Logic

School of EECS
Washington State University

Knowledge-based Agent

- Knowledge Base (KB) and Knowledge Engineering
  - TELL agent about the environment
- Knowledge representation and logic formalisms
  - **Propositional logic**
  - **First-order logic**
  - Temporal logics, modal logics (esp. in DAI context!)
  - Many others…

- Reasoning via logical inference
  - ASK agent how to achieve goal based on current knowledge about the task and the world

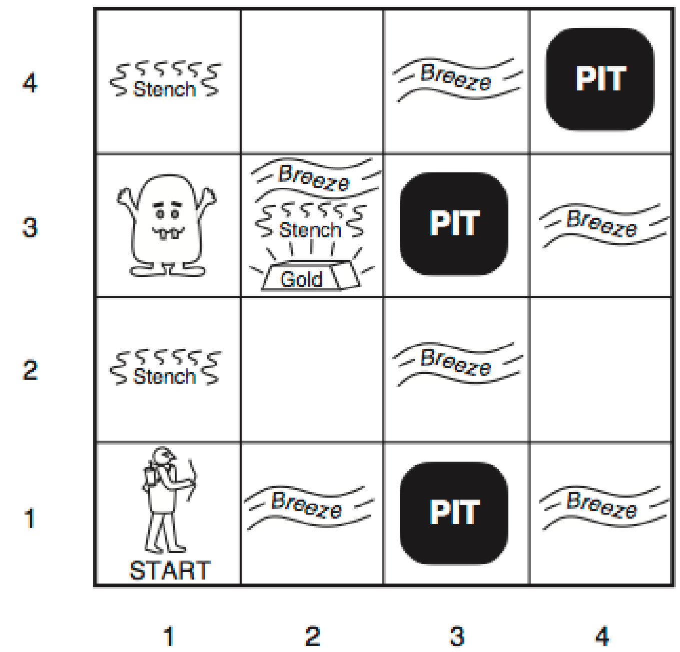
Knowledge-based Agent

function KB-AGENT (*percept*) **returns** an action
  persistent: *KB*, a knowledge base
        t, a counter, initially 0, indicating time
  TELL (KB, MAKE-PERCEPT-SENTENCE (*percept*, *t*))
  action ← ASK (KB, MAKE-ACTION-QUERY (*t*))
  TELL (KB, MAKE-ACTION-SENTENCE (*action*, *t*))
  t ← *t* + 1
  return *action*

Wumpus World (PEAS)

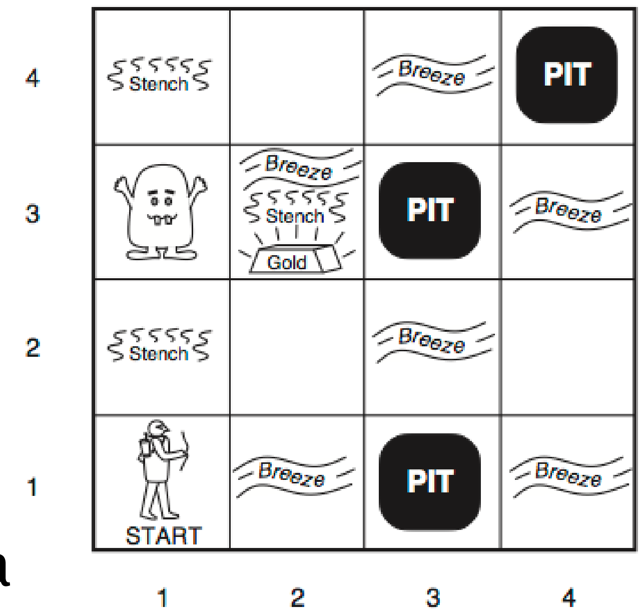
- **Performance** measure
  - +1000 for leaving cave with gold
  - −1000 for falling in pit or being eaten by Wumpus
  - −1 for each action taken

    (here, action = move or turn)
  - −10 for using the arrow
  - Game ends when agent either gets killed or leaves the cave



Wumpus World (PEAS)

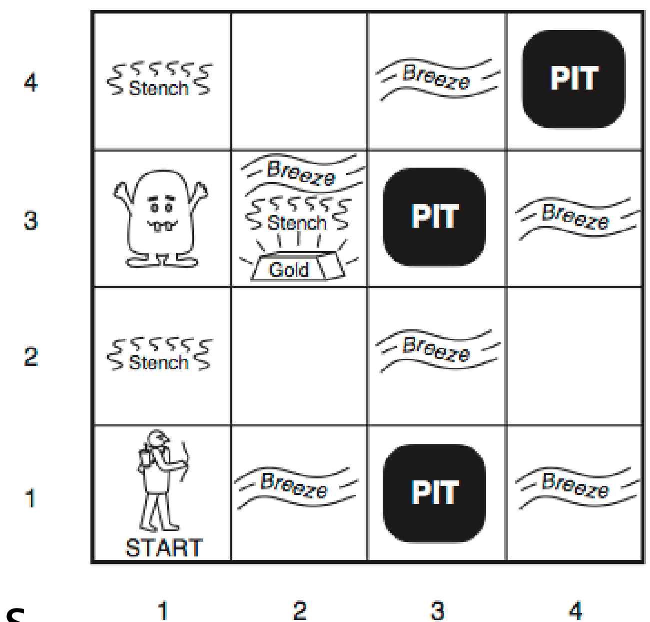
- **Environment**
  - 4x4 grid of rooms
  - Agent starts in square [1,1] facing right
  - Location of Wumpus and gold chosen at random
    - any other than [1,1]
  - Each square other than [1,1] has a containing a pit



  - What if… (location of pits vs. location of gold/Wumpus)

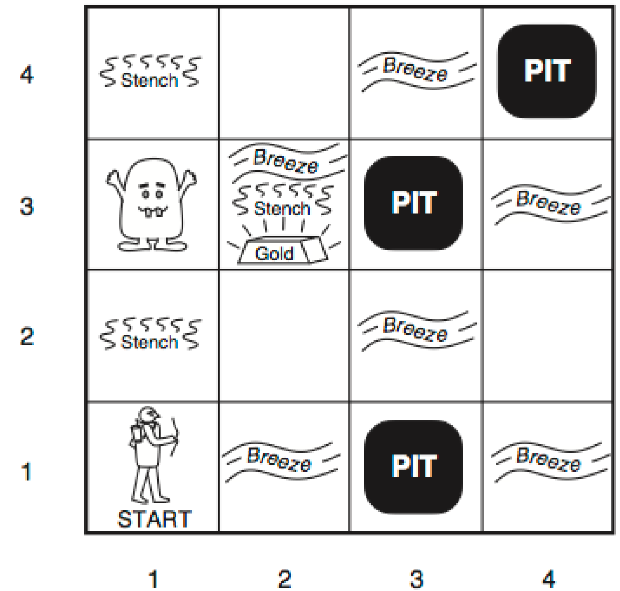
Wumpus World (PEAS)

- **Actuators**
  - Forward
  - TurnLeft by 90°
  - TurnRight by 90°
  - Grab picks up gold if agent in gold location
  - Shoot shoots arrow in direction agent is facing
    - Arrow continues straight until hits Wumpus or wall
  - Climb leaves cave if agent in [1,1]



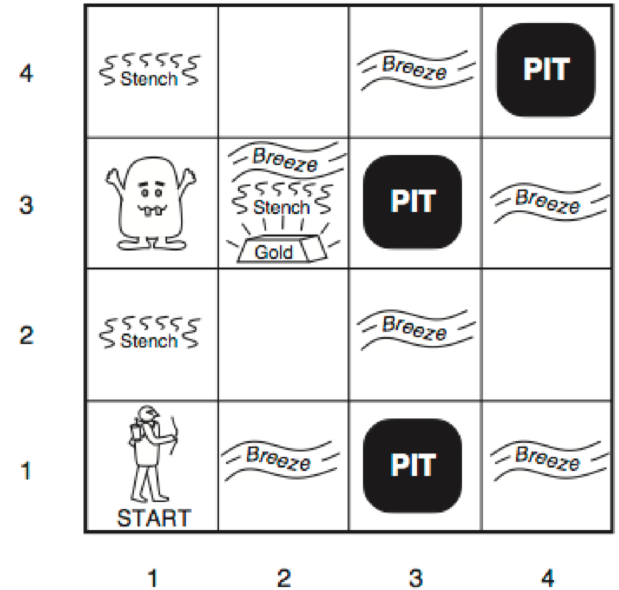
Wumpus World (PEAS)

- **Sensors** (Boolean in our WW)
  - Stench if wumpus in directly (not diagonally) adjacent square
  - Breeze if pit in directly adjacent square
  - Glitter if gold in agent's current square
  - Bump if agent walks into a wall
  - Scream if wumpus is killed

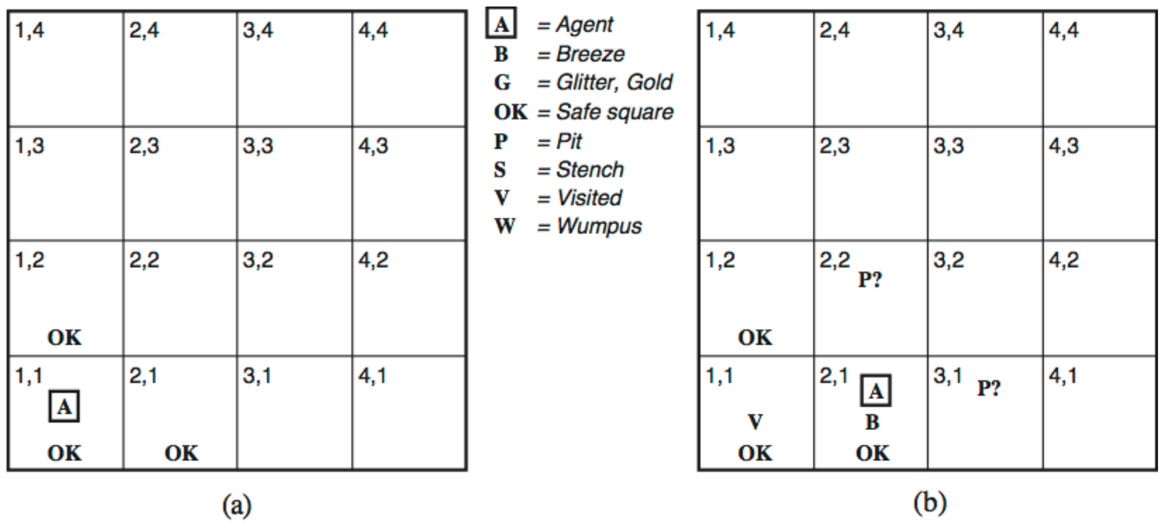


Acting Logically in Wumpus World

- **Goals**
  - Visit safe locations
  - Grab gold if present
  - If have gold or no more safe spots / unvisited locations, then move to [1,1] and Climb out



