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华南理工大学《线性代数与解析几何》习题解答

（2014~2015学年，适用专业：新生各专业）

第1章　行列式（习题1）

2.　证明下列等式

(1) $\begin{vmatrix} a & b+x \\ c & d+y \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & x \\ c & y \end{vmatrix}$

证明：$\begin{vmatrix} a & b+x \\ c & d+y \end{vmatrix} = a(d+y) - c(b+x) = (ad-cb) + (ay-cx)$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & x \\ c & y \end{vmatrix}$

(2) $\begin{vmatrix} 0 & b & a \\ 1 & e & f \\ 0 & d & c \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

证明：$\begin{vmatrix} 0 & b & a \\ 1 & e & f \\ 0 & d & c \end{vmatrix} = da - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(3) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$

证明：$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} \end{matrix}$

$= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{matrix} -[a_{31}a_{22}a_{13} + a_{21}a_{12}a_{33} + a_{11}a_{32}a_{23} \\ -a_{11}a_{22}a_{33} - a_{21}a_{32}a_{13} - a_{31}a_{12}a_{23}] \end{matrix}$

$= -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$

4. 求相应的i,j值:

(1) $17i52j6$成偶排列;

解: $\because \tau(1735246)=0+0+1+1+3+2+1=8,$

$\therefore i=3,j=4$

(2) $246i891j7$为奇排列.

解: $\because \tau(246389157)=0+0+0+2+0+0+6+3+2=13$

$\therefore i=3,j=5$

6. 计算下列各排列的逆序数并判断排列的奇偶性.

(1) 26538417;

解: $\tau(26538417)=0+0+1+2+0+3+6+1=13$，奇

(2) $n(n-1)\cdots 21$;

解: $\tau(n(n-1)\cdots 21)=0+1+\cdots+(n-1)=\dfrac{n(n-1)}{2},$

n为$4k$或$4k+1$时为偶，否则为奇.

(3) $2n(2n-2)\cdots 2(2n-1)(2n-3)\cdots 1.$

解: $\tau(2n(2n-2)\cdots 2(2n-1)(2n-3)\cdots 1)$

$$=0+1+\cdots+(n-1)+1+3+\cdots+(2n-1)$$

$$=\frac{n(n-1)}{2}+\frac{n(1+2n-1)}{2}=\frac{n(3n-1)}{2}$$

n为$4k$或$4k+3$为偶，否则为奇.

7. 写出5阶行列式$|a_{ij}|$中含有因子$a_{12}a_{35}a_{41}$的项.

解: $\tau(23514)=0+0+0+3+1=4, \tau(24513)=5$

$$(-1)^{\tau(23514)}a_{12}a_{23}a_{35}a_{41}a_{54}=a_{12}a_{23}a_{35}a_{41}a_{54}$$

$$(-1)^{\tau(24513)}a_{12}a_{24}a_{35}a_{41}a_{53}=-a_{12}a_{24}a_{35}a_{41}a_{53}$$

8. 在多项式$f(x)=\begin{vmatrix} x & 7 & 3 & -1 \\ 1 & 4 & x & 0 \\ 0 & x & -1 & 5 \\ 2 & 1 & 2 & 3 \end{vmatrix}$中，求二次项$x^2$的系数.

解：记$f(x)=|a_{ij}|$，对应的二次项记为$g(x)$，则

$$
\begin{aligned}
g(x)&=(-1)^{\tau(1342)}a_{11}a_{32}a_{43}a_{24}+(-1)^{\tau(1342)}a_{11}a_{23}a_{34}a_{42}\\
&\quad+(-1)^{\tau(4321)}a_{14}a_{23}a_{32}a_{41}\\
&=[(-1)^{\tau(1342)}a_{43}a_{24}+(-1)^{\tau(1342)}a_{34}a_{42}+(-1)^{\tau(4321)}a_{14}a_{41}]x^2\\
&=[(-1)^2\times 2\times 0+(-1)^2\times 5\times 1+(-1)^6\times 2\times(-1)]x^2\\
&=3x^2
\end{aligned}
$$

所以，二次项x^2的系数为3.

