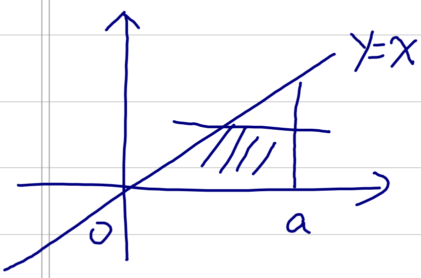
二重积分（交换积分顺序）.

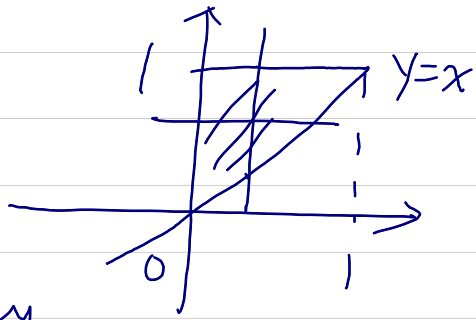
1. $\int_0^a dx \int_0^x f(y)\,dy = \int_0^a \int_y^a f(y)\,dx\,dy$



$= \int_0^a (a-y) f(y)\,dy.$

$= \int_0^a (a-x) f(x)\,dx.$

2. $\int_0^1 dx \int_x^1 f(x) f(y)\,dy.$



$= \int_0^1 \int_0^y f(x) f(y)\,dx\,dy$

$= \int_0^1 \int_0^x f(y) f(x)\,dy\,dx. \triangleq I$

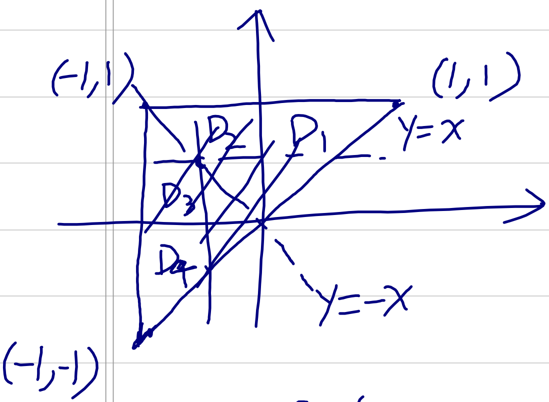
$\Rightarrow 2I = \int_0^1 \int_0^1 f(x) \underline{f(y)}\,dx\,dy$

$= \int_0^1 f(y) \underline{\int_0^1 f(x)\,dx}\,dy$

$= \left[\int_0^1 f(x)\,dx\right] \left[\int_0^1 f(y)\,dy\right] = A^2.$

$\Rightarrow I = \frac{A^2}{2}$

二重积分 (利用对称性)



$$\iint_D (xy + \cos x \sin y)\,dxdy.$$

$$= \iint_{D_1 \cup D_2} (xy + \cos x \sin y)\,dxdy + \iint_{D_3 \cup D_4} (xy + \cos x \sin y)\,dxdy$$

$$= 2\iint_{D_1} \cos x \sin y\,dxdy$$

$$= 2 \int_0^1 \int_0^y \cos x \sin y\,dx\,dy$$

$$= 2\int_0^1 \sin^2 y\,dy. \qquad \cos 2y = 1 - 2\sin y$$

$$= 2\int_0^1 \frac{1-\cos 2y}{2}\,dy$$

$$= 2 \cdot \frac{1}{2} - \frac{\sin 2y}{4}\Big|_0^1 = 1 - \frac{\sin 1}{4}$$

总结：(1) 如果区域关于 $y=0$ 对称.（D_1 为 D 的上半部分）

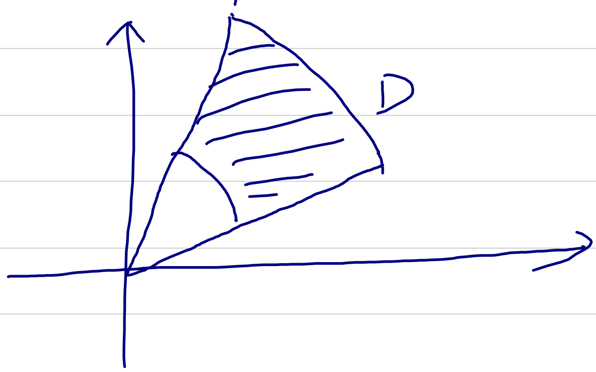
$$\iint_D f(x,y)\,dx\,dy = \begin{cases} 0 & \text{关于 } y \text{ 是奇函数.} \\ 2\iint_{D_1} f(x,y)\,dx\,dy & \text{关于 } y \text{ 是偶函数.} \end{cases}$$