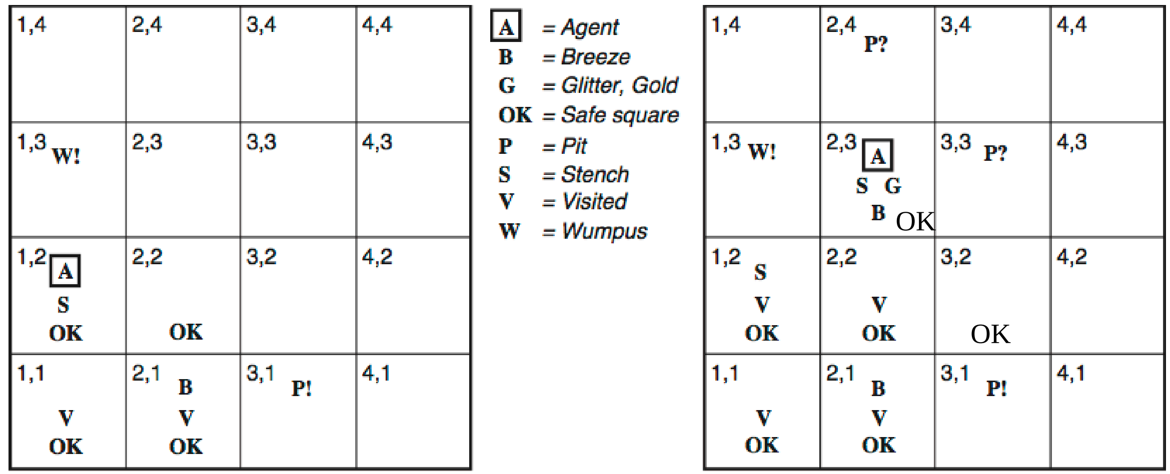
Acting Logically in Wumpus World



(a) (b)

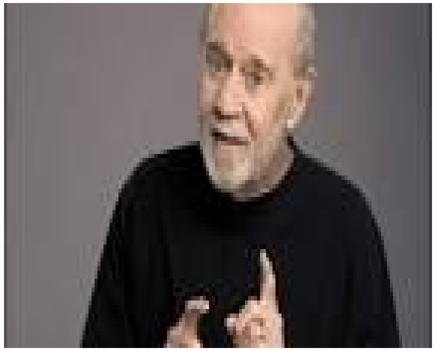
- Percept$_1$ = [None,None,None,None,None]
  - [1,2] and [2,1] safe
- Action = Forward
- Percept$_2$ = [None,Breeze,None,None,None]
- Either [2,2] or [3,1] or both has a pit
- Execute TurnLeft, TurnLeft, Forward, TurnRight, Forward

Acting Logically in Wumpus World



(a) (b)

- Percept_7 = [Stench,None,None,None,None]
  - Wumpus in [1,3]
  - No pit in [2,2] (safe), so pit in [3,1]
- Could Shoot, but <TurnRight,Forward> to [2,2]
- Percept_9 = [None,None,None,None,None]
  - [3,2] and [2,3] are safe
- <TurnLeft,Forward> to [2,3]
- Percept_{11} = [Stench,Breeze,Glitter,None,None]
- Grab gold, head home, and Climb (score: 1000 – 17 = 983)



An Appropriate George Carlin Quote

(… on this very special day in history of the free world…)

"In America, *anyone* can become president.

That's the problem."

Intro to Logic

- A **Knowledge Base (KB)** consists of "sentences"
- Syntax specifies a well-formed sentence
- Semantics specifies the meaning of a sentence
- A model **m** specifies whether each sentence is true or false
  - There can be many (possibly infinitely many) models
  - E.g., Model m_1 may say "wumpus in [2,3]" is true, m_1(wumpus in [2,3])
  - E.g., Model m_2 may say "wumpus in [2,3]" is false, or "wumpus not in [2,3]", m_2(wumpus not in [2,3])
  - Model m_3 may agree with m_1 on above but differ on smt. else
- **m(α)** says "**m** satisfies **α**" or "**m** is a model of **α**"

Logical Entailment

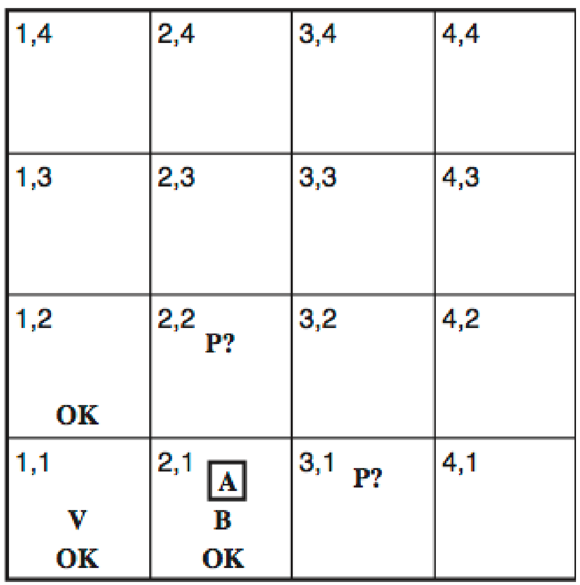
- Entailment between sentences implies that one sentence follows logically from another
- $\alpha \vDash \beta$ means α entails β
  - Or, for every model in which α is true, β is also true
  - $\alpha \vDash \beta$ if and only if $M(\alpha) \subseteq M(\beta)$

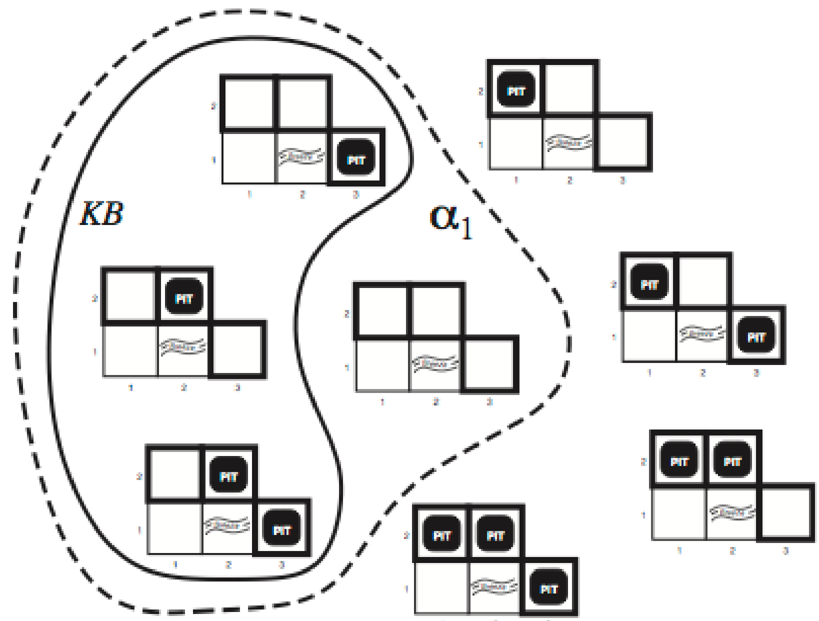
  where $M(\alpha)$ is the set of all models in which α is true
- Example: Pit([3,1]) $\vDash$ Breeze([2,1])

Logical Entailment (continued)

- E.g., KB = {¬Breeze([1,1]), Breeze([2,1]), *rules of Wumpus World*}

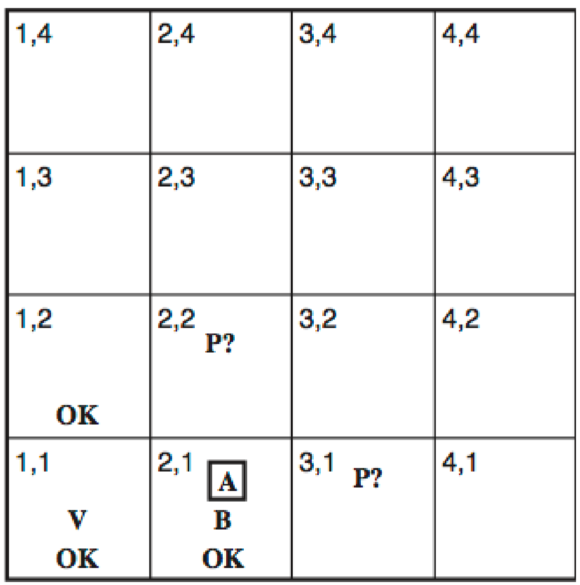
- Pit([1,2]) ?, Pit([2,2] ?, Pit([3,1]) ?
- $\alpha_1 = \neg$Pit([1,2]), KB $\models \alpha_1$, M(KB) $\subseteq$ M(α_1)

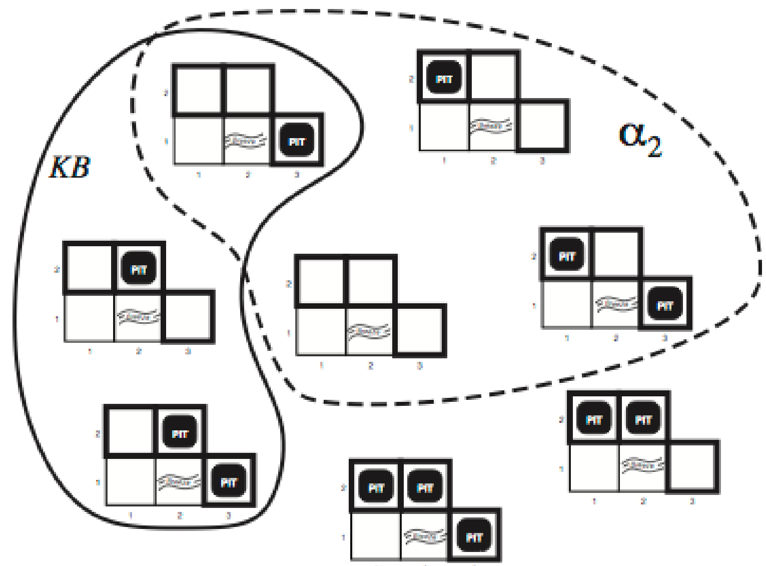




Logic: Entailed vs. Not Entailed

- E.g., KB = {¬Breeze([1,1]), Breeze([2,1]), *rules of Wumpus World*}
- $\alpha_2 = \neg Pit([2,2])$, $KB \nvDash \alpha_2$, $M(KB) \nsubseteq M(\alpha_2)$