10. 用行列式的定义计算下列行列式：

(1) $\begin{vmatrix} 0 & a & 0 & a\\ b & 0 & 0 & 0\\ 0 & c & 0 & d\\ 0 & 0 & e & 0\end{vmatrix}$

解：$\begin{vmatrix} 0 & a & 0 & a\\ b & 0 & 0 & 0\\ 0 & c & 0 & d\\ 0 & 0 & e & 0\end{vmatrix}=\sum\limits_{j_1j_2j_3j_4}(-1)^{\tau(j_1j_2j_3j_4)}a_{1j_1}a_{2j_2}a_{3j_3}a_{4j_4}$

$$
\begin{aligned}
&=\sum_{j_1,j_3}(-1)^{\tau(j_11j_33)}a_{1j_1}ba_{3j_3}e\\
&=(-1)^{\tau(2143)}abde+(-1)^{\tau(4123)}abce\\
&=(-1)^{0+1+0+1}abde+(-1)^{0+1+1+1}abce=abde-abce
\end{aligned}
$$

(2) $\begin{vmatrix} 0 & 0 & 0 & a\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & c\\ g & f & e & d\end{vmatrix}$

解：$\begin{vmatrix} 0 & 0 & 0 & a\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & c\\ g & f & e & d\end{vmatrix}=\sum\limits_{j_1j_2j_3j_4}(-1)^{\tau(j_1j_2j_3j_4)}a_{1j_1}a_{2j_2}a_{3j_3}a_{4j_4}$

$$
=\sum_{j_2j_3j_4}(-1)^{\tau(4j_2j_3j_4)}a\times 0\times 0\times a_{4j_4}=0
$$

(3) $\begin{vmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{vmatrix}$

解: $\begin{vmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{vmatrix} = \begin{array}{l} \sum\limits_{j_2 j_3} (-1)^{\tau(1 j_2 j_3 4)} a\ a_{2j_2} a_{3j_3} a \\ + \sum\limits_{i_2 i_3} (-1)^{\tau(4 j_2 j_3 1)} b\ a_{i_2 2} a_{i_3 3} b \end{array}$

$= a^2(a^2 - b^2) - b^2(a^2 - b^2) = (a^2 - b^2)^2$

(4) $D = \begin{vmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$

解: $D = (-1)^{\tau(1,n,n-1,\cdots,2)} a_{11} a_{n2} a_{n-1,3} \cdots a_{2n}$

$= (-1)^{\frac{(n-1)(n-2)}{2}} a_{11} a_{n2} a_{n-1,3} \cdots a_{2n}$

(5) $D = \begin{vmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix}$

解: $D = (-1)^{\tau(n12\cdots(n-1))} a_n \times 1 \times 1 \times \cdots \times 1 = (-1)^{n-1} a_n$

11. 利用行列式的性质计算下列行列式:

(1) $\begin{vmatrix} 101 & 2 & 201 \\ 325 & 6 & 525 \\ 13 & 1 & 113 \end{vmatrix}$

解: $\begin{vmatrix} 101 & 2 & 201 \\ 325 & 6 & 525 \\ 13 & 1 & 113 \end{vmatrix} \xlongequal[c3 \div 100]{c_3 + (-1) c_1} 100 \begin{vmatrix} 101 & 2 & 1 \\ 325 & 6 & 2 \\ 13 & 1 & 1 \end{vmatrix}$

$$\xlongequal[r3+(-1)r_1]{r_2+(-2)r_1} 100\begin{vmatrix} 101 & 2 & 1 \\ 123 & 2 & 0 \\ -88 & -1 & 0 \end{vmatrix} = -5300$$

(2) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 12 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 4 & 7 & 9 \end{vmatrix}$

解: $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 12 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 4 & 7 & 9 \end{vmatrix} \xlongequal[r_3 \leftrightarrow r_2]{\substack{r_2+(-3)r_1 \\ r_4+(-4)r_3}} - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 3 & -7 \\ 0 & 0 & -5 & -11 \end{vmatrix} = 68$

(3) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$

解: $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \xlongequal[r_1 \div 10]{r_1+\sum_{i=2}^{4} r_i} 10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \xlongequal{\substack{r_2+(-2)r_1 \\ r_3+(-3)r_1 \\ r_4+(-4)r_1}}$

$10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & -3 & -2 & -1 \end{vmatrix} \xlongequal{\substack{r_3+(-1)r_2 \\ r_4+3r_2}} 10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 2 & -4 \end{vmatrix} = 160$

