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Three-Dimensional Moment Invariants

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Abstract—Recognition of three-dimensional objects independent of size, position, and orientation is an important and difficult problem of scene analysis. The use of three-dimensional moment invariants is proposed as a solution. The generalization of the results of two-dimensional moment invariants which had linked two-dimensional moments to binary quantics is done by linking three-dimensional moments to ternary quantics. The existence and number of nth order moments in two and three dimensions is explored. Algebraic invariants of several ternary forms under different orthogonal transformations are derived by using the invariant property of coefficients of ternary forms. The result is a set of three-dimensional moment invariants which are invariant under size, orientation, and position change. This property is highly significant in compressing the data which are needed in three-dimensional object recognition. Empirical examples are also given.

Index Terms—Algebraic invariants, binary quantics, cogradient sets, contragradient sets, eccentricity of quadratics, image processing, invariant features, moment invariants, moment generating functions, object recognition, pattern recognition.

I. INTRODUCTION

RECOGNITION of three-dimensional (3-D) objects independent of size, position, and orientation is an important and difficult problem in scene analysis. In this paper an approach is presented which uses the concept of three-dimensional moment invariants as a step toward the solution of the recognition problem.

The use of moment invariants was first proposed in 1962 by Hu [1] in two-dimensional character recognition. Extending the application of moment invariants to more complex two-dimensional scenes was done by Hall [2] and others [3].

The concept of moment invariants relies on the theory of invariant algebra which deals with the properties of a certain class of algebraic expressions which remain invariant under general linear transformations.

The case of ternary quantics, whose properties will be used in the three-dimensional moment invariants, is not as fully explored as the case of binary quantics. In this paper the ternary quantics case is explored and ternary algebraic invariants are derived for several orthonormal, linear transformations. A key step in the derivation is the use of the invariant property of the coefficients in a ternary form.

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In Section II the three-dimensional moment generating function and central moments are defined, and a uniqueness theorem relating moments and distributing functions is stated. Section III gives the definitions of algebraic forms, algebraic invariants, cogradient set, contragradient set, discriminant, and Hessian of an algebraic form. In Section IV, the existence of a three-dimensional moment invariant is discussed when the axis of reference undergoes a linear orthonormal transformation. The actual derivation of several ternary invariants is discussed in Section V. Experimental results are discussed in Section VI. Finally, Section VII deals with the summary of important results and conclusions about the potential of the three-dimensional moment invariants.

II. THREE-DIMENSIONAL MOMENTS

A. A Uniqueness Theorem

The three-dimensional moments of order $p + q + r$ of a density $\rho(x_1, x_2, x_3)$ are defined in terms of the Riemann integral as

$$m_{pqr} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^p x_2^q x_3^r \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3. \tag{1}$$

It is assumed that $\rho(x_1, x_2, x_3)$ is piecewise continuous and therefore bounded and it is zero in R^3 space except at a finite part. Based on this assumption it can be proved that the sequence $\{m_{pqr}\}$ determines uniquely $\rho(x_1, x_2, x_3)$.

B. Moment Generating Functions

The moment generating function for three-dimensional moments may be defined as

$$M(u_1, u_2, u_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\,[u_1 x_1 + u_2 x_2 + u_3 x_3] \cdot \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{2}$$

which can be expanded into a power series,

$$M(u_1, u_2, u_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{p=0}^{\infty} \cdot \frac{1}{p!} (u_1 x_1 + u_2 x_2 + u_3 x_3)^p \cdot \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3. \tag{3}$$

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C. Central Moments

The central moments μ_{pqr} are defined as

$$\mu_{pqr} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \bar{x}_1)^p (x_2 - \bar{x}_2)^q (x_3 - \bar{x}_3)^r \cdot \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{4}$$

where

$$\bar{x}_1 = \frac{m_{100}}{m_{000}}; \quad \bar{x}_2 = \frac{m_{010}}{m_{000}}; \quad \bar{x}_3 = \frac{m_{001}}{m_{000}}. \tag{5}$$

The central moments do not change under the translation of coordinates, i.e., if

$$x_1' = x_1 + \alpha$$
$$x_2' = x_2 + \beta$$
$$x_3' = x_3 + \gamma, \tag{6}$$

then $\mu_{pqr}(x_1', x_2', x_3') = \mu_{pqr}(x_1, x_2, x_3)$, where α, β, γ, are constants.

From here on, for the sake of simplicity of description, only central moments will be considered, assuming that the centroid $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is at the origin, and they are expressed as

$$\mu_{pqr} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^p x_2^q x_3^r \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{7}$$

and the moment generating function also will be referred to the central moments.

