BUEC 333, Test 1

June 10, 2015, 14:30-17:20

Important!

- On the front page of **your answer sheet**, write (i) your name; (ii) your student ID.
- On the front page of **this document** (the questions), write: (i) your name; and (ii) your student ID.
- Once you **finish** this exam, **signal** it to us, and we will come to pick up your exam. Stay seated until somebody comes to cllect your exam.
- No explanation = no points. A correct answer **with correct explanation** earns 1 point for each subquestion.
- For a "compute" question, an explanation can consist of starting from **an appropriate formula**, and working towards the correct numerical answer.

1 Probability

1. Consider a random variable X with sample space $\{-a, a\}$ and probability distribution given by $P(X=-a)$=P$(X=a)=0.5$.
   (a) Compute $E(X)$. **0.5*(-a)+0.5*a=0.**
   (b) Compute Var(X). **0.5*(-a-0)^2 + 0.5*(a-0)^2=a^2**
   (c) Compute Var$\left(-\frac{1}{a}X\right)$. **From KC 2.3, Var(-1/a X)=1/(a^2) * Var(X)=1.**

2. Let X and Y be two RVs, both with sample space $\{0,1\}$. The joint probabilities are all equal:

$$P(X=0, Y=0) = P(X=0, Y=1) = P(X=1, Y=0) = P(X=1, Y=1) = 0.25.$$

   (a) Compute $E(X)$. **P(X=0)=P(X=0,Y=0)+P(X=0,Y=1)=0.25+0.25=0.5. E(X)=0*0.5+1*0.5=0.5**
   (b) Compute $E(X|Y=0)$. **P(X=0|Y=0)=P(X=0,Y=0) / P(Y=0) = 0.25/0.5=0.5. E(X|Y=0)=0*0.5+1*0.5.**
   (c) Compute $E(X)$ using the law of iterated expectations. **E(X)=E(X|Y=0)P(Y=0) + E(X|Y=1)P(Y=1)=0.5*0.5+0.5*0.5.**
   (d) Compute Cov(X,Y). **Using KC 2.3, Cov(X,Y)=E(XY)-E(X)E(Y). E(X)=E(Y)=0.5. XY has sample space {0,1}, with P(XY=0)=0.75) and P(XY=1)=0.25. So E(XY)=0.25. So Cov(X,Y)=E(XY)-E(X)E(Y)=0.25-0.5*0.5=0. An alternative answer first shows that X and Y are independent or mean-independent, and then uses the fact that independent variables have zero covariance.**

3. Let X and Y be two RVs, both with sample space $\{0, 10\}$. You know that $P(X = 10) = 0.3$ and $P(Y = 10) = 0.4$.

   (a) Assume that X and Y are independent. Compute $P(X = 10, Y = 0)$. **Because of independence, this equals P(X=10)*P(Y=0)=0.3*0.6=0.18**

   (b) Assume that X and Y are independent. Compute $P(X = 10|Y = 0)$. **Because of independent, this equals P(X=10)=0.3.**

   (c) If X and Y are not independent, can you compute $P(X = 10, Y = 0)$? Explain. **Nope!**

   (d) Assume that $P(X = 10, Y = 10) = 0.1$. Compute $E[X|Y = 10]$. **Using the marginals, you can deduce that P(10,0)=0.2, P(0,10)=0.3, P(0,0)=0.4.**
   Then P(X=0|Y=10)=P(0,10)/P(Y=10)=0.3/0.4=0.75, and P(X=10|Y=10)=0.25. So E(X|Y=10)=0.75*0+0.25*10=2.5.

4. Let $(X_1, \cdots, X_n)$ be a random sample from a distribution with mean $E(X) = \mu_x$ and $\text{Var}(X) = \sigma^2$.

   (a) Can you express $E(X_1)$ in terms of μ_x? **E(X1)=mu_X**

   (b) Express $\text{Var}(X_1 + X_2)$ in terms of σ^2. **Var(X1+X2)=2**σ^2

2 Statistics

5. Assume that you have a random sample $(7, 8, 9)$ for a RV X with mean μ_x and variance σ_x^2.

   (a) For this sample, compute $\bar{X}$. **8**

   (b) For this sample, compute the sample variance. **(7-8)^2 + (8-8)^2 + (9-8)^2=2, so sample variance is 2/(3-1)=1.**

   (c) Construct a 95%-CI for μ_x based on the sample mean and sample variance. Assume that the Central Limit Theorem applies, so that

$$0.95 = P\left(-1.96 < \frac{\bar{X} - \mu_x}{SE(\bar{X})} < 1.96\right)$$

   CI = [8 - 1.96 *1/sqrt(3), 8 + 1.96/sqrt(3)]

   (d) What is the coverage probability of your confidence interval? **95%**

   (e) Interpret the numbers in your answer to question (c). **If I constructed infinitely many such confidence intervals, they would capture** μ **with 95% probability. Not correct: mu is in that interval with 95% probability. Not correct: I am 95% confident that mu is in that interval.**

6. We have a random sample of size $n = 10$, $(X_1, X_2, \cdots, X_{10})$, from a distribution with mean μ_x and variance σ_x^2. Consider two estimators for μ_x. First, the sample average $\bar{X} = \frac{1}{10}(X_1 + \cdots + X_{10})$. We know that $E(\bar{X}) = \mu_x$ and that $\text{Var}(\bar{X}) = \sigma^2/10$. Second, the odd sample average: $\tilde{X} = \frac{1}{5}(X_1 + X_3 + X_5 + X_7 + X_9)$.

(a) Is $\tilde{X}$ a random variable? Explain your answer! **YES! It is a function of random variables. It is random because of random sampling.**

(b) Is $\bar{X}$ unbiased? Explain your answer! **Yes, as it is stated that $E(\bar{X}) = \mu$, which is the definition of unbiasedness.**

(c) Is $\tilde{X}$ unbiased? Explain your answer! **Yes.**
1/5*E(X1+X3+X5+X7+X9)=1/5(E(X1)+E(X3)+E(X5)+E(X7)+E(X9))=...
...=1/5*5*mu=mu.

(d) Which one of the two estimators is more efficient? Explain your answer! **First, show that Var(X-tilde)=sigma^2/5. Then conclude that Var(X-tilde)>Var(Xbar) so that Xbar is more efficient than X-tilde.**