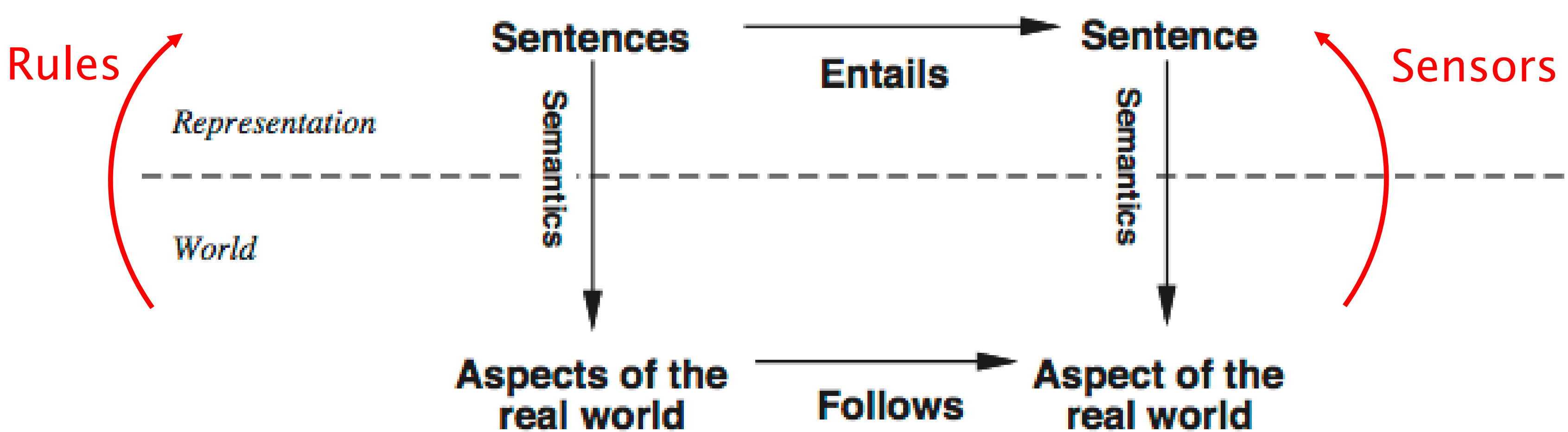


Logical Inference

- <u>Logical inference</u> is the process by which we infer one sentence is true from others
- E.g., **model checking** (previous two slides)
  - Enumerate all possible models in which KB is true and check if α is also true
- $KB \vdash_i \alpha$ means that α can be derived from KB using inference algorithm **i**
- Inference algorithm **i** is ***sound*** or <u>truth-preserving</u> if everything derived is also entailed
- Inference algorithm **i** is ***complete*** if it can derive everything that is entailed

Logic in the Real World

- Grounding is the connection between logic and the "real world"
- From an engineering / practitioner standpoint, grounding (together w/ design & implementation complexity) is among most important aspect of AI models and theories



Propositional Logic

- **Syntax**
  - Atomic sentences consist of a single propositional symbol, which can be true or false

Sentence	$\rightarrow$ AtomicSentence $\mid$ ComplexSentence	
AtomicSentence	$\rightarrow$ True $\mid$ False $\mid$ P $\mid$ Q $\mid$ R $\mid$ North $\mid$ $W_{1,3}$ $\mid$ ...	
ComplexSentence	$\rightarrow$ (Sentence) $\mid$ [Sentence]	
	$\mid$ $\neg$ Sentence	"not"
	$\mid$ Sentence $\land$ Sentence	"and"
	$\mid$ Sentence $\lor$ Sentence	"or"
	$\mid$ Sentence $\Rightarrow$ Sentence	"implies"
	$\mid$ Sentence $\Leftrightarrow$ Sentence	"if and only if"

Operator Precedence: $\neg$, $\land$, $\lor$, $\Rightarrow$, $\Leftrightarrow$

Propositional Logic: Syntax

- Syntax
  - Not (¬) is a <u>negation</u>
  - <u>Literal</u> is either an atomic sentence (<u>positive literal</u>) or a negated atomic sentence (<u>negative literal</u>)
  - And (∧) is a <u>conjunction</u>; its parts are <u>conjuncts</u>
  - Or (∨) is a <u>disjunction</u>; its parts are <u>disjuncts</u>
  - Implies (⇒) is an <u>implication</u>
    - Its lefthand side is the <u>antecedent</u> or <u>premise</u>
    - Its righthand side is the <u>consequent</u> or <u>conclusion</u>
  - If and only if (⇔) is a <u>biconditional</u>

Propositional Logic: Semantics

- **Semantics**
  - How to determine the truth value (true or false) of every proposition in a model

- True is true in every model
- False is false in every model

- Truth values of every other proposition must be specified directly in the model
  - E.g., $m_1 = \{W_{1,3}=true, P_{3,1}=true, P_{2,2}=false, \ldots\}$

Propositional Logic

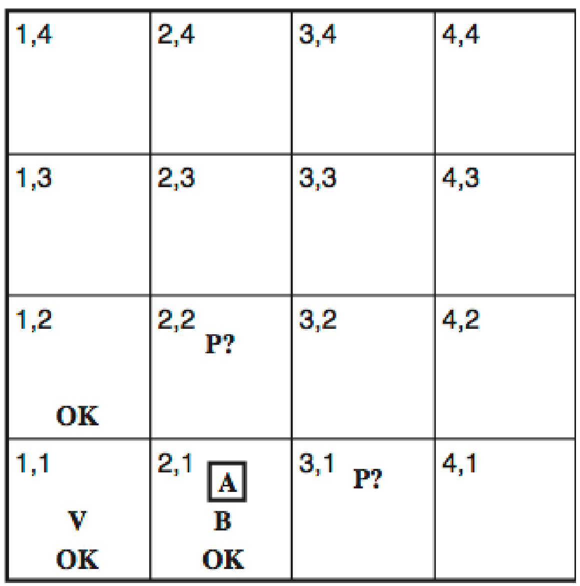
- Semantics for complex sentences in model **m**
  - $\neg P$ is true iff P is false in m
  - $P \wedge Q$ is true iff both P and Q are true in m
  - $P \vee Q$ is true iff either P or Q is true in m
  - $P \Rightarrow Q$ is true unless P is true and Q is false in m
  - $P \Leftrightarrow Q$ is true iff P and Q are both true or both false in m

Truth Table

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	false	true	false	false	true	true
false	true	true	false	true	true	false
true	false	false	false	true	false	false
true	true	false	true	true	true	true

Propositional Wumpus World KB

- $P_{x,y}$ is true if there is a pit in [x,y]
- $W_{x,y}$ is true if there is a wumpus in [x,y], alive or dead
- $B_{x,y}$ is true if the agent perceives a breeze in [x,y]
- $S_{x,y}$ is true if the agent perceives a stench in [x,y]



R_1: $\neg P_{1,1}$
R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$
R_4: $\neg B_{1,1}$
R_5: $B_{2,1}$

Simple Propositional Inference

```
function TT-ENTAILS? (KB, α) returns true or false   // KB ⊨ α ?
  symbols ← a list of the proposition symbols in KB and α
  return TT-CHECK-ALL (KB, α, symbols, {})

function TT-CHECK-ALL (KB, α, symbols, model) returns true or false
  if EMPTY? (symbols) then
    if PL-TRUE? (KB, model) then return PL-TRUE? (α, model)
    else return true
  else do
    P ← FIRST (symbols)
    rest ← REST (symbols)
    return (TT-CHECK-ALL (KB, α, rest, model ∪ {P = true}) and
                TT-CHECK-ALL (KB, α, rest, model ∪ {P = false}))
```

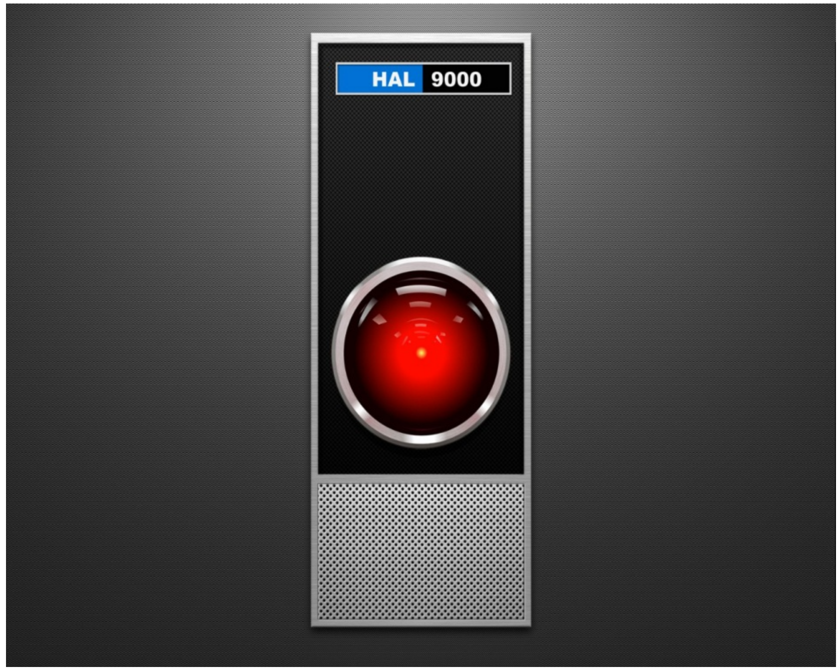
if KB false, return true?

and?

PL-TRUE? (*s*, *m*) returns true if sentence *s* true in model *m*.

TT-ENTAILS? sound and complete, but $O(2^n)$ time complexity.

What Happened to HAL?



"2001: A Space Odyssey" (1968)

Propositional Theorem Proving

- Theorem proving
  - Applying rules of inference on sentences in KB to derive sentence α

- A sentence is <u>valid</u> if it is true in all models

- **Deduction Theorem:**
  - For any sentences α and β, $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid

Propositional Theorem Proving

- A sentence is <u>satisfiable</u> if it is true in some model
  - $\alpha \models \beta$ if and only if the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable
- Proof by refutation or contradiction
  - Prove β from α by checking if $(\alpha \wedge \neg\beta)$ is unsatisfiable
  - I.e., assume β to be false and show this leads to a contradiction with α
  - Also implies that an inconsistent α can prove anything

Logical Equivalences

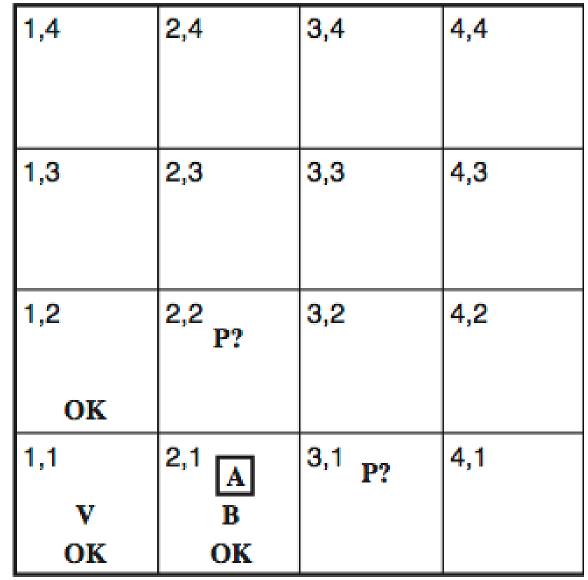
$$
\begin{array}{rcll}
(\alpha \wedge \beta) & \equiv & (\beta \wedge \alpha) & \text{commutativity of } \wedge \\
(\alpha \vee \beta) & \equiv & (\beta \vee \alpha) & \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) & \equiv & (\alpha \wedge (\beta \wedge \gamma)) & \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) & \equiv & (\alpha \vee (\beta \vee \gamma)) & \text{associativity of } \vee \\
\neg(\neg \alpha) & \equiv & \alpha & \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg \beta \Rightarrow \neg \alpha) & \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg \alpha \vee \beta) & \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) & \equiv & ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) & \equiv & (\neg \alpha \vee \neg \beta) & \text{de Morgan} \\
\neg(\alpha \vee \beta) & \equiv & (\neg \alpha \wedge \neg \beta) & \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) & \equiv & ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) & \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) & \equiv & ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) & \text{distributivity of } \vee \text{ over } \wedge
\end{array}
$$

Inference Rules

- Notation: $$\frac{\text{sentences given}}{\text{sentences inferred}}$$
- Modus Ponens: $$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$
- And-Elimination: $$\frac{\alpha \wedge \beta}{\alpha}$$
- Bi-conditional elimination:

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)} \quad \text{and} \quad \frac{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}{\alpha \Leftrightarrow \beta}$$

Sample Proof



R_1: $\neg P_{1,1}$
R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$
R_4: $\neg B_{1,1}$
R_5: $B_{2,1}$

Prove: $\neg P_{1,2}$

Apply biconditional elimination to R_2
R_6: $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

Apply And-Elimination to R_6
R_7: $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

Apply Contraposition to R_7
R_8: $(\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1}))$

Apply Modus Ponens to R_8 and R_4
R_9: $\neg(P_{1,2} \vee P_{2,1})$

Apply De Morgan's rule to R_9
R_{10}: $\neg P_{1,2} \wedge \neg P_{2,1}$

Apply And-Elimination to R_{10}
$\neg P_{1,2}$ **(done)**

Proof by Search

- Use search to perform propositional inference
- Initial State: KB
- Actions: Apply inference rules to sentences matching top of rule
- Result: Add sentences in bottom of rule to KB
- Goal: KB contains sentence to be proved
- Sound?
- Complete?
- Efficient?

