Problem Set 6

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1. **(50pts) Relation between circular and standard convolution.**

   (a) (5 pts) Consider the causal signals $x[n]$ and $h[n]$ of length N_1 and N_2 respectively. Show that $y[n] = x[n] \circledast h[n]$ is also a causal sequence of length $N_1 + N_2 - 1$.

(a) for $x[n]$, n only take values in the set $\{0, 1, \cdots, N_1 - 1\}$

$h[n]$, n only take values in the set $\{0, 1, \cdots, N_2 - 1\}$

$\therefore y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$

$= \sum_{k=0}^{N_1-1} x[k] h[n-k]$

$\therefore 0 \le n-k \le N_2 - 1$

$\therefore \begin{cases} \text{when } k=0, & 0 \le n \le N_2 - 1 \text{ take values} \\ \text{when } k = N_1 - 1, & N_1 - 1 \le n \le N_1 + N_2 - 2 \text{ take values} \end{cases}$

$\therefore$ for $y[n]$, n only take values when

$0 \le n \le N_1 + N_2 - 2$

$\therefore y[n]$ is causal

length $= (N_1 + N_2 - 2) - 0 + 1 = N_1 + N_2 - 1$

(b) (15pts) Let $\tilde{x}[n]$ and $\tilde{h}[n]$ be the zero-padded N length versions of $x[n]$ and $h[n]$, respectively. Prove that if $N \geq N_1 + N_2 - 1$, then

$$y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{1}$$

is equal to

$$y_c[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[\langle n-k \rangle_N] = \tilde{x}[n] Ⓝ \tilde{h}[n] \tag{2}$$

for $0 \leq n \leq N-1$. Explain why this equivalence is not true when $N < N_1 + N_2 - 1$.

(b) ① $\because$ zero-padding

$\therefore y[n] = x[n] \circledast h[n] = \sum_{k=0}^{N_1-1} x[k]h[n-k]$

$= \sum_{k=0}^{N_1-1} \tilde{x}[k]\tilde{h}[n-k]$

$= \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k]$

$\because$ when $N \geq N_1 + N_2 + 1$, $N_2 \leq N - N_1 - 1$

$\therefore$ for $h[n-k]$, it only takes values when

$0 \leq n-k \leq N_2 - 1 \leq (N - N_1 - 1) - 1 = N - N_1 - 2 < N$

$\therefore$ for $\tilde{h}[n-k]$, it must be zero unless

$0 \leq n-k < N$

$\therefore \tilde{h}[n-k] = \tilde{h}[<n-k>_N]$

$\therefore y[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[<n-k>_N] = y_c[n]$

② when $N < N_1 + N_2 - 1$, will occure $n-k \geq N$

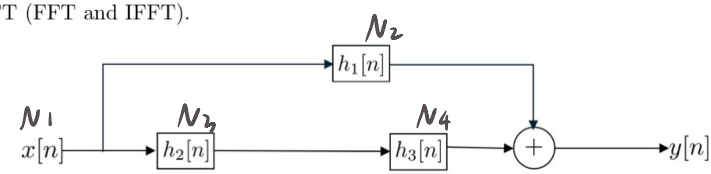
$\therefore <n-k>_N \neq n-k$

$\therefore y_c[n] = \sum_{k=0}^{N} \tilde{x}[k]\tilde{h}[<n-k>_N]$ will wrap around

$\therefore y[n] \neq y_c[n]$

(for example, if $N = 4$, length of linear convolution is $N_1 + N_2 - 1 = 5$
$y[4]$ will wrap around and add to $y[0]$
which is not equal to $y[n]$)

(c) Now consider four causal signals $x[n]$, $h_1[n]$, $h_2[n]$ and $h_3[n]$ of lengths N_1, N_2, N_3 and N_4, respectively. We want to compute the output $y[n]$ shown in the figure below using forward and inverse FFT (FFT and IFFT).



i. (5pts) Express the input-output relation of the above system using standard convolution.

ii. (10pts) We want to replace *all* the standard convolutions in (i) with M-modulo circular convolution. Using (a) and (b), find the minimum value of M that gives the same output as in (i).

iii. (15pts)Derive an efficient algorithm to compute $y[n]$ using FFT and IFFT. The algorithm should use the IFFT function only once. (Assume the FFT and IFFT functions are available.) Indicate the complexity of this algorithm using $\mathcal{O}(\cdot)$ notation.

