# UCSB, Physics 129AL, Computational Physics: Problem Set 2

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### GitHub Submission Guideline

We will use GitHub for problem set submissions. By the due date, you should have a single public repository on GitHub containing all the work you have done for this problem set. Finally, upload a screenshot or a .txt file to Canvas with your GitHub username and repository name so the TA knows who you are and which repository you are using for the problem set.

**Remember: talk to your fellow students, work together, and use GPTs. You will find it much easier than working alone. Good luck! All work should be done in the Docker container, and don't forget to commit it to Git!**

## Problem 1: Coordinate transformation and Parallel Transport

In this problem, we will be looking at the coordinate transformation and parallel transports on a sphere.

**a)** Write down the coefficients of three coordinate systems that describe a point $(1, \theta, \phi)$ on the unit sphere,

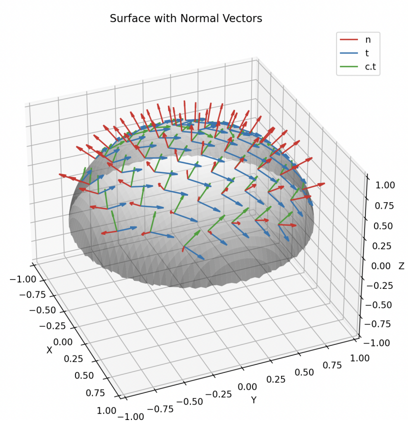
- Spherical coordinate $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\} \to (1, \theta, \phi)$.
- Cartesian coordinate $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\} \to (\sin(\theta)\cos(\phi), \dots)$
- Cylindrical coordinate $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\psi, \hat{\mathbf{z}}\} \to (\dots)$

Write down the spherical and cylindrical basis in terms of the Cartesian basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$. Write a python function or multiple python functions that convert coordinates and basis between the three.

**b)** A position vector in the unit sphere is $r = \hat{\mathbf{e}}_r$ such that the coordinate $(1, \theta, \phi)$. Create local **orthonormal** coordinate systems on the unit sphere and represent them as vectors in Cartesian coordinate system. You should reproduce something like the plot below.

**c)** Can you plot the unit sphere in spherical basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ $(1, \theta, \phi)$? If so, plot it. If not, explain why. It should be very simple, and don't over think. Hint: You can see that in Cartesian coordinate system, basis are independent of coordinates. However, when we change to spherical basis, the basis itself becomes coordinate-dependent. (you can move around a vector for free, i.e. trivial parallel transport in the Cartesian coordinate, but in the spherical coordinate system, parallel transports of vectors will take some efforts.

**d)** Create a function that generates the local coordinate system on a given mesh, parametrized by a general surface $z = f(x, y)$. Hint: use "np.gradient(f(x, y), dx(or, dy), axis=1(or, 0))". You should realize that on the surface, the normals $r = \hat{\mathbf{e}}_r$ are purely extrinsic: it extends to the ambient space that depends on how a surface is situated within the ambient space.



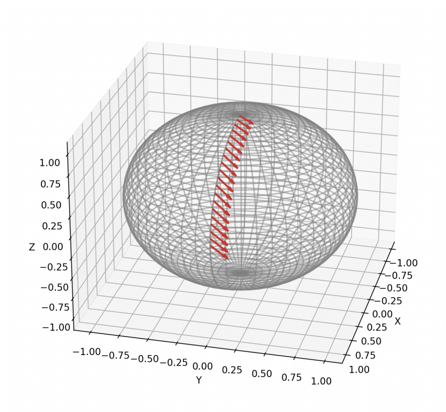

**e)** As what I discussed and demonstrated in class, parallel transports on spheres are very important and illustrative. First, follow my lecture (assume you took notes), write a code that demonstrate the parallel transport of a vector $\mathbf{n}(\theta_0, \phi = 0)$ near the north pole $[r = 1, \phi = 0, \theta = \pi/5]$ to the equator at $[r = 1, \phi = 0, \theta = \frac{\pi}{2}]$ following the unit-speed parametrization,

$$\gamma(t) = t\hat{\mathbf{e}}_\theta(t) = \theta\hat{\mathbf{e}}_\theta = \gamma(\theta). \tag{1}$$