III. Algebraic Invariants

A. Algebraic Forms and Invariants

A homogeneous polynomial in n variables $x_1, x_2, x_3, \cdots, x_n$ of order m in those variables is called an n-ary quantic, or form, of order m. For example,

$$f(x_1, x_2) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3 \tag{8}$$

is a binary quantic of order three, while

$$f(x_1, x_2, x_3) = a_{200} x_1^2 + 2a_{110} x_1 x_2 + 2a_{020} x_2^2 + 2a_{101} x_1 x_3 + 2a_{011} x_2 x_3 + a_{002} x_3^2 \tag{9}$$

is a ternary quantic of order 2 which is commonly called a quadratic form. With reference to the number of variables n, a quantic is called binary if $n = 2$, ternary if $n = 3$, and n-ary if there are n variables. The most general notation for a binary form of order m is

$$f(x_1, x_2) = (a_{m0}; a_{m-1,1}; \cdots; a_{0m})(x_1, x_2)^m = a_{m0} x_1^m + m a_{m-1,1} x_1^{m-1} x_2 + \cdots + \frac{m!}{(m-n)!n!} \cdot a_{m-n,n} x_1^{m-n} x_2^n + \cdots + a_{0m} x_2^m \tag{10}$$

and the most general notation for a ternary form of order m is

$$f(x_1, x_2, x_3) = \sum_{q,p,r=0}^{m} \frac{m!}{p!q!r!} a_{pqr} x_1^p x_2^q x_3^r \tag{11}$$

where $p + q + r = m$ and p, q, r are all positive integers.

A homogeneous polynomial $I(a)$ of coefficients is called an invariant of an algebraic form f if, after transforming the set of variables of $f(x)$ into a new set of variables $f(x')$, and constructing a corresponding function $I(a')$, where $\{a'\}$ are the coefficients of the new form, which is called f', the following would be true:

$$I(a) = \Lambda I(a') \tag{12}$$

where Λ depends only upon the transformation, that is, it is free from any relationship with f.

Only invariants of homogeneous polynomials under linear transformations will be considered in the sequel. In this case $\Lambda = \Delta^{\omega}$, where Δ is the determinant of the transformation and ω is called the weight of the invariant. When $\omega = 0$, the invariant is called absolute invariant.

B. Cogradient and Contragradient Sets

In many invariant theory problems two sets of variables are brought under consideration simultaneously. If these sets are subject to the same transformations they are said to be cogradient sets of variables. For example, if the two sets of variables (x_1, x_2, x_3) and (u_1, u_2, u_3) are both subject to the following transformations,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{13}$$

or more compactly if $X' = TX$ where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

will also undergo the same transformation, namely

$$U' = TU; \tag{14}$$

then $\{x_1, x_2, x_3\}$ and $\{u_1, u_2, u_3\}$ are called cogradient sets. If whenever X undergoes the transformation T, namely

$$X' = TX$$

another set of quantities namely $W = \{w_1, w_2, w_3\}$ will go under a transformation T^{-t}, namely

$$W' = T^{-t} W \tag{15}$$

where

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

Then $\{w_1, w_2, w_3\}$ and $\{x_1, x_2, x_3\}$ are called contragradient sets. They satisfy the conditions that the inner products of W and X and W' and X' are equal,

$$W' \cdot X' = W \cdot X \tag{16}$$

C. Hessian of an Algebraic Form

If f is an n-ary form, the determinant

$$H = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{vmatrix}$$

is called the Hessian of f. It is proved [5] that H is an invariant of f, that is, if Δ is the determinant of the transformation in (13), then

$$H' = \Delta^2 H. \tag{17}$$

D. Discriminant of Algebraic Form

If a quantic in n variables can be differentiated with respect to each of the variables, the determinant of the coefficients of the n differentials, which is called eliminant, is known as the discriminant of the given quantic. The discriminant is an invariant of f [5].

IV. Three-Dimensional Moment Invariants

A. Relationship of 3-D Moments and Algebraic Forms

If the moment generating function is expanded into a Taylor series

$$M(u_1, u_2, u_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \cdot \frac{1}{m!} (u_1 x_1 + u_2 x_2 + u_3 x_3)^m \cdot \rho(x_1, x_2, x_3)\, dx_1 dx_2 dx_3, \tag{18}$$

interchanging the integration and summation processes, and using the definition of moment gives

$$M(u_1, u_2, u_3) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{p,q,r=0}^{m} \mu_{pqr} \frac{m!}{p!q!r!} u_1^p u_2^q u_3^r. \tag{19}$$

Now if (u_1, u_2, u_3) and (x_1, x_2, x_3) undergo the following transformations,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} \tag{20}$$

and also

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{21}$$

then $\{x\}$ and $\{u\}$ are two contragradient sets and the following relation holds:

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = u_1' x_1' + u_2' x_2' + u_3' x_3'. \tag{22}$$

In addition,

$$M_1(u_1', u_2', u_3') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \cdot \frac{1}{m!} (u_1' x_1' + u_2' x_2' + u_3' x_3')^m \, dx_1' dx_2' dx_3' \cdot \frac{1}{|J|} \rho'(x_1', x_2', x_3') \tag{23}$$

where J is the Jacobian of transformation for $\{x\}$ and $\rho'(x_1', x_2', x_3') = \rho(x_1, x_2, x_3)$. Calling the moments after transformation μ'_{pqr} and defining them as

$$\mu'_{pqr} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1'^p x_2'^q x_3'^r \rho'(x_1', x_2', x_3')\, dx_1' dx_2' dx_3' \qquad p, q, r = 0, 1, \cdots, \tag{24}$$

then

$$M(u_1', u_2', u_3') = \frac{1}{|J|} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{p,q,r=0}^{m} \frac{m!}{p!q!r!} \mu'_{pqr} u_1'^p u_2'^q u_3'^r. \tag{25}$$

Since the transformation for the (a) coefficients in

$$f = \sum_{p,q,r=0}^{m} \frac{m!}{p!q!r!} a_{pqr} u_1^p u_2^q u_3^r \tag{26}$$

is the same as the transformation for the $x_1^p x_2^q x_3^r$ monomial in the expansion of

$$g = (u_1 x_1 + u_2 x_2 + u_3 x_3)^m, \tag{27}$$

then from (18), (19), (23), and (25) it can be seen that the same relationship also holds between the three-dimensional moments of order m and the monomial except the $1/|J|$ factor.

