HW03 — Handout — Q1

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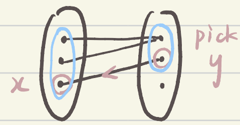
(a) proof: pick $y \in f(f^{-1}\{B\})$, $y = f(x)$, $x = f^{-1}(y)$

$\therefore x \in f^{-1}\{B\}$ (def of f)

$\therefore y \in B$ (def of f^{-1})

$\therefore f(f^{-1}\{B\}) \subset B$ (def of $\subset$)

QED



(b) proof: Claim 1: $f(f^{-1}\{B\}) \subset B$

(Question (a))

Claim 2: $B \subset f(f^{-1}\{B\})$

prof: pick $y \in B$

$\therefore y = f(x)$ (def of surjective)

$\therefore x = f^{-1}(y)$ (def of f^{-1})

$\therefore x \in f^{-1}\{B\}$ (def of f^{-1})

$\therefore y \in f(f^{-1}\{B\})$ (def of f)

$\therefore B \subset f(f^{-1}\{B\})$ (def of $\subset$)

$\therefore f(f^{-1}\{B\}) = B$ (def of $=$)

QED

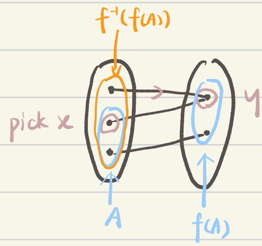
(c) proof: pick $x \in A$, $y = f(x)$, $x = f^{-1}\{y\}$

$\therefore y \in f(A)$ (def of f^{-1})

$\therefore x \in f^{-1}\{f(A)\}$ (def of f^{-1})

$\therefore A \subset f^{-1}\{f(A)\}$ (def of $\subset$)

QED



(d) proof: Claim 1: $A \subset f^{-1}\{f(A)\}$

(Question (c))

Claim 2: $f^{-1}\{f(A)\} \subset A$

prof: pick $x \in f^{-1}\{f(A)\}$, $x = f^{-1}(y)$

$\therefore f^{-1}(y) \in f^{-1}\{f(A)\}$

$\therefore y \in f(A)$

$\therefore x \in A$ (def of injective)

$\therefore f^{-1}\{f(A)\} \subset A$ (def of $\subset$)

$\therefore A = f^{-1}\{f(A)\}$ (def of $=$)

QED

(e) Claim 1: $f: 2^X \to 2^Y$ is injective

prof: Suppose $X_1 \in 2^X$, $X_2 \in 2^X$, $f(X_1) = Y_1$, $f(X_2) = Y_2$, $X_1 \cap X_2 = \phi$

$\therefore X_1 \subset X$, $X_2 \subset X$, and $X_1 \cap X_2 = \phi$ (def of 2^X)

$\because f: X \to Y$ is injective

$\therefore Y_1 \subset Y$, $Y_2 \subset Y$, and $Y_1 \cap Y_2 = \phi$

$\therefore Y_1 \in 2^Y$, $Y_2 \in 2^Y$, and $Y_1 \cap Y_2 = \phi$ (def of 2^X)

$\therefore f: 2^X \to 2^Y$ is injective

QED - Claim 1

Claim 2: $f: 2^X \to 2^Y$ is surjective

prof: Suppose $Y_1 \in 2^Y$

$\therefore Y_1 \subset Y$ (def of 2^X)

$\because f: X \to Y$ is surjective

$\therefore$ must have $f(X_1) = Y_1$, $X_1 \subset X$

$\therefore X_1 \in 2^X$ (def of 2^X)

$\therefore f: 2^X \to 2^Y$ is surjective

QED - Claim 2

$\therefore f: 2^X \to 2^Y$ is bijective

QED

HW03 — Handout — Q2

(a) $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\pi^{n+1}}{(-3)^n} \cdot \frac{(-3)^{n-1}}{n\pi^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{\pi}{-3} \right| = \frac{\pi}{3} > 1$

$\therefore$ series diverge

(b) $L = \lim_{n\to\infty} \left| \frac{(n+1)^2 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 2^{n+1}} \right| = \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{3} \right| = \frac{2}{3} < 1$

$\therefore$ series converge absolutely

(c) $L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\sqrt{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n\sqrt{n}} \right| = \lim_{n\to\infty} \left| (-1)\sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2} \right| = 1$

$\therefore$ test faild

(d) $L = \lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+2)4^{2(n+1)+1}} \cdot \frac{(n+1)4^{2n+1}}{10^n} \right| = \lim_{n\to\infty} \left| \frac{10}{4^2} \cdot \frac{n+1}{n+2} \right| = \frac{5}{8} < 1$

$\therefore$ series converge absolutely

HW03 — Handout — Q3

(a) use radio test:

$$L=\lim_{n\to\infty}\left|\frac{(n+2)(x-3)^{n+1}}{8^{n+1}}\cdot\frac{8^n}{(n+1)(x-3)^n}\right|=\lim_{n\to\infty}\left|\frac{x-3}{8}\cdot\frac{n+2}{n+1}\right|=\frac{|x-3|}{8}$$

$\because$ series converge

$\therefore \frac{|x-3|}{8}<1 \Rightarrow |x-3|<8 \Rightarrow -8<x-3<8$

$\therefore -5<x<11$

when $x=-5$: $\sum_{n=1}^{\infty}\frac{n+1}{8^n}(x-3)^n=\sum_{n=1}^{\infty}\frac{n+1}{8^n}(-8)^n=\sum_{n=1}^{\infty}(-1)^n(n+1)$

$\because \lim_{n\to\infty}(n+1)=\infty$

$\therefore$ series diverge

when $x=11$: $\sum_{n=1}^{\infty}\frac{n+1}{8^n}(x-3)^n=\sum_{n=1}^{\infty}\frac{n+1}{8^n}\cdot 8^n=\sum_{n=1}^{\infty}(n+1)$