(4) $\begin{vmatrix} 1^3 & 2^3 & 3^3 & 4^3 \\ 4^3 & 1^3 & 2^3 & 3^3 \\ 3^3 & 4^3 & 1^3 & 2^3 \\ 2^3 & 3^3 & 4^3 & 1^3 \end{vmatrix} = D$

解: $D \xlongequal[r_1 \div 100]{r_1+\sum_{i=2}^{4} r_i} 100\begin{vmatrix} 1 & 1 & 1 & 1 \\ 4^3 & 1^3 & 2^3 & 3^3 \\ 3^3 & 4^3 & 1^3 & 2^3 \\ 2^3 & 3^3 & 4^3 & 1^3 \end{vmatrix} \xlongequal[r_4 \div 4]{c_4+\sum_{i=1}^{3} c_i} 400\begin{vmatrix} 1 & 1 & 1 & 1 \\ 4^3 & 1^3 & 2^3 & 25 \\ 3^3 & 4^3 & 1^3 & 25 \\ 2^3 & 3^3 & 4^3 & 25 \end{vmatrix}$

$$\xlongequal[1\leqslant i\leqslant 3]{c_i-c_4} 400\begin{vmatrix} 0 & 0 & 0 & 1 \\ 39 & -24 & -17 & 25 \\ 2 & 39 & -24 & 25 \\ -17 & 2 & 39 & 25 \end{vmatrix} = -400\begin{vmatrix} 39 & -24 & -17 \\ 2 & 39 & -24 \\ -17 & 2 & 39 \end{vmatrix}$$

$$\xlongequal[r_1\div 22]{r_1+1\times r_3} -8800\begin{vmatrix} 1 & -1 & -1 \\ 2 & 39 & -24 \\ -17 & 2 & 39 \end{vmatrix} \xlongequal[c_3\div(-2)]{\substack{c_2+1\times c_1 \\ c_3+1\times c_1}} 17600\begin{vmatrix} 1 & 0 & 0 \\ 2 & 41 & 13 \\ -17 & -15 & -28 \end{vmatrix}$$

$$= 17600\times(-953) = -16772800$$

(5) $$\begin{vmatrix} x & x & \cdots & x & a \\ x & x & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & a & \cdots & x & x \\ a & x & \cdots & x & x \end{vmatrix}$$

解: $$\begin{vmatrix} x & x & \cdots & x & a \\ x & x & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & a & \cdots & x & x \\ a & x & \cdots & x & x \end{vmatrix} \xlongequal[r_1\div\left((n-1)x+a\right)]{r_1+\sum\limits_{i=2}^{n} r_i} [(n-1)x+a]\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x & x & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & a & \cdots & x & x \\ a & x & \cdots & x & x \end{vmatrix}$$

$$\xlongequal[2\leqslant i\leqslant n]{r_i-x\times r_1} [(n-1)x+a]\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & a-x & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a-x & \cdots & 0 & 0 \\ a-x & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\tau(n,n-1,\cdots,1)}[(n-1)x+a](x-a)^{n-1}$$

$$= (-1)^{\frac{n(n-1)}{2}}[(n-1)x+a](x-a)^{n-1}$$

(6) $$\begin{vmatrix} x & x & \cdots & x & a \\ 0 & 0 & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a & \cdots & 0 & x \\ a & 0 & \cdots & 0 & x \end{vmatrix} = D$$

解：若$a=0$，则$D=0$；若$a\neq 0$，则

$$\begin{vmatrix} x & x & \cdots & x & a \\ 0 & 0 & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a & \cdots & 0 & x \\ a & 0 & \cdots & 0 & x \end{vmatrix} \xlongequal{r_1-\sum\limits_{i=2}^{n}\frac{x}{a}r_i} \begin{vmatrix} 0 & 0 & \cdots & 0 & a-(n-1)\frac{x^2}{a} \\ 0 & 0 & \cdots & a & x \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a & \cdots & 0 & x \\ a & 0 & \cdots & 0 & x \end{vmatrix}$$

$$=(-1)^{\frac{n(n-1)}{2}}a^{n-1}(a-\frac{(n-1)x^2}{a})$$

12. 证明下列等式：

(1) $$\begin{vmatrix} 2 & \lg 4 & 2\lg 5 \\ 1 & \cos^2\alpha & \sin^2\alpha \\ -1 & -\sqrt{2} & \dfrac{1}{\sqrt{2}+1} \end{vmatrix}$$