Fundamental Theorem of Moment Invariants: Given an algebraic invariant of the form

$$I(a_{n00}; \cdots; a_{00n}) = \Lambda^{\omega} I(a'_{n00}; \cdots; a'_{00n}), \tag{28}$$

there exists a moment invariant of the form

$$I(\mu_{n00}; \cdots; \mu_{00n}) = \Lambda^{\omega} \frac{1}{|J|} I(\mu'_{n00}; \cdots; \mu'_{00n}). \tag{29}$$

Proof: Consider the moment generating functions

$$M(u_1, u_2, u_3) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p,q,r=0}^{n} \frac{n!}{p!q!r!} u_1^p u_2^q u_3^r \mu_{pqr}.$$

Define

$$f = \sum_{p,q,r}^{n} \frac{n!}{p!q!r!} u_1^p u_2^q u_3^r \mu_{pqr}$$

after transformation the moment generating function is

$$M'(u_1', u_2', u_3') = \frac{1}{|J|} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p,q,r=0}^{n} \frac{n!}{p!q!r!} u_1'^p u_2'^q u_3'^r \mu_{pqr}'$$

and

$$f' = \sum_{p,q,r=0}^{n} \frac{n!}{p!q!r!} u_1'^p u_2'^q u_3'^r \mu_{pqr}'.$$

Since for f and f' there exists an invariant form, then for M and M' from the previous relations can be seen that there is an invariant of the form

$$I(\mu_{n00}; \cdots; \mu_{00n}) = \Lambda^{\omega} \frac{1}{|J|} I(\mu_{n00}'; \cdots; \mu_{00n}'). \qquad \text{Q.E.D.}$$

Example: Consider the following linear transformation which performs scaling as shown in Fig. 1:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \qquad \alpha_1, \alpha_2, \alpha_3 \text{ constant} \tag{30}$$

and the ternary m-ic,

$$f = a_{m00} x_1^m + 2a_{m-1,1,0} x_1^{m-1} x_2 + \cdots + a_{00m} x_3^m. \tag{31}$$

Every coefficient of f will be an algebraic invariant by using the definitions of invariant,

$$a_{m00} \alpha_1^m = a_{m00}' \tag{32}$$

$$a_{pqr} \alpha_1^p \alpha_2^q \alpha_3^r = a_{pqr}'; \qquad p + q + r = m. \tag{33}$$

Then for the moment invariant

$$\mu_{pqr} \alpha_1^{p+1} \alpha_2^{q+1} \alpha_3^{r+1} = \mu_{pqr}'. \tag{34}$$

For the zeroth order relation, (34) would be

$$\mu' = \alpha_1 \alpha_2 \alpha_3 \mu. \tag{35}$$

Now assuming for the sake of simplicity $\alpha = \alpha_1 = \alpha_2 = \alpha_3$, then

$$\mu_{pqr} \alpha^{m+3} = \mu_{pqr}' \tag{36}$$

$$p, q, r = 1, 2, 3, 4, 5, \cdots$$

$$\mu_{pqr} \left(\frac{\mu'}{\mu} \right)^{(m/3)+1} = \mu_{pqr}' \tag{37}$$

and

$$\mu_{010}' = \mu_{100}' = \mu_{001}' \equiv 0.$$

B. General Linear Transformation and the Moment Invariants

Only *absolute moment invariants* are valuable as far as object recognitions are concerned, because these expressions are independent of any parameter of transformation. In a general linear transformation the algebraic form of order m,

$$f = a_{\underbrace{m0\cdots0}_{n-1}} x_1^m + \cdots + a_{\underbrace{0\cdots0m}_{n-1}} x_n^m, \tag{38}$$

will be transformed into f',

$$f' = a'_{\underbrace{m0\cdots0}_{n-1}} x_1'^m + \cdots + a'_{\underbrace{0\cdots0m}_{n-1}} x_n'^m. \tag{39}$$

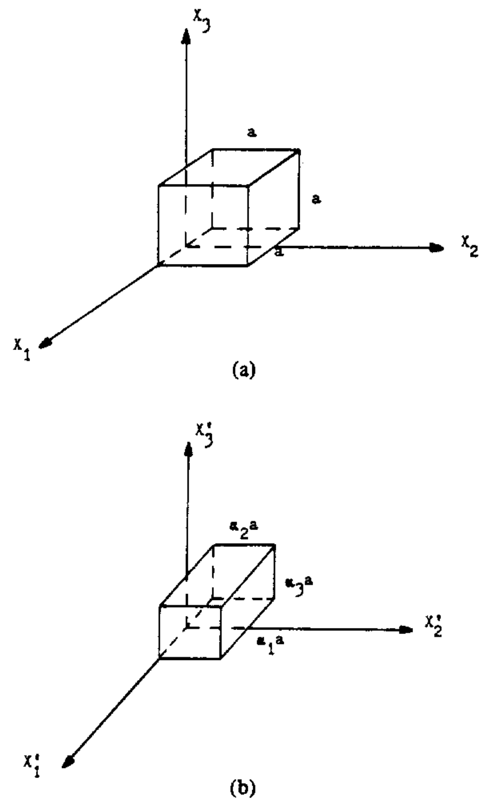

Fig. 1. Transforming (a) into (b) by scaling.