(c) i. $y[n] = x[n] \circledast h_1[n] + x[n] \circledast h_2[n] \circledast h_3[n]$

ii. for $x[n] \circledast h_1[n]$, length $= N_1 + N_2 - 1$

for $x[n] \circledast h_2[n] \circledast h_3[n]$, length $= (N_1 + N_3 - 1) + N_4 - 1 = N_1 + N_3 + N_4 - 2$

$\therefore$ for $y[n]$. length $= \max(N_1 + N_2 - 1,\ N_1 + N_3 + N_4 - 2)$

$\therefore$ according to (b), only when

$M \geqslant \max(N_1 + N_2 - 1,\ N_1 + N_3 + N_4 - 2)$

linear convolution equals to circular convolution

$\therefore$ minimum value of M is $\max(N_1 + N_2 - 1,\ N_1 + N_3 + N_4 - 2)$

iii. Step 1: zero-padding $x[n], h_1[n], h_2[n], h_3[n]$ to length M — $\theta(M)$

Step 2: compute FFT of each zero-padding signal

$\tilde{X}[k] = FFT(\tilde{x}[n])$, $\tilde{H}_1[k] = FFT(\tilde{h}_1[n])$,

$\tilde{H}_2[k] = FFT(\tilde{h}_2[n])$, $\tilde{H}_3[k] = FFT(\tilde{h}_3[n])$, — $\theta(M \log M)$

Step 3: compute DFT $\tilde{Y}[k] = \tilde{X}[k] \cdot \tilde{H}_1[k] + \tilde{X}[k] \cdot \tilde{H}_2[k] \cdot \tilde{H}_3[k]$ — $\theta(M)$

Step 4: compute inverse FFT of $\tilde{Y}[k]$

$y[n] = IFFT(\tilde{Y}[k])$ — $\theta(M \log M)$

$\therefore$ total complexity: $O(M \log M)$

2. **(20 pts) Fast DCT using FFT** Let $x[n]$ be a N-length sequence. The N-length DCT-II coefficients $C[k]$ of the sequence $x[n]$ are given by

$$C[k] = \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right] \qquad \text{for } k = 0, \ldots N-1.$$

A naive implementation of the above formula requires $\mathcal{O}(N^2)$ computations. We are interested in finding a computationally efficient way to compute $C[k]$ using FFT, which has a computational complexity of $\mathcal{O}(N \log N)$ for signals of length N. Consider the sequence $y[n]$ of length $2N$ defined as

$$y[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ x[2N-n-1] & N \le n \le 2N-1 \end{cases}.$$

Let $Y[k]$ be the $2N$-length DFT of $y[n]$.

(a) (10 pts)Prove that

$$Y[k] = 2e^{\frac{j\pi k}{2N}} \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right]$$

for $k = 0, \ldots, N-1$.

(a) $Y[k] = \sum_{n=0}^{2N-1} y[n] e^{-j(\frac{2\pi}{2N} \cdot k)n} = \sum_{n=0}^{2N-1} y[n] e^{-j(\frac{\pi}{N} k)n}$

$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi k}{N}n} + \sum_{n=N}^{2N-1} x[2N-n-1] e^{-j\frac{\pi k}{N}n}$

$\downarrow$ let $m = 2N-n-1$, $n = 2N-m-1$

$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi k}{N}n} + \sum_{m=0}^{N-1} x[m] e^{-j\frac{\pi k}{N}(2N-m-1)}$

$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi k}{N}n} + e^{-j\frac{\pi k}{N}(2N-1)} \sum_{m=0}^{N-1} x[m] e^{j\frac{\pi k}{N}m}$

$\because = e^{-j\frac{\pi k}{N}\cdot 2N} \cdot e^{-j\frac{\pi k}{N}(-1)}$

$= \underbrace{\cancel{e^{-j2\pi k}}}_{=1} \cdot e^{j\frac{\pi k}{N}}$

$= e^{j\frac{\pi k}{N}}$

let $m=n$

$\therefore = \sum_{n=0}^{N-1} x[n] \left(e^{-j\frac{\pi k}{N}n} + e^{j\frac{\pi k}{N}} e^{j\frac{\pi k}{N}n}\right)$

$= \sum_{n=0}^{N-1} x[n] e^{j\frac{\pi k}{2N}} \left(e^{-j\frac{\pi k}{2N}} e^{-j\frac{\pi k}{N}n} + e^{j\frac{\pi k}{2N}} e^{j\frac{\pi k}{N}n}\right)$