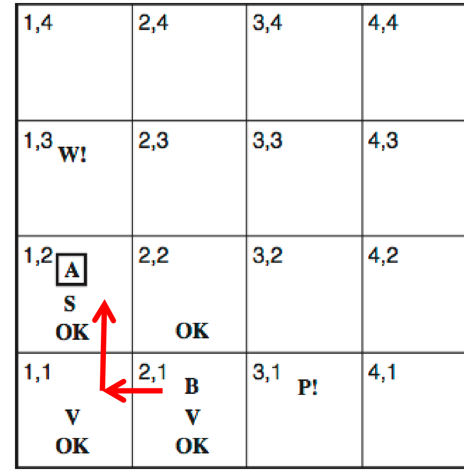
Resolution

- $\{(A \Rightarrow B), A\} \models B$ *“modus ponens”*
- $\{(\neg A \vee B), A\} \models B$ *“unit resolution”*

- $(A \wedge B \wedge C) \Rightarrow D$
  - $\neg(A \wedge B \wedge C) \vee D$
  - $\neg A \vee \neg B \vee \neg C \vee D$

- *Full resolution*
  - $\{(\neg A \vee \neg B \vee \neg C \vee D), (A \vee E)\} \models \neg B \vee \neg C \vee D \vee E$

Proof by Resolution



- Example (cont.)
  - Agent moves from [2,1] to [1,2]

    Add percept information to KB
    R_{11}: $\neg B_{1,2}$
    R_{12}: $B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3})$

    By earlier process, can eliminate pits in [2,2] and [1,3]
    R_{13}: $\neg P_{2,2}$
    R_{14}: $\neg P_{1,3}$

    Apply Biconditional Elimination to R_3, then Modus Ponens with R_5
    R_{15}: $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

    Apply **Resolution** to R_{13} and R_{15}
    R_{16}: $P_{1,1} \vee P_{3,1}$

    Apply **Resolution** to R_1 and R_{16}
    R_{17}: $P_{3,1}$

Unit Resolution

- Given literals $l_1,\ldots,l_k$ and m, where l_i and m are complementary
- Unit resolution inference rule:

$$\frac{l_1 \vee \cdots \vee l_k, \quad m}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k}$$

- <u>Clause</u> is a disjunction of literals
  - $l_1 \vee \ldots \vee l_k$ is a clause
  - m is a unit clause

Full Resolution

- Given literals $l_1,\ldots,l_k$ and $m_1,\ldots,m_n$, where l_i and m_j are complementary
- Resolution inference rule:

$$\frac{l_1 \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

- Resolution is sound and complete!
- Requires sentences to be clauses
- Luckily, every sentence in propositional logic can be expressed as a conjunction of clauses

Conjunctive Normal Form (CNF)

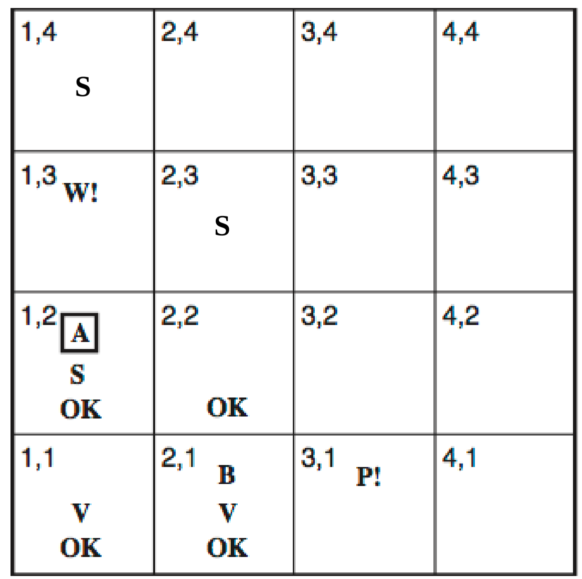
- Procedure for converting propositional logic sentence to CNF
  1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
  2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
  3. Move $\neg$ inward to appear only in literals
     - $\neg(\neg\alpha) \equiv \alpha$ (double-negation elimination)
     - $\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$ (De Morgan)
     - $\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$ (De Morgan)
  4. Apply distributive law of $\vee$ over $\wedge$
     - $(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

CNF Conversion Example

- $W_{1,3} \Leftrightarrow S_{1,4} \wedge S_{1,2} \wedge S_{2,3}$
- Step 1: Eliminate $\Leftrightarrow$
  - $(W_{1,3} \Rightarrow S_{1,4} \wedge S_{1,2} \wedge S_{2,3}) \wedge (S_{1,4} \wedge S_{1,2} \wedge S_{2,3} \Rightarrow W_{1,3})$
- Step 2: Eliminate $\Rightarrow$
  - $(\neg W_{1,3} \vee (S_{1,4} \wedge S_{1,2} \wedge S_{2,3})) \wedge (\neg(S_{1,4} \wedge S_{1,2} \wedge S_{2,3}) \vee W_{1,3})$
- Step 3: Move $\neg$ inward
  - $(\neg W_{1,3} \vee (S_{1,4} \wedge S_{1,2} \wedge S_{2,3})) \wedge (\neg S_{1,4} \vee \neg S_{1,2} \vee \neg S_{2,3} \vee W_{1,3})$
- Step 4: Apply distributivity of $\vee$ over $\wedge$
  - $(\neg W_{1,3} \vee S_{1,4}) \wedge (\neg W_{1,3} \vee S_{1,2}) \wedge (\neg W_{1,3} \vee S_{2,3}) \wedge (\neg S_{1,4} \vee \neg S_{1,2} \vee \neg S_{2,3} \vee W_{1,3})$

Another Resolution Example

- From previous conversion plus And-Elimination:
  - C1: $(\neg W_{1,3} \vee S_{1,4})$
  - C2: $(\neg W_{1,3} \vee S_{1,2})$
  - C3: $(\neg W_{1,3} \vee S_{2,3})$
  - C4: $(\neg S_{1,4} \vee \neg S_{1,2} \vee \neg S_{2,3} \vee W_{1,3})$
- Assume we have observed the surrounding stenches:
  - C5: $S_{1,4}$
  - C6: $S_{1,2}$
  - C7: $S_{2,3}$
- Resolving C4 with C5:
  - C8: $(\neg S_{1,2} \vee \neg S_{2,3} \vee W_{1,3})$
- Resolving C8 with C6:
  - C9: $(\neg S_{2,3} \vee W_{1,3})$
- Resolving C9 with C7:
  - $W_{1,3}$

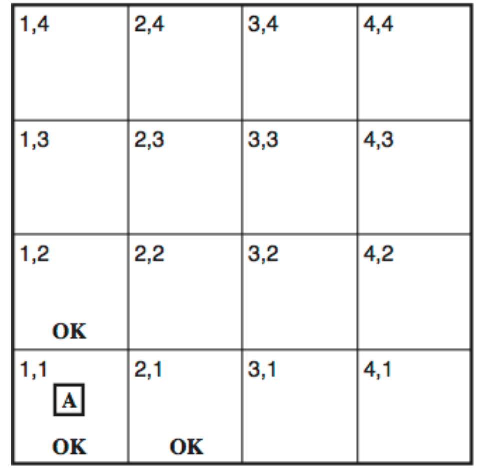


Propositional Logic Resolution

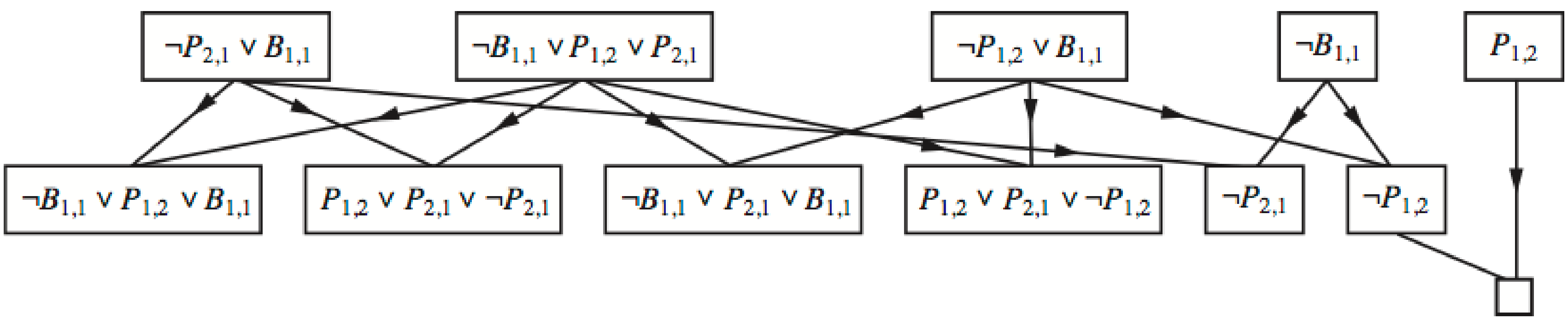
function PL-RESOLUTION? (KB, α) **returns** *true* or *false*
  clauses ← convert ($KB \wedge \neg\alpha$) to CNF
  new ← {}
  loop do
    for each pair of clauses C_i, C_j **in** *clauses* **do**
      resolvents ← PL-RESOLVE (C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      new ← *new* ∪ *resolvents*
    if *new* ⊆ *clauses* **then return** *false*
    clauses ← *clauses* ∪ *new*

- Proof by contradiction
- Empty clause result of resolving A with ¬A
- If iteration produces no new clauses, then $KB \not\models \alpha$

Example



- $KB = R_2 \land R_4 = (B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



Efficiency of PL-Resolution

- Branching factor $|KB|^2$
- Many heuristics apply
  - E.g., prefer unit clauses
- Local search works surprisingly well (e.g., WALKSAT)
- Horn clause is a clause with at most one positive literal
  - E.g., $(A \wedge B \wedge C \Rightarrow D) \equiv (\neg A \vee \neg B \vee \neg C \vee D)$
  - Entailment with Horn clauses is linear in $|KB|$
  - Early *Computational Learning Theory* was considerably based on learning Horn clauses

