1 Mechanics and Fit

1. Let GPA be Y and ACT be X.

   a) $\bar{Y} = 3.21$, $\bar{X} = 25.88$, $s_{XY} = 5.8125$, $s_X^2 = 56.875$. Recall that

   $$\hat{\beta}_1 = \frac{s_{XY}}{s_X^2} = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} \qquad \text{and} \qquad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \tag{1}$$

   Thus $\hat{\beta}_1 = 0.1022$ and $\hat{\beta}_0 = 0.5681$

   b) No

   c) A one unit increase in ACT is, on average, associated with a $\hat{\beta}_1$ unit increase in GPA

   d) Observation specific values are in the table below

Student	$\hat{Y}_i$	$\hat{u}_i$
1	2.7143	0.0857
2	3.0209	0.3791
3	3.2253	-0.2252
4	3.3275	0.1725
5	3.5319	0.0681
6	3.1231	-0.1231
7	3.1231	-0.4231
8	3.6341	0.0659
		$\sum_i \hat{u}_i = -0.0001$

   e) $\hat{Y}(X = 20) = 0.5681 + 0.1022(20) = 2.6121$

   f) $ESS = 0.5940$, $TSS = 1.0288$. Recall that

   $$R^2 = \frac{ESS}{TSS} = \frac{\sum_i (\hat{Y}_i - \bar{Y})^2}{\sum_i (Y_i - \bar{Y})^2}$$

   Thus $R^2 = 0.5774$.

2. $\bar{X} = \bar{Y} = 0$

   a) If $k = 2$ for $i = 1, \ldots, 6$, $Y_i = X_i$ (i.e. $\hat{\beta}_0 = 0$, $\hat{\beta}_1 = 1$), $\hat{u}_i = Y_i - \hat{Y}_i = Y_i - X_i = 0$, $SSR = \sum_i \hat{u}_i = 0$, $R^2 = 1 - \frac{SSR}{TSS} = 1 - 0 = 1$

b) Given the formulae in equation (1) above,

$$\hat{\beta}_1 = \frac{5+k}{7}, \qquad \text{and} \qquad \hat{\beta}_0 = 0$$

c) If $k = 1$, $\hat{\beta}_0 = 0, \hat{\beta}_1 = \frac{6}{7}, \hat{Y}_i = \frac{6}{7}X_i$. Thus $R^2 = \frac{ESS}{TSS} = \frac{144/7}{22} = \frac{72}{77}$

d) If $k = 3$, $\hat{\beta}_0 = 0, \hat{\beta}_1 = \frac{8}{7}, \hat{Y}_i = \frac{8}{7}X_i$. $R^2 = \frac{256/7}{38} = \frac{128}{133}$

e) If $k = 10$, $\hat{\beta}_0 = 0, \hat{\beta}_1 = \frac{15}{7}, \hat{Y}_i = \frac{15}{7}X_i$. $R^2 = \frac{900/7}{220} = \frac{45}{77}$

f) Given that $\hat{\beta}_0 = 0$ and $\hat{\beta}_1 = \frac{5+k}{7}$, $\hat{\beta}_1 \to \infty$ as $k \to \infty$. That is, the fitted line converges to a vertical line going through the origin. And since $\hat{Y}_i = \frac{5+k}{7}X_i$, $ESS = \frac{1}{7}(100 + 4k^2 + 40k)$, $TSS = 20 + 2k^2$, and

$$R^2 = \frac{50 + 20k + 2k^2}{70 + 7k^2}$$

$R^2 \to \frac{2}{7}$ as $k \to \infty$

3. Stock & Watson 4.1

a) The predicted average test score is

$$\widehat{TestScore} = 520.4 - 5.82 \times 22 = 392.36$$

b) The predicted change in the classroom average test score is

$$\Delta\widehat{TestScore} = (-5.82 \times 19) - (-5.82 \times 23) = 23.28$$

c) Using the formula for $\hat{\beta}_0$ in equation (4.8), we know the sample average of the test score across the 100 classroom is

$$\overline{TestScore} = \hat{\beta}_0 + \hat{\beta}_1\overline{CS} = 520.4 - 5.82 \times 21.4 = 395.85$$

2 Estimation and Inference

1. Stock & Watson 4.4

2. Stock & Watson 4.6

$$\begin{aligned}E(Y_i \mid X_i) &= E(\beta_0 + \beta_1 X_i + u_i \mid X_i)\\ &= E(\beta_0 \mid X_i) + E(\beta_1 X_i \mid X_i) + E(u_i \mid X_i)\\ &= \beta_0 + \beta_1 X_i\end{aligned}$$

3. Stock & Watson 4.7

$$\begin{aligned}E(\hat{\beta}_0) &= E(\bar{Y} - \hat{\beta}_1 \bar{X})\\ &= E\left[\left(\beta_0 + \beta_1 \bar{X} + \frac{1}{n}\sum_i u_i\right) - \hat{\beta}_1 \bar{X}\right]\\ &= \beta_0 + \underbrace{E(\beta_1 - \hat{\beta}_1)}_{\substack{=0 \text{ since } \hat{\beta}_1\\ \text{is unbiased}}} \bar{X} + \frac{1}{n}\sum_i \underbrace{E(u_i)}_{\substack{=0\\ \text{by assumption}}}\\ &= \beta_0\end{aligned}$$

4. Stock & Watson 5.2 (a)-(d)

a)

$$\begin{aligned}\beta_1 &= E(Wage_i \mid Male_i = 1) - E(Wage_i \mid Male_i = 0)\\ &= \text{difference in mean earnings between men and women}\\ &= \text{wage gender gap}\end{aligned}$$