$= e^{j\frac{\pi k}{2N}} \sum_{n=0}^{N-1} x[n] \left(e^{-j\frac{\pi k}{N}(\frac{1}{2}+n)} + e^{j\frac{\pi k}{N}(\frac{1}{2}+n)}\right)$

$= e^{j\frac{\pi k}{2N}} \sum_{n=0}^{N-1} x[n] \cdot 2\cos\left[\frac{\pi k}{N}(\frac{1}{2}+n)\right]$

$= 2e^{j\frac{\pi k}{2N}} \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{N}(\frac{1}{2}+n)k\right]$

(b) (10 pts)Give an algorithm that takes $x[n]$ as input and computes $C[k]$ using $Y[k]$. What is the computational complexity of your algorithm? Express your answer using $\mathcal{O}(\cdot)$ notation. (You don't need to explain FFT implementation in your algorithm. Assume the FFT function is available.)

(b) Step 1: use input $x[n]$ (length = N) to construct $y[n]$ of length 2N

$$y[n] = \begin{cases} x[n] & , 0 \le n < N \\ x[2N-1-n] & , N \le n \le 2N \end{cases} \qquad \Theta(N)$$

Step 2: compute FFT of $y[n]$, $Y[k] = FFT(y[n])$ $\qquad \Theta(N\log N)$

Step 3: $\because$ according to (a) , $Y[k] = 2e^{j\frac{\pi k}{2N}} \cdot C[k]$

$\therefore$ For $k = 0, 1, \cdots, N-1$, compute $C[k] = \frac{1}{2} e^{-j\frac{\pi k}{2N}} Y[k]$ $\qquad \Theta(N)$

$\therefore$ total complexity : $\mathcal{O}(N\log N)$

3. **(30 pts)DFT and inverse DFT** If we denote the DFT operation by $\mathcal{F}$, and its inverse by $\mathcal{F}^{-1}$. That is, for any signal $x[n]$ of length N its DFT is $X[k] = \mathcal{F}\{x[n]\}$ and $x[n] = \mathcal{F}^{-1}\{X[k]\}$.

(a) (15 pts) Prove that

$$\mathcal{F}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{F}\{X^*[k]\}\right)^*, \tag{3}$$

and

$$x[n] = \frac{1}{N}\mathcal{F}\{X[\langle -k\rangle_N]\} \tag{4}$$

We saw in lecture that the complexity of computing $\mathcal{F}\{x[n]\}$ is $\mathcal{O}(N\log(N))$ by using FFT algorithms. What is the complexity of computing the inverse DFT, i.e., $\mathcal{F}^{-1}\{X[k]\}$?

1a) ① $\because F\{X^*[k]\} = \sum_{k=0}^{N-1} X^*[k]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$

$\therefore (F\{X^*[k]\})^* = \left(\sum_{k=0}^{N-1} X^*[k]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}\right)^* = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}\cdot k\cdot n}$

$\therefore \frac{1}{N}(F\{X^*[k]\})^* = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}\cdot k\cdot n}$

$= F^{-1}\{X[k]\}$

② $\because \frac{1}{N}F\{X[\langle -k\rangle_N]\} = \frac{1}{N}\sum_{k=0}^{N-1} X[\langle -k\rangle_N]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$ and $k\in[0,N-1]$

$\therefore \langle -k\rangle_N = N-k$

$\therefore$ let $m = \langle -k\rangle_N = N-k \in [0,N-1]$, $k = N-m$

$\therefore \frac{1}{N}F\{X[\langle -k\rangle_N]\} = \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{-j\frac{2\pi}{N}(N-m)\cdot n}$

$= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, \underbrace{\cancel{e^{-j2\pi n}}}_{=1}\, e^{j\frac{2\pi}{N}m\cdot n}$

$= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j\frac{2\pi}{N}m\cdot n}$

$= F^{-1}\{X[k]\}$

$= x[n]$

③ If we want to compute inverse DFT

Step 1: compute complex conjugate of $X[k]$ — $\theta(N)$

Step 2: compute DFT of $X^*[k]$ — $\theta(N)$

Step 3: compute complex conjugate of $F\{X^*[k]\}$

if use FFT $\longrightarrow \theta(N\log N)$, if don't use FFT $\longrightarrow \theta(N^2)$

Step 4: multiply $(F\{X^*[k]\})^*$ by $\frac{1}{N}$ — $\theta(N)$

$\therefore$ total complexity of computing $F^{-1}\{X[k]\}$ is $\underset{\text{use FFT}}{O(N\log N)}$ (/ $\underset{\text{don't use FFT}}{O(N^2)}$)

(b) (15 pts) Compute the N length signal corresponding to applying the DFT operation $L \geq 1$ times to a signal $x[n]$. That is, compute DFT of $x[n]$ then apply the DFT to the result, and then apply DFT to the result and so on. Your result should not have any unresolved sums, and should only depend on the signal $x[n]$, its DFT $X[k]$ and N Hint: first compute for the case $L = 2$, i.e., $\mathcal{F}\{\{\mathcal{F}\{x[n]\}\}$, and then generalize your result for larger L.