PL-Based Agent

- KB includes initial state and axioms
  - $\neg W_{1,1} \wedge \neg P_{1,1} \wedge L_{1,1} \wedge FacingEast \wedge HaveArrow \wedge \neg HaveGold \wedge AgentAlive \wedge InCave \wedge WumpusAlive$
  - $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
  - $S_{1,1} \Leftrightarrow (W_{1,2} \vee W_{2,1})$
  - …
  - At most one wumpus
    - $\neg(W_{1,1} \wedge W_{1,2}), \neg(W_{1,1} \wedge W_{1,3}), \ldots$
- Would prefer
  - $B_{x,y} \Leftrightarrow (P_{x-1,y} \vee P_{x+1,y} \vee P_{x,y-1} \vee P_{x,y+1})$
  - **First-Order Logic** makes expressing these properties much easier, less cumbersome

PL-Based Agent: More Ideas

- Movement
  - E.g., TurnLeft
    - Add ¬FacingEast and FacingNorth to KB
    - Remove FacingEast; add FacingNorth
  - E.g., Forward
    - Remove $L_{1,1}$; add $L_{2,1}$
    - Bump?
  - E.g., Shoot
    - Remove HaveArrow, add ¬HaveArrow
    - WumpusAlive?
- Monotonic Logic
  - Set of entailed sentences can only increase
- History? (e.g., $L_{1,1}$ at t=1)

PL-Based Agent: Frames

- Frame problem
  - Describing what changes and what stays the same after taking an action
- Frame Axioms
  - E.g., $HaveArrow^{t+1} \Leftrightarrow HaveArrow^{t} \wedge \neg Shoot^{t}$
  - Requires sets of axioms for each time transition
  - Or, first-order-ness (“FOLness”)

PL-Based Agent: Summary

- Using intermediate information
  - $OK_{2,1}$
  - $\neg Visited_{2,1}$
- Specifying Goals in PL
  - $HaveGold \wedge \neg InCave$
  - $OK_{x,y} \Rightarrow Visited_{x,y}$
  - $\neg WumpusAlive$

Summary on PL Agents:

- In general, pure PL-based agent cumbersome
- Need hybrid logic/search-based agent

First-Order Logic

- Propositional logic insufficient to express even commonsense knowledge
- First-order logic (FOL) or First-order predicate calculus (FOPC)
- Borrowing from elements of natural language
  - <u>Objects</u>: nouns, noun phrases (e.g., wumpus, pit)
  - <u>Relations</u>: verbs, verb phrases (e.g., shoot)
    - <u>Properties</u>: adjectives (e.g., smelly)
    - <u>Functions</u>: map input to single output (e.g., location(wumpus))

FOL Syntax

$$
\begin{aligned}
\text{Sentence} &\rightarrow \text{AtomicSentence} \mid \text{ComplexSentence} \\[1em]
\text{AtomicSentence} &\rightarrow \text{Predicate} \mid \text{Predicate (Term,...)} \mid \text{Term} = \text{Term} \\[1em]
\text{ComplexSentence} &\rightarrow \text{(Sentence)} \mid \text{[Sentence]} \\
&\quad \mid \neg\ \text{Sentence} \\
&\quad \mid \text{Sentence} \wedge \text{Sentence} \\
&\quad \mid \text{Sentence} \vee \text{Sentence} \\
&\quad \mid \text{Sentence} \Rightarrow \text{Sentence} \\
&\quad \mid \text{Sentence} \Leftrightarrow \text{Sentence} \\
&\quad \mid \text{Quantifier Variable,... Sentence} \\[1em]
\text{Term} &\rightarrow \text{Function (Term,...)} \mid \text{Constant} \mid \text{Variable} \\
\text{Quantifier} &\rightarrow \forall \mid \exists \\
\text{Constant} &\rightarrow A \mid B \mid \text{Wumpus} \mid 1 \mid 2 \mid \ldots \\
\text{Variable} &\rightarrow a \mid x \mid s \mid \ldots \\
\text{Predicate} &\rightarrow \text{True} \mid \text{False} \mid \text{Adjacent} \mid \text{At} \mid \text{Alive} \mid \ldots \\
\text{Function} &\rightarrow \text{Location} \mid \text{RightOf} \mid \ldots
\end{aligned}
$$

Operator Precedence: $\neg, =, \wedge, \vee, \Rightarrow, \Leftrightarrow$

FOL Semantics

- Constant symbols stand for objects
- Predicate symbols stand for relations
- Function symbols stand for functions
- R&N convention: Above symbols begin with uppercase letters
  - E.g., Wumpus, Adjacent, RightOf
- Arity is the number of arguments to a predicate or function
  - E.g., Adjacent (loc_1, loc_2), RightOf (location)

FOL Semantics

- Terms represent objects with constants, variables or functions
- Note: Functions do not return an object, but represent that object
  - E.g., Action(Forward,t) ∧ Orientation(Agent, Right, t) ∧ At(Agent, loc, t) ⇒ At(Agent, RightOf(loc), t+1)
- R&N convention: variables begin with lowercase letters

- **Pause: what are the main differences between PL and FOL?**
- Anything in PL that is not in FOL?
- Anything in FOL that's not in PL?

Quantifiers

- Express properties of collections of objects
- Universal quantification ($\forall$)
  - A statement is true <u>for all</u> objects represented by quantified variables
  - E.g., $\forall$ x,y At(Wumpus,x,y) $\Rightarrow$ Stench(x+1,y)
  - Same as $\forall$ x,y At(Wumpus,x,y) $\wedge$ Stench(x+1,y) ?
  - Same as $\forall$ x,y $\neg$At(Wumpus,x,y) $\vee$ Stench(x+1,y) ?
- $\forall$x P(x) $\equiv$ P(A) $\wedge$ P(B) $\wedge$ P(Wumpus) $\wedge$ ...

Quantifiers

- Existential quantification ($\exists$)
  - There exists at least one set of objects, represented by quantified variables, for which a statement is true
  - E.g., $\exists$ w,x,y At(w,x,y) $\wedge$ Wumpus(w)
  - Same as $\exists$ w,x,y At(w,x,y) $\Rightarrow$ Wumpus(w) ?
- $\exists$x P(x) $\equiv$ P(A) $\vee$ P(B) $\vee$ P(Wumpus) $\vee$ ...

Properties of Quantifiers

- Nested quantifiers
- $\forall x\ \forall y$ same as $\forall y\ \forall x$ same as $\forall x,y$
- $\exists x\ \exists y$ same as $\exists y\ \exists x$ same as $\exists x,y$
- $\exists x\ \forall y$ same as $\forall y\ \exists x$?
  - $\exists x\ \forall y\ Likes(x,y)$?
  - $\forall y\ \exists x\ Likes(x,y)$?
  - $\forall x\ \exists y\ Likes(x,y)$?
  - $\exists y\ \forall x\ Likes(x,y)$?

Properties of Quantifiers

- Negation and quantifiers
- $\exists x\ P(x) \equiv \neg\forall x\ \neg P(x)$
  - "If P is true for some x, then P can't be false for all x"
- $\forall x\ P(x) \equiv \neg\exists x\ \neg P(x)$
  - "If P is true for all x, then there can't be an x for which P is false."
- $\forall x\ \neg P(x) \equiv \neg\exists x\ P(x)$
  - "If P is false for all x, then there can't be an x for which P is true."
- $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$
  - "If P is not true for all x, then there must be an x for which P is false."

Equality

- Equality symbol (Term1 = Term1) means Term1 and Term2 refer to the same object
  - E.g., RightOf(Location(1,1)) = Location(2,1)

- Useful for constraining two terms to be different
- E.g., Sibling relation:
  - $Sibling(x,y) \Leftrightarrow Parent(p,x) \wedge Parent(p,y)$
  - $Sibling(x,y) \Leftrightarrow Parent(p,x) \wedge Parent(p,y) \wedge \neg(x = y)$
  - $\forall x,y\ Sibling(x,y) \Leftrightarrow \exists p\ Parent(p,x) \wedge Parent(p,y) \wedge \neg(x = y)$

Closed-World Assumption

- How do we express that there is only one wumpus?
  - $Wumpus(x,y) \Rightarrow \neg Wumpus(w,z) \wedge (\neg(w = x) \vee \neg(z=y))$
- How about one arrow? One gold? At least one pit?
- <u>Closed-world assumption</u>
  - Atomic sentences not known to be true are assumed false
  - (Note: closed world assumption in other contexts, e.g. distributed computing, may mean smt. different!)
- <u>Unique-names assumption</u>
  - Every constant symbol refers to a distinct object
  - (Note: remember that Midterm problem? Crypt-arithmetic?)
- <u>Domain closure</u>
  - If not named by a constant symbol, then doesn't exist

Using FOL

- TELL (KB, α)
- ASK (KB, β)
- TELL (KB, Percept([st,br,Glitter,bu,sc],5))
- ASK (KB, ∃a Action(a,5))
- I.e., does KB entail any particular actions at time 5?
- Answer: Yes, {a/Grab} ← substitution (binding list)
- ASKVARS (KB, α)
  - Returns answers (variable bindings) that make α true
  - Or, use Answer literal (later)

FOL for the Wumpus World

- Percepts
  - Percept(p,t) = predicate that is true if percept p observed at time t
  - Percept is a list of five terms
  - E.g., Percept([Stench,Breeze,Glitter,None,None],5)
- Actions
  - Forward, TurnLeft, TurnRight, Grab, Shoot, Climb
- AskVars (∃a BestAction(a,5)) → {a/Grab}

FOL for the Wumpus World

- "Perception"
  - $\forall$ t,s,g,m,c Percept([s,Breeze,g,m,c],t) $\Rightarrow$ Breeze(t)
  - $\forall$ t,s,b,m,c Percept([s,b,Glitter,m,c],t) $\Rightarrow$ Glitter(t)
- Reflex agent
  - $\forall$ t Glitter(t) $\Rightarrow$ BestAction(Grab,t)

FOL for the Wumpus World

- Location list term [x,y] (e.g., [1,2])
  - Pit(s) or Pit([x,y])
  - At(Wumpus,[x,y],t)
  - At(Agent,[1,1],1)
- Definition of Breezy(s), where s is a location
  - $\forall s\ Breezy(s) \Leftrightarrow \exists r\ Adjacent(s,r) \wedge Pit(r)$
- Definition of Adjacent
  - $\forall x,y,a,b\ Adjacent([x,y],[a,b]) \Leftrightarrow (x=a \wedge (y=b-1 \vee y=b+1)) \vee (y=b \wedge (x=a-1 \vee x=a+1))$

FOL for the Wumpus World

- Movement
- Wumpus never moves
  - $\forall t \; At(Wumpus,[1,3],t)$
- Nothing can be in two places at once
  - $\forall x,s_1,s_2,t \; At(x, s_1,t) \wedge At(x, s_2,t) \Rightarrow s_1=s_2$
- Successor-state axioms for each action
  - Describes what's true before and after action
  - $\forall t \; HaveArrow(t+1) \Leftrightarrow (HaveArrow(t) \wedge \neg Action(Shoot,t))$
  - $\forall t \; HaveGold(t+1) \Leftrightarrow (HaveGold(t) \vee (Glitter(t) \wedge Action(Grab,t)))$
  - ...