证明：$$\begin{vmatrix} 2 & \lg 4 & 2\lg 5 \\ 1 & \cos^2\alpha & \sin^2\alpha \\ -1 & -\sqrt{2} & \dfrac{1}{\sqrt{2}+1} \end{vmatrix} \xlongequal{c_2+1\times c_3} \begin{vmatrix} 2 & 2 & 2\lg 5 \\ 1 & 1 & \sin^2\alpha \\ -1 & -1 & \dfrac{1}{\sqrt{2}+1} \end{vmatrix} = 0$$

(2) $$D_1=\begin{vmatrix} x_1+y_1 & x_2+y_2 & x_3+y_3 \\ y_1+z_1 & y_2+z_2 & y_3+z_3 \\ z_1+x_1 & z_2+x_2 & z_3+x_3 \end{vmatrix} = 2\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 2D$$

证明：$$D_1 \xlongequal[r_1\div 2]{r_1+\sum\limits_{i=2}^{3}r_i} 2\begin{vmatrix} x_1+y_1+z_1 & x_2+y_2+z_2 & x_3+y_3+z_3 \\ y_1+z_1 & y_2+z_2 & y_3+z_3 \\ z_1+x_1 & z_2+x_2 & z_3+x_3 \end{vmatrix}$$

$$\xlongequal[2\leqslant i\leqslant 3]{r_i-r_1} 2\begin{vmatrix} x_1+y_1+z_1 & x_2+y_2+z_2 & x_3+y_3+z_3 \\ -x_1 & -x_2 & -x_3 \\ -y_1 & -y_2 & -y_3 \end{vmatrix}$$

$$\xlongequal[r_2\div(-1),r_3\div(-1)]{r_1+\sum\limits_{i=2}^{3}r_i} 2\begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \qquad \xlongequal[r_2\leftrightarrow r_3]{r_1\leftrightarrow r_2} 2\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

$$\xlongequal{D=D^T} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

(3) $$D=\begin{vmatrix} a & 2 & 3 & \cdots & n \\ 1 & a+1 & 3 & \cdots & n \\ 1 & 2 & a+2 & \cdots & n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & a+n-1 \end{vmatrix}$$

$$=\left[a+\frac{(n-1)(n+2)}{2}\right](a-1)^{n-1}$$

证明: $$D\xlongequal[c_n\div\left(a+\frac{(n-1)(n+2)}{2}\right)]{c_n+\sum\limits_{i=1}^{n-1}c_i}\left(a+\frac{(n-1)(n+2)}{2}\right)\begin{vmatrix} a & 2 & 3 & \cdots & 1 \\ 1 & a+1 & 3 & \cdots & 1 \\ 1 & 2 & a+2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & 1 \end{vmatrix}$$

$$\xlongequal[1\leqslant i\leqslant n-1]{c_i-i\times c_n}\left(a+\frac{(n-1)(n+2)}{2}\right)\begin{vmatrix} a-1 & 0 & 0 & \cdots & 1 \\ 0 & a-1 & 0 & \cdots & 1 \\ 0 & 0 & a-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$=\left[a+\frac{(n-1)(n+2)}{2}\right](a-1)^{n-1}$$

(4) $$D_n=\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}=\sum_{i=0}^{n}a^ib^{n-i}$$

证明: (1)$n=1,2$时等式成立;(2)若$n=k$时成立，则

$$D_k=\sum_{i=0}^{k}a^ib^{k-i}$$

从而$n=k+1$时，按第一列展开得到

$$D_{k+1}=(a+b)D_k+abD_{k-1}=(a+b)\sum_{i=0}^{k}a^ib^{k-i}-ab\sum_{i=0}^{k-1}a^ib^{k-1-i}$$

$$=\sum_{i=0}^{k}a^{i+1}b^{k-i}+\sum_{i=0}^{k}a^ib^{k+1-i}-\sum_{i=0}^{k-1}a^{i+1}b^{k-i}$$

$$=\sum_{i=1}^{k+1} a^i b^{k+1-i} + \sum_{i=0}^{k} a^i b^{k+1-i} - \sum_{i=1}^{k} a^i b^{k+1-i} = \sum_{i=0}^{k+1} a^i b^{k+1-i}$$

所以，$n = k + 2$时结论也对，原命题为真.