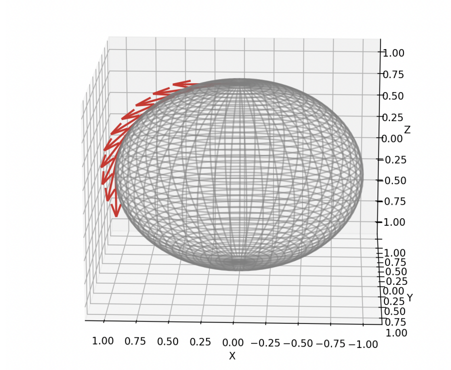
The We have the following initial condition:

$$\mathbf{n}(\theta_0, \phi = 0) = \alpha\, \hat{\mathbf{e}}_\theta + \beta\, \hat{\mathbf{e}}_\phi = \alpha\, \hat{\mathbf{e}}_\theta + \beta \sin(\theta_0)\, \mathbf{e}_\phi, \qquad \alpha^2 + \beta^2 \sin^2(\theta_0) = |\mathbf{n}|^2. \tag{2}$$

Hint: Check your vector normalization during the parallel transport. You should reproduce something like the plot below,



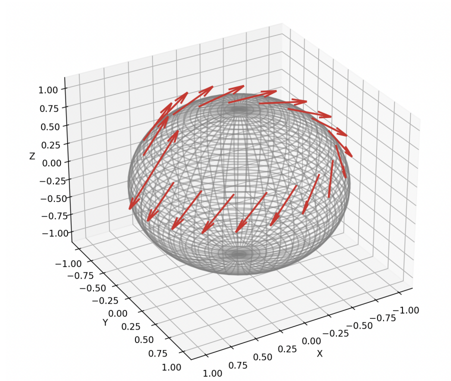

and,

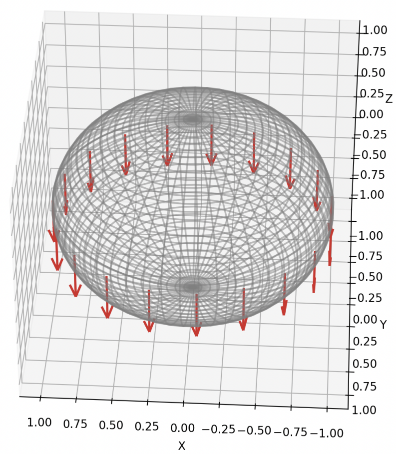


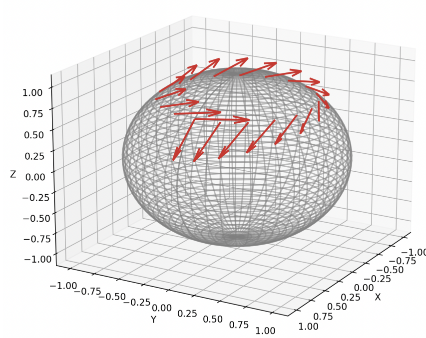

**f)** Also follow my lecture (assume you took notes for this), write a code that demonstrate the parallel transport of a vector from $[r = 1, \phi = 0, \theta = \theta_0]$ to $[r = 1, \phi = 2\pi, \theta = \theta_0]$ with the following the unit-speed parametrization,

$$\gamma(t) = t\hat{\mathbf{e}}_\phi(t) = \phi\hat{\mathbf{e}}_\phi = \gamma(\phi). \tag{3}$$

You should reproduce something like the plot below.



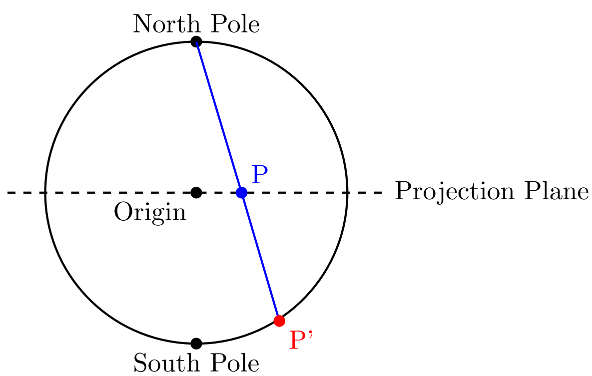

**g)** The difference in transported vector is call the **holonomy**: It describes how a vector changes when it is parallel transported around a closed loop on a manifold. Make a plot that measure the strength for different $\theta_0$. Hint: x will be $\theta_0$, y will be the inner product between vector before the parallel transport and after the transport (one loop, $0 \to 2\pi$).

