Lecture #9: Ensemble Learning

What is Ensemble Learning?

In ensemble learning, the idea is to combine multiple classifiers into a single one. Ensemble learning usually works very well in practice.

Two methods (for this class):

1. Bagging
2. Boosting

Bagging

BAGGING: (Bootstrap AGGregatING)

1. Input: n labelled training examples $(x_1, y_1), .., (x_n, y_n)$

2. Algorithm:

   Repeat k times:

   (a) Select m samples out of n with replacement from the training set to get training set S_i

   (b) Train classifier h_i on S_i (usually, h_i's are the same type of classifier)

3. Output: Classifiers $h_1, .., h_k$

Testing: Given test example x, output the majority of $h_1(x), h_2(x), .., h_k(x)$ (break ties at random as usual)

Choice Points in Bagging

1. How to pick k?

Higher k is better, but also increases training time, storage requirement and classification time. So pick a k which is feasible.

Choice Points in Bagging

2. How to pick m?

Popular choice for $m = n$.

<u>But this is still very different from working with the entire training set!</u>

$$\Pr(S_i = S) = \frac{n!}{n^n} \quad \left(\begin{array}{l} \#\text{ ways of choosing } n \text{ samples} \\ \text{with replacement} = n^n \end{array} \right)$$

↦ only $n!$ of these ways give you the entire training set!

↦ $\frac{n!}{n^n}$ = a very tiny number $\ll 2^{-n/2}$

For any (x_j, y_j), $\Pr((x_j, y_j) \text{ is not in } S_i) = \left(1 - \frac{1}{n}\right)^n \approx \frac{1}{e}$ (for large n)

$1/e \approx 0.37$; so about 37% of the data is left out of S_i.

Why does Bagging work?

It can be shown that bagging decreases the variance of a classifier. (it doesn't help much with bias).

Thus it prevents overfitting.

Boosting

Boosting

Sometimes it is:

- easy to come up with simple, easy to use, rules of thumb classifiers
- but hard to come up with a single highly accurate rule.

Examples:

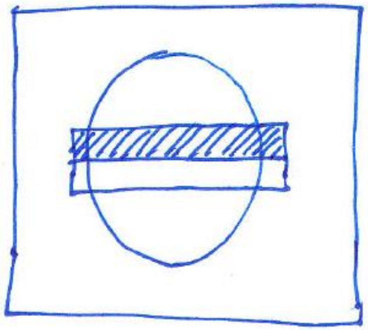
(1) Spam classification, based on email text.

Certain words, eg. "Nigeria", "Online Pharmacy", etc. typically are a good indicator of spam.

Rule-of-thumb: Does email contain word "Nigeria"?

Boosting (contd.)

(2) Detect if an image has a face in it.



On an average, pixels around the eyes are darker than those below.

Rule of thumb: Is the (average darkness in the shaded region) − (average darkness in the white rectangular region below) > 0 ?

Boosting gives us a way to combine these ~~weak~~ rules ~~into~~ of thumb into good classifiers.

Definitions:

1. Weak Learner: A simple rule of thumb that doesn't necessarily work very well.
2. Strong Learner: A good classifier (with high accuracy)

Boosting Framework

Boosting Procedure:

1. Design method to find a good rule of thumb.
2. Repeat:
   - Find a good rule of thumb
   - Modify training data to get a second data set
   - Apply method ~~of~~ to new data set to get a good rule of thumb, and so on.

1. How to get a good rule of thumb? Application specific (more later)
2. How to modify training data set?
   - Give highest weight to the hardest examples – those that were misclassified more often by previous rules of thumb.
3. How to combine the rules of thumb into a prediction rule?

   Take a weighted majority of the rules.