Then if

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \tag{40}$$

by equating coefficients,

$$a_{\underbrace{m0\cdots0}_{n-1}} = \alpha_{11}^m a'_{\underbrace{m0\cdots0}_{n-1}} \tag{41}$$

$$\vdots$$

If there are two variables as in a binomial expansion, then $m + 1$ coefficients exist and hence $m + 1$ relationships of the type (41). For example, if the number of variables is two and $m = 3$ (i.e., binary cubic) there are four independent relationships involving the four coefficients and parameters of the transformation. Since there are four parameters of the transformation for a general linear transformation, then there is no absolute invariant for a binary cubic form. In general, for binary m-ics there are $m + 1$ relationships involving four parameters of the transformation, so $m + 1 - 4 = m - 3$ independent relations free of the parameters exist. Hence, there are at most $m - 3$ absolute invariants. It can be concluded that there are no absolute invariants for quadratic and cubic binary forms (see Table I).

For ternary quantics, the number of relationships is six and

TABLE I

n \ m	2	3	4
BINARY	0	0	1
TERNARY	0	1	6
QUATERNARY	1	11	15

the number of transformation parameters is $3^2 = 9$. Hence, there are no ordinary (nonabsolute) invariants except the discriminant, because if there were two invariants, such as ϕ and Ψ, then

$$\phi = \Delta^P \phi' \quad \text{and} \quad \Psi = \Delta^q \Psi' \tag{42}$$

then

$$\phi = \left(\frac{\Psi}{\Psi^1}\right)^{P/q} \phi' \tag{43}$$

which is an absolute invariant contrary to the above argument which proves no absolute invariant can exist.

C. Orthogonal Transformation and Moment Invariants

Consider the following orthogonal transformation:

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{44}$$

or more compactly

$$x' = Ax \tag{45}$$

where due to orthogonality

$$AA^T = I \tag{46}$$

and the following relationships hold:

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 + \cdots + a_{1n}^2 = 1 \\ a_{21}^2 + a_{22}^2 + \quad \cdots \quad = 1 \\ \vdots \qquad \cdots \qquad \vdots \end{cases} \tag{47}$$

There are n of these relationships. Similarly, the inner products of the columns lead to

$$a_{11}a_{21} + a_{12}a_{22} + \cdots = 0 \tag{48}$$

$$a_{21}a_{11} + a_{12}a_{22} + \cdots = 0 \tag{49}$$

$$\vdots$$

but (48) and (49) are the same so the number of the relationships of this type is $(n(n-1))/2$. Overall, the number of independent parameters is

$$N = n^2 - \frac{n(n-1)}{2} - n = \frac{n(n-1)}{2}. \tag{50}$$

For the $n = 2$ orthogonal transformation the number of independent parameters is one, and for $n = 3$ the number of independent parameters is three. Now for a ternary cubic the number of independent relationships is ten, so there exist at most $10 - 3 = 7$ absolute invariants involving only the moments of order 3 by using the fundamental theorem of the moment invariant. In the case of the ternary quadratic there are at most $6 - 3 = 3$ absolute invariants. It can be said that there are at most $((n+1)(n+2))/2 - 3$ absolute invariant moments of order n only, because the number of relationships in this case is $((n+1)(n+2))/2$ so the number of absolute invariants are at most $((n+1)(n+2))/2 - 3$ (see Table II).

D. Evaluation of Invariants of Ternary Quadratic

Consider the general ternary quadratic

$$f = a_{200}x_1^2 + 2a_{110}x_1x_2 + 2a_{011}x_2x_3 + 2a_{101}x_1x_3 + a_{020}x_2^2 + a_{002}x_3^2. \tag{51}$$

To form the discriminant of f

$$\frac{\partial f}{\partial x_1} = 2a_{200}x_1 + 2a_{110}x_2 + 2a_{101}x_3 \tag{52}$$

$$\frac{\partial f}{\partial x_2} = 2a_{110}x_1 + 2a_{020}x_2 + 2a_{011}x_3$$

$$\frac{\partial f}{\partial x_3} = 2a_{101}x_1 + 2a_{011}x_2 + 2a_{002}x_3$$

$$\Delta = 8\{a_{200}a_{020}a_{002} + a_{110}a_{011}a_{101} + a_{110}a_{011}a_{101} - a_{101}a_{020}a_{101} - a_{011}a_{011}a_{200} - a_{110}a_{110}a_{002}\} \tag{53}$$

$$\Delta = 8(a_{200}a_{020}a_{002} + 2a_{110}a_{011}a_{101} - a_{020}a_{101}^2 - a_{011}^2a_{200} - a_{110}^2a_{002}). \tag{54}$$

This is one of the invariants of (51).

