EE 510 HW #2

1ⓐ 2.3 #3

①

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \qquad E_{21} \begin{cases} R_1 : R_1 \\ R_2 : R_2 \quad 4R_1 \\ R_3 : R_3 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31} \begin{cases} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 + 2R_1 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$E_{32} \cdot \begin{cases} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - 2R_2 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore \quad E_{32} E_{31} E_{21} A = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad L = E_{32}^{-1} E_{31}^{-1} E_{21}^{-1}$$

$$\underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \underset{E_{31}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ +2 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = LU.$$

CHECK: $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

#25 . $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}}^{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

$M = [A \vdots b]$

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right]$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 3 \end{bmatrix} \qquad \begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - R_2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \longleftarrow$$

∴ The last column of the Echelon form gives a degenerate equation:

$0x + 0y + 0z = 3 \longrightarrow$ There is no solution.

☆

Changing b to b′ where $b' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we have .

$$Ax = b' : \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{b'}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

∴ $M' = [A \vdots b']$ and its Echelon form is as follows:

$$M' = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 3 \end{array}\right]$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \qquad \begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - R_2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftarrow$$

$$\Downarrow$$

$$\underbrace{0x + 0y + 0z = 0}$$

There is a solution.

1(b) 2·5 #25

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad M = [A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 - \tfrac{1}{2}R_1 \\ R_3 : R_3 - \tfrac{1}{2}R_1 \end{array} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - \tfrac{1}{3}R_2 \end{array} \left[\begin{array}{ccc|ccc} \textcircled{2} & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right]$$

∴ We have 3 pivots. This implies that the rows are linearly ind.
So A is INVERTIBLE. Let us find A^{-1}:

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \tfrac{3}{4}R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \qquad \begin{array}{l} R_1 : R_1 - R_3 \\ R_2 : R_2 - \tfrac{1}{2}R_3 \\ R_3 : R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : \tfrac{2}{3}R_2 \\ R_3 : R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \qquad \begin{array}{l} R_1 : R_1 - R_2 \\ R_2 : R_2 \\ R_3 : R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$\begin{array}{l} R_1 : \tfrac{1}{2}R_1 \\ R_2 : R_2 \\ R_3 : R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

CHECK:

$$A A^{-1} \qquad \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is INVERTIBLE

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad M = [B \,\vdots\, I]$$

$$M = \begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 + \frac{1}{2}R_1 \\ R_3 : R_3 + \frac{1}{2}R_2 \end{array} \begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 + R_2 \end{array} \left[\begin{array}{ccc:ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \multicolumn{3}{c:}{\underbrace{0 \quad\; 0 \quad\; 0}} & 1 & 1 & 1 \end{array}\right]$$

∴ B has linearly independent rows. So B is not INVERTIBLE.

#30

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

$\therefore$

$$\begin{matrix} R_1 : R_1 \\ R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \end{matrix} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \qquad \begin{matrix} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - R_2 \end{matrix} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

$\therefore$ For A to have linearly independent row, then we need $a \neq 0$ and $a-b \neq 0 \Rightarrow a \neq b$.

Let us find A^{-1}:

$$M \equiv [A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_1 : R_1 \\ R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \end{matrix} \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & 0 & -1 & 1 \end{array}\right] \qquad \begin{matrix} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - R_2 \end{matrix} \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$$\begin{matrix} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : \frac{1}{a-b} R_3 \end{matrix} \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right] \qquad \begin{matrix} R_1 : R_1 \\ R_2 : \frac{1}{a-b} R_2 \\ R_3 : R_3 \end{matrix} \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$$\begin{matrix} R_1 : R_1 - bR_3 \\ R_2 : R_2 \\ R_3 : R_3 \end{matrix} \left[\begin{array}{ccc|ccc} a & b & 0 & 1 & \frac{b}{a-b} & \frac{-b}{a-b} \\ 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right] \qquad \begin{matrix} R_1 : R_1 - bR_2 \\ R_2 : R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|ccc} a & 0 & 0 & \frac{a}{(a-b)} & 0 & \frac{-b}{(a-b)} \\ 0 & 1 & 0 & \frac{-1}{(a-b)} & \frac{1}{(a-b)} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{(a-b)} & \frac{1}{(a-b)} \end{array}\right]$$

$$\begin{matrix} R_1 : \frac{1}{a} R_1 \\ R_2 : R_2 \\ R_3 : R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{(a-b)} & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & -\frac{1}{(a-b)} & \frac{1}{(a-b)} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{(a-b)} & \frac{1}{(a-b)} \end{array}\right]$$

A^{-1}

$$A^{-1} = \begin{bmatrix} \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & -\frac{1}{a-b} & \frac{1}{(a-b)} \end{bmatrix}$$

$\therefore$ This exists if $a \neq 0$ and $a \neq b$.