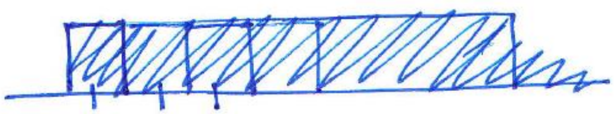
Weak Learner: Example

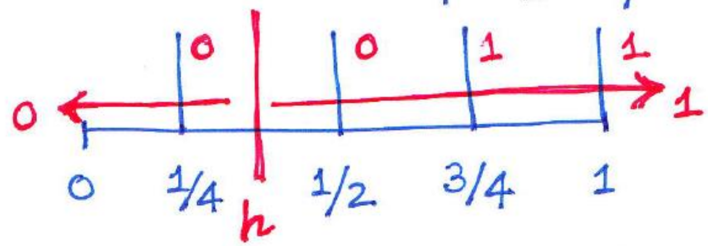
Let D be a distribution over labelled examples, and let h be a classifier.

Error of h wrt D is:

$$err_D(h) = \Pr_{(x,y)\sim D}[h(x) \neq y]$$

<u>Example:</u> D: X: takes values $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, each w.p. $1/4$.





$Y = 1$ if X has value $> 1/2$, o/w $Y = 0$.

Then if h is the rule:

$$h(x) = 1 \text{ if } x > 1/4$$
$$= 0 \text{ o/w.}$$

Then, $err_D(h) = \frac{1}{4}$.

Basic Definitions

→ h is called a <u>weak learner</u> if $err_D(h) < 0.5$

→ Error of random guessing is 0.5 (with 2 labels)

Given training examples $(x_1, y_1), \ldots, (x_n, y_n)$, we can assign weights $w_1, \ldots, w_n$ to these examples. If $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$, we can think of these weights as a probability distribution over the examples.

Error of a classifier h wrt W is:

$$err_W(h) = \sum_{i=1}^{n} w_i \, 1(h(x_i) \neq y_i)$$

1 is the indicator function, where $1(P) = 1$ if P is true
$= 0$ otherwise.

Boosting Algorithm

Input: Training set $S = \{(x_1, y_1), .., (x_n, y_n)\}$, $y_i = \pm 1$

$D_1(i) = 1/n$ for all $i = 1, .., n$

For $t = 1, 2, 3, \ldots$

h_t = weak-learner wrt D_t. (so, $err_{D_t}(h_t) < 0.5$)

$\varepsilon_t = err_{D_t}(h_t)$

$\alpha_t = \frac{1}{2} \ln \frac{1-\varepsilon_t}{\varepsilon_t}$ (so, α_t is high when ε_t is low, and almost 0 when ε_t is close to 0.5

$$D_{t+1}(i) = \frac{D_t(i)\, e^{-\alpha_t y_i h_t(x_i)}}{Z_t}$$

(D_{t+1} goes ↑ if i is misclassified by h_t; so higher D_t means harder example.

where Z_t is a normalization constant to ensure that

$$\sum_i D_{t+1}(i) = 1.$$

Final classifier: $H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right)$ (weighted majority of $h_t(x)$'s)

Boosting Examples

Example of Weighted Error:

Suppose training data is: $((0,0),1)$, $((1,0),1)$, $((0,1),-1)$

weights W: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{4}$

classification rule: Predict 1 if $x_1 \le 1/2$, -1 otherwise.

$$err_W(h) = \frac{1}{2}\times 0 + \frac{1}{4}\times 1 + \frac{1}{4}\times 1 = \frac{1}{2}$$

(The usual (unweighted) error would be $2/3$).

Boosting Algorithm Example:

Training data:

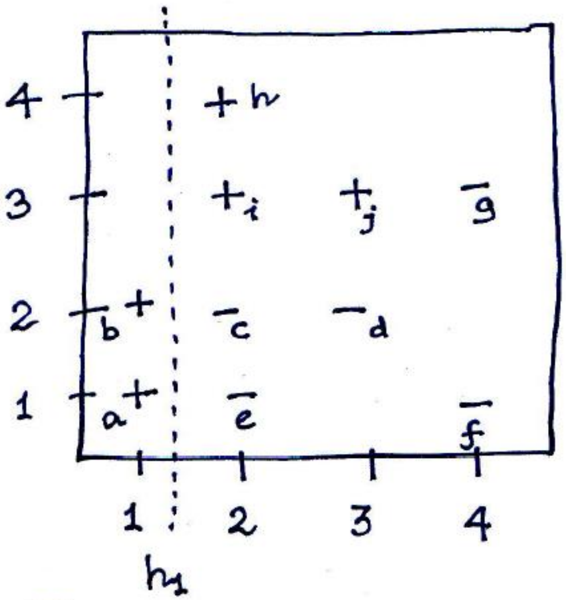
$((1,1),+)$ $((2,1),-)$ $((4,1),-)$

$((1,2),+)$ $((2,2),-)$ $((3,2),-)$

$((2,3),+)$ $((3,3),+)$ $((4,3),-)$

$((2,4),+)$

Boosting Examples



Initially: $D_1(i) = 0.1$ (for all i)

~~Suppose~~

Weak Learners: Set of vertical and horizontal thresholds.

(1) Suppose we pick $h_1(x) = +$ if $x_1 \leq 1.5$

$= -$ otherwise

Name the points: a, b, .., j (for ease of understanding)

Then:

$$err_{D_1}(h_1) = \varepsilon_1 = 0.3 \qquad \alpha_1 = 0.42$$

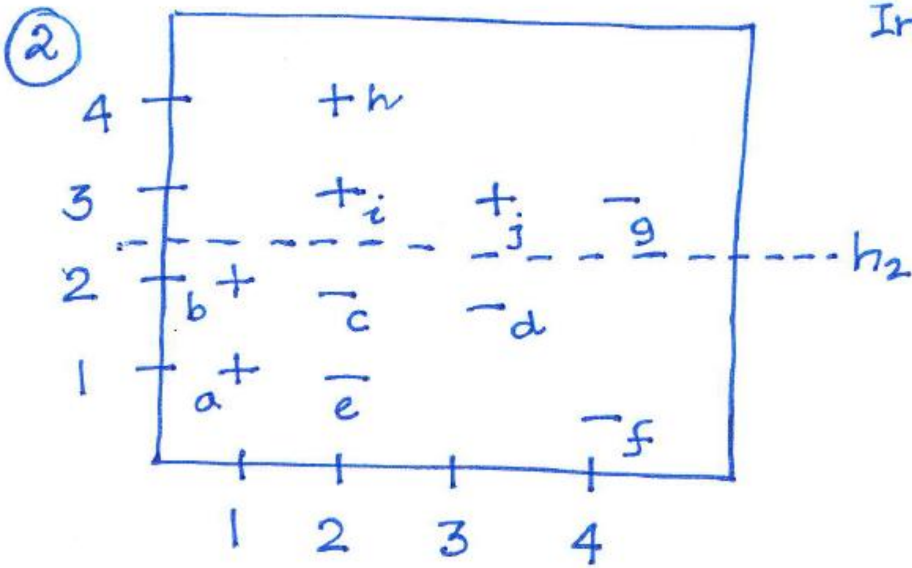
Weights of a, b, c, d, e, f, g: $D_2 = 0.07$

Weights of h, i, j: $D_2 = 0.17$

$$Z_2 = 7 \cdot e^{-0.42} \cdot 0.1 + 3 \cdot 0.1 \cdot e^{0.42}$$
$$= 0.92$$

Note: Calculations rounded to 2 decimal places.