(2) 如果区域关于 $x=0$ 对称.（D_1 为 D 的右半部分）

$$\iint_D f(x,y)\,dx\,dy = \begin{cases} 0 & f(x,y) \text{ 关于 } x \text{ 是奇函数.} \\ 2\iint_{D_1} f(x,y)\,dx\,dy & f(x,y) \text{ 关于 } x \text{ 是偶函数.} \end{cases}$$

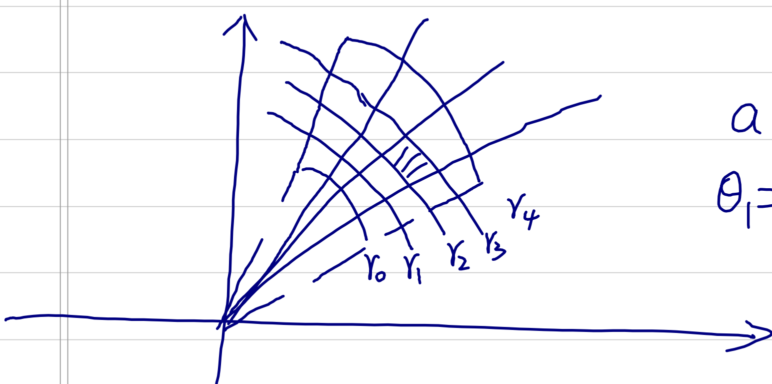
极坐标下二重积分的计算. $\iint_D f(x,y)\,dx\,dy$

积分区域如下：



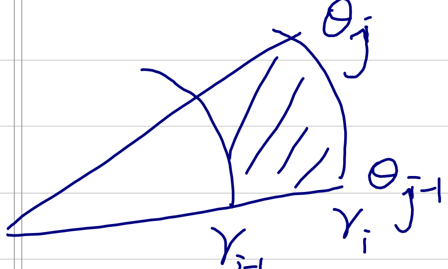
$a \le r \le b$

$\theta_1 \le \theta \le \theta_2$.



$a = r_0 < r_1 < r_2 < r_4 < r_i < \cdots < r_n = b$

$\theta_1 = \theta_0 < \theta_1 < \theta_2 < \cdots \theta_{j-1} < \theta_j < \cdots \theta_m = \theta_2$



$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i, \eta_j)\Delta\sigma_{ij}$$

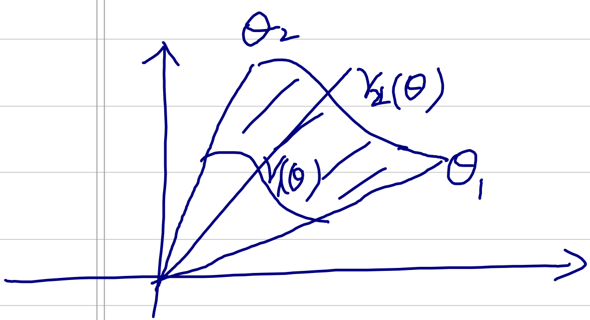
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{r}_i\cos\bar{\theta}_j,\ \bar{r}_i\sin\bar{\theta}_j)\Delta\sigma_{ij}$$

$$\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{r}_i\cos\bar{\theta}_j,\ \bar{r}_i\sin\bar{\theta}_j)\bar{r}_i\,\underline{\Delta\theta_i\Delta r_j}$$

$$\longrightarrow \iint_{\substack{a\le r\le b\\ \theta_1\le\theta\le\theta_2}} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

$$\Delta\sigma_{ij} = \frac{1}{2}r_i^2\Delta\theta_j - \frac{1}{2}r_{i-1}^2\Delta\theta_j = 2\cdot\frac{1}{2}(r_i - r_{i-1})\left(\frac{r_i + r_{i-1}}{2}\right)\Delta\theta_j$$