$$C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 - \frac{c}{2} R_1 \\ R_3 : R_3 - 4R_1 \end{array} \begin{bmatrix} 2 & c & c \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 0 & 7 - 4c & c - 4c \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - \frac{(7-4c)}{c - \frac{c^2}{2}} R_2 \end{array} \begin{bmatrix} 2 & c & c \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 0 & 0 & c - 7 \end{bmatrix} = \begin{bmatrix} 2 & c & 0 \\ 0 & \frac{c}{2}(2 - c) & \frac{c}{2}(2 - c) \\ 0 & 0 & (c - 7) \end{bmatrix}$$

$\therefore$ If, A is not invertible, then one of the pivots is 0.

$\Rightarrow \frac{c}{2}(2 - c) = 0$ OR $(c - 7) = 7$

$\therefore$ For $c = 0$ or $c = 2$, or $c = 7$, A is not invertible.

2.6 $EA = u$ $A = \begin{bmatrix} 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$ $EA = u$.

$R_1 : R_1$
$R_2 : R_2$
$R_3 : R_3 - 3R_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\therefore\ EA = u \implies A = \underbrace{E^{-1}}_{L} u \qquad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = L$$

CHECK $A = Lu$

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \underset{u}{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}}$$

13 $A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$

$E_1 \begin{cases} R_1 : R_1 \\ R_2 : R_2 - R_1 \\ R_3 : R_3 - R_2 \\ R_4 : R_4 - R_3 \end{cases} \begin{bmatrix} a & a & a & a \\ 0 & (b-a) & (b-a) & (b-a) \\ 0 & (b-a) & (c-a) & (c-a) \\ 0 & (b-a) & (c-a) & (d-a) \end{bmatrix}$

$E_2 \begin{cases} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 - R_2 \\ R_4 : R_4 - R_2 \end{cases} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$

$E_3 \begin{cases} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 \\ R_4 : R_4 - R_3 \end{cases} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$

$$\therefore\ E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \implies E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \implies E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \implies E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U \quad \Rightarrow \quad \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} E_3 E_2 E_1}_{I} A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

$$L = \underset{E_1^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}} \underset{E_2^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \underset{E_3^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

To have four pivots, we need the following:

$a \neq 0$, $b \neq a$, $c \neq b$ and $d \neq c$.

(22) $A = uL$ $\qquad A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\left\{ \begin{array}{l} R_1 : R_1 - R_3 \\ R_2 : R_2 - R_3 \\ R_3 : R_3 \end{array} \right. \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad E_2 \left\{ \begin{array}{l} R_1 : R_1 - R_2 \\ R_2 : R_2 \\ R_3 : R_3 \end{array} \right. \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L}$$

$$\therefore \quad E_2 E_1 A = L \qquad E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} L$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{u}$$

<u>CHECK</u>:

$$uL = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A \checkmark$$

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2)

$Ax = b \qquad M = [A \,\vdots\, b]$

$$M = \begin{bmatrix} 2 & 4 & 1 & 3 & a \\ -3 & 1 & 2 & -2 & b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 + \frac{3}{2} R_1 \\ R_3 : R_3 + \frac{13}{2} R_1 \\ R_4 : R_4 - 6 R_1 \end{array} \begin{bmatrix} 2 & 4 & 1 & 3 & a \\ 0 & 7 & \frac{7}{2} & \frac{5}{2} & b + \frac{3a}{2} \\ 0 & -21 & -\frac{21}{2} & -\frac{15}{2} & c - \frac{13a}{2} \\ 0 & -14 & -7 & -5 & d - 6a \end{bmatrix}$$

$$\begin{array}{l} R_1 : R_1 \\ R_2 : R_2 \\ R_3 : R_3 + 3R_2 \\ R_4 : R_4 + 2R_2 \end{array} \begin{bmatrix} 2 & 4 & 1 & 3 & a \\ 0 & 7 & \frac{7}{2} & \frac{5}{2} & b + \frac{3a}{2} \\ 0 & 0 & 0 & 0 & c + 3b - 2a \\ 0 & 0 & 0 & 0 & d + 2b - 3a \end{bmatrix}$$

$\therefore \quad c + 3b = 2a \quad$ and $\quad d + 2b = 3a$.

∴ ① $b_{1j} = 0$ for all j where $1 < j \le n$ — Subclaim 1

② $b_{2j} = 0$ for all j where $2 < j \le n$ — subclaims 1 & 2.

