UCSB, Physics 129AL, Computational Physics
Lecture notes, Week 2

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0.0.1 Embedding

The distinction between intrinsic and extrinsic properties of surfaces is fundamental in differential geometry and has significant implications in various fields. Intrinsic properties, such as Gaussian curvature, depend solely on measurements made within the surface and are invariant under isometric deformations (transformations of surfaces that preserve the lengths of curves on the surface). In contrast, extrinsic properties, like mean curvature, rely on how the surface is embedded in a higher dimensional space, or the **ambient space**.

This is particular useful if we want to study interactions between embedded object and environment (such as electric fields outside of a solid sphere). In numerical calculations, it is usually used to calculate surface dynamics of fields where interactions (as the field is 3D) are important. Let's consider example, a line in $\mathbb{R}$ with length L. We embed it in $\mathbb{R}^n$. Regardless how the line is bended, the length of the line remains the same. Therefore, distance between two points on the line is the intrinsic. On the other hand, the degrees of freedom allows the line to move around and bend (isometries are transformations that preserve the distance between points on the manifold). The curvature induced is extrinsic. This has great implications in graphics. Let's say you want to project a wall paper to a cup, cylinder. What can you do? You need to find a isometry that preserves the distance between any given two points. You can only do that without "connecting the back", locally isometric to a plane. What about a sphere? You can't find a continuous bijective mapping between them with a continuous inverse. This defines the concept of topology.

The **first fundamental form** describes the intrinsic property of the embedded surface within an **ambient space**,

$$\mathrm{I} = ds^2 = [\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}]dudv = g_{ij}dudv, \tag{1}$$

where g_{ij} are the components of the surface metric.

We should note that the inner product used here is the metric of the ambient space. Explicitly, for a surface parameterized by coordinates $\mathbf{r}(u,v)$, it becomes, $ds^2 = (\partial_u \mathbf{r})^2 du^2 + 2(\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) dudv + (\partial_v \mathbf{r})^2 dv^2$. This can be written in a matrix form,

$$\mathrm{I} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} (\partial_u \mathbf{r})^2 & (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) \\ (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) & (\partial_v \mathbf{r})^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{2}$$

where you should note that the change in the metric from a (0,2) tensor to a (1,1) tensor, that takes one vector and one covector.

On the other hand, let's consider the surface variation $\delta \mathbf{r}(u,v)$ with respect to a unit normal direction with strength g, namely the deformed surface,

$$\mathbf{R}(u,v,g) = \mathbf{r}(u,v) \pm g\mathbf{N}(u,v). \tag{3}$$

The first fundamental form of this deformed surface is given by,

$$\begin{aligned} dS^2 =& (\partial_u \mathbf{R})^2 du^2 + 2(\partial_u \mathbf{R}) \cdot (\partial_v \mathbf{r}) dudv + (\partial_v \mathbf{R})^2 dv^2 \\ \approx& ds^2 \pm 2g[(\partial_u \mathbf{r})(\partial_u \mathbf{N})du^2 + (\partial_v \mathbf{r})(\partial_v \mathbf{N})dv^2 + (\partial_u \mathbf{r})(\partial_v \mathbf{N})dudv + (\partial_v \mathbf{r})(\partial_u \mathbf{N})dudv] \\ =& ds^2 \pm 2g\mathrm{II}, \end{aligned} \tag{4}$$

such that,

$$\frac{dS^2 - ds^2}{2g} = \mathrm{II} = [(\partial_u^2 \mathbf{r}) \cdot \mathbf{N} du^2 + (\partial_v^2 \mathbf{r}) \cdot \mathbf{N} dv^2 + 2(\partial_u \partial_v \mathbf{r}) \cdot \mathbf{n} dudv], \tag{5}$$

where we use the fact that $\partial_v \mathbf{r} \cdot \mathbf{n} = 0$. II is the second fundamental form that captures the variation with respect to local orthogonal deformations. It is a measure of the extrinsic property of the embedded surface,

$$\mathrm{II} = d\mathbf{N}(u,v) \cdot [\partial_u \mathbf{r} du + \partial_v \mathbf{r} dv] = [\partial_u \mathbf{N} du + \partial_v \mathbf{N} dv] \cdot [\partial_u \mathbf{r} du + \partial_v \mathbf{r} dv], \tag{6}$$