14. 计算下列行列式

(1) $$\begin{vmatrix} 7 & 49 & 1 & 1 \\ 0 & 20 & 0 & 0 \\ -3 & 6 & -1 & 5 \\ -2 & 11 & -3 & 1 \end{vmatrix}$$

解：$$\begin{vmatrix} 7 & 49 & 1 & 1 \\ 0 & 20 & 0 & 0 \\ -3 & 6 & -1 & 5 \\ -2 & 11 & -3 & 1 \end{vmatrix} = 20 \begin{vmatrix} 7 & 1 & 1 \\ -3 & -1 & 5 \\ -2 & -3 & 1 \end{vmatrix} \overset{c_1-7c_3}{\underset{c_2-c_1}{=\!=}} 20 \begin{vmatrix} 0 & 0 & 1 \\ -38 & -6 & 5 \\ -9 & -4 & 1 \end{vmatrix}$$

$$= 20 \begin{vmatrix} -38 & -6 \\ -9 & -4 \end{vmatrix} = 1960$$

(2) $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix}$$

解：$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

$$= (a_{11}a_{22} - a_{12}a_{21}) \times (a_{33}a_{44} - a_{34}a_{43})$$

(3) $$D = \begin{vmatrix} x & 0 & 0 & \cdots & 0 & y \\ y & x & 0 & \cdots & 0 & 0 \\ 0 & y & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & y & x \end{vmatrix}$$

解：$D = x^n + (-1)^{\tau(n12\cdot(n-1))} y^n = x^n + (-1)^{n-1} y^n$

18. 解下列线性方程组：

(1) $$\begin{cases} x_1 + 3x_2 + 2x_3 & = 7 \\ 2x_1 + x_2 + 4x_3 & = 6 \\ 3x_1 + 2x_2 + x_3 & = 4 \end{cases}$$

解：$\because D=\begin{vmatrix}1&3&2\\2&1&4\\3&2&1\end{vmatrix}=25, D_1=\begin{vmatrix}7&3&2\\6&1&4\\4&2&1\end{vmatrix}=-3,$

$$D_2=\begin{vmatrix}1&7&2\\2&6&4\\3&4&1\end{vmatrix}=40, D_3=\begin{vmatrix}1&3&7\\2&1&6\\3&2&4\end{vmatrix}=29$$

$\therefore x_1=\dfrac{D_1}{D}=\dfrac{-3}{25}, x_2=\dfrac{D_2}{D}=\dfrac{8}{5}, x_3=\dfrac{D_3}{D}=\dfrac{29}{25}.$

(2) $\begin{cases}3x_1+x_2+2x_3 &=0\\5x_2+x_3+4x_4 &=1\\4x_1+3x_3+x_4 &=0\\2x_1+x_2+x_4 &=1\end{cases}$

解：$\because D=\begin{vmatrix}3&1&2&0\\0&5&1&4\\4&0&2&1\\2&1&0&1\end{vmatrix}=39,$

$$D_1=13, D_2=13, D_3=-26, D_4=0$$

$\therefore x_1=\dfrac{1}{3}, x_2=\dfrac{1}{3}, x_1=\dfrac{-2}{3}, x_4=0$

(3) $\begin{cases}x_1+6x_2+2x_3-x_4+x_5 &=0\\x_1-2x_2-x_3 &=0\\2x_1-3x_3+x_4+3x_5 &=0\\x_3+2x_5 &=0\\x_2+x_3-3x_4 &=0\end{cases}$

解：$\because D_1=D_2=D_3=D_4=D_5=0$

$\therefore x_1=x_2=x_3=x_4=x_5=0$

21. 设$a_1,a_2,\cdots,a_n$是互不相同的实数，$b_1,b_2,\cdots,b_n$ 是任意实数，用克拉默法则证明：存在唯一的次数小于n的多项式$f(x)$使得

$$f(a_i)=b_i$$

证明：设$f(x)=\sum\limits_{j=0}^{n-1}k_jx^j$，则

$$f(a_j)=b_j(1\leqslant i\leqslant n)$$

为n元一次方程组，其中$k_j(0 \leqslant j \leqslant n-1)$未知.其对应的系数行列式为

$$D=\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix}=\prod_{1 \leqslant j<i \leqslant n}(a_i-a_j) \neq 0$$

所以，存在唯一解.