UCSB, Physics 129AL, Computational Physics
Lecture notes, Week 2

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January 13, 2025

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0.0.1 Embedding

The distinction between intrinsic and extrinsic properties of surfaces is fundamental in differential geometry and has significant implications in various fields. Intrinsic properties, such as Gaussian curvature, depend solely on measurements made within the surface and are invariant under isometric deformations (transformations of surfaces that preserve the lengths of curves on the surface). In contrast, extrinsic properties, like mean curvature, rely on how the surface is embedded in a higher dimensional space, or the **ambient space**.

This is particular useful if we want to study interactions between embedded object and environment (such as electric fields outside of a solid sphere). In numerical calculations, it is usually used to calculate surface dynamics of fields where interactions (as the field is 3D) are important. Let's consider example, a line in $\mathbb{R}$ with length L. We embed it in $\mathbb{R}^n$. Regardless how the line is bended, the length of the line remains the same. Therefore, distance between two points on the line is the intrinsic. On the other hand, the degrees of freedom allows the line to move around and bend (isometries are transformations that preserve the distance between points on the manifold). The curvature induced is extrinsic. This has great implications in graphics. Let's say you want to project a wall paper to a cup, cylinder. What can you do? You need to find a isometry that preserves the distance between any given two points. You can only do that without "connecting the back", locally isometric to a plane. What about a sphere? You can't find a continuous bijective mapping between them with a continuous inverse. This defines the concept of topology.

The **first fundamental form** describes the intrinsic property of the embedded surface within an **ambient space**,

$$\mathrm{I} = ds^2 = [\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}]dudv = g_{ij}dudv, \tag{1}$$

where g_{ij} are the components of the surface metric.

We should note that the inner product used here is the metric of the ambient space. Explicitly, for a surface parameterized by coordinates $\mathbf{r}(u, v)$, it becomes, $ds^2 = (\partial_u \mathbf{r})^2 du^2 + 2(\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) dudv + (\partial_v \mathbf{r})^2 dv^2$. This can be written in a matrix form,

$$\mathrm{I} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} (\partial_u \mathbf{r})^2 & (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) \\ (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) & (\partial_v \mathbf{r})^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{2}$$

where you should note that the change in the metric from a (0,2) tensor to a (1,1) tensor, that takes one vector and one covector.

On the other hand, let's consider the surface variation $\delta \mathbf{r}(u, v)$ with respect to a unit normal direction with strength g, namely the deformed surface,

$$\mathbf{R}(u, v, g) = \mathbf{r}(u, v) \pm g\mathbf{N}(u, v). \tag{3}$$

The first fundamental form of this deformed surface is given by,

$$\begin{aligned} dS^2 =& (\partial_u \mathbf{R})^2 du^2 + 2(\partial_u \mathbf{R}) \cdot (\partial_v \mathbf{r}) dudv + (\partial_v \mathbf{R})^2 dv^2 \\ \approx& ds^2 \pm 2g[(\partial_u \mathbf{r})(\partial_u \mathbf{N}) du^2 + (\partial_v \mathbf{r})(\partial_v \mathbf{N}) dv^2 + (\partial_u \mathbf{r})(\partial_v \mathbf{N}) dudv + (\partial_v \mathbf{r})(\partial_u \mathbf{N}) dudv] \\ =& ds^2 \pm 2g\mathrm{II}, \end{aligned} \tag{4}$$

such that,

$$\frac{dS^2 - ds^2}{2g} = \mathrm{II} = [(\partial_u^2 \mathbf{r}) \cdot \mathbf{N} du^2 + (\partial_v^2 \mathbf{r}) \cdot \mathbf{N} dv^2 + 2(\partial_u \partial_v \mathbf{r}) \cdot \mathbf{n} dudv], \tag{5}$$

where we use the fact that $\partial_v \mathbf{r} \cdot \mathbf{n} = 0$. II is the second fundamental form that captures the variation with respect to local orthogonal deformations. It is a measure of the extrinsic property of the embedded surface,

$$\mathrm{II} = d\mathbf{N}(u, v) \cdot [\partial_u \mathbf{r} du + \partial_v \mathbf{r} dv] = [\partial_u \mathbf{N} du + \partial_v \mathbf{N} dv] \cdot [\partial_u \mathbf{r} du + \partial_v \mathbf{r} dv], \tag{6}$$

The inner product used here is the metric of the ambient space.

$$\mathrm{II} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} (\partial_u^2 \mathbf{r}) \cdot \mathbf{N} & (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} \\ (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} & (\partial_v^2 \mathbf{r}) \cdot \mathbf{N} \end{pmatrix} \begin{pmatrix} du \\ dv. \end{pmatrix} \tag{7}$$

The second fundamental form describes how the surface normal varies in a neighborhood, which has an alternative form.

$\mathcal{S}$ is the **shape operator** (Weingarten Map), defined via the ratio of the following form,

$$\mathcal{S}(u, v) = \begin{pmatrix} (\partial_u \mathbf{r})^2 & (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) \\ (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) & (\partial_v \mathbf{r})^2 \end{pmatrix}^{-1} \begin{pmatrix} (\partial_u^2 \mathbf{r}) \cdot \mathbf{N} & (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} \\ (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} & (\partial_v^2 \mathbf{r}) \cdot \mathbf{N}. \end{pmatrix} \tag{8}$$

The **principle curvatures** at (u, v) are the eigenvalues of the shape operator and the principal directions are the eigenvectors.

The **Gaussian curvature** K is the product of the principal curvatures, or the determinant of the shape operator.

The **mean curvature** H is the average of the principal curvatures, or the trace of the shape operator.

These forms are crucial in differential geometry and have important applications in general relativity and other areas of theoretical physics, and mesh generations.