The inner product used here is the metric of the ambient space.

$$\mathrm{II} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} (\partial_u^2 \mathbf{r}) \cdot \mathbf{N} & (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} \\ (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} & (\partial_v^2 \mathbf{r}) \cdot \mathbf{N} \end{pmatrix} \begin{pmatrix} du \\ dv. \end{pmatrix} \tag{7}$$

The second fundamental form describes how the surface normal varies in a neighborhood, which has an alternative form.

$\mathcal{S}$ is the **shape operator** (Weingarten Map), defined via the ratio of the following form,

$$\mathcal{S}(u,v) = \begin{pmatrix} (\partial_u \mathbf{r})^2 & (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) \\ (\partial_u \mathbf{r}) \cdot (\partial_v \mathbf{r}) & (\partial_v \mathbf{r})^2 \end{pmatrix}^{-1} \begin{pmatrix} (\partial_u^2 \mathbf{r}) \cdot \mathbf{N} & (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} \\ (\partial_u \partial_v \mathbf{r}) \cdot \mathbf{N} & (\partial_v^2 \mathbf{r}) \cdot \mathbf{N}. \end{pmatrix} \tag{8}$$

The **principle curvatures** at (u, v) are the eigenvalues of the shape operator and the principal directions are the eigenvectors.

The **Gaussian curvature** K is the product of the principal curvatures, or the determinant of the shape operator.

The **mean curvature** H is the average of the principal curvatures, or the trace of the shape operator.

These forms are crucial in differential geometry and have important applications in general relativity and other areas of theoretical physics, and mesh generations.

0.0.2 Geodesic

Let's say there is a curve $\gamma(t)$ (vector), parameterized by t on a surface S. We want to know paths with extrema distance between two end points t_0, t_1. It is called **geodesic** if at every point in between, the acceleration $\partial_t^2\gamma(t)$ is either zero or parallel to its unit normal. In other words, the transport is at constant speed along geodesics $\partial_t(\partial_t\gamma(t))^2 = \partial_t^2\gamma(t) \cdot \partial_t\gamma(t) = 0$. Let's consider the arc length functional,

$$E[\gamma(t)] = \int \sqrt{ds^2} = \int \sqrt{g_{ij}dx^i dx^j} = \int dt\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = \int_{t_0}^{t_1} dt\sqrt{g_{ij}[\gamma(t)][\partial_t\gamma(t)]^i[\partial_t\gamma(t)]^j} \tag{9}$$

To find the geodesics, we look for critical points of $E[\gamma]$ under small variations of the curve. Consider a variation of the curve $\gamma(t)$:

$$\gamma^\mu(t, \epsilon) = \gamma^\mu(t) + \epsilon\eta^\mu(t),$$

where, $\eta^\mu(t)$ is the variation of the μ component, and ϵ is a small parameter.

The first variation of $E[\gamma^\mu]$ is:

$$\delta E[\gamma^\mu] = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} E[\gamma^\mu(t, \epsilon)].$$

Expanding $E[\gamma]$ under variation:

$$E[\gamma^\mu(t, \epsilon)] = \int_{t_0}^{t_1} g_{ij}[\gamma(t) + \epsilon\eta(t)]\left(\frac{d(\gamma^i + \epsilon\eta^i)}{dt}\right)\left(\frac{d(\gamma^j + \epsilon\eta^j)}{dt}\right)dt. \tag{10}$$

Differentiating with respect to ϵ and evaluating at $\epsilon = 0$:

$$\begin{aligned} E[\gamma(t, \epsilon)] &= E[\gamma(t)] + \int_{t_1}^{t_2} \frac{\partial g_{ij}}{\partial[\gamma^g - \epsilon\eta^g]}\eta^g\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}dt + 2\int_{t_1}^{t_2} g_{ij}\frac{d\gamma^i}{dt}\frac{d\eta^j}{dt} + \mathcal{O}(\epsilon) \\ \delta E[\gamma(t, 0)] &= \int_{t_1}^{t_2} \frac{\partial g_{ij}}{\partial\gamma^g}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\eta^g dt - 2\int_{t_1}^{t_2} g_{ij}\frac{d^2\gamma^i}{dt^2}\eta^j + \mathcal{O}(\epsilon) \end{aligned} \tag{11}$$

where the second line we use the integration by parts. Since for any η^j, this equation must hold, we have,

$$\frac{\partial g_{ij}}{\partial \gamma^g}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} - 2g_{ij}\frac{d^2\gamma^i}{dt^2} = 0 \tag{12}$$