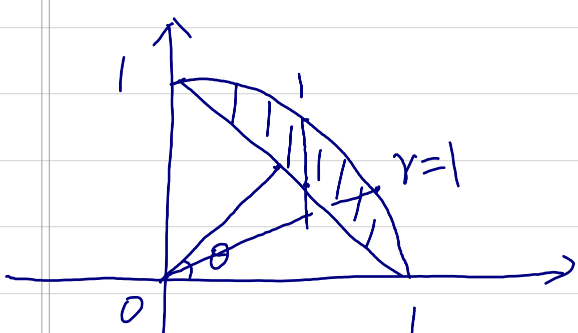
$$\approx \bar{r}_i\,\Delta r_i\,\Delta\theta_j.$$



$$\iint_D f(x,y)\,dx\,dy = \iint_{\substack{\theta_1\le\theta\le\theta_2 \\ r_1(\theta)\le r\le r_2(\theta)}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta_1}^{\theta_2}\left(\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\right) d\theta.$$

例: $y=1-x$, $y=\sqrt{1-x^2}$



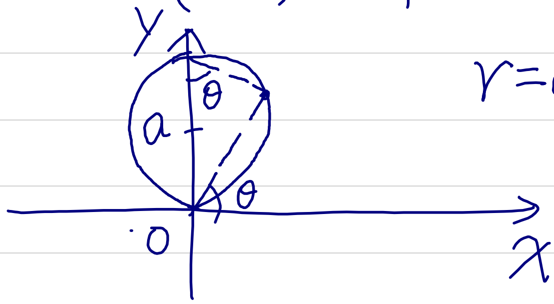
$r\sin\theta + r\cos\theta = 1.$

$\Rightarrow r = \dfrac{1}{\sin\theta+\cos\theta}.$

$$\int_0^{\frac{\pi}{2}}\left(\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} f(r\cos\theta, r\sin\theta)\, r\, dr\right) d\theta$$

课上写成了 $\frac{\pi}{4}$.

(2) $x^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{4}$

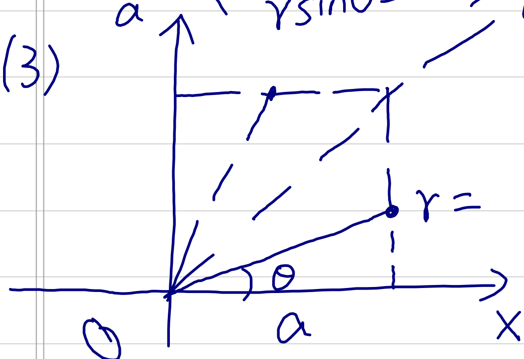


$r=a\sin\theta,\quad 0\le\theta\le\pi$.

$$\int_0^{\pi}\left(\int_0^{a\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\right)d\theta.$$

$r\sin\theta=a \Rightarrow r=\frac{a}{\sin\theta}$.

(3)

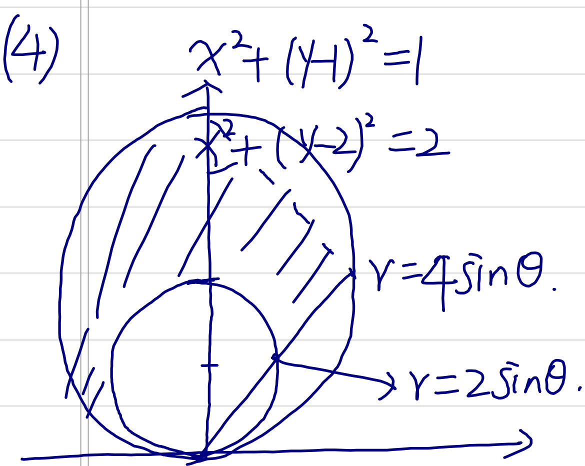


$r\cos\theta=a\Rightarrow r=\frac{a}{\cos\theta}$.

