UCSB, Physics 129AL, Computational Physics: Section Worksheet, Week 5A

Zihang Wang (UCSB), zihangwang@ucsb.edu

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Section Participation and Submission Guidelines

Section attendance is required, but you do not need to complete all the work during the section. At each section, the TA will answer any questions that you might have, and you are encouraged to work with others and look for online resources during the section and outside of sections. Unless otherwise stated, the work will be due one week from the time of assignment. The TA will give you 1 point for each task completed. You can see your grades on Canvas.

We will use GitHub for section worksheet submissions. By the due date, you should have a single public repository on GitHub containing all the work you have done for the section work. Finally, upload a screenshot or a .txt file to Canvas with your GitHub username and repository name so the TA knows who you are and which repository you are using for the section.

Remember: talk to your fellow students, work together, and use GPTs. You will find it much easier than working alone. Good luck! All work should be done in the Docker container, and don't forget to commit it to Git!

Task 1:Density of States

In the class, we discussed how to find density of states. In this task, you are asked to calculate the density of states for the following given Hamiltonian, numerically and analytically. Hint: for numerical approach, you can mesh the phase space volume and numerically calculate the variation in energy when extending the area. Then, you can use polynomial least-square to approximate the scaling.

Consider a classical particle in a **2D harmonic oscillator potential** with Hamiltonian:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2),$$

where m is the mass, ω is the oscillator frequency, and $\hbar = 1$.

a) Density of States

Derive the density of states $g(E)$, defined as the number of microstates per unit energy.
Hint: The phase space volume $\Omega(E)$ (for $H \leq E$) is a 4D hyper-ellipsoid. Use coordinate transformations.

b) Partition Function via Density of States

Compute the canonical partition function $Z(\beta)$:

$$Z(\beta) = \int_0^\infty g(E) e^{-\beta E} dE.$$

c) Density of States of 2D non-linear harmonic oscillator potential

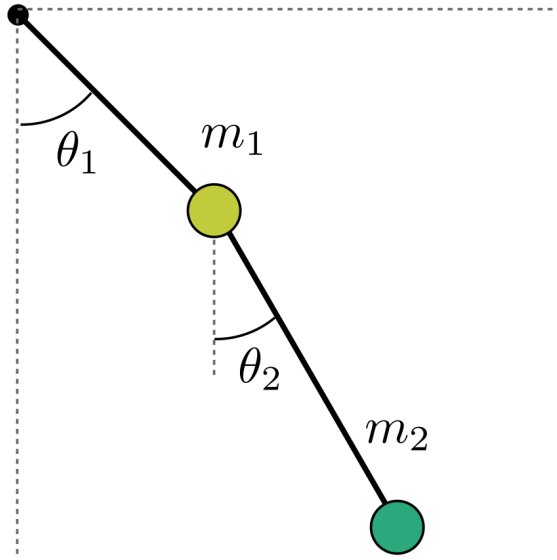
Find the density of states of the following ,

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2) + \lambda (x^2 + y^2)^2,$$

When solving it analytically, do the momentum integral first.

Task 2: Double Pendulum Dynamics

You are given a double pendulum system with the following definition,



- Two masses: m_1, m_2
- Two rigid massless rods of lengths L_1, L_2
- Gravitational acceleration: g

Define the state variables:

- θ_1: Angle of the first pendulum with the vertical
- $\dot{\theta}_1 \sim p_1$: Angular velocity of the first pendulum
- θ_2: Angle of the second pendulum with the vertical
- $\dot{\theta}_2 \sim p_2$: Angular velocity of the second pendulum

a) Lagrangian

Find the Lagrangian and equation of motions in a matrix form. Do not use p_1, p_2 as those are designed for the Hamiltonian.

b) Hamiltonian

Use Legendre transform to find the corresponding Hamiltonian,

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t). \tag{1}$$

What is the generalized momentum defined above? Write the Hamiltonian in a matrix form.

c) Hamiltonian and Phase space dynamics

With a given initial condition, white a program that plots the phase space trajectory (θ_2, p_2) and double pendulum dynamics in real space (Cartesian coordinates). You can use any integrator you like, e.g. *from scipy.integrate import solve_ivp*. We will soon be looking at backend numerical methods used in those packages.

d) Phase space density

Generalized the above formulation, initialize a point cloud with similar initial conditions, then plot the change in **partial phase space volume** enclosed by those point clouds on (θ_2, p_2), using any convex hull technique we developed previously.