Another invariant can be formed by using the Hessian

$$H = \det \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix} \tag{55}$$

where

$$f_{x_ix_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad i,j = 1,2,3 \tag{56}$$

$$\begin{array}{lll} f_{x_1x_1} = 2a_{200} & f_{x_2x_1} = 2a_{110} & f_{x_3x_1} = 2a_{101} \\ f_{x_1x_2} = 2a_{110} & f_{x_2x_2} = 2a_{020} & f_{x_3x_2} = 2a_{011} \\ f_{x_1x_3} = 2a_{101} & f_{x_2x_3} = 2a_{011} & f_{x_3x_3} = 2a_{002} \end{array} \tag{57}$$

$$H = 8 \begin{vmatrix} a_{200} & a_{110} & a_{101} \\ a_{110} & a_{020} & a_{011} \\ a_{101} & a_{011} & a_{002} \end{vmatrix} = \Delta.$$

Hence the Hessian is the same as the discriminant in this case. According to the fundamental theorem of moment invariants, there exists the following relation for an orthogonal transformation:

TABLE II

n \ m	2	3	4
BINARY	2	3	4
TERNARY	3	7	12
QUATERNARY	4	14	20

$$\mu_{200}\mu_{020}\mu_{002} + 2\mu_{110}\mu_{011}\mu_{101} - \mu_{020}\mu_{101}^2 - \mu_{011}^2\mu_{200} - \mu_{110}^2\mu_{002} = 8(\mu'_{200}\mu'_{020}\mu'_{002} + 2\mu'_{110}\mu'_{011}\mu'_{101} - \mu'_{020}\mu'^2_{101} - \mu'^2_{011}\mu'_{200} - \mu'^2_{110}\mu'_{002}).$$

Hence,

$$I_1 = \mu_{200}\mu_{020}\mu_{002} + 2\mu_{110}\mu_{011}\mu_{101} - \mu_{020}\mu_{101}^2 - \mu_{011}^2\mu_{200} - \mu_{110}^2\mu_{002}$$

as a moment invariant form.

V. Invariants of Ternary Quantics

A. *Invariants of Ternary Quadratics: Examples*

Example 1: Consider a ternary form composed only of quadratic terms,

$$f = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 \tag{58}$$

and an orthogonal transformation in the form of a rotation about the x_1 axis,

$$R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{59}$$

$$\begin{aligned} x &= R_{23}y \\ x_1 &= y_1 \\ x_2 &= y_2\cos\theta - y_3\sin\theta \\ x_3 &= y_2\sin\theta + y_3\cos\theta \end{aligned}$$

$$f(y) = a_1y_1^2 + y_2^2(a_2\cos^2\theta + a_3\sin^2\theta) + y_2y_3(a_3 - a_2)\sin^2\theta + y_3^2(a_3\cos^2\theta + a_2\sin^2\theta). \tag{60}$$

The three invariants are

$$\begin{aligned} A_1 &= a_1 = I_1 \\ A_2 + A_3 &= a_2 + a_3 = I_2 \\ (A_3 - A_2)^2 + A_4^2 &= (a_3 - a_2)^2 + a_4^2 = I_3 \end{aligned} \tag{61}$$

where $a_4 = 0$ in this case.

Example 2:

$$f(x) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 \tag{62}$$

and the rotation about the x_3 axis,

$$R_{12} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{63}$$

$$\begin{aligned} x_1 &= \cos\theta\, y_1 - \sin\theta\, y_2 \\ x_2 &= \sin\theta\, y_1 + \cos\theta\, y_2 \\ x_3 &= y_3 \end{aligned} \tag{64}$$

$$f(y) = y_1^2(a_1\cos^2\theta + \sin^2\theta\, a_2) + y_2^2(a_1\sin^2\theta + a_2\cos^2\theta) + 2\sin\theta\cos\theta(a_2 - a_1)y_1y_2 + a_3y_3^2. \tag{65}$$

The invariants are

$$\begin{aligned} I_1 &= a_3 = A_3 \\ I_2 &= a_1 + a_2 = A_1 + A_2 \\ I_3 &= (a_1 - a_2)^2 + a_4^2 = (A_1 - A_2)^2 + A_4^2 \end{aligned} \tag{66}$$

where $a_4 = 0$ in this case. In general, for a conic form

$$f = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_2x_3 + 2b_2x_1x_3 + 2b_3x_1x_2 + 2c_1x_1 + 2c_2x_2 + 2c_3x_3 + d. \tag{67}$$

The invariants are derived to be (see Section V-B)

$$\Delta_1 = \det\begin{bmatrix} a_1 & b_3 & b_2 & c_1 \\ b_3 & a_2 & b_1 & c_2 \\ b_2 & b_1 & a_3 & c_3 \\ c_1 & c_2 & c_3 & d \end{bmatrix} \tag{68}$$

$$\Delta_2 = \det\begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \tag{69}$$

$$J_1 = a_1 + a_2 + a_3 \tag{70}$$

$$J_2 = a_2a_3 - b_1^2 + a_1a_3 - b_2^2 + a_1a_2 - b_3^2. \tag{71}$$

These invariants are relative invariants because

$$\begin{aligned} \Delta_1 &= \rho^4\Delta' \\ \Delta_2 &= \rho^3\Delta'_2 \\ J_1 &= \rho J'_1 \\ J_2 &= \rho^2 J'_2 \end{aligned} \tag{72}$$

which will lead to the following three absolute invariants under the rotation:

$$\frac{J_1J_2}{\Delta_2}, \frac{J_1^2}{J_2}, \frac{J_1\Delta_2}{\Delta_1}.$$

Now for the two previous examples these absolute invariants will be evaluated.

