Seat No.______ Major/Class______ School______ Student ID □□□□□□□□□□□□ Name______

(DONNOT WRITE YOUR ANSWER IN THIS AREA)

Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

Mathematical Analysis I Exam Paper A（2019-2020-1）

Notice:

1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	I	II	III	IV	Sum
Score					

Score

I. Please fill the correct answers in the following blanks. $(4'\times 5=20')$

1. $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{n}=$ e^2 .

2. If $\sin(x+y)=y^2\cos x$, then $dy=$ $\dfrac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)}dy$.

3. The inflection points of the curve $f(x)=\dfrac{1}{2}x^2+\dfrac{9}{10}(x-1)^{\frac{5}{3}}$ are $(1,\frac{1}{2}),\ (0,-\frac{9}{10})$

4. Suppose f is continuous with the property that $|f(x)|\le x^2$ for all x, then $f'(0)=$ 0 .

5. If $f(x)$ is continuous, and $f(x)=\cos^4 x+\dfrac{1}{\pi}\int_0^{\frac{\pi}{2}} f(x)\,dx$, then $f(x)=$ $\cos^4 x+\frac{3}{8}$.

II. **Finish the following calculations.** (6-11: $6' \times 6 = 36'$)

Score

6. $\lim_{x\to+\infty}\left(x - x^2\ln\left(1+\frac{1}{x}\right)\right)$.

Solution: By Taylor's formula,

$$\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x^2} + \frac{1}{3}\cdot\frac{1}{x^3} + o\left(\frac{1}{x^3}\right).$$

$$x - x^2\ln\left(1+\frac{1}{x}\right)$$

$$= x - x^2\left(\frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x^2} + \frac{1}{3}\cdot\frac{1}{x^3} + o\left(\frac{1}{x^3}\right)\right)$$

$$= x - x + \frac{1}{2} - \frac{1}{3}\cdot\frac{1}{x} + o\left(\frac{1}{x}\right)$$

$$\lim_{x\to+\infty}\left(x - x^2\ln\left(1+\frac{1}{x}\right)\right)$$

$$= \lim_{x\to+\infty}\left(\frac{1}{2} - \frac{1}{3x} + o\left(\frac{1}{x}\right)\right) = \frac{1}{2}$$

7. $\lim_{x\to3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt\right)$

$$= \left(\lim_{x\to3} x\right)\left(\lim_{x\to3}\frac{\int_3^x \frac{\sin t}{t}dt}{x-3}\right)$$

Let $F(x) = \int_3^x \frac{\sin t}{t}\,dt = F(x) - F(3)$

Then $\lim_{x\to3}\frac{\int_3^x\frac{\sin t}{t}dt}{x-3} = \lim_{x\to3}\frac{F(x)-F(3)}{x-3}$

$$= F'(3)$$

$F'(x) = \frac{\sin x}{x}. \Rightarrow F'(3) = \frac{\sin 3}{3}.$

Then $\lim_{x\to3}\left(\frac{x}{x-3}\int_3^x\frac{\sin t}{t}dt\right) = \left(\lim_{x\to3}x\right)\left(\lim_{x\to3}\frac{\int_3^x\frac{\sin t}{t}dt}{x-3}\right)$

$$= 3\times\frac{\sin 3}{3} = \sin 3.$$

Or They can use L'Hospital Rule.

8. If $y = \arctan x$, find $y^{(n)}(0)$.

Solution: $y = \arctan x$, $y(0) = 0$

$y' = \frac{1}{1+x^2}$ $y'(0) = 1$

$(1+x^2)y' = 1$

Take $(n-1)$th derivatives on the both side with respect to x, by Leibniz formula,

$$y^{(n)} + C_{n-1}^1 (2x) y^{(n-1)} + C_{n-1}^2 \cdot 2\, y^{(n-2)} = 0.$$

Let $x = 0$.