Inference in First-Order Logic

- Now that we have FOL, how can we perform sound, complete and efficient inference?
- Approaches
  - Convert to propositional logic
  - Generalized Modus Ponens
  - Forward and backward chaining
  - Logic programming (Prolog)
  - Resolution
- State of the art

Propositionalization

- Convert FOL problem to PL problem
- Main challenge: Remove quantifiers
- Universal instantiation

$$\frac{\forall v \quad \alpha}{\mathrm{SUBST}(\{v/g\},\alpha)}$$

- Substitute every ground term g for any variable v in α
- E.g., ∀s,t,r At(Wumpus,s,t) ∧ Adjacent(s,r) ⇒ Stench(r)
  - At(Wumpus,[1,3],1) ∧ Adjacent([1,3],[2,3]) ⇒ Stench([2,3])
  - At(Wumpus,[2,2],1) ∧ Adjacent([2,2],[2,3]) ⇒ Stench([2,3])
  - ...

Propositionalization

- Existential instantiation

$$\frac{\exists v \quad \alpha}{\text{SUBST}(\{v/k\},\alpha)}$$

  - Substitute new constant symbol k for variable v in α
- E.g., ∃s,t At(Wumpus,s,t)
  - At(Wumpus,S1,T1)
  - S1 and T1 are new constant symbols
- After removing quantifiers, perform PL inference
- Complete, but inefficient
- FOL is semi-decidable
  - Some algorithms can say yes to every entailed sentence, but no algorithm can say no to every non-entailed sentence

Generalized Modus Ponens

- Substitution (binding) θ = {x/y}
  - Replace all occurrences of x with y
  - E.g., α = At(Wumpus,s,t), θ = {s/[1,3], t/5}
    - α θ = At(Wumpus,[1,3],5)
- Generalized Modus Ponens

$$\frac{p_1', p_2', \ldots, p_n', \quad (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{\mathrm{SUBST}(\theta, q)}$$

  - where $\mathrm{S{\small UBST}}(\theta,p_i')$ = $\mathrm{S{\small UBST}}(\theta,p_i)$ for all i
- Find θ via unification

Generalized Modus Ponens

Example:

- $\forall s,r\ Pit(s) \wedge Adjacent(s,r) \Rightarrow Breeze(r)$
- Pit([3,1]), Adjacent([3,1],[2,1])
- $p_1 = Pit(s)$, $p_2 = Adjacent(s,r)$, $q = Breeze(r)$
- $p_1' = Pit([3,1])$, $p_2' = Adjacent([3,1],[2,1])$
- $\theta = \{s/[3,1], r/[2,1]\}$
- $SUBST(\theta,q) = Breeze([2,1])$

Unification

- Unification determines if two sentences match given some substitution (unifier)
- UNIFY(p,q) = θ where SUBST(θ,p) = SUBST(θ,q)

- Examples
  - UNIFY (At(Wumpus,s,t), At(Wumpus,[1,3],5)) = {s/[1,3], t/5}
  - UNIFY (At(Wumpus,s,t), At(Wumpus,r,5)) = {s/r, t/5}
  - UNIFY (At(Wumpus,s,t), At(Wumpus,AgentLoc(t),5)) = {s/AgentLoc(t), t/5} = {s/AgentLoc(5), t/5}

Unification Issues

- Standardizing apart
  - Use unique variable names in each sentence
  - UNIFY (At(x,[1,3],t), At(Wumpus,x,t)) = failure
  - UNIFY (At(x_{17},[1,3],t), At(Wumpus,x_{21},5)) = {x_{17}/Wumpus, x_{21}/[1,3], t/5}
- Most General Unifier (MGU)
  - Unifier returned by Unify should place the least possible restrictions on variables
  - UNIFY (Adjacent(r,s),Adjacent([1,3],x)) = {r/[1,3], s/[2,3], x/[2,3]} works
  - But so does {r/[1,3], s/x} (more general)

Unification Algorithm

- Recursively compare two expressions
- Build up substitutions along the way
- Details
  - Compound expression of the form F(A,B)
  - If x = F(A,B), then x.OP = F, x.ARGS = [A,B] (a list)
  - List decomposed using First and Rest
  - If x = [A,B,C], then x.FIRST = A and x.REST = [B,C]

Unification

```
function UNIFY (x, y, θ) returns a substitution to make x and y identical
  inputs: x, a variable, constant, list, or compound expression
          y, a variable, constant, list, or compound expression
          θ, the substitution built up so far (optional, defaults to empty)
  if θ = failure then return failure
  else if x = y then return θ
  else if VARIABLE?(x) then return UNIFY-VAR(x,y,θ)
  else if VARIABLE?(y) then return UNIFY-VAR(y,x,θ)
  else if COMPOUND?(x) and COMPOUND?(y) then
    return UNIFY(x.ARGS, y.ARGS, UNIFY(x.OP, y.OP, θ ))
  else if LIST?(x) and LIST?(y) then
    return UNIFY(x.REST, y.REST, UNIFY(x.FIRST, y.FIRST, θ ))
  else return failure
```

Unification

function UNIFY-VAR (*var, x,* θ) **returns** a substitution
  if $\{var\ /\ val\} \in \theta$ **then return** UNIFY(*val, x,* θ)
  else if $\{x\ /\ val\} \in \theta$ **then return** UNIFY(*var, val,* θ)
  else if OCCUR-CHECK?(*var, x*) **then return** failure
  else return add $\{var\ /\ x\}$ to θ

- Occur check
  - When matching variable and term, check if variable occurs in term
  - If so, failure (e.g., P(x) does not unify with P(P(x))
  - Makes UNIFY quadratic in size of expression
  - Some inference systems omit occur check

Forward Chaining

- Start with atomic sentences in KB
- Apply Modus Ponens where possible to infer new atomic sentences
- Continue until goal is proven or no new inferences can be made
- Assume first-order definite clauses for now
  - Disjunction of literals with exactly one positive literal
  - E.g., $\neg A(x) \vee \neg B(y) \vee C(x,y) \equiv A(x) \wedge B(y) \Rightarrow C(x,y)$

An Example
(or: how to make this material almost fun…)

- The law says that it is a crime for an American to sell weapons to "hostile nations"
- The "country" of NoNo, a "hostile nation" per Fox News, has some missiles, and all of its missiles were sold to it by Colonel West, who is an American

  (…although nobody has seen his birth certificate and he hasn't paid much taxe$ to the IRS)

- Prove that **Col. West is a criminal !**
- (… without assuming anything about his affiliation with the DNC ☺)

Example

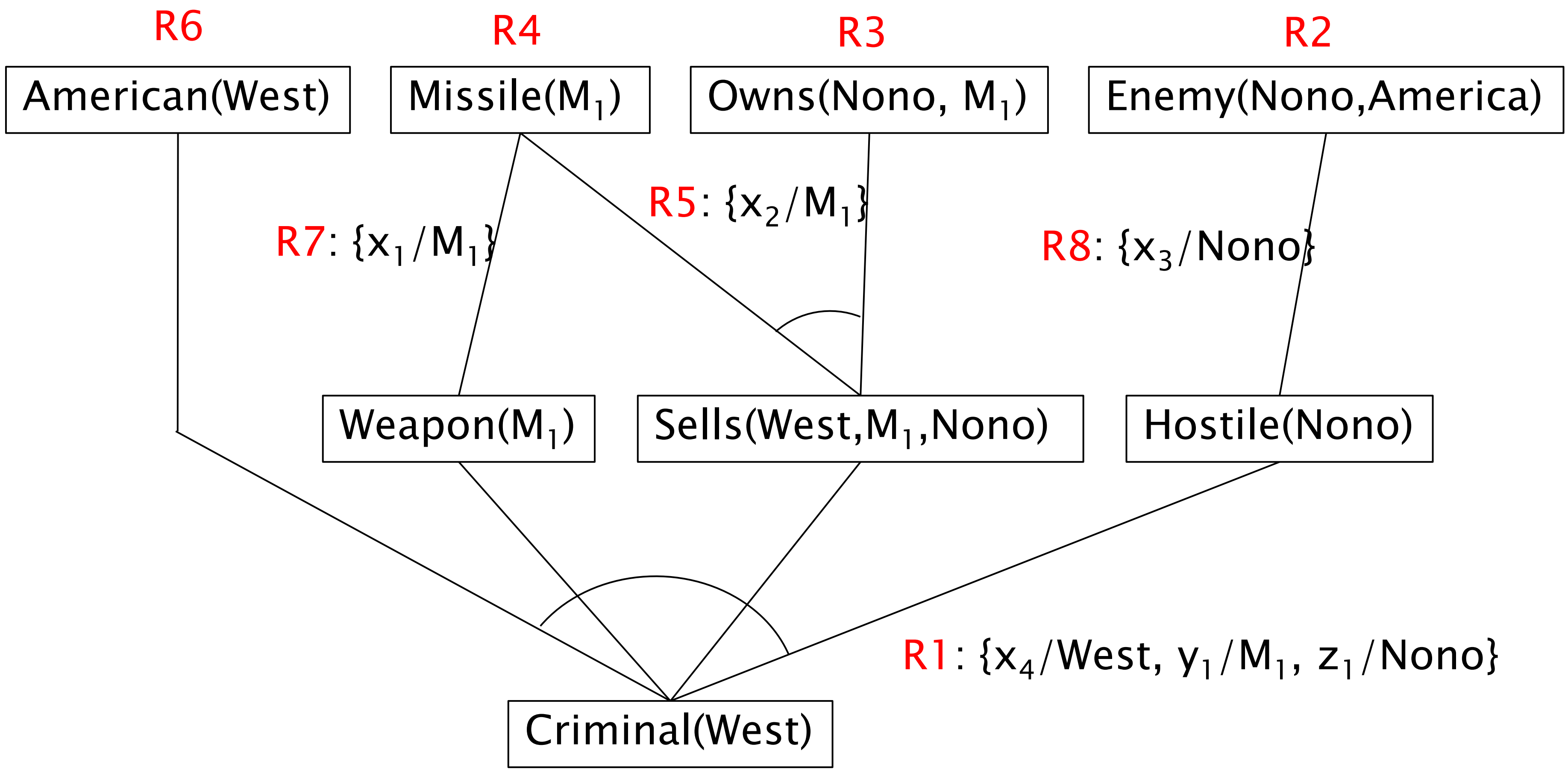
- "… it is a crime for an American to sell weapons to hostile nations."
  - R1: $American(x) \wedge Weapon(y) \wedge Sells(x,y,z) \wedge Hostile(z) \Rightarrow Criminal(x)$
- "… Nono, an enemy of America, …"
  - R2: $Enemy(Nono,America)$
- "… Nono … has some missiles"
  - $\exists x\ Owns(Nono,x) \wedge Missile(x)$
  - R3: $Owns(Nono,M_1)$
  - R4: $Missle(M_1)$

Existential Instantiation

Example

- "… all of its missiles were sold to it by Colonel West"
  - R5: Missile(x) ∧ Owns(Nono,x) ⇒ Sells(West,x,Nono)
- "… Colonel West, who is American."
  - R6: American(West)
- A few more rules…
  - R7: Missile(x) ⇒ Weapon(x)
  - R8: Enemy(x,America) ⇒ Hostile(x)
- Note: All variables universally quantified