$$\int_0^{\frac{\pi}{4}}\left(\int_0^{\frac{a}{\cos\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\right)d\theta$$

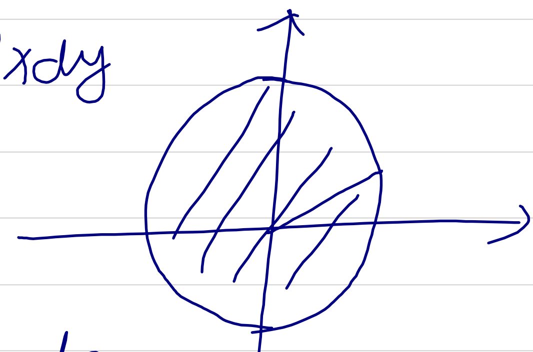
$$+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^{\frac{a}{\sin\theta}} f(r\cos\theta, r\sin\theta)\,r\right)d\theta.$$

(4)



$$\int_0^{\pi}\left(\int_{2\sin\theta}^{4\sin\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\right) d\theta$$

例1. $$\iint_{x^2+y^2\le a^2} e^{-(x^2+y^2)}\, dx\, dy$$



$$= \int_0^{2\pi}\left(\int_0^{a} e^{-r^2}\, r\, dr\right) d\theta$$

$$= \int_0^{2\pi} \left.\frac{-e^{-r^2}}{2}\right|_0^a d\theta$$

$$= 2\pi \cdot \frac{-e^{-a^2}+1}{2}$$

$$= \pi\left(1-e^{-a^2}\right)$$

$$I = \int_0^{+\infty} e^{-x^2} dx \qquad 2I = \int_{-\infty}^{+\infty} e^{-x^2} dx.$$

$$4I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right)$$

$$= \int_{-\infty}^{+\infty} e^{-x^2}\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx\,dy$$

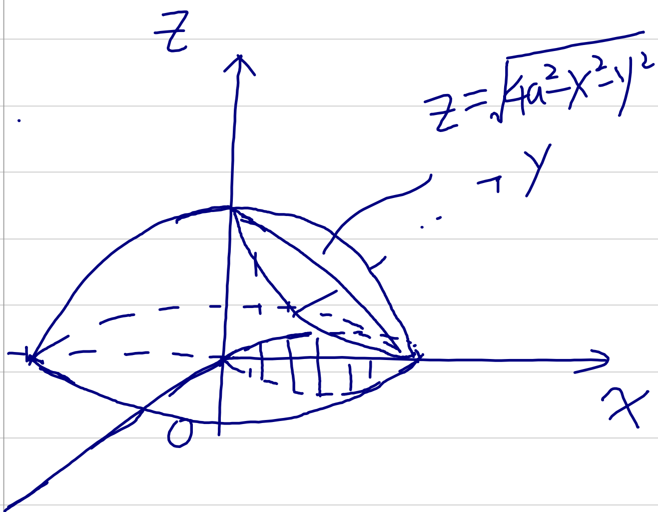
$$= \lim_{a \to +\infty} \iint_{x^2+y^2 \leq a^2} e^{-(x^2+y^2)} dx\,dy$$

$$= \lim_{a \to +\infty} \pi(1-e^{-a^2}) = \pi.$$

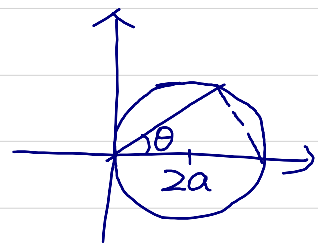
$$\Rightarrow \quad I^2 = \frac{\pi}{4}$$

$$\Rightarrow \quad I = \frac{\sqrt{\pi}}{2}.$$

例12.