$$y^{(n)}(0) + (n-1)(n-2) y^{(n-2)}(0) = 0.$$

Then we have

$$y^{(n)}(0) = -(n-1)(n-2) y^{(n-2)}(0)$$

$$y^{(2k+1)}(0) = -(2k)(2k-1) y^{(2k-1)}(0)$$

$$= (2k)(2k-1)(2k-2)(2k-3) y^{(2k-3)}(0)$$

$$= \cdots = (-1)^k (2k)!\, y'(0) = (-1)^k (2k)!$$

$$y^{(2k)}(0) = 0.$$

$$y^{(n)}(0) = \begin{cases} (-1)^k (2k)! & n = 2k+1 \\ 0 & n = 2k \end{cases}$$

9. Evaluate the indefinite integral $\int \frac{1}{x^2\sqrt{x^2+1}}dx$

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$$

$\underline{\underline{x = \tan t}}$

$dx = \sec^2 t\,dt$

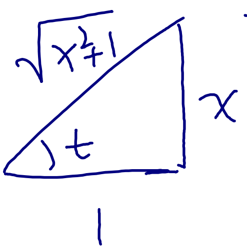
$$\int \frac{\sec^2 t}{\tan^2 t \sec t}\,dt$$

$$= \int \frac{\sec t}{\frac{\sin^2 t}{\cos^2 t}}\,dt$$

$$= \int \frac{\cos t}{\sin^2 t}\,dt$$

$$= -\frac{1}{\sin t} + C$$

$$= -\frac{\sqrt{x^2+1}}{x} + C$$



where C is an arbitrary constant.

10. (a) Find the tangent to the cycloid $x = a(\theta - \sin\theta), y = a(1-\cos\theta)$ ($a > 0$ is a constant) at the point where $\theta = \frac{\pi}{3}$.

(b) Find the area under one arch of the cycloid.

(a) $\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{a\sin\theta}{a(1-\cos\theta)}$

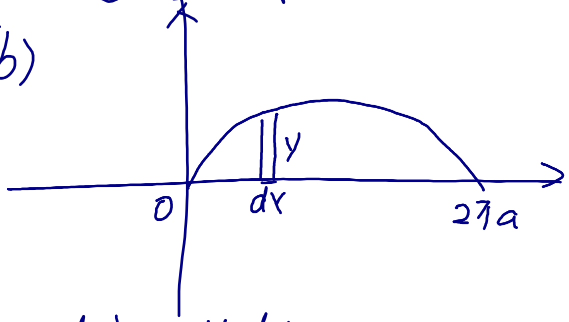
$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{a\sin\frac{\pi}{3}}{a(1-\cos\frac{\pi}{3})} = \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}} = \sqrt{3}$.

$x|_{\theta=\frac{\pi}{3}} = a(\frac{\pi}{3} - \frac{\sqrt{3}}{2})$ $\quad y|_{\theta=\frac{\pi}{3}} = a(1-\cos\frac{\pi}{3}) = \frac{a}{2}$

So the tangent line is

$y - \frac{a}{2} = \sqrt{3}\left(x - a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$

(b)



$dA = y\,dx$.

$A = \int_0^{2\pi a} y\,dx$

$= \int_0^{2\pi} a(1-\cos\theta)\, a\cdot(1-\cos\theta)\,d\theta$

$= a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$.

$= 2\pi a^2 + a^2\int_0^{2\pi}\frac{\cos 2\theta + 1}{2}\,d\theta$

$= 3\pi a^2$

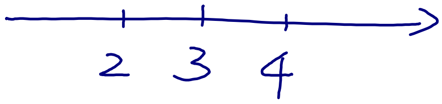
III. Prove the following conclusions. ($8' \times 3 = 24$)

Score

11. Prove that $\lim_{x \to 3} x^2 = 9$ by using the ε, δ definition of limit.

proof: We can restrict x lies in the 1-neighborhood of 3. i.e., $|x-3|<1$.

Then $|x+3| = |x-3+6| < 7$.



$\forall \varepsilon > 0$.

$\exists \delta = \min\{\frac{\varepsilon}{7}, 1\}$.

if $0 < |x-3| < \delta$.

Then $|x^2-9| = |x-3||x+3| < 7|x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$.

12. Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$.

Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Show your reasons.

$f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x-0} = \lim_{x \to 0} x \sin\frac{1}{x} = 0$.

$f'(x) = 2x\sin\frac{1}{x} + x^2 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right)$

$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$. $(x \neq 0)$.

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}. & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(x)$ is not continuous at $x=0$.

$x_n' = \frac{1}{2n\pi}$. $x_n' \to 0$, as $n \to \infty$.

$x_n'' = \frac{1}{\frac{\pi}{2} + 2n\pi}$ $x_n'' \to 0$, as $n \to \infty$.

