What I did in class

We denote the OLS estimators for β_0 and β_1 by $\left(\hat{\beta}_0, \hat{\beta}_1\right)$. These estimators minimize the sum of squares of residuals, i.e. they solve

$$\text{argmin}_{b_0,b_1} \sum_{i=1}^{n} \hat{u}_i^2,$$

where $\hat{u}_i = Y_i - b_0 - b_1 X_i$ are the residuals given estimates b_0, b_1. The objective function is perfectly behaved for minimization, so that we can find the minimum by solving at the FOC conditions. In particular, note that

$$\frac{\partial \hat{u}_i^2}{\partial b_0} = \frac{\partial \hat{u}_i^2}{\partial \hat{u}_i} \cdot \frac{\partial \hat{u}_i}{\partial b_0} = 2\hat{u}_i \cdot (-1) = -2\left(Y_i - b_0 - b_1 X_i\right)$$

and that the OLS estimator solves the FOC (assuming that $\hat{\beta}_1$ is set to the solution of its own FOC):

$$\begin{aligned} -2\sum_{i=1}^{n}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right) &= 0 \\ &\Leftrightarrow \\ \hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X}. \end{aligned}$$

To obtain an expression for $\hat{\beta}_1$, note that in the FOC for $\hat{\beta}_1$ we can use the expression for $\hat{\beta}_0$ just obtained, so that

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \\ &= Y_i - \left(\bar{Y} - \hat{\beta}_1 \bar{X}\right) - \hat{\beta}_1 X_i \\ &= \left(Y_i - \bar{Y}\right) - \hat{\beta}_1\left(X_i - \bar{X}\right) \end{aligned}$$

and the derivative at $\hat{\beta}_1$ is

$$\frac{\partial \hat{u}_i^2}{\partial \hat{\beta}_1} = \frac{\partial \hat{u}_i^2}{\partial \hat{u}_i} \cdot \frac{\partial \hat{u}_i}{\partial \beta_1} = 2\hat{u}_i \cdot \left(-\left(X_i - \bar{X}\right)\right) = -2\left(\left(Y_i - \bar{Y}\right) - \hat{\beta}_1\left(X_i - \bar{X}\right)\right)\left(\left(X_i - \bar{X}\right)\right).$$

Summing over i and solving for $\hat{\beta}_1$ gives

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$

For the homework question

Anything that makes use of the derivation above is fine. The way I would go is to replace the expression for $\hat{\beta}_0$ above by whatever the question asks for, and then solving for $\hat{\beta}_1$ with a derivation analogous to that above.