$$V=\iint\limits_{(x-a)^2+y^2\le a^2}\sqrt{4a^2-x^2-y^2}\,dx\,dy$$



$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{2a\cos\theta} r\sqrt{4a^2-r^2}\,dr\right)d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\cdot\int_0^{2a\cos\theta}(4a^2-r^2)^{\frac{1}{2}}d(-r^2)\ d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{2}{3}(4a^2-r^2)^{\frac{3}{2}}\Big|_0^{2a\cos\theta}\ d\theta$$

$$=-\frac{1}{3}\cdot 2\int_0^{\frac{\pi}{2}}\left[(4a^2-4a^2\cos^2\theta)^{\frac{3}{2}}-8a^3\right]d\theta$$

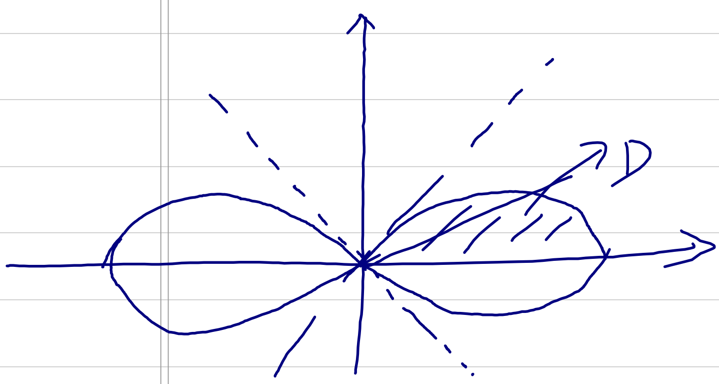
$$=-\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}}\left[(2a\sin\theta)^3-8a^3\right]d\theta$$

$$=-\frac{2}{3}\,8a^3\cdot\left(\frac{2}{3}\cdot 1-\frac{\pi}{2}\right)=\frac{16}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

例3. 求双纽线 $(x^2+y^2)^2=2a^2(x^2-y^2)$ 所围成区域的面积.

解: 引入极坐标. $x=r\cos\theta$, $y=r\sin\theta$.

$$\Rightarrow r^2=2a^2\cos 2\theta.$$



$$S=4\iint_D 1\,dxdy$$

$$=4\int_0^{\frac{\pi}{4}}\left(\int_0^{r(\theta)} r\,dr\right)d\theta$$

$$=4\int_0^{\frac{\pi}{4}} \frac{r^2}{2}\Big|_0^{r(\theta)}d\theta$$

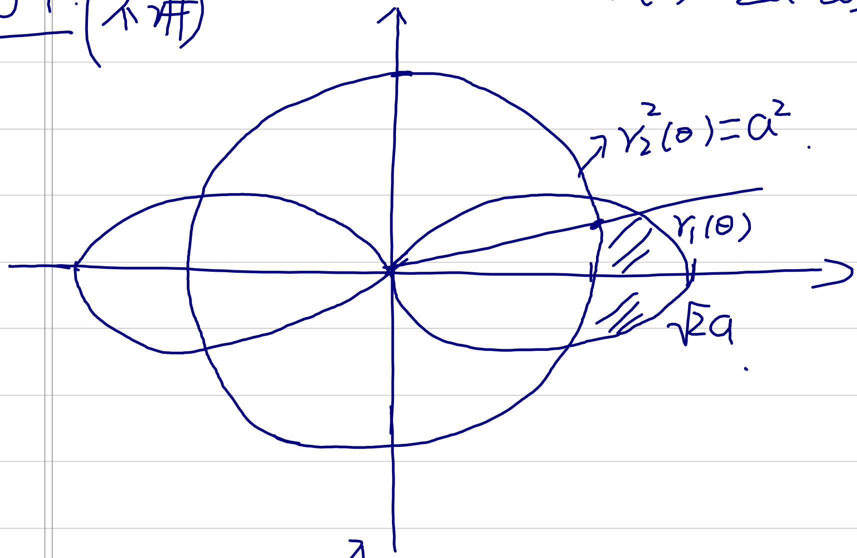
$$=2\int_0^{\frac{\pi}{4}} r^2(\theta)\,d\theta$$

$$=2\int_0^{\frac{\pi}{4}} 2a^2\cos 2\theta\,d\theta$$

$$=4a^2\cdot\frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{4}}=2a^2.$$

例4.（不讲）

$r^2(\theta)=2a^2\cos2\theta$



$\cos2\theta=\frac{1}{2}$

$2\theta=\frac{\pi}{3}$.