In the case of Example 1

$$\Delta_1 = \det\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0 \tag{73}$$

$$\Delta_2 = \det\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = a_1a_2a_3 \tag{74}$$

$$J_1 = a_1 + a_2 + a_3 \tag{75}$$

$$J_2 = a_2a_3 + a_1a_3 + a_1a_2 - b_1^2 \tag{76}$$

$$\frac{J_1J_2}{\Delta_2} = \frac{(a_1 + a_2 + a_3)(a_2a_3 + a_1a_3 + a_1a_2 - b_1^2)}{a_1a_2a_3} \quad \text{first invariant} \tag{77}$$

$$\frac{J_1^2}{J_2} = \frac{(a_1 + a_2 + a_3)^2}{a_1a_2 + a_1a_3 + a_2a_3 - b_1^2} \quad \text{second invariant} \tag{78}$$

$$\Delta_1 = 0 \quad \text{third invariant} \tag{79}$$

$$\begin{cases} I_1 + I_2 = J_1 \\ I_2^2 + 2I_1^2 + I_3 = 2(J_1^2 - 2J_2). \end{cases}$$

In the case of Example 2

$$\begin{aligned} &I_1 + I_2 = J_1 \\ &I_2^2 + I_3 + 2I_1^2 = 2(J_1^2 - 2J_2) \end{aligned} \tag{80}$$

$$\begin{aligned} 4J_2 &= 2J_1^2 - I_2^2 - I_3 - 2I_1^2 \\ &= 2I_1^2 + 2I_2^2 - 2I_1I_2 - I_2^2 - I_3 - 2I_1^2 \end{aligned}$$

$$J_2 = 1/4(I_2^2 - 2I_1I_3 - I_3) \tag{81}$$

B. Invariants of Ternary Quadratics: Conic Surfaces

Consider the quadratic forms

$$f \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_2x_3 + 2b_2x_1x_3 + 2b_3x_1x_2. \tag{82}$$

For all rotations

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 \tag{83}$$

where $y_i; i = 1, 3$ are the new coordinates. Hence,

$$\begin{aligned} U_\lambda = {} & a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_2x_3 + 2b_2x_1x_3 \\ & + 2b_3x_1x_2 - \lambda(x_1^2 + x_2^2 + x_3^2) \end{aligned} \tag{84}$$

will be transformed into

$$\begin{aligned} V_\lambda = {} & A_1y_1^2 + A_2y_2^2 + A_3y_3^2 + 2B_1y_2y_3 + 2B_2y_1y_3 \\ & + 2B_3y_1y_2 - \lambda(y_1^2 + y_2^2 + y_3^2). \end{aligned} \tag{85}$$

›w the theory of conics states that the condition that U_λ ›resents a plane-pair is

$$\det\begin{bmatrix} a_1 - \lambda & b_3 & b_2 \\ b_3 & a_2 - \lambda & b_1 \\ b_2 & b_1 & a_3 - \lambda \end{bmatrix} = 0 \tag{86}$$

which leads to

$$\lambda^3 - J_1\lambda^2 + J_2\lambda + \Delta_2 = 0 \tag{87}$$

and similarly for V_λ

$$\lambda^3 - J_1'\lambda^2 + J_2'\lambda + \Delta_2' = 0.$$

Hence,

$$\begin{aligned} &J_1 = a_1 + a_2 + a_3 \\ &J_2 = A_1 + A_2 + A_3; \quad \text{where } A_i; i = 1, 3 \text{ is cofactor of } a_i; i = 1, 3 \text{ in } \Delta_2. \end{aligned} \tag{88}$$

$$\Delta_2 = \det\begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}.$$

Now consider a general form of quadratic equations

$$\begin{aligned} F = {} & a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_2x_3 + 2b_2x_1x_3 \\ & + 2b_3x_1x_2 + 2c_1x_1 + 2c_2x_2 + 2c_3x_3 + d. \end{aligned} \tag{89}$$

Proceeding in a similar manner as in the previous case it is observed that

$$x_1^2 + x_2^2 + x_3^2 + 1 = y_1^2 + y_2^2 + y_3^2 + 1 \tag{90}$$

for all rotations. Defining

$$F_\lambda = F - \lambda(x_1^2 + x_2^2 + x_3^2 + 1), \tag{91}$$

it can be stated that F_λ represents a cone if

$$\det\begin{bmatrix} a_1 - \lambda & b_3 & b_2 & c_1 \\ b_3 & a_2 - \lambda & b_1 & c_2 \\ b_2 & b_1 & a_3 - \lambda & c_3 \\ c_1 & c_2 & c_3 & d \end{bmatrix} = 0 \tag{92}$$

which will lead to invariancy of $J_3 = B_1 + B_2 + B_3 + \Delta_2$ where B_i, $i = 1, 4$ are cofactors of a_1, a_2, a_3, d in Δ_1 which is defined in (68).