Boosting Examples



In Round 2, suppose we pick

$$h_2(x) = + \text{ if } x_2 > 2.5$$
$$= - \text{ otherwise.}$$

$err_{D_2}(h_2) = \varepsilon_2 = 0.21$

$\alpha_2 = 0.66$

Weights of a, b: $D_3 := 0.07 \times e^{0.66} / Z_3 = 0.17$

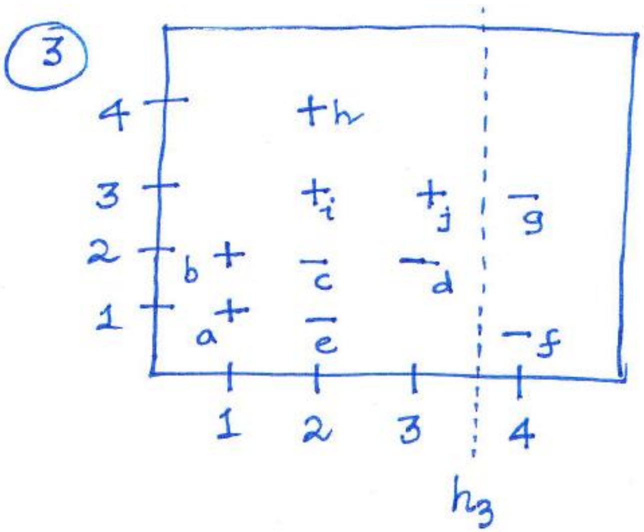
Weights of c, d, e, f: $D_3 := 0.07 \times e^{-0.66} / Z_3 = 0.04$

Weights of h, i, j: $D_3 := 0.17 \times e^{-0.66} / Z_3 = 0.11$

Weight of g: $D_3 := 0.07 \times e^{0.66} / Z_3 = 0.17$

$Z_3 = 0.81$

Boosting Examples



In Round 3, suppose we pick:

$$h_3(x) = + \text{ if } x_1 \le 3.5$$
$$= - \text{ otherwise.}$$

$err_{D_3}(h_3) = \varepsilon_3 = 0.12$

$\alpha_3 = 0.99$

Weights of a, b: $D_4 := 0.17 \times e^{-0.99} / Z_4 = 0.1$

" " c, d, e: $D_4 := = 0.04 e^{0.99} / Z_4 = 0.17$

" " h, i, j: $D_4 := 0.11 \times e^{-0.99} / Z_4 = 0.06$

" " f : $D_4 := 0.04 e^{-0.99} / Z_4 = 0.02$ $\quad Z_4 = 0.65$

" " g : $D_4 := 0.17 e^{-0.99} / Z_4 = 0.1$

Final classifier: $\text{sign}(\alpha_1 h_1(x) + \alpha_2 h_2(x) + \alpha_3 h_3(x))$

$$= \text{sign}(0.42 h_1(x) + 0.66 h_2(x) + 0.99 h_3(x))$$

Boosting and Overfitting

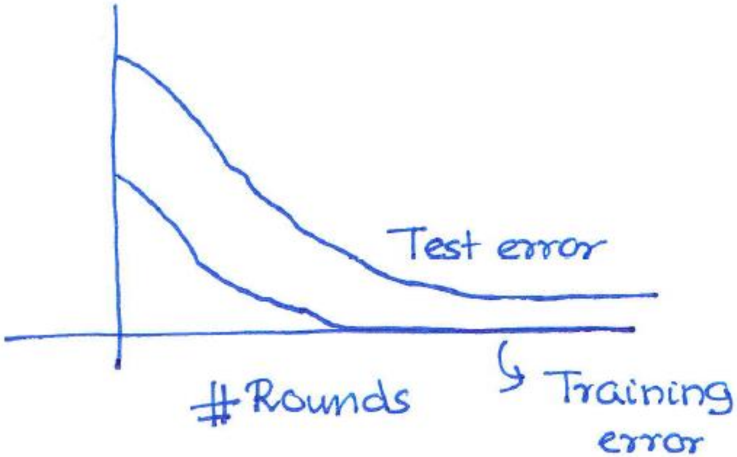
When to stop boosting? Use a validation dataset to find a stopping time.

Stop when validation error does not improve.

Boosting and Overfitting:

Overfitting can happen with boosting, but often does not.

Typical boosting run:



Reason is that the margin of classification often increases with boosting.

Boosting Margin

Intuitively, margin of classification measures how far the + labels are from the − labels.

Small Margin Large Margin

Note: Notion of margin for boosting is a little different from the exact way we defined margin for perceptron, but the difference is fairly technical.

For boosting:

- think of each $h_t()$ as a feature
- Feature space is:

  $$[\,h_1(x), h_2(x), \ldots, h_T(x)\,]$$

- Margin of example x is: $\left|\sum_{t=1}^{T} \alpha_t h_t(x)\right|$.
- If you have large margin data, then classifiers need less training examples to avoid overfitting. (This is also why kernels work, even if they are very high dimensional feature spaces.)