Expectation of linear transformations of continuous random variables

Our goal is to show that for a continuous random variable X, and known scalars a and b, the following equality holds:

$$E(aX+b) = aE(X) + b.$$

Assume that the random variable X has sample space $\mathbb{R}$ and continuously differentiable cumulative distribution function

$$F_X(x) \equiv P(X \leq x).$$

Then X has a probability density function with respect to the Lebesgue measure, called $f(x)$:

$$f_X(x) = \frac{\partial F(x)}{\partial x}.$$

The expectation of X can then be written as

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Next, take two constants a, b and consider the transformation

$$Y = aX + b.$$

The cumulative density function of Y is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(aX + b \leq y) \\ &= P\left(X \leq \frac{y-b}{a}\right) \\ &= F_X\left(\frac{y-b}{a}\right). \end{aligned}$$

The density function of Y is the derivative of its cumulative density function, i.e.

$$\begin{aligned} f_Y(y) &= \frac{\partial F_X\left(\frac{y-b}{a}\right)}{\partial y} \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right). \end{aligned}$$

Therefore, we can write the expectation of the transformed random variable Y in terms of the expectation of X, using a change of variables

$$\begin{aligned} y &= ax + b \\ x &= \frac{y-b}{a}. \end{aligned}$$

Note that the limits of integration do not change. We obtain

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} (ax+b) \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \frac{dy}{dx} dx \\ &= \int_{-\infty}^{\infty} (ax+b) \frac{1}{a} f_X(x)\, a dx \\ &= a \int_{-\infty}^{\infty} x f_X(x)\, dx + b \int_{-\infty}^{\infty} f_X(x)\, dx \\ &= aE(X) + b \end{aligned}$$

using $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.