Thus, $J_1, J_2, J_3, \Delta_1, \Delta_2$ are invariant under rotations. These invariants are relative invariants because

$$\begin{aligned} &J_1 = \rho J_1' \\ &J_2 = \rho^2 J_2' \\ &J_3 = \rho^3 J_3' \qquad \rho \text{ is a constant of proportionality} \\ &\Delta_1 = \rho^4 \Delta_1' \\ &\Delta_2 = \rho^3 \Delta_2'. \end{aligned} \tag{93}$$

Discarding J_3 which is not invariant under translations, the absolute invariants for any transformations are

$$\frac{J_1J_2}{\Delta_2}, \ \frac{J_1^2}{J_2}, \ \frac{J_1\Delta_2}{\Delta_1} \quad \text{or} \quad \frac{J_1^2}{J_2}, \ \frac{\Delta_2}{J_1^3} \quad \text{and} \quad \frac{\Delta_1}{J_2^2}.$$

Using the fundamental theorem of moment invariants and the fact that when f is homogeneous $\Delta_1 = 0$ then the moment invariants are

$$\frac{J_{1\mu}^2}{J_{2\mu}} \quad \text{and} \quad \frac{\Delta_{2\mu}}{J_{1\mu}^3}$$

where

$$J_{1\mu} = \mu_{200} + \mu_{020} + \mu_{002}$$

$$J_{2\mu} = \mu_{020}\mu_{002} - \mu_{011}^2 + \mu_{200}\mu_{002} - \mu_{101}^2 + \mu_{200}\mu_{020} - \mu_{110}^2$$

$$\Delta_{2\mu} = \det\begin{bmatrix} \mu_{200} & \mu_{110} & \mu_{101} \\ \mu_{110} & \mu_{020} & \mu_{011} \\ \mu_{101} & \mu_{011} & \mu_{002} \end{bmatrix}.$$

C. Geometrical Interpretations of Invariants

A geometrical interpretation of the three-dimensional quadratic algebraic invariants will contribute to the conceptual understandings of these algebraic forms.

Any geometrical property which remains invariant under rotations and translations can be expressed in terms of

$$\frac{J_1 J_2}{\Delta_2}, \ \frac{J_1^2}{J_2}, \ \text{and} \ \frac{J_1 \Delta_2}{\Delta_1}.$$

Example 3–Radius of a Sphere: Consider

$$a_1x_1^2 + a_1x_2^2 + a_1x_3^2 - 2c_1a_1x_1 - 2c_2a_1x_2 - 2c_3a_1x_3 - a_1d = 0 \tag{94}$$

writing

$$a_1R^2 = a_1d + a_1(c_1^2 + c_2^2 + c_3^2)$$

$$R^2 = d + c_1^2 + c_2^2 + c_3^2. \tag{95}$$

But

$$\Delta_1 = -a_1^4 d - a_1^4(c_1^2 + c_2^2 + c_3^2) \tag{96}$$

$$\Delta_2 = a_1^3 \tag{97}$$

$$J_2 = 3a_1^2, \ J_1 = 3a_1 \tag{98}$$

then

$$R^2 = -3\left(\frac{\Delta_1}{J_1\Delta_2}\right). \tag{99}$$

Hence,

$$\text{area of sphere} = -3\left(\frac{\Delta_1}{J_1\Delta_2}\right)2\pi \tag{100}$$

$$\text{volume of sphere} = 3/4\pi\left(-3\frac{\Delta_1}{J_1\Delta_2}\right)^{3/2} = -9/4\pi\left(\frac{\Delta_1}{J_1\Delta_2}\right)^{3/2} \tag{101}$$

Eccentricity of Quadratics: By definition a quadratic is the locus of a point such that the square of its distance from a fixed point is proportional to the product of its distance from two fixed planes [4].

The proportionality constant is called the eccentricity of the quadratic.

Example 4: Consider a case where the two planes coincide; then

$$(x_1 - \alpha)^2 + (x_2 - \beta)^2 + (x_3 - \gamma)^2 - \frac{e}{k^2 + l^2 + m^2}(kx_1 + lx_2 + mx_3 + n)^2 = 0 \tag{102}$$

where (α, β, γ) is the fixed point's coordinate and $kx_1 + lx_2 + mx_3 + n = 0$ is the equation of the plane (see Fig. 2).

Then it can be concluded that

$$J_1 = (3 - e)\mu$$

$$J_2 = (3 - 2e)\mu^2$$

$$\frac{J_2}{J_1^2} = \frac{3 - 2e}{(3 - e)^2}; \tag{103}$$

but J_2/J_1^2 is an absolute invariant, so the eccentricity is given by an equation in terms of only one absolute invariant.

VI. Experimental Results

A rectangular solid [Fig. 3(a)] was placed in nine different positions and for each case the coordinates of each of its eight vertices were obtained. The resulting moment invariants for each position are shown in Table III. These results overwhelmingly verify the invariancy of these moments irrespective of position and orientation. The small variations are due to measurement error and the discrete nature of the data. In two similar experiments a cylinder and a pyramid [Figs. 3(b) and (c)] were used and for each case the set of moments in four different positions was obtained. The resulting moment invariants are shown in Tables IV and V. The mean and standard deviations of moment invariants which are shown in Table VI display the discriminatory property of the moments.

These experiments show that a solid object can be represented by a set of two moment invariants irrespective of any specific coordinate system. A robot using its own preferred coordinate system could identify this object by measuring the coordinates of its vertices (or other points which represent the object discretely) and use that data to obtain and compare the moments.