0.0.2 Geodesic

Let's say there is a curve $\gamma(t)$ (vector), parameterized by t on a surface S. We want to know paths with extrema distance between two end points t_0, t_1. It is called **geodesic** if at every point in between, the acceleration $\partial_t^2\gamma(t)$ is either zero or parallel to its unit normal. In other words, the transport is at constant speed along geodesics $\partial_t(\partial_t\gamma(t))^2 = \partial_t^2\gamma(t)\cdot\partial_t\gamma(t) = 0$. Let's consider the arc length functional,

$$E[\gamma(t)] = \int\sqrt{ds^2} = \int\sqrt{g_{ij}dx^i dx^j} = \int dt\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = \int_{t_0}^{t_1} dt\sqrt{g_{ij}[\gamma(t)][\partial_t\gamma(t)]^i[\partial_t\gamma(t)]^j} \tag{9}$$

To find the geodesics, we look for critical points of $E[\gamma]$ under small variations of the curve. Consider a variation of the curve $\gamma(t)$:

$$\gamma^\mu(t,\epsilon) = \gamma^\mu(t) + \epsilon\eta^\mu(t),$$

where, $\eta^\mu(t)$ is the variation of the μ component, and ϵ is a small parameter.

The first variation of $E[\gamma^\mu]$ is:

$$\delta E[\gamma^\mu] = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} E[\gamma^\mu(t,\epsilon)].$$

Expanding $E[\gamma]$ under variation:

$$E[\gamma^\mu(t,\epsilon)] = \int_{t_0}^{t_1} g_{ij}[\gamma(t)+\epsilon\eta(t)]\left(\frac{d(\gamma^i+\epsilon\eta^i)}{dt}\right)\left(\frac{d(\gamma^j+\epsilon\eta^j)}{dt}\right)dt. \tag{10}$$

Differentiating with respect to ϵ and evaluating at $\epsilon = 0$:

$$\begin{aligned} E[\gamma(t,\epsilon)] &= E[\gamma(t)] + \int_{t_1}^{t_2}\frac{\partial g_{ij}}{\partial[\gamma^g-\epsilon\eta^g]}\eta^g\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}dt + 2\int_{t_1}^{t_2} g_{ij}\frac{d\gamma^i}{dt}\frac{d\eta^j}{dt} + \mathcal{O}(\epsilon) \\ \delta E[\gamma(t,0)] &= \int_{t_1}^{t_2}\frac{\partial g_{ij}}{\partial\gamma^g}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\eta^g dt - 2\int_{t_1}^{t_2} g_{ij}\frac{d^2\gamma^i}{dt^2}\eta^j + \mathcal{O}(\epsilon) \end{aligned} \tag{11}$$

where the second line we use the integration by parts. Since for any η^j, this equation must hold, we have,

$$\frac{\partial g_{ij}}{\partial \gamma^g}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} - 2g_{ij}\frac{d^2\gamma^i}{dt^2} = 0 \tag{12}$$

Since we can change index,

$$-\frac{1}{2}g^{ij}\left[\frac{\partial g_{uj}}{\partial \gamma^g}\frac{d\gamma^v}{dt}\frac{d\gamma^j}{dt}\right] + \frac{d^2\gamma^i}{dt^2} = 0 \tag{13}$$

the geodesic equation becomes,

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{uj}\frac{d\gamma^u}{dt}\frac{d\gamma^j}{dt} = 0,$$

and it describes the paths that minimize "an energy functional", which correspond to the shortest (or extrema) paths between two points on the manifold. We will talk more when we working with classical phase space evolutions.

0.0.3 parallel transport

Metric compatibility ensures the inner product remain consistent when **parallel transported** to different parts of the region. For example, let's move vectors along a curve $\gamma(t)$ such that the vector $\mathbf{n}$ does not change with time,

$$\frac{\partial}{\partial t}\mathbf{n} = \frac{\partial \gamma}{\partial t}\cdot\frac{\partial}{\partial \gamma}\mathbf{n} = \frac{\partial \gamma^j}{\partial t}[\partial_j n^i + n^k\Gamma^i_{jk}]\hat{\mathbf{e}}_{\mathbf{i}} = [\frac{\partial n^i}{\partial t} + n^k\Gamma^i_{jk}\gamma^j]\hat{\mathbf{e}}_{\mathbf{i}} = 0. \tag{14}$$

0.0.4 Tensors

In a given coordinate system with basis vectors $\{e_i\}$ and dual basis vectors $\{\omega^i\}$, a tensor is mutilinear map that expressed as:

$$\mathcal{T} = T^{i_1 i_2\cdots i_q}_{j_1 j_2\cdots j_p}\, e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes\omega^{j_1}\otimes\omega^{j_2}\otimes\cdots, \tag{15}$$

where $T^{i_1 i_2\cdots i_q}_{j_1 j_2\cdots j_p}$ is the tensor component, and

- Upper indices $(i_1, i_2, \ldots)$ denote contravariant components.
- Lower indices $(j_1, j_2, \ldots)$ denote covariant components.

In physics, it is common to write a tensor as a sum of tensor products between m vector spaces $v \in V\otimes V\otimes V\otimes\cdots$ and/or n dual spaces $\omega \in V^*\otimes V^*\otimes V^*\otimes\cdots$,

$$\mathcal{T} = \sum_i v^i\otimes\omega_i. \tag{16}$$

For example, the simplest case is the density matrix,

$$\rho = \sum_i |\phi_i\rangle\langle\psi_j| = \sum_i U^\dagger |\phi_i'\rangle\langle\psi_j'|\, U, \tag{17}$$

which transforms in their respective spaces.