$\vdots$

$b_{n-1,j} = 0$ for all j where $n-1 < j \le n$ —

∴ <u>QED claim</u> — Subclaims 1 and 2

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④ $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 9 \\ 2 & -1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$ $Ax = b$

What is x ?

$$M = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 3 & 8 & 9 & 10 \\ 2 & -1 & 2 & 2 \end{array}\right]$$

$R_1 : R_1$
$R_2 : R_2 - 3R_1$
$R_3 : R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 2 & 12 & -8 \\ 0 & -5 & 4 & -10 \end{array}\right]$$

$R_1 : R_1$
$R_2 : R_2$
$R_3 : R_3 + \frac{5}{2} R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 2 & 12 & -8 \\ 0 & 0 & 34 & -30 \end{bmatrix}$$

Backward Substitution :

$34x_3 = -30 \Rightarrow x_3 = -\frac{30}{34} = -\frac{15}{17}$.

$\therefore 2x_2 + 12x_3 = -8 \Rightarrow 2x_2 = -8 - 12 \times \left(-\frac{15}{17}\right)$

$\Rightarrow x_2 = -4 + 6 \times \frac{15}{17}$

$= -4 + \frac{90}{17} = -\frac{68}{17} + \frac{90}{17} = \frac{22}{17}$

$x_1 + 2x_2 - x_3 = 6$

$\therefore x_1 = 6 - 2x_2 + x_3 = 6 - 2\left[\frac{22}{17}\right] + -\left[\frac{30}{34}\right]$

$= 6 - \frac{88}{34} - \frac{30}{34} = 6 - \frac{118}{34}$

$= \frac{86}{34} = \frac{43}{17}$

$$\therefore \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix} = \frac{43}{17} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \frac{22}{17} \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix} - \frac{15}{17} \begin{bmatrix} -1 \\ 9 \\ 2 \end{bmatrix}$$

3) Let $\underset{n\times n}{A} = [a_{ij}]$ be LOWER TRIANGULAR such that $a_{ii} \neq 0$ for $1 \leq i \leq n$.

$\therefore$ A is invertible —— $a_{ii} \neq 0$ for $1 \leq i \leq n$

$\therefore$ There exists $A^{-1} = B = [b_{ij}]$ such that $AA^{-1} = AB = I$.

Claim: $b_{ij} = 0$ for all i such that $1 \leq i \leq j \leq n$.

Subclaim 1: $b_{ij} = 0$ for $j \geq 2$

$\therefore AB = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ $\therefore$ For $j \geq 2$, $I_{ij} = 0$

$$\Rightarrow 0 = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{11} b_{ij} + \sum_{k=2}^{n} a_{ik} b_{kj}$$

$$= a_{11} b_{ij} + \underbrace{\sum_{k=2}^{n} \overset{0}{\cancel{a_{1k}}} b_{kj}}_{0}$$ — A is lower triangular $\therefore a_{1k} = 0$ for $k > 1$

$$= a_{11} b_{ij}$$

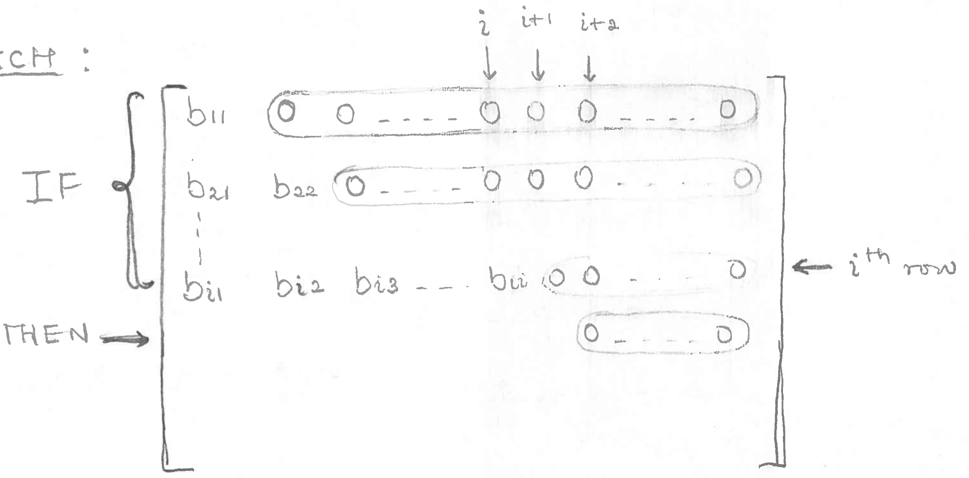
$\therefore b_{ij} = 0$ because $a_{11} \neq 0$ —— Assumption

$\therefore \forall j : 2 \leq j \leq n, \; b_{ij} = 0$

QED Subclaim 1.

Subclaim 2: Let $\underset{n\times n}{A}$ be lower triangular with inverse $\underset{n\times n}{B}$. If $b_{ij} = 0$ for all j such that $1 \leq i \leq j \leq n$, then $b_{i+1,m} = 0$ for all m such that $i+1 < m \leq n$.