Since we can change index,

$$-\frac{1}{2}g^{ij}\left[\frac{\partial g_{uj}}{\partial \gamma^g}\frac{d\gamma^v}{dt}\frac{d\gamma^j}{dt}\right] + \frac{d^2\gamma^i}{dt^2} = 0 \tag{13}$$

the geodesic equation becomes,

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{uj}\frac{d\gamma^u}{dt}\frac{d\gamma^j}{dt} = 0,$$

and it describes the paths that minimize "an energy functional", which correspond to the shortest (or extrema) paths between two points on the manifold. We will talk more when we working with classical phase space evolutions.

0.0.3 parallel transport

Metric compatibility ensures that inner products remain consistent when vectors are parallel transported to different regions. In the local frame, where vectors are expressed relative to an **orthonormal basis** $\mathbf{e}_i$ that moves along with the curve, the angle between a parallel-transported vector and the local tangent vector remains constant. This means that if you hold a rod pointing in one direction (not necessarily normal), you will not notice any change from your perspective in the local frame with an orthonormal basis. However, an external observer may see that the direction of your rod changes as you move along the curve. If your rod happens to point along the tangent vector of your path and remains aligned with it, then your path is a geodesic curve.

It is important to note that this property holds only in the local orthonormal basis $\mathbf{e}_i$, not in the coordinate basis $\hat{\mathbf{e}}_i$. The coordinate basis is not normalized and scales according to the metric, while the vector components counteract this scaling to preserve consistency. Specifically, in the coordinate basis, even locally, there are changes in the vector components, i.e., $\hat{\mathbf{e}}_i \sim \sqrt{g_{ii}}\mathbf{e}_i$.

For example, let's move vectors along a curve $\gamma(t)$ such that the vector $\mathbf{n}$ does not change with time at the local frame,

$$\frac{\partial}{\partial t}\mathbf{n} = \frac{\partial \gamma}{\partial t}\cdot\frac{\partial}{\partial \gamma}\mathbf{n} = \frac{\partial \gamma^j}{\partial t}[\partial_j n^i + n^k\Gamma^i_{jk}]\hat{\mathbf{e}}_\mathbf{i} = [\frac{\partial n^i}{\partial t} + n^k\Gamma^i_{jk}\frac{\partial \gamma^j}{\partial t}]\hat{\mathbf{e}}_\mathbf{i} = 0. \tag{14}$$

The equation for parallel transport is then given by,

$$\frac{\partial n^i}{\partial t} + n^k\Gamma^i_{jk}\frac{\partial \gamma^j}{\partial t} = 0, \tag{15}$$

and you should note that $\mathbf{V} = \frac{\partial \gamma^j}{\partial t}$ is the velocity. Sometimes, you might see people use the notation with directional derivative along the tangent vector $\mathbf{V}$,

$$\nabla_V \mathbf{n} = \mathbf{V} \cdot (\nabla_j n^i)\hat{\mathbf{e}}_i = 0. \tag{16}$$

Let's consider what happens when parallel transporting a vector $\mathbf{n}$ on the unit sphere. We have two natural parameters, $\phi(t), \theta(t)$ with the standard parametrization t. Non-zero Christoffel Symbols on the sphere are given by,

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta, \tag{17}$$

with metric,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{18}$$

Let's first consider the transport of a vector near the north pole $[r = 1, \phi = 0, \theta > 0]$ to the equator at $[r = 1, \phi = 0, \theta = \frac{\pi}{2}]$ following the unit-speed parametrization,

$$\gamma(t) = 0 + t\hat{\mathbf{e}}_\theta(t) = \theta\hat{\mathbf{e}}_\theta = \gamma(\theta). \tag{19}$$

the curve is unit-speed since $|d\gamma(t)/dt| = 1$ (**noted the norm**, not the vector), and you are following a curve that is in the tangent direction of the unit sphere.