(b) when $L=1$, $F\{x[n]\} = X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\cdot k\cdot n}$

when $L=2$, $F\{F\{x[n]\}\} = Y[m] = \sum_{k=0}^{N-1} X[k] e^{-j\frac{2\pi}{N}\cdot m\cdot k}$

$$= \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\cdot k\cdot n}\right)\cdot e^{-j\frac{2\pi}{N}\cdot m\cdot k}$$

$$= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(n+m)}$$

i) when $\langle n+m\rangle_N = 0$, $n+m = aN$, $a \in \mathbb{Z}$, $n = aN-m = \langle -m\rangle_N$

$$\sum_{k=0}^{N-1} \underbrace{\cancel{e^{-j2\pi k\cdot a}}}_{=1} = N \sum_{n=0}^{N-1} x[n]$$

ii) when $\langle n+m\rangle_N \neq 0$,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(n+m)k} = \frac{1-e^{-j\frac{2\pi}{N}(n+m)\cdot N}}{1-e^{-j\frac{2\pi}{N}(n+m)}} = \frac{1-\overbrace{\cancel{e^{-j2\pi(n+m)}}}^{=1}}{1-e^{-j\frac{2\pi}{N}(n+m)}} = 0$$

$$\therefore Y[m] = \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(n+m)} = N x[\langle -m\rangle_N]$$

$$\therefore F^2\{x[n]\} = N x[\langle -n\rangle_N]$$

when $L=3$, $F\{Y[m]\} = Z[q] = F\{N x[\langle -m\rangle_N]\} = N F\{x[\langle -m\rangle_N]\}$

$$= N \sum_{m=0}^{N-1} x[\langle -m\rangle_N] e^{-j\frac{2\pi}{N}\cdot m\cdot q}$$

let $p = \langle -m\rangle_N$. $m = N - \langle p\rangle_N = N-p$

$$Z[q] = N \sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}(N-p)\cdot q}$$

$$= N \sum_{p=0}^{N-1} x[p] e^{j\frac{2\pi}{N}p\cdot q}$$

$$= N \left(\sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}p\cdot q}\right)^*$$

$$= N X^*[q]$$

$$\therefore F^3\{x[n]\} = N X^*[k]$$

when $L=4$, $F\{Z[q]\} = H[s] = F\{N X^*[q]\} = N F\{X^*[q]\}$

$$= N \sum_{q=0}^{N-1} X^*[q] e^{-j\frac{2\pi}{N}qs}$$

$$= N^2 \cdot \frac{1}{N} \sum_{q=0}^{N-1} X^*[q] e^{-j\frac{2\pi}{N}qs}$$

$$= N^2 x^*[s]$$

$$\therefore \boxed{F^4\{x[n]\} = N^2 x^*[n]}$$

$\vdots$

$\therefore$ if $x[n]$ is Complex signal, conjugation is non-negligible
the output cycles through $x[\langle -n\rangle_k]$, $x^*[n]$, $X[k]$, $X^*[\langle -k\rangle_N]$ ....
it's hard to find a loop

But if $x[n]$ is Real signal:
when $L=3$, $F^3\{x[n]\} = NX^*[k] = NX[\langle -k\rangle_N]$
when $L=4$, $F^4\{x[n]\} = N^2 x^*[n] = N^2 x[n]$

$\vdots$

$$\therefore \boxed{F^L\{x[n]\} = \begin{cases} N^{\frac{L}{2}} x[n] & , L \bmod 4 = 0 \\ N^{\frac{L-1}{2}} X[k] & , L \bmod 4 = 1 \\ N^{\frac{L}{2}} x[\langle -n\rangle_N] & , L \bmod 4 = 2 \\ N^{\frac{L-1}{2}} X[\langle -k\rangle_N] & , L \bmod 4 = 3 \end{cases}}$$