Example: Forward Chaining



function FOL-FC-ASK (KB, α) returns a substitution or *false*

inputs: *KB*, the knowledge base, a set of first-order definite clauses

α, the query, an atomic sentence

local variables: *new*, the new sentences inferred at each iteration

```
repeat until new is empty
  new ← { }
  for each rule in KB do
    (p1 ∧ … ∧ pn ⇒ q) ← STANDARDIZE-VARIABLES(rule)
    for each θ such that SUBST(θ, p1 ∧…∧ pn) = SUBST(θ, p1' ∧…∧ pn')
                for some p1', …, pn' in KB
      q' ← SUBST(θ, q)
      if q' does not unify with some sentence already in KB or new then
        add q' to new
        φ = UNIFY(q', α)
        if φ is not fail then return φ
  add new to KB
return false
```

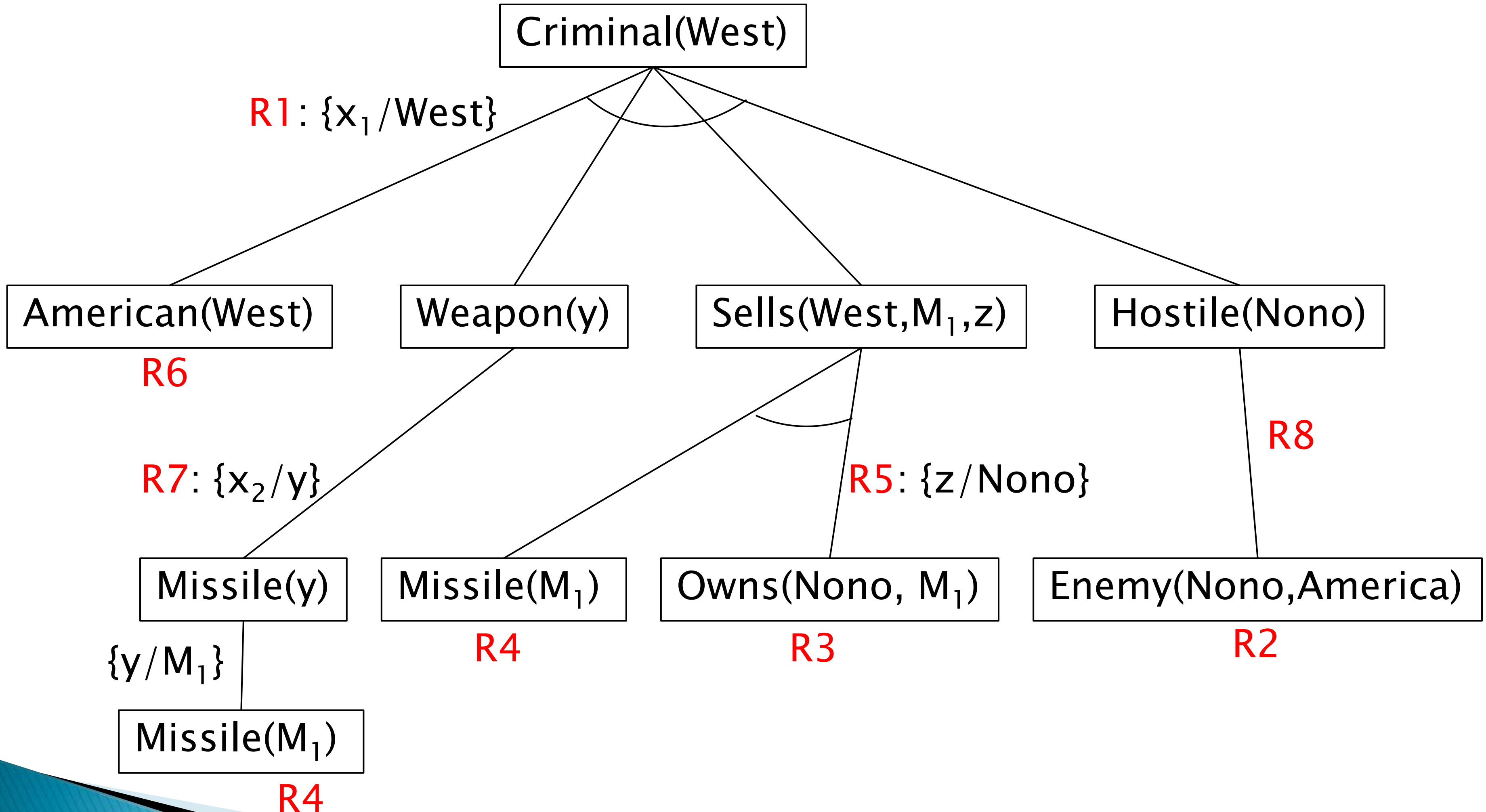
Forward Chaining

- Sound?
- Complete?
- Efficient?
  - Matching all rules against all known facts
    - Conjunct ordering
    - R5: $Missile(x) \land Owns(Nono,x) \Rightarrow Sells(West,x,Nono)$
  - Recheck every rule on every iteration
    - Every new fact inferred on iteration t must be derived from at least one new fact inferred on iteration t-1
    - Incremental forward chaining
  - Irrelevant facts (e.g., Enemy(Wumpus,America))

Backward Chaining

- Work backwards from the goal
- For rules concluding goal, add premises as new goals
- Continue until all open goals supported by known facts
- Again, assume first-order definite clauses for now

Example: Backward Chaining



Backward Chaining

function FOL-BC-ASK($KB, goals, \theta$) **returns** a set of substitutions
  inputs: KB, a knowledge base
    $goals$, a list of conjuncts forming a query (θ already applied)
    θ, the current substitution, initially the empty substitution $\{\ \}$
  local variables: $answers$, a set of substitutions, initially empty

  if $goals$ is empty **then return** $\{\theta\}$
  $q' \leftarrow \text{SUBST}(\theta, \text{FIRST}(goals))$
  for each sentence r **in** KB
    where $\text{STANDARDIZE-APART}(r) = (\ p_1\ \wedge\ \ldots\ \wedge\ p_n\ \Rightarrow\ q)$
    and $\theta' \leftarrow \text{UNIFY}(q, q')$ succeeds
   $new\_goals \leftarrow [\,p_1, \ldots, p_n | \text{REST}(goals)]$
   $answers \leftarrow \text{FOL-BC-ASK}(KB, new\_goals, \text{COMPOSE}(\theta', \theta)) \cup answers$
  return $answers$

Backward Chaining

- Sound?
- Complete?
- Efficient?
  - Matching all rules against all **open goals**
    - More constraints
    - R5: $Missile(x) \land Owns(Nono,x) \Rightarrow Sells(West,x,Nono)$
  - Recheck every rule on every iteration
    - Yes, but only those whose consequent unifies with an open goal
  - Irrelevant facts (e.g., Enemy(Wumpus,America))
    - Excluded

FOL Resolution

- Resolution using refutation (proof by contradiction) is sound and complete
  - $(\neg A(x) \vee B(x), A(Wumpus)) \vdash B(x)\ \{x/Wumpus\}$
- Convert FOL to clausal form (CNF)
- Efficient? Resolution strategies

IMPORTANT:

- **We will skip the rest of this material**
- Grad. students and those UG's with deeper interest in AI (and/or Logic) advised to read Ch. 9 in [AIMA]

The Material starting with the next slide, up to the End of Lecture Notes on Logic, IS OPTIONAL … and won't be covered on the homework or the final exam!

Note: if skipping reading the rest, please take a look at the last two slides of this deck

Logic Programming (optional)

- A program is a set of logic statements defining the constraints of the problem
- Theorem-proving used to “run” the program

- Prolog is a logic programming language
  - Closed-world assumption
  - Supports arithmetic “X is 1+2” binds X to 3
  - Allows assertion and retraction of sentences
  - No occur check
  - Depth-first backward chaining (incomplete)

Convert FOL to CNF

- Conjunctive Normal Form (CNF)
  - Conjunction of clauses
  - Each clause is a disjunction of literals
  - Variables assumed to be universally quantified
- Example
  - $\forall x,y,z$ American(x) $\wedge$ Weapon(y) $\wedge$ Sells(x,y,z) $\wedge$ Hostile(z) $\Rightarrow$ Criminal(x)
  - $\neg$American(x) $\vee$ $\neg$Weapon(y) $\vee$ $\neg$Sells(x,y,z) $\vee$ $\neg$Hostile(x) $\vee$ Criminal(x)
- Same as propositional logic, but need to eliminate existential quantifiers

Convert FOL to CNF

- Step 1: Eliminate implications $\Rightarrow$
  - From: $\forall x\ A(x) \land B(x) \Rightarrow C(x)$
  - To: $\forall x\ \neg A(x) \lor \neg B(x) \lor C(x)$
- Step 2: Move $\neg$ inwards
  - $\neg \forall x\ A(x)$ becomes $\exists x\ \neg A(x)$
  - $\neg\ \exists x\ A(x)$ becomes $\forall x\ \neg A(x)$
- Step 3: Standardize variables
  - From: $(\forall x\ A(x)) \land (\forall x\ B(x))$
  - To: $(\forall x_1\ A(x_1)) \land (\forall x_2\ B(x_2))$

Convert FOL to CNF

- Step 4: Skolemize (Skolemization)
  - Eliminate existential quantifiers by replacing them with a new constant or function
  - Skolem constant, Skolem function
  - Arguments of the Skolem function are all the universally quantified variables in whose scope the existential quantifier appears
  - From: $\forall x,y\ \exists z\ P(x,y,z)$
  - To: $\forall x,y\ P(x,y,F1(x,y))$

Thoralf Skolem (1887–1963)
Norwegian mathematician

Convert FOL to CNF

- Step 5: Drop universal quantifiers
  - All remaining variables universally quantified
  - So, just drop the $\forall x,y,\ldots$
- Step 6: Distribute $\vee$ over $\wedge$
  - From: $(A(x) \wedge B(x)) \vee C(x)$
  - To: $(A(x) \vee C(x)) \wedge (B(x) \vee C(x))$

Example (FOL → CNF)

- What is a brick?
  - A brick is on something that is not a pyramid
  - There is nothing that a brick is on and that is on the brick as well
  - There is nothing that is not a brick and also is the same thing as a brick.

$$\forall x\ [Brick(x) \Rightarrow (\exists y\ [On(x,y) \wedge \neg Pyramid(y)] \wedge \neg\exists y\ [On(x,y) \wedge On(y,x)] \wedge \forall y\ [\neg Brick(y) \Rightarrow \neg Equal(x,y)])]$$

Example (FOL → CNF)

- Step 1: Eliminate implications

$$\forall x\ [\neg Brick(x) \lor (\exists y\ [On(x,y) \land \neg Pyramid(y)] \land$$
$$\neg\exists y\ [On(x,y) \land On(y,x)] \land$$
$$\forall y\ [\neg\neg Brick(y) \lor \neg Equal(x,y)])]$$

Example (FOL → CNF)

- Step 2: Move ¬ inwards

$$\forall x\ [\neg Brick(x) \vee (\exists y\ [On(x,y) \wedge \neg Pyramid(y)] \wedge$$
$$\forall y\ \neg[On(x,y) \wedge On(y,x)] \wedge$$
$$\forall y\ [Brick(y) \vee \neg Equal(x,y)])]$$

$$\forall x\ [\neg Brick(x) \vee (\exists y\ [On(x,y) \wedge \neg Pyramid(y)] \wedge$$
$$\forall y\ [\neg On(x,y) \vee \neg On(y,x)] \wedge$$
$$\forall y\ [Brick(y) \vee \neg Equal(x,y)])]$$