Let's start with a vector $\mathbf{n}_0$ near the north pole on the tangent plane (initial condition), and we want to transport it to the equator. Let's look at the transport equations,

$$\begin{aligned} \partial_\phi n^i + n^k \Gamma^i_{\phi k}\frac{\partial \gamma^\phi}{\partial t} &= 0, \\ \partial_\theta n^i + n^k \Gamma^i_{\theta k}\frac{\partial \gamma^\theta}{\partial t} &= 0. \end{aligned} \tag{20}$$

We write the non-zero terms explicitly,

$$\begin{aligned} \partial_\phi n^\phi + n^\theta \Gamma^\phi_{\phi\theta}\frac{\partial \gamma^\phi}{\partial t} &= \partial_\phi n^\phi + n^\theta \cot\theta \frac{\partial \gamma^\phi}{\partial t} = 0 \\ \partial_\phi n^\theta + n^\phi \Gamma^\theta_{\phi\phi}\frac{\partial \gamma^\phi}{\partial t} &= \partial_\phi n^\theta - \sin\theta\cos\theta n^\phi \frac{\partial \gamma^\phi}{\partial t} = 0 \\ \partial_\theta n^\phi + n^\phi \Gamma^\phi_{\theta\phi}\frac{\partial \gamma^\theta}{\partial t} &= \partial_\theta n^\phi + n^\phi \cot\theta \frac{\partial \gamma^\theta}{\partial t} = 0. \end{aligned} \tag{21}$$

The unit-speed condition $\frac{\partial \gamma^\theta}{\partial t} = 1$ gives,

$$\partial_\theta n^\phi + n^\phi \cot\theta = 0, \tag{22}$$

with solution,

$$n^\phi(\theta) = \frac{A}{\sin(\theta)}, \tag{23}$$

where A is a constant. You might find problematic at $\theta = 0$ because $\hat{\mathbf{e}}_\phi$ is undefined. It is one reason that when transport vectors, we avoid at the exact $\theta = 0$ point for the unit-speed parametrization if there are non-zero ϕ components. When at $\pi/2$, it seems like they will have the same component, but this does not mean that the overall direction of the vector is unchanged since they have different local basis.

Let's summarize the concept of parallel transport. 1) The vector is parallel (does not change) in the local frame at every point along the curve, e.g. the angle between the vector and the local tangent is unchanged in the local coordinate system. Those local coordinate are not normalized, and will accumulate changes as we move to different regions. For example, the basis in the $\hat{\mathbf{e}}_\phi \sim \sqrt{g_{\phi\phi}}\mathbf{e}_\phi = \sin(\theta)\mathbf{e}_\phi$ related to $\mathbf{e}_\phi$, an orthonormal basis we discussed previously. To compensate the scaling in the basis so that the vector will remain invariant, we must introduce a change in vector components. Using our previous example, $n^\phi(\theta) = \frac{A}{\sin(\theta)}$, it is now obvious that the scaling factor $\sin(\theta)$ will cancel the scaling introduced by the non-normalized basis (coordinate basis $\hat{\mathbf{e}}_\phi$). Therefore, with the local orthonormal basis $\mathbf{e}_\phi$, the vector component is fixed $n^\phi(\theta) = A$.

The inner product will be preserved if two vectors are parallel transported along the same curve. However, unless it is on geodestics, the inner product between one parallel transported vector and the tangent vector will not be the same. Therefore, you will see the vector being parallel transported "moves".