The estimated gender gap $= \hat{\beta}_1 = \$2.12$ /hour

b) $H_0 : \beta_1 = 0$, $H_1 : \beta \neq 0$. $t = \frac{2.12-0}{0.36} = 5.89$, p-value $= \Pr(Z \leq -5.89) \times 2 \approx 0 < 0.01$. Thus reject H_0 at $\alpha = 1\%$. That is, the estimated gender gap is significantly different from 0.

c) $\{2.12 \pm 1.96 \times 0.36\} = (1.4144, 2.8256)$

d) $\overline{Wage} = \hat{\beta}_0 + \hat{\beta}_1 \overline{Male}$.
For women:

$$Male_i = 0 \Rightarrow \overline{Male} = 0 \Rightarrow \overline{Wage} = \hat{\beta}_0 + \hat{\beta}_1 \times 0 = \$12.52/\text{hour}$$

For men:

$$Male_i = 1 \Rightarrow \overline{Male} = 1 \Rightarrow \overline{Wage} = \hat{\beta}_0 + \hat{\beta}_1 = \$14.64/\text{hour}$$

5. Stock & Watson 5.3

   The 99% confidence interval is $1.5 \times \{3.94 \pm 2.58 \times 0.31\}$ or

$$4.71\text{lbs} \leq WeightGain \leq 7.11\text{lbs}.$$

6.

   a. 130

   b. 124

   c. The R^2 never decreases. This question relies on Chapter 6 knowledge.

   d. No, the estimator is a random variable.

7.

   a-c. See book.

   d. No. Yes. Yes. Last one: yes, because it depends on $\hat{\beta}_1$.

3 Prediction and Forecasting

1. Stock & Watson 14.1

   a) Y_t is stationary so its probability distribution does not change over time. As a results expected value of Y_t and Y_{t-1} are the same.

   b)

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t \Rightarrow E(Y_t) = \beta_0 + \beta_1 E(Y_{t-1})$$

   From (a) we know $E(Y_t) = E(Y_{t-1})$

$$\Rightarrow E(Y_t) = \beta_0 + \beta_1 E(Y_t)$$

$$\Rightarrow (1 - \beta_1) E(Y_t) = \beta_0$$

$$\Rightarrow E(Y_t) = \beta_0/(1-\beta_1)$$

2.

a) Using the result from 1(b), $E(Y_t) = \frac{\beta_0}{1-\beta_1} = \frac{2.5}{1-0.7} = 8.33$.

$Var(Y_t) = Var(0.7Y_{t-1} + u_t) = 0.7^2 Var(Y_{t-1}) + Var(u_t) = 0.49Var(Y_t) + Var(u_t)$ (Note that $Var(Y_t) = Var(Y_{t-1})$ because Y_t is stationary.)

Thus $(1-0.49)Var(Y_t) = 9,$ and therefore $Var(Y_t) = 9/(1-0.49) = 17.647$.

b) $Cov(Y_t, Y_{t-1}) = Cov(2.5 + 0.7Y_{t-1} + u_t, Y_{t-1}) = 0.7Var(Y_{t-1}) + Cov(u_t, Y_{t-1})$
$= 0.7Var(Y_t) = 0.7 \times 17.647 = 12.3529$

$Cov(Y_t, Y_{t-2}) = Cov(2.5 + 0.7(2.5 + 0.7Y_{t-2} + u_{t-1}) + u_t, Y_{t-2})$
$= Cov(0.49Y_{t-2} + 0.7u_{t-1} + u_t, Y_{t-2})$
$= 0.49Cov(Y_{t-2}, Y_{t-2}) + 0.7Cov(u_{t-1}, Y_{t-2}) + Cov(u_t, Y_{t-2})$
$= 0.49Var(Y_{t-2}) = 0.49Var(Y_t) = 0.49 \times 17.647 = 8.64703$

c) (not assigned)

d) $Y_{T+1|T} = E(Y_{T+1}|Y_T, Y_{T-1}, ...) = E(2.5 + 0.7Y_T + u_t|Y_T, Y_{T-1,...}) = 2.5 + 0.7Y_T$
$= 2.5 + 0.7 \times 102.3 = 74.11$

3. Let $(Y_1, Y_2, Y_3) = (-1, 1, -1)$.

a) Recall that

$$\widehat{\text{cov}(Y_t, Y_{t-j})} = \frac{1}{T}\sum_{t=j+1}^{T}(Y_t - \bar{Y}_{j+1:T})(Y_{t-j} - \bar{Y}_{1:T-j})$$

where $T = 3$ and $j = 1$ for this question (and note that both $\bar{Y}$s equal zero). Applying this formula we get

$$\widehat{\text{cov}(Y_t, Y_{t-1})} = \frac{1}{3}\Big[(Y_2 - 0)(Y_1 - 0) + (Y_3 - 0)(Y_2 - 0)\Big] = -\frac{2}{3}.$$

b) Recall that

$$\hat{\rho}_j = \frac{\widehat{\text{cov}(Y_t, Y_{t-j})}}{\widehat{\text{var}(Y_t)}}$$

where $j = 1$ for this question. Since $\bar{Y} = \frac{1}{3}(-1+1-1) = -\frac{1}{3}$,

$$\widehat{\text{var}(Y_t)} = \frac{1}{2}\left[\left(-1+\frac{1}{3}\right)^2 + \left(1+\frac{1}{3}\right)^2 + \left(-1+\frac{1}{3}\right)^2\right] = \frac{4}{3}$$

Thus,

$$\hat{\rho}_1 = \frac{-2/3}{4/3} = -\frac{1}{2}.$$

c) Yes, consider $(0, 0, 0, 0, 0)$.

4. See "forecasts and predicted values" on p.533 of Stock & Watson.