## Problem 2: Geometric Transformations

You will be looking at a famous geometric transformation: the **Stereographic projection** of the unit sphere, defined by the map $S$,

$$\mathbf{P}' = S(\mathbf{P}) = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{1-z}, \quad z < \infty, \tag{4}$$

where the $\mathbf{P} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ is a point on the unit sphere, and $\mathbf{P}'$ is the projected point on the x-y plane. You should note the coordinate system: It is in the Cartesian coordinate system.




**a)** Numerically show this transformation is conformal: you should demonstrate and show a contour map with matplotlib that angle between any two curves on the unit sphere is preserved. You should have two subplots: one shows a few pairs of curves on the unit sphere and the other shows those pairs after the stereographic projection. Label the angles and curves.

Hint: it will be a good practice to first generate the unit sphere mesh, and define one curve and measure the orthogonality of its tangent vectors with other curves that have intersections different points.

**b)** As we discussed in the class, the geodesics of a unit sphere are "great circles". What are those look like after the stereographic projection? You should have two subplots: one shows great circles on the unit sphere and the other shows those curves after the stereographic projection.

**c)**: Use your result from Problem 1, for each unit-speed parametrization, plot parallel transport trajectories of a closed loop for various initial vectors under the stereographic projection.

**d)** Plot inner products between two vectors at the same point after the stereographic projection. Does it preserved the inner product after the stereographic projection? Hint: conformal transformation.

**f)** Can the stereographic projection alter the **holonomy** on the unit sphere when parallel transport (use one loop, $0 \to 2\pi$).

# Problem 3: Lifting Map and Delaunay triangulation

In the section 2, you have made some progress in constructing convex hull in 2D and mesh in 3D. In this problem, you are asked to study some intrinsic and extrinsic properties of 3D meshes generated via various lifting maps, e.g. parabolic lifting map we discussed in the lecture. You will also look at how the convex hull transformed in 3D.

**a)** Use the point cloud I gave in the section, plot the convex hull and generate the Delaunay triangulation.

**b)** Now, lift the 2D mesh to the third dimension by the following lifting map:

$$z = f(x, y) = x^2 + y^2, \quad x, y \in [-2, 2], \tag{5}$$

and for each triangle, calculate the change in its area. Plot it as a 2D heatmap. Hint: the grid will be the non-transformed x,y grid, and the color will be the area ratio between two triangles before and after lifting.

**c)** Calculate the induced metric, resulted by the lifting. Hint: while you can find it numerically (try it, and it is not hard), you can use the analytic form I gave above. Do not use the metric of the surface when doing the inner product. The inner product is defined trivially in the ambient space.

**d)** Calculate the surface normal of the lifted mesh and plot it on top of the the mesh.

**e)** Calculate the vertex normal of the lifted mesh and plot it on top of the the mesh.

**f)** Compute the second fundamental form (look at the lecture notes) using the vertex normal. You do not need to calculate the second derivative numerically: use the analytical form. Hint: make sure you specify the vector direction when preform the inner product with the vertex normal, and do not use the metric of the surface. The inner product is defined trivially in the ambient space.

**g)** Calculate the shape operator for all vertexes on the lifted surface numerically (look at the lecture notes). Calculate the principle, Gaussian, mean curvatures for those points and visualize them. Hint: you need to diagonalize a 2 by 2 matrix at each point.

**h)** Now, repeat **b-g** for a different lifting map:

$$z = f(x, y) = x^2 + xy + y^2, \quad x, y \in [-2, 2]. \tag{6}$$

**i)** While **i) is not required** for this problem set, if you have time want to think about it, think about how to numerically determine the geodesics on a given lifted surface without explicitly solving the geodesic equation? Think about it.