Example (FOL → CNF)

- Step 3: Standardize variables

$$
\begin{aligned}
&\forall x\ [\neg Brick(x) \lor (\exists y\ [On(x,y) \land \neg Pyramid(y)] \land \\
&\qquad \forall a\ [\neg On(x,a) \lor \neg On(a,x)] \land \\
&\qquad \forall b\ [Brick(b) \lor \neg Equal(x,b)])]
\end{aligned}
$$

Example (FOL → CNF)

- Step 4: Skolemization

$$\forall x\ [\neg Brick(x) \lor [On(x,F(x)) \land \neg Pyramid(F(x))] \land \\ \forall a\ [\neg On(x,a) \lor \neg On(a,x)] \land \\ \forall b\ [Brick(b) \lor \neg Equal(x,b)]])]$$

- Step 5: Drop universal quantifiers

$$\neg Brick(x) \lor [On(x,F(x)) \land \neg Pyramid(F(x))] \land \\ [\neg On(x,a) \lor \neg On(a,x)] \land \\ [Brick(b) \lor \neg Equal(x,b)]])]$$

Example (FOL → CNF)

- Step 6: Distribute ∨ over ∧

$(\neg Brick(x) \vee On(x,F(x))) \wedge$
$(\neg Brick(x) \vee \neg Pyramid(F(x))) \wedge$
$(\neg Brick(x) \vee \neg On(x,a) \vee \neg On(a,x)) \wedge$
$(\neg Brick(x) \vee Brick(b) \vee \neg Equal(x,b))$

Resolution Inference Rule

$$\frac{l_1 \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_n}{\text{SUBST}(\theta, l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)}$$

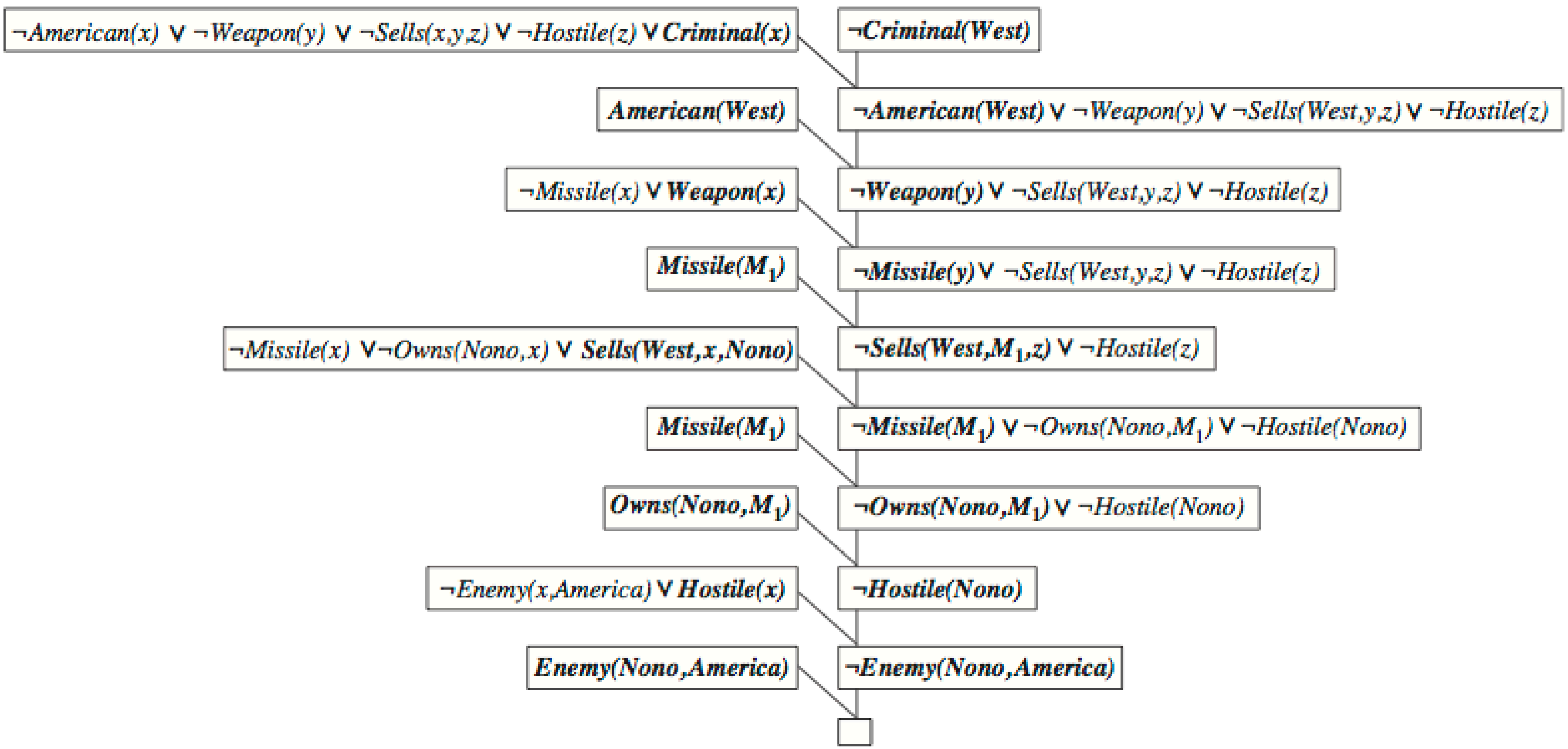
where $\text{UNIFY}(l_i, \neg m_j) = \theta$

- Binary resolution resolves one pair of complementary literals
- Full resolution can resolve multiple pairs of complementary literals
- Resolution plus proof by refutation is sound and complete

Example Proof: Criminal(West)

- CNF
  - $\neg$American(x) $\vee$ $\neg$Weapon(y) $\vee$ $\neg$Sells(x,y,z) $\vee$ $\neg$Hostile(z) $\vee$ Criminal(x)
  - $\neg$Missile(x) $\vee$ $\neg$Owns(Nono,x) $\vee$ Sells(West,x,Nono)
  - $\neg$Missile(x) $\vee$ Weapon(x)
  - $\neg$Enemy(x,America) $\vee$ Hostile(x)
  - Enemy(Nono,America)
  - Owns(Nono,M_1)
  - Missile(M_1)
  - American(West)
- Prove: Criminal(West)
  - Add $\neg$Criminal(West) to KB and derive empty clause

Example Proof: Criminal(West)



Is There a Criminal?

- Prove: ∃c Criminal(c)
  - Add ¬∃c Criminal(c) to KB
  - I.e., add ¬Criminal(c) to KB
- Generated clauses
  - ¬American(c) ∨ ¬Weapon(y) ∨ ¬Sells(c,y,z) ∨ ¬Hostile(z)
  - ¬Weapon(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)
  - ¬Missile(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)
  - ¬Sells(West,M_1,z) ∨ ¬Hostile(z)
  - ¬Missile(M_1) ∨ ¬Owns(Nono,M_1) ∨ ¬Hostile(Nono)
  - ¬Owns(Nono,M_1) ∨ ¬Hostile(Nono)
  - ¬Hostile(Nono)
  - ¬Enemy(Nono,America)
  - □

Answer Literal

- Add clause with negated goal and answer literal to KB
- Search for clause containing only answer literal
- Prove: $\exists x,y,z\ Goal(x,y,z)$
- Add $[\neg Goal(x,y,z) \vee Answer(x,y,z)]$ to KB
- Final clause $Answer(x,y,z)$ will have variables bound to answers

Who is the Criminal?

- Prove: $\exists c$ Criminal(c) and retrieve c
  - Add [¬Criminal(c) ∨ Answer(c)] to KB
- Generated clauses
  - ¬American(c) ∨ ¬Weapon(y) ∨ ¬Sells(c,y,z) ∨ ¬Hostile(z) ∨ Answer(c)
  - ¬Weapon(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z) ∨ Answer(West)
  - ¬Missile(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z) ∨ Answer(West)
  - ¬Sells(West,M_1,z) ∨ ¬Hostile(z) ∨ Answer(West)
  - ¬Missile(M_1) ∨ ¬Owns(Nono,M_1) ∨ ¬Hostile(Nono) ∨ Answer(West)
  - ¬Owns(Nono,M_1) ∨ ¬Hostile(Nono) ∨ Answer(West)
  - ¬Hostile(Nono) ∨ Answer(West)
  - ¬Enemy(Nono,America) ∨ Answer(West)
  - Answer(West)

Equality

- Handle (x = y) terms in resolution theorem proving
- Let SUB(x,y,m) mean to replace x with y everywhere x occurs within m
- Demodulation

$$\frac{x = y, \quad m_1 \vee \cdots \vee m_n}{\mathrm{SUB}(\mathrm{SUBST}(\theta, x), \mathrm{SUBST}(\theta, y), m_1 \vee \cdots \vee m_n)}$$

$$\text{where } \mathrm{UNIFY}(x, \mathrm{z}) = \theta \text{ and } z \text{ appears in literal } m_i$$

- Example
  - Given: Father(Father(x)) = Grandfather(x)
    Birthdate(Father(Father(Bob)),1941)
  - Infer: Birthdate(Grandfather(Bob),1941)

Equality

- Paramodulation

$$\frac{l_1 \vee \ldots \vee l_k \vee x = y, \quad m_1 \vee \cdots \vee m_n}{\mathrm{SUB}(\mathrm{SUBST}(\theta, x), \mathrm{SUBST}(\theta, y), \mathrm{SUBST}(\theta, l_1 \vee \ldots \vee l_k \vee m_1 \vee \cdots \vee m_n))}$$

where $\mathrm{UNIFY}(x, \mathrm{z}) = \theta$ and z appears in literal m_i

- Paramodulation is complete for FOL with equality

Strategies for Efficient Resolution

- Unit preference
  - Prefer resolutions where one clause contains a single literal (unit clause)
  - New sentence always shorter
  - Incomplete (but complete for Horn clauses)
- Set of support
  - Subset of clauses, one of which must always be used for resolution
  - New clauses added to set of support
  - Complete if remaining clauses satisfiable
  - Initially, SoS = negated goal

Strategies for Efficient Resolution

- Input resolution
  - Every resolution combines sentence from (KB+goal) and some other sentence
  - E.g., Criminal(West) proof
  - Incomplete (but complete for Horn clauses)
- Linear resolution
  - Input resolution, but also allowing ancestor sentences
  - Complete
- Subsumption
  - Eliminate sentences more specific than (subsumed by) others
  - E.g., if P(x) in KB, then don't add P(A)

Theorem Proving: State of the Art

- Vampire (www.vprover.org)
- E (www.eprover.org)
- iProver (www.cs.man.ac.uk/~korovink/iprover)
- Conference on Automated Deduction (CADE) ATP System Competition (CASC)
  - www.cs.miami.edu/~tptp/CASC/
- Applications
  - Mathematical theorem proving
  - Hardware and software verification and synthesis

Summary

- Knowledge-based agent
- Logic
- Propositional logic
- First-order logic
- Inference
  - Unification
  - Resolution
  - Proof by contradiction