Let's say we initially start with a vector,

$$\mathbf{n}(\theta_0, \phi = 0) = \alpha\,\hat{\mathbf{e}}_\theta + \beta\,\hat{\mathbf{e}}_\phi = \alpha\,\mathbf{e}_\theta + \beta\sin(\theta_0)\,\mathbf{e}_\phi, \qquad \alpha^2 + \beta^2\sin^2(\theta_0) = |\mathbf{n}|^2. \tag{24}$$

and we transport it from somewhere near the north pole to the equator at $\phi = 0$. We first note that n^θ is independent of θ,

$$\partial_\theta n^\theta + n^\phi \Gamma^\theta_{\theta\phi} \frac{\partial \gamma^\theta}{\partial t} = 0 \to \partial_\theta n^\theta = 0, \tag{25}$$

and independent of ϕ,

$$\partial_\phi n^\theta - \sin\theta\cos\theta n^\phi \frac{\partial \gamma^\phi}{\partial t} = 0 \to \partial_\phi n^\theta = 0. \tag{26}$$

This implies that n_0^θ will remain the same as it transports to the equator. We know from the previous result that the ϕ component of the vector transform like the following:

$$n^\phi(\theta) = \frac{A}{\sin(\theta)}, \tag{27}$$

Then, the transformed vector becomes,

$$\mathbf{n}(\theta, \phi = 0) = \alpha\,\hat{\mathbf{e}}_\theta + \frac{\beta\,\sin(\theta_0)}{\sin(\theta)}\,\hat{\mathbf{e}}_\phi = \alpha\,\mathbf{e}_\theta + \beta\,\sin(\theta_0)\,\mathbf{e}_\phi. \tag{28}$$

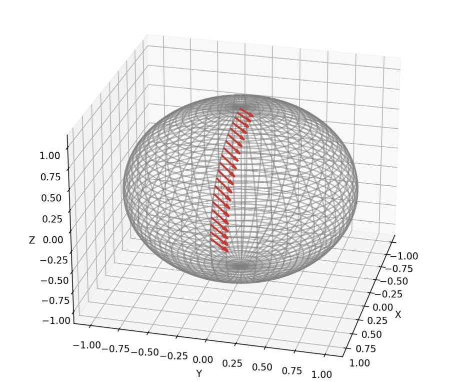
Since we know,

$$\begin{aligned}\mathbf{e}_\theta &= (\cos\theta\cos\phi)\hat{\mathbf{x}} + (\cos\theta\sin\phi)\hat{\mathbf{y}} - (\sin\theta)\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x}} - (\sin\theta)\hat{\mathbf{z}} \\ \mathbf{e}_\phi &= (-\sin\phi)\hat{\mathbf{x}} + (\cos\phi)\hat{\mathbf{y}} = \hat{\mathbf{y}}.\end{aligned} \tag{29}$$

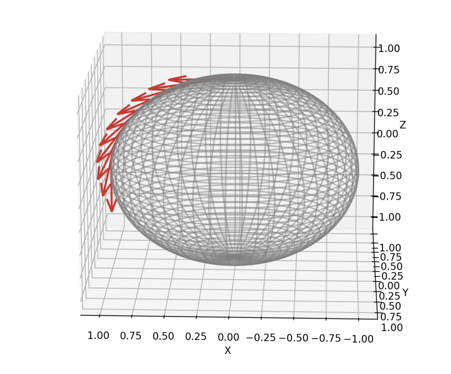
$$\mathbf{n}(\theta, \phi = 0) = \alpha\cos\theta\hat{\mathbf{x}} - \alpha(\sin\theta)\hat{\mathbf{z}} + \beta\,\sin(\theta_0)\hat{\mathbf{y}}, \tag{30}$$

with origin $\hat{\mathbf{e}}_r$ on the unit sphere since the full vector is at $(1, \theta, \phi) = \hat{\mathbf{e}}_r + \cdots)$. You see the rotation in the Cartesian coordinate system when traveling along the unit-speed curve (in those well-know parallel transport figures): it is because we are using a global fixed basis $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ The parallel transport of a vector is performed with respect to a local basis.

In this example example, if there is initial non-zero $n^\phi(\theta)\hat{\mathbf{e}}_\phi$, this component will remain there during the parallel transport, something like the figure below.



If $\mathbf{n}_0 = \hat{\mathbf{e}}_\theta$, the term $n^\phi(\theta) = 0$ for all θ. This is a part of the **geodesics**: it parallel transports its own tangent vector, something like the figure below.



