{\ln m - 1}{\ln^2 m} \tag{51}$$

Thus, the term $\frac{m}{\ln m}$ is increasing when $m > e$.

Then, given that $\ln(z+3) \geq \ln\ln(z+3)$, and $z > 0 \Rightarrow \ln(z+3) > \ln 3$. Hence,

$$\begin{aligned} \frac{m}{\ln m} &\geq \frac{3(z+3)\ln(z+3)}{\ln 3 + \ln(z+3) + \ln\ln(z+3)} \\ &> \frac{3(z+3)\ln(z+3)}{3\ln(z+3)} = z + 3 > z \end{aligned} \tag{52}$$

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