$\theta=\frac{\pi}{6}$

$$D=4\int_0^{\frac{\pi}{6}}\int_a^{r_1(\theta)} r\,dr\,d\theta$$

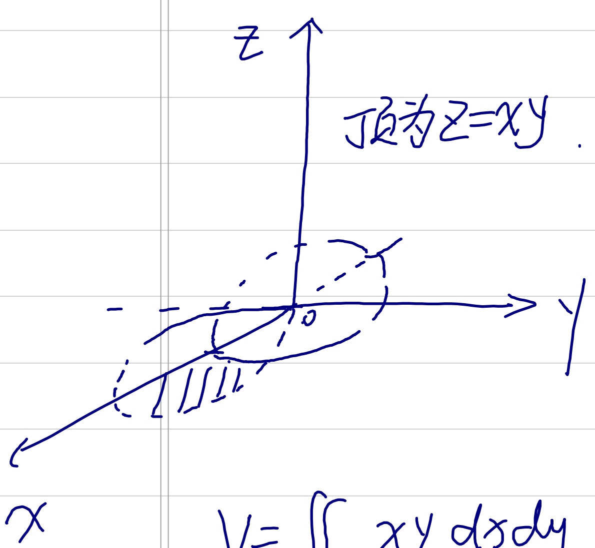
$$=4\int_0^{\frac{\pi}{6}}\frac{r_1^2(\theta)-a^2}{2}\,d\theta.$$

$$=2\int_0^{\frac{\pi}{6}}\left(2a^2\cos2\theta-a^2\right)d\theta.$$

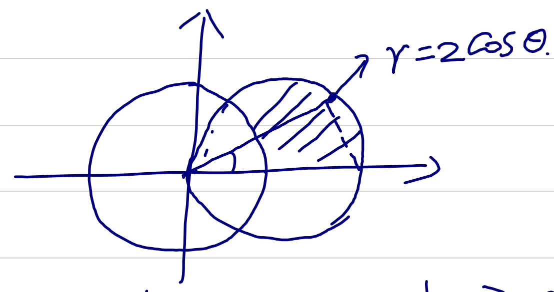
$$=2a^2\cdot\left(\sin2\theta\cdot\Big|_0^{\frac{\pi}{6}}-\frac{\pi}{6}\right)$$

$$=2a^2\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)=a^2\left(\sqrt{3}-\frac{\pi}{3}\right)$$

例 设曲顶柱体以xoy平面为底，曲面$z=xy$为顶，侧面由平面$y=0$在$y\geq 0$的一侧，圆柱面$x^2+y^2=1$的外侧和圆柱面$x^2+y^2=2x$的内侧围成，求此曲顶柱体的体积.



积分区域



交点 $2\cos\theta=1 \Rightarrow \cos\theta=\frac{1}{2}$

$$V=\iint_D xy\,dxdy$$

$$=\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\cos\theta\cdot r\sin\theta\cdot r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{3}} \frac{r^4}{4}\Big|_1^{2\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{3}}\frac{16\cos^4\theta-1}{4}\sin\theta\cos\theta\,d\theta$$

$$=-4\int_0^{\frac{\pi}{3}}\cos^5\theta\,d(\cos\theta)-\frac{1}{4}\int_0^{\frac{\pi}{3}}\sin\theta\,d\sin\theta$$