You might wondering if you treat the sphere as embedded in 3D Euclidean space and simply "slide" the vector so that it points in the same direction in 3D, it looks like there is no change (the arrow visually remains parallel in ambient space). However, this is not the same as parallel transport on the sphere itself, because the transported vector may no longer lie entirely within the tangent spaces of the sphere at each point, and in local basis, it is very different!

To reaffirm our discuss above, let's consider the transport of a vector from $[r=1, \phi=0, \theta=\theta_0]$ to $[r=1, \phi=\pi/2, \theta=\theta_0]$ with the following the unit-speed parametrization,

$$\gamma(t) = t\hat{\mathbf{e}}_\phi(t) = \phi\hat{\mathbf{e}}_\phi = \gamma(\phi). \tag{31}$$

the curve is also unit-speed since $|d\gamma(t)/dt| = 1$, and you are following a curve that is in the tangent direction of the unit sphere.

We write the equation again,

$$\begin{aligned}
\partial_\phi n^\phi + n^\theta \Gamma^\phi_{\phi\theta} \frac{\partial \gamma^\phi}{\partial t} &= \partial_\phi n^\phi + n^\theta \cot\theta \frac{\partial \gamma^\phi}{\partial t} = 0 \\
\partial_\phi n^\theta + n^\phi \Gamma^\theta_{\phi\phi} \frac{\partial \gamma^\phi}{\partial t} &= \partial_\phi n^\theta - \sin\theta\cos\theta n^\phi \frac{\partial \gamma^\phi}{\partial t} = 0 \\
\partial_\theta n^\phi + n^\phi \Gamma^\phi_{\theta\phi} \frac{\partial \gamma^\theta}{\partial t} &= \partial_\theta n^\phi + n^\phi \cot\theta \frac{\partial \gamma^\theta}{\partial t} = 0.
\end{aligned} \tag{32}$$

The unit-speed condition $\frac{\partial \gamma^\theta}{\partial t} = 1$ gives,

$$\begin{aligned}
\partial_\phi n^\phi + n^\theta \cot\theta &= 0 \\
\partial_\phi n^\theta - \sin\theta\cos\theta n^\phi &= 0 \\
\partial_\theta n^\phi &= 0 \\
\partial_\theta n^\theta &= 0,
\end{aligned} \tag{33}$$

following the previous result. From the last two equations, we know n^θ, n^ϕ are independent of θ. Now, we left with the differential equation,

$$\begin{aligned}
\partial_\phi n^\phi + n^\theta \cot\theta_0 &= 0 \\
\partial_\phi n^\theta - \sin\theta_0\cos\theta_0 n^\phi &= 0.
\end{aligned} \tag{34}$$

Let's take the derivative of the first (second) equation with respect to ϕ, and plug in the second (first) equation,

$$\begin{aligned}
\partial_\phi^2 n^\phi + \cos^2\theta_0 n^\phi &= 0, \\
\partial_\phi^2 n^\theta + \cos^2\theta_0 n^\theta &= 0,
\end{aligned} \tag{35}$$

note the constant θ_0 with general solution,

$$\begin{aligned}
n^\phi(\phi) &= A\cos[\cos(\theta_0)\phi] + B\sin[\cos(\theta_0)\phi] \\
n^\theta(\phi) &= A'\cos[\cos(\theta_0)\phi] + B'\sin[\cos(\theta_0)\phi]
\end{aligned} \tag{36}$$

and A, B, A', B' are constants. Let's consider the vector we have previously,

$$\mathbf{n}(\theta_0, \phi = 0) = \alpha\, \hat{\mathbf{e}}_\theta + \beta\, \hat{\mathbf{e}}_\phi = \alpha\, \mathbf{e}_\theta + \beta \sin(\theta_0)\, \mathbf{e}_\phi, \qquad \alpha^2 + \beta^2 \sin^2(\theta_0) = |\mathbf{n}|^2. \tag{37}$$

Since the initial condition,

$$n^\phi(\phi = 0) = \beta = A, \qquad n^\theta(\phi = 0) = \alpha = A'. \tag{38}$$