$\therefore$ series diverge

$\therefore$ the interval of convergence is $(-5,11)$

(b) $L=\lim_{n\to\infty}\left|(n+1)!\left(\frac{x}{2}\right)^{n+1}\cdot\frac{1}{n!}\cdot\left(\frac{2}{x}\right)^n\right|=\lim_{n\to\infty}\left|(n+1)\cdot\frac{x}{2}\right|=\infty$

$\therefore$ series diverge

$\therefore$ series doesn't have the interval of convergence

(c) $L=\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\cdot\frac{5^n\cdot\sqrt{n}}{(2x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{2x-1}{5}\cdot\sqrt{\frac{n}{n+1}}\right|=\frac{|2x-1|}{5}$

$\because$ series converge

$\therefore \frac{|2x-1|}{5}<1 \Rightarrow |2x-1|<5 \Rightarrow -5<2x-1<5 \Rightarrow -4<2x<6$

$\therefore -2<x<3$

when $x=-2$: $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$

$\because \lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, $\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$

$\therefore$ series converge

when $x=3$: $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$

$\because \frac{1}{2}<1$

$\therefore$ series diverge

$\therefore$ the interval convergence for series is $[-2,3)$

HW03 — Handout — Q4

(a) $L = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$

$\therefore |a_n - L| = |\sqrt{n+1} - \sqrt{n} - 0| = |\sqrt{n+1} - \sqrt{n}| < 10^{-6}$

$\because \sqrt{n+1} > \sqrt{n}$

$\therefore \sqrt{n+1} - \sqrt{n} < 10^{-6}$

$\therefore (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) < 10^{-6}(\sqrt{n+1} + \sqrt{n})$

$\therefore n+1-n < 10^{-6}(\sqrt{n+1} + \sqrt{n})$

$\therefore \sqrt{n+1} + \sqrt{n} > 10^{6}$

$\therefore 2\sqrt{n+1} > 10^{6}$

$\therefore n+1 > 0.25 \cdot 10^{12}$

$\therefore n > 2.5 \times 10^{11} - 1$

$\therefore n_0 = 2.5 \times 10^{11}$

(b) $L = \lim_{n\to\infty} 2^{-n}\cos(n\pi) = \lim_{n\to\infty} 2^{-n} \cdot (-1)^n = \lim_{n\to\infty} (-\frac{1}{2})^n = 0$

$\therefore |a_n - L| = |2^{-n}\cos(n\pi) - 0| = |2^{-n} \cdot (-1)^n| = |(-\frac{1}{2})^n| = |\frac{1}{2^n}| < 10^{-6}$

$\therefore 2^n > 10^6$

$\therefore \lg(2^n) > \lg(10^6)$

$\therefore n \lg 2 > 6$

$\therefore n > \frac{6}{\lg 2} \approx 19.93$

$\because$ n is a positive integar

$\therefore n_0 = 20$

(c) $L = \lim_{n\to\infty} (1+\frac{2}{n})^n = e^{\lim_{n\to\infty} \ln(1+\frac{2}{n})^n} = e^{\lim_{n\to\infty} n\ln(1+\frac{2}{n})}$

$\because$ when $x \to 0$, $\ln(1+x) \sim x$

$\therefore$ when $x \to \infty$, $\ln(1+\frac{1}{x}) \sim \frac{1}{x}$

$\therefore L = e^{\lim\limits_{n\to\infty} n \cdot \frac{2}{n}} = e^2$

$\therefore |a_n - L| = |(1+\frac{2}{n})^n - e^2| < 10^{-6}$

$\therefore n_0 = 14603984$ (calculated by computer)

HW03 — Handout — Q5

① $A \times B = \{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}$

② $\because |A \times B| = 2 \times 3 = 6$

$\therefore |2^{A \times B}| = 2^6 = 64$

there are 64 elements in $2^{A \times B}$

③ $\mathcal{A}_1 = \{\phi, \{(a_1,b_1)\}, \{(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}, X\}$

$\mathcal{A}_2 = \{\phi, \{(a_1,b_1),(a_1,b_2)\}, \{(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}, X\}$

$\mathcal{A}_3 = \{\phi, \{(a_1,b_1),(a_2,b_1),(a_3,b_1)\}, \{(a_2,b_1),(a_2,b_2),(a_3,b_2)\}, X\}$

$\mathcal{A}_4 = \{\phi, \{(a_1,b_1)\}, \{(a_1,b_2)\}, \{(a_1,b_1),(a_1,b_2)\}, \{(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\},$

$\{(a_1,b_1),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}$

$\{(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}, X\}$

HW03 — Handout — Q6

(a) Proof: pick $(a,b) \in A \times B$

$\therefore a \in A$ and $b \in B$ (def of $\times$)

$\because A \subset X$ and $B \subset Y$

$\therefore a \in X$ and $b \in Y$ (def of $\subset$)

$\therefore (a,b) \in X \times Y$ (def of $\times$)

$\therefore A \times B \subset X \times Y$ (def of $\subset$)

QED

(b) Disprove: Suppose (x,y), $x \in B$, $x \notin A \cup C$, $y \in C$

① $\therefore (x,y) \in B \times C$ (def of $\times$)

$\therefore (x,y) \in (A \cup B) \times C$ (def of $\cup$)

② $\because x \notin A \cup C$

$\therefore (x,y) \notin (A \cup C) \times (B \cup C)$ (def of $\times$)

$\therefore (A \cup B) \times C \neq (A \cup C) \times (B \cup C)$ (def of $=$)

$\therefore$ this statement is false

QED