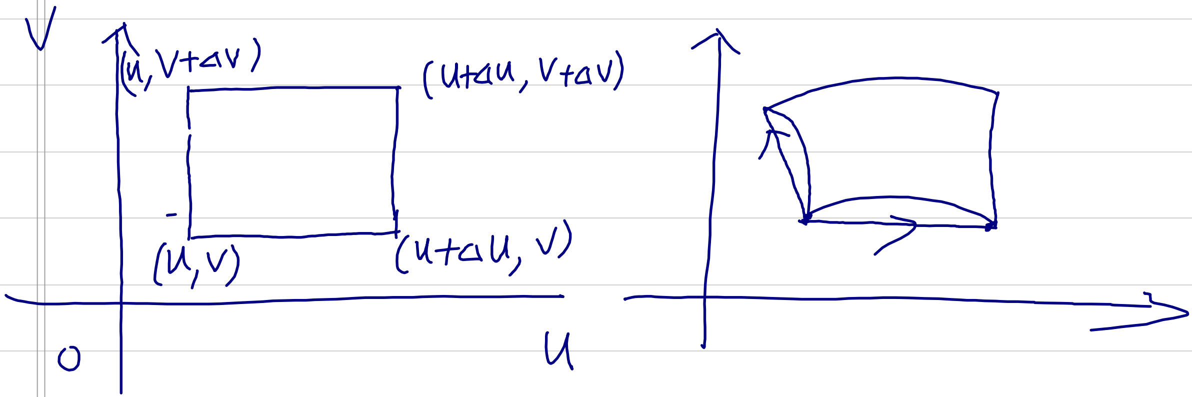
$$=-4\cdot\frac{\cos^6\theta}{6}\Big|_0^{\frac{\pi}{3}}-\frac{1}{4}\cdot\frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{3}}$$

$$=-\frac{2}{3}\left(\left(\frac{1}{2}\right)^6-1\right)-\frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)^2$$

$$=\frac{\cancel{2}^{1}}{\cancel{3}_{1}}\cdot\frac{\cancel{63}^{21}}{\cancel{64}_{32}}-\frac{3}{32}=\frac{18}{32}=\frac{9}{16}.$$

二重积分的换元法

$$\iint_D f(x,y)\,dx\,dy.$$



平行于u轴的线以及平行于v轴的直线将D'进行分割。
相应的变换 $\begin{cases} x(u,v) \\ y(u,v) \end{cases}$ 把 xoy 平面上的区域D进行了分割

$$\vec{a} = \big(x(u+\Delta u, v) - x(u,v)\big)\vec{i} + \big(y(u+\Delta u, v) - y(u,v)\big)\vec{j} + 0\vec{k}$$

$$= \frac{\partial x}{\partial u}\Delta u\,\vec{i} + \frac{\partial y}{\partial u}\Delta u\,\vec{j} + 0\vec{k}$$

$$\vec{b} = \big(x(u, v+\Delta v) - x(u,v)\big)\vec{i} + \big(y(u, v+\Delta v) - y(u,v)\big)\vec{j} + 0\vec{k}$$

$$= \frac{\partial x}{\partial v}\Delta v\,\vec{i} + \frac{\partial y}{\partial v}\Delta v\,\vec{j} + 0\vec{k}.$$

$$\Delta\sigma \approx \left\| \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta v & 0 \\ \frac{\partial x}{\partial v}\Delta u & \frac{\partial y}{\partial v}\Delta v & 0 \end{vmatrix} \right\| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v.$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

例：作变换 $u=\frac{y^2}{x}$, $v=xy$. $p\le u\le q$, $a\le v\le b$.

$x^2y^2 \cdot \frac{x}{y^2}$

$$\Rightarrow \begin{cases} y=(uv)^{\frac{1}{3}} \\ x=\left(\frac{v^2}{u}\right)^{\frac{1}{3}}. \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v^2}{u}\right)^{-\frac{2}{3}}\cdot\left(-\frac{v^2}{u^2}\right) & \frac{1}{3}\left(\frac{v^2}{u}\right)^{-\frac{2}{3}}\cdot\frac{2v}{u} \\ \frac{1}{3}(uv)^{-\frac{2}{3}}\cdot v. & \frac{1}{3}(uv)^{-\frac{2}{3}}\cdot u \end{vmatrix}$$

$$= -\frac{1}{9}v^{-2}\cdot\left(-\frac{v^2}{u}\right) - \frac{1}{9}v^{-2}\,2\frac{v^2}{u}$$