We can then match the condition,

$$\begin{aligned} \partial_\phi n^\phi &= -n^\theta \cot\theta_0 \\ -A\cos(\theta_0)\sin[\cos(\theta_0)\phi] + B\cos(\theta_0)\cos[\cos(\theta_0)\phi] &= -A'\cot\theta_0\cos[\cos(\theta_0)\phi] - B'\cot\theta_0\sin[\cos(\theta_0)\phi] \\ -\beta\sin\theta_0\sin[\cos(\theta_0)\phi] + B\sin\theta_0\cos[\cos(\theta_0)\phi] &= -\alpha\cos[\cos(\theta_0)\phi] - B'\sin[\cos(\theta_0)\phi] \end{aligned} \tag{39}$$

$$\begin{aligned} \partial_\phi n^\theta &= \sin\theta_0\cos\theta_0 n^\phi \\ -\alpha\sin[\cos(\theta_0)\phi] + B'\cos[\cos(\theta_0)\phi] &= \beta\sin(\theta_0)\cos[\cos(\theta_0)\phi] + B\sin(\theta_0)\sin[\cos(\theta_0)\phi] \end{aligned} \tag{40}$$

and at $\phi = 0$, it reduces to,

$$B' = \beta\sin\theta_0, \tag{41}$$

and

$$B = -\frac{\alpha}{\sin\theta_0}. \tag{42}$$

We finally have,

$$\begin{aligned} n^\phi(\phi) &= \beta\cos[\cos(\theta_0)\phi] - \frac{\alpha}{\sin\theta_0}\sin[\cos(\theta_0)\phi] \\ n^\theta(\phi) &= \alpha\cos[\cos(\theta_0)\phi] + [\beta\sin\theta_0]\sin[\cos(\theta_0)\phi]. \end{aligned} \tag{43}$$

We then must verify that the vector norm,

$$\begin{aligned} (n^\phi(\phi)\sin\theta_0)^2 &= (\beta\sin\theta_0\cos[\cos(\theta_0)\phi] - \alpha\sin[\cos(\theta_0)\phi])^2 \\ &= \beta^2\sin^2\theta_0\cos^2[\cos(\theta_0)\phi] + \alpha^2\sin^2[\cos(\theta_0)\phi] - 2\alpha\beta\sin\theta_0\cos[\cos(\theta_0)\phi]\sin[\cos(\theta_0)\phi] \end{aligned} \tag{44}$$

$$\begin{aligned} (n^\theta(\phi))^2 &= (\alpha\cos[\cos(\theta_0)\phi] + [\beta\sin\theta_0]\sin[\cos(\theta_0)\phi])^2 \\ &= \alpha^2\cos^2[\cos(\theta_0)\phi] + 2\alpha\cos[\cos(\theta_0)\phi][\beta\sin\theta_0]\sin[\cos(\theta_0)\phi] + [\beta\sin\theta_0]^2\sin^2[\cos(\theta_0)\phi] \end{aligned} \tag{45}$$

and if you sum the two,

$$(n^\theta(\phi))^2 + (n^\phi(\phi)\sin\theta_0)^2 = \alpha^2 + \beta^2\sin^2\theta_0, \tag{46}$$

you will get the same norm as those before preforming the parallel transport. The vector becomes,

$$\begin{aligned} \mathbf{n}(\theta_0, \phi) &= n^\theta(\phi)\,\mathbf{e}_{\theta_0}(\phi) + n^\phi(\phi)\sin\theta_0\,\mathbf{e}_\phi \\ &= (\alpha\cos[\cos(\theta_0)\phi] + [\beta\sin\theta_0]\sin[\cos(\theta_0)\phi])[(\cos\theta_0\cos\phi)\hat{\mathbf{x}} + (\cos\theta_0\sin\phi)\hat{\mathbf{y}} - (\sin\theta_0)\hat{\mathbf{z}}] \\ &+ \left(\beta\cos[\cos(\theta_0)\phi] - \frac{\alpha}{\sin\theta_0}\sin[\cos(\theta_0)\phi]\right)[(-\sin\phi)\hat{\mathbf{x}} + (\cos\phi)\hat{\mathbf{y}}], \end{aligned} \tag{47}$$