SKETCH:



Proof: Let $m \in \{i+2, \dots, n\}$, and

$b_{ij} = 0$ for $j \in \{i+1, \dots, n\}$ —— Assumption

$$\therefore I_{i+1,m} = 0 = (AB)_{i+1,m}$$

$$= \sum_{k=1}^{n} a_{i+1,k} b_{km} = \sum_{k=1}^{i} a_{i+1,k} b_{km} + \sum_{k=i+1}^{n} a_{i+1,k} b_{km}$$

$$= \sum_{k=1}^{i} a_{i+1,k} b_{km} + \sum_{k=i+1}^{n} a_{i+1,k} b_{km}$$ — $m > k$; $b_{ij} = 0$ for $i < j$

$$= a_{i+1,i+1} b_{i+1,m} + \sum_{k=i+2}^{n} a_{i+1,k} b_{km}$$ —— A is lower triangular

$$= a_{i+1,i+1} b_{i+1,m} \Rightarrow a_{i+1,i+1} b_{i+1,m} = 0 \text{ and } a_{i+1,i+1} \neq 0$$

$\therefore b_{i+1,m} = 0$

$\therefore$ QED Subclaim 2

⑤ $Ax = b$

Case 1 : $Ax = b \neq 0$ (Non-homogeneous)

$\therefore S = \{x : Ax = b \neq 0\}$

For $x = \underline{0}$, $Ax = A\underline{0} = \underline{0} \neq b$

$\therefore \underline{0} \notin S$

So the solution set for $Ax = b \neq 0$ is NOT a subspace.

Case 2 : $Ax = b = \underline{0}$

$\therefore S = \{x ; Ax = b = \underline{0}\}$

① $A\underline{0} = \underline{0} \Rightarrow \underline{0} \in S$

② Let $u, v \in S$ and $a, b \in \mathbb{R}$.

$$\begin{aligned} A(au + bv) &= A(au) + A(bv) \\ &= aAu + bAv \\ &= a\underline{0} + b\underline{0} \\ &= \underline{0} \end{aligned}$$

$\therefore au + bv \in S$

So the solution set for $Ax = b = \underline{0}$ is a subspace

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⑥ Claim : The intersection of any number of subspaces of a vector space V is a subspace of V

Proof : By mathematical induction :

Basis Step : Let A_1 and A_2 are subspaces of V

$W = A_1 \cap A_2$

B_1
$\therefore \underline{0} \in A_1$ and $\underline{0} \in A_2$ —— A_1 is a subspace, B is a subspace
$\therefore \underline{0} \in A_1 \cap A_2$ —— definition of $\cap$

B_2
Let $u, v \in W$, and $a, b \in F$
$\therefore u, v \in A_1$ and $u, v \in A_2$ —— def of $\cap$, $W = A_1 \cap A_2$
$\therefore \underbrace{au + bv}_{} \in A_1$ and $\underbrace{au + bv}_{} \in A_2$ —— A_1 is a subspace
A_2 is a subspace.
$\therefore au + bv \in A_1 \cap A_2 = W$

$\therefore$ QED Basis step.

Induction Step :

Claim : Induction - Hypothesis

Assume that $A_1, A_2, \ldots, A_n$ are subspaces in V, $A_1 \cap A_2 \cap \ldots \cap A_n$ is a subspa of V, and A_{n+1} is a subspace of V. Then $A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}$ is a subspace of V.

Proof : $A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}$

$$= (A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}$$

$$= \underbrace{\left(\bigcap_{k=1}^{n} A_k \right)} \cap A_{n+1}$$

NOTE : $\bigcap_{k=1}^{n} A_k$ is a subspace of V — Assumption

A_{n+1} is a subspace of V — Assumption

$\therefore \left(\bigcap_{k=1}^{n} A_k \right) \cap A_{n+1}$ is a subspace of V — Basis step

QED Induction step.

$\therefore$ QED — Basis step & Induction step

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7) $S = \{\alpha_1, \ldots, \alpha_n\}$

① S is a subset of vector space V.

② $0 \in F$

$$\therefore \sum_{k=1}^{n} 0 \, \alpha_k = \underline{0} \in L(S)$$

For $a, b \in F$, $u, v \in L(S)$ where $u = \sum_{i=1}^{n} c_i \alpha_i$ and $v = \sum_{i=1}^{n} d_i \alpha_i$

$$\therefore a\left(\sum_{k=1}^{n} c_i \alpha_i \right) + b\left(\sum_{k=1}^{n} d_i \alpha_i \right) = \sum_{k=1}^{n} a c_i \alpha_i + \sum_{k=1}^{n} b d_i \alpha_i$$

$$= \sum_{k=1}^{n} \underbrace{(a c_i + b d_i)}_{\in F} \alpha_i \in L(S).$$

$\therefore$ L(S) is a subspace of V.