$$= -\frac{1}{3u}$$

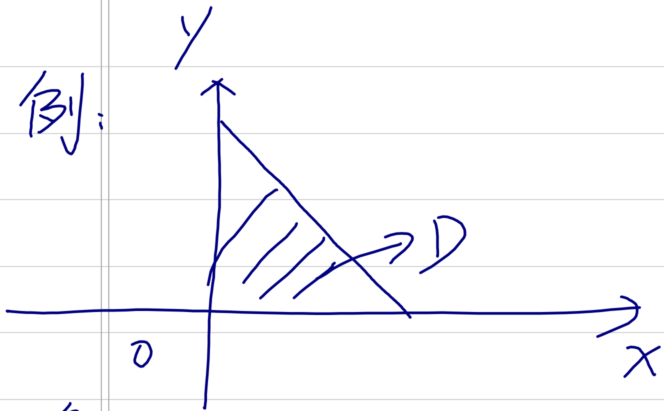
$$S = \iint_D 1\,dx\,dy = \iint_{D'} \frac{1}{3u}\,du\,dv.$$

$$= \int_a^b \left(\int_p^q \frac{1}{3u}\,du\right)dv$$

$$= \int_a^b \frac{1}{3}(\ln q - \ln p)\,dv$$

$$= \frac{1}{3}(b-a)(\ln q - \ln p)$$

例:



解: 作变换

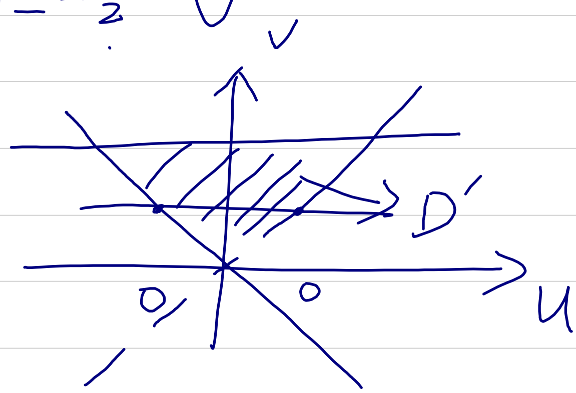
$u=y-x,\quad v=x+y$, 则 $x=\frac{v-u}{2}$, $y=\frac{u+v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$x=0 \Rightarrow u=v.$

$y=0 \Rightarrow u=-v.$

$x+y=2 \Rightarrow v=2.$



$$\iint_D e^{\frac{y-x}{x+y}}\,dx\,dy$$

$$=\iint_{D'} e^{\frac{u}{v}}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

$$=\frac{1}{2}\int_0^2\int_{-v}^{v} e^{\frac{u}{v}}\,du\,dv$$

$$=\frac{1}{2}\int_0^2 \left(ve^{\frac{u}{v}}\Big|_{-v}^{v}\right) dv$$

$$=\frac{1}{2}\int_0^2 (ve-ve^{-1})\,dv = \frac{e-e^{-1}}{2}\,\frac{v^2}{2}\Big|_0^2 = e-e^{-1}$$

广义极坐标变换.

$$\begin{cases} x = ar\cos\theta. \\ y = br\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr.$$

除了 $r=0$ 外, Jacobi 行列式不为 0.

$$\iint_D f(x,y)dxdy = \iint_{D'} f(ar\cos\theta,\ br\sin\theta)abr\,dr\,d\theta.$$

例. $$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$$

$$\xlongequal[y=br\sin\theta]{x=ar\cos\theta} \int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}\,dr\,d\theta.$$

$$= 2\pi\cdot\left(-\frac{1}{2}\right)\frac{2}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1$$

$$= 2\pi\cdot\frac{1}{3} = \frac{2}{3}\pi.$$

例: $\iint_D 1\,dx\,dy$

$$\xlongequal[y=br\sin\theta]{x=ar\cos\theta} \int_0^{\frac{\pi}{2}}\int_0^{r(\theta)} abr\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} ab\cdot\frac{r^2}{2}\Big|_0^{r(\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}} ab\cdot\frac{ab\sin 2\theta}{4c^2}\,d\theta$$

$$= \frac{a^2b^2}{4c^2}\cdot\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{a^2b^2}{8c^2}$$

$$(r^2)^2 = \frac{abr^2\cos\theta\sin\theta}{c^2}$$

$$\Rightarrow r^2 = \frac{ab\sin 2\theta}{2c^2}$$