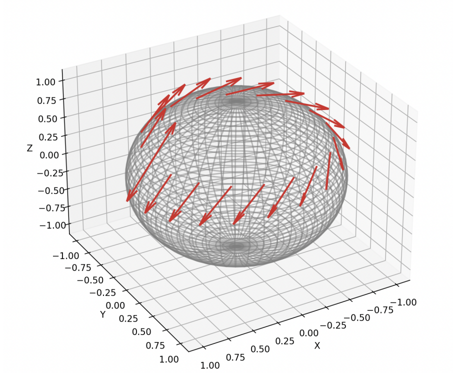
where we use the following for basis conversion,

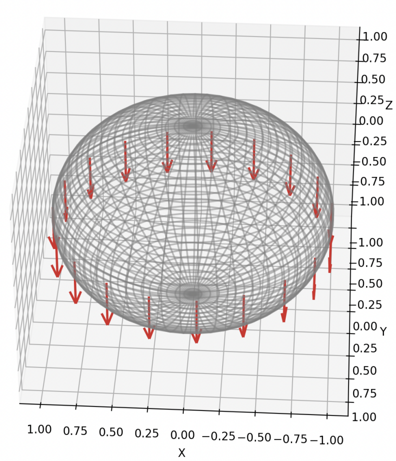
$$\begin{aligned} \mathbf{e}_{\theta_0}(\phi) &= (\cos\theta_0\cos\phi)\hat{\mathbf{x}} + (\cos\theta_0\sin\phi)\hat{\mathbf{y}} - (\sin\theta_0)\hat{\mathbf{z}} \\ \mathbf{e}_{\phi} &= (-\sin\phi)\hat{\mathbf{x}} + (\cos\phi)\hat{\mathbf{y}}. \end{aligned} \tag{48}$$

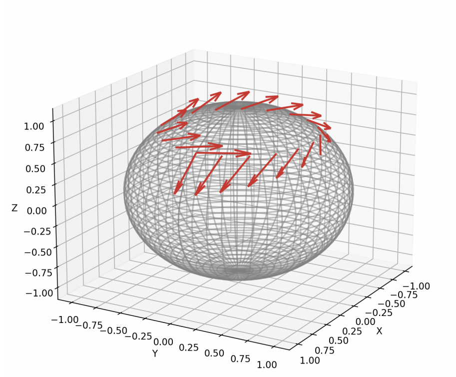
You may realize that after a complete loop (closed) from $\phi = 0$ to $\phi = 2\pi$, we cannot get back to the original vector direction in both local coordinate system and fixed Cartesian coordinate system because of the $\cos(\theta_0)$ term, i.e. it is intrinsically path dependent, unless, we are on geodesics $\theta_0 = \pi/2$. The time is defined by

However, we will not see this in the first example.

You will see something like the figure below, $\theta_0 = \pi/3, \pi/2, \pi/4$,







0.0.4 Tensors

In a given coordinate system with basis vectors $\{e_i\}$ and dual basis vectors $\{\omega^i\}$, a tensor is mutilinear map that expressed as:

$$\mathcal{T} = T^{i_1 i_2 \cdots i_q}_{j_1 j_2 \cdots j_p} \, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes \omega^{j_1} \otimes \omega^{j_2} \otimes \cdots \,, \tag{49}$$

where $T^{i_1 i_2 \cdots i_q}_{j_1 j_2 \cdots j_p}$ is the tensor component, and

- Upper indices $(i_1, i_2, \ldots)$ denote contravariant components.
- Lower indices $(j_1, j_2, \ldots)$ denote covariant components.

In physics, it is common to write a tensor as a sum of tensor products between m vector spaces $v \in V \otimes V \otimes V \otimes \cdots$ and/or n dual spaces $\omega \in V^* \otimes V^* \otimes V^* \otimes \cdots$,

$$\mathcal{T} = \sum_i v^i \otimes \omega_i. \tag{50}$$

For example, the simplest case is the density matrix,

$$\rho = \sum_i |\phi_i\rangle \langle \psi_j| = \sum_i U^\dagger |\phi_i'\rangle \langle \psi_j'| U, \tag{51}$$

which transforms in their respective spaces.