VII. Conclusion

Extension of two-dimensional moment invariants to three dimensions was studied in this paper. This generalization is not trivial because of the difficulties which are present in the derivation of general ternary quantic invariant forms upon which three-dimensional moment invariants rely.

As a special but important subset of general ternary quantics, the class of ternary quadratic forms was explored and several geometrical interpretations of invariants were given. It was stated that every geometrical property of a quadratic

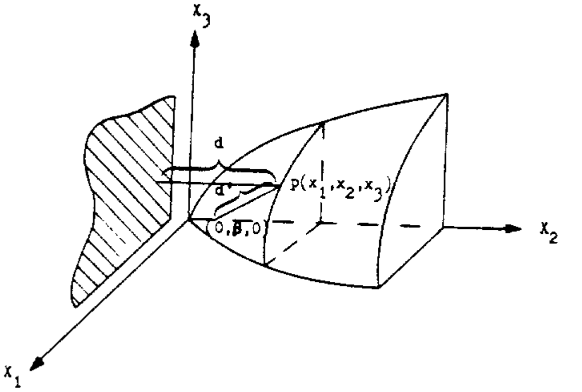

Fig. 2. Eccentricity is d'/d.

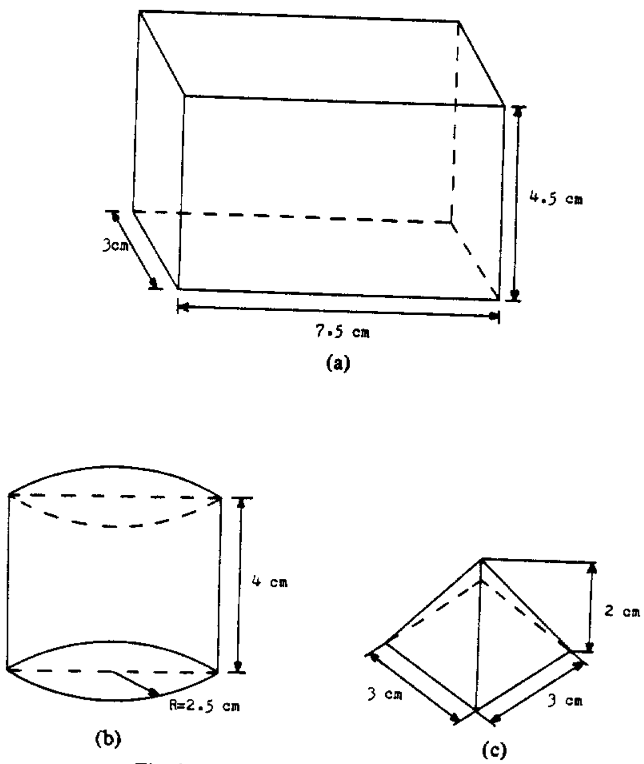

Fig. 3. Objects used in the experiment.

surface which remains invariant under rotation and translation can be given in terms of its absolute invariant forms.

The experimental results confirm the invariancy of moment invariants, and their utility in classifying and identifying various three-dimensional objects irrespective of size or frame of reference.

Thus, moments exhibit a potential for greatly reducing the amount of data and processing needed for three-dimensional object recognition.

TABLE III

POSITION NUMBER	$\frac{J_{1\mu}^2}{J_{2\mu}}$	$\frac{\Delta_{2\mu}}{J_{1\mu}^3}$
1	4.000	0.016
2	3.956	0.018
3	3.993	0.044
4	3.898	0.023
5	3.988	0.015
6	3.984	0.015
7	3.957	0.016
8	3.941	0.050
9	3.957	0.016

TABLE IV

POSITION NUMBER	$\frac{J_{1\mu}^2}{J_{2\mu}}$	$\frac{\Delta_{2\mu}}{J_{1\mu}^3}$
1	3.242	0.0272
2	3.362	0.0519
3	3.242	0.0272
4	3.242	0.0272

TABLE V

POSITION NUMBER	$\frac{J_{1\mu}^2}{J_{2\mu}}$	$\frac{\Delta_{2\mu}}{J_{1\mu}^3}$
1	3.000	0.037
2	3.002	0.037
3	3.000	0.037
4	3.000	0.037

TABLE VI

	$\frac{J_{1\mu}^2}{J_{2\mu}}$		$\frac{\Delta_{2\mu}}{J_{1\mu}^3}$	
	MEAN	STANDARD DEVIATION	MEAN	STANDARD DEVIATION
SOLID RECTANGLE	3.970	0.0098	0.020	0.0094
PYRAMID	3.272	0.0150	0.033	0.0030
CYLINDER	3.000	0.0002	0.037	0.0000

References

[1] M.-K. Hu, "Visual pattern recognition by moment invariants," *IRE Trans. Inform. Theory*, vol. IT-8, pp. 179–187, Feb. 1962.

[2] E. L. Hall, W. O. Crawford, Jr., and F. E. Roberts, "Computer classification of pneumoconiosis from radiographs of coal workers," *IEEE Trans. Biomed. Eng.*, vol. BME-22, pp. 518–527, Nov. 1975.

[3] F. A. Sadjadi and E. L. Hall, "Numerical computation of moment invariants for scene analysis," in *Proc. IEEE Conf. on Pattern Recognition and Image Processing*, Chicago, IL, 1978.