$f(x_n') \to -1$, $f(x_n'') \to 0$. as $n \to \infty$.

By Heine Theorem, $\lim_{x \to 0} f'(x)$ does not exist.

Then $f'(x)$ is not continuous at $x=0$.

13. Assume that $f(x)$ is continuous on the closed interval $[a,b]$, and f is differentiable in the open interval (a,b), $0<a<b$. Use Cauchy's mean value theorem to prove that there exists $\xi \in (a,b)$ such that

$$\frac{af(b)-bf(a)}{a-b} = f(\xi) - \xi f'(\xi).$$

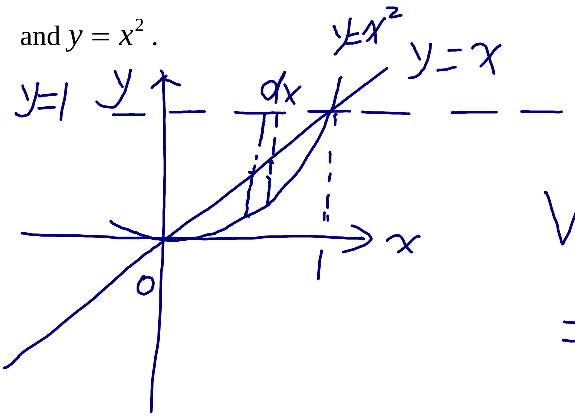
Proof: Let $G(x)=\frac{1}{x}$. Then $G(x)$ is continuous on $[a,b]$ and is differentiable in (a,b). $F(x)=\frac{f(x)}{x}$

$$\frac{af(b)-bf(a)}{a-b} = \frac{\frac{f(b)}{b}-\frac{f(a)}{a}}{\frac{1}{b}-\frac{1}{a}} = \frac{F(b)-F(a)}{G(b)-G(a)} \overset{\exists \xi\in(a,b)}{=\!=\!=} \frac{F'(\xi)}{G'(\xi)} = \frac{\frac{f'(\xi)\xi-f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi)-\xi f'(\xi).$$

Score

IV. Finish the following questions. ($10' \times 2 = 20$)

14. Find the volume of the solid obtained by rotating about $y=1$ the region between $y=x$ and $y=x^2$.



$dV = \pi(1-x^2)^2dx - \pi(1-x)^2dx$

$V=\int_0^1 [\pi(1-x^2)^2 - \pi(1-x)^2]dx$

$=\int_0^1 \pi(1-2x^2+x^4-1+2x-x^2)\,dx$

$=\int_0^1 \pi(x^4-3x^2+2x)dx$

$=\pi\cdot\left(\frac{x^5}{5}-x^3+x^2\right)\Big|_0^1 = \frac{1}{5}\pi.$

15. A sequence $\{a_n\}$ is given by $a_1=\frac{1}{2}\left(a+\frac{1}{a}\right)$, $a_{n+1}=\frac{1}{2}\left(a_n+\frac{1}{a_n}\right)$ $(n=1,2,3\cdots)$, $(a>0)$.

Show that $\lim\limits_{n\to+\infty} a_n$ exists and find it.

Solution: step 1. $\{a_n\}$ is a positive sequence.

$a_1=\frac{1}{2}(a+\frac{1}{a})>0$.

suppose $a_k>0$.

Then $a_{k+1}=\frac{1}{2}(a_k+\frac{1}{a_k})>0$.

By mathematical induction, we have for any n,

$a_n>0$.

step 2. $\{a_n\}$ is bounded below.

$a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n})\geq\sqrt{a_n\cdot\frac{1}{a_n}}=1$.

Step 3. $\{a_n\}$ is decreasing.

$a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1}{2a_n}-\frac{a_n}{2}$

$=\frac{1-a_n^2}{2a_n}\leq 0$.

Then $a_{n+1}\leq a_n$ for any n.

By the monotonic sequence theorem, we know that $\lim\limits_{n\to+\infty}a_n$ exists.

Step 4. We assume that $\lim\limits_{n\to+\infty}a_n=A$.

Take limit on the both side of $a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n})$

$A=\frac{1}{2}(A+\frac{1}{A})\Rightarrow\frac{1}{2}A=\frac{1}{2A}\Rightarrow A^2=1$.

$A=1$. or $A=-1$

Since $\{a_n\}$ is positive, Then $\lim\limits_{n\to+\infty}a_n=1$.