HW 01 — Handout

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Q1:

(a) $\because A=\{5,6,14,21\}$ $B=\{9,14\}$

$\therefore A\cup B=\{5,6,9,14,21\}$ $A\cap B=\{14\}$

① $f^{-1}(A)=\{z\in\mathbb{Z}: f(z)\in A\}=\{z\in\mathbb{Z}: z^2+5\in A\}$ (def of f^{-1}, f)

$$\therefore \begin{cases} z^2+5=5 \Rightarrow z=0 \\ z^2+5=6 \Rightarrow z=\pm 1 \\ z^2+5=14 \Rightarrow z=\pm 3 \\ z^2+5=21 \Rightarrow z=\pm 4 \end{cases}$$

$\therefore f^{-1}(A)=\{-4,-3,-1,0,1,3,4\}$

② $f^{-1}(B)=\{z\in\mathbb{Z}: f(z)\in B\}=\{z\in\mathbb{Z}: z^2+5\in B\}$

$$\therefore \begin{cases} z^2+5=9 \Rightarrow z=\pm 2 \\ z^2+5=14 \Rightarrow z=\pm 3 \end{cases}$$

$\therefore f^{-1}(B)=\{-3,-2,2,3\}$

③ $f^{-1}(A\cup B)=\{z\in\mathbb{Z}: f(z)\in A\cup B\}=\{z\in\mathbb{Z}: z^2+5\in A\cup B\}$

$$\therefore \begin{cases} z^2+5=5 \Rightarrow z=0 \\ z^2+5=6 \Rightarrow z=\pm 1 \\ z^2+5=9 \Rightarrow z=\pm 2 \\ z^2+5=14 \Rightarrow z=\pm 3 \\ z^2+5=21 \Rightarrow z=\pm 4 \end{cases}$$

$\therefore f^{-1}(A\cup B)=\{-4,-3,-2,-1,0,1,2,3,4\}$

④ $f^{-1}(A\cap B)=\{z\in\mathbb{Z}: f(z)\in A\cap B\}=\{z\in\mathbb{Z}: z^2+5\in A\cap B\}$

$\therefore z^2+5=14 \Rightarrow z=\pm 3$

$\therefore f^{-1}(A\cap B)=\{-3,3\}$

(b) Verify: According to (a)

① $f^{-1}(A)\cup f^{-1}(B)=\{-4,-3,-1,0,1,3,4\}\cup\{-3,-2,2,3\}$

$=\{-4,-3,-2,-1,0,1,2,3,4\}$

$\therefore f^{-1}(A)\cup f^{-1}(B)=f^{-1}(A\cup B)$

② $f^{-1}(A) \cap f^{-1}(B) = \{-4,-3,-1,0,1,3,4\} \cap \{-3,-2,2,3\} = \{-3,3\}$

$\therefore f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$

(c) Verify: $f(A) = \{z^2+5 : z \in A\} = \{30,41,201,446\}$

$f(B) = \{z^2+5 : z \in B\} = \{86,201\}$ (def of f)

$f(A \cup B) = \{z^2+5 : z \in A \cup B\} = \{30,41,86,201,446\}$

$f(A \cap B) = \{z^2+5 : z \in A \cap B\} = \{201\}$

$\therefore$ ① $f(A) \cup f(B) = \{30,41,201,446\} \cup \{86,201\} = \{30,41,86,201,446\}$

$\therefore f(A) \cup f(B) = f(A \cup B)$

② $f(A) \cap f(B) = \{30,41,201,446\} \cap \{86,201\} = \{201\}$

$\therefore f(A) \cap f(B) = f(A \cap B)$

Q2:

(a) Proof: $A \subset B$ IFF $A - B = \phi$

Claim 1: $A \subset B \rightarrow A - B = \phi$

prof: Suppose $A \subset B$

$\therefore \forall x \in A,\ x \in B$ (def of $\subset$)

$\therefore \forall x \in A,\ x \notin B^c$ (def of ^{c})

$\therefore A \cap B^c = \phi$ (def of $\cap$)

$\therefore A - B = \phi$ (def of $-$)

$\therefore A \subset B \rightarrow A - B = \phi$ QED

Claim 2: $A - B = \phi \rightarrow A \subset B$

prof: Suppose $A - B = \phi$

$\therefore A \cap B^c = \phi$ (def of $-$)

$\therefore \forall x \in A,\ x \notin B^c$ (def of $\cap$)

$\therefore \forall x \in A,\ x \in B$ (def of ^{c})

$\therefore A \subset B$ (def of $\subset$)

$\therefore A - B = \phi \rightarrow A \subset B$ QED

$\therefore A \subset B$ IFF $A - B = \phi$

QED

(b) Proof: $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

* Lemma 1: $P \& Q \longrightarrow P$ * Lemma 2: $P \longrightarrow P \vee Q$

prof:

P	Q	P	$\&$	Q	$\longrightarrow$	P	P	$\longrightarrow$	P	$\vee$	Q
1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	0	0	1	1	1	1	1	1	0
0	1	0	0	1	1	0	0	1	0	1	1
0	0	0	0	0	1	0	0	1	0	0	0

∴ valid   QED − Lemma 1 . Lemma 2

* Communitive Properties:

① $A \cup B = B \cup A$

Claim 1: $A \cup B \subseteq B \cup A$

prof: pick $x \in A \cup B$

∴ $x \in A$ OR $x \in B$ (def of $\cup$)

∴ $x \in B$ OR $x \in A$ (def of OR)

∴ $x \in B \cup A$ (def of $\cup$)

∴ $A \cup B \subseteq B \cup A$ (def of $\subseteq$)

Claim 2: $B \cup A \subseteq A \cup B$

prof: pick $x \in B \cup A$

∴ $x \in B$ OR $x \in A$

∴ $x \in A$ OR $x \in B$

∴ $x \in A \cup B$

∴ $B \cup A \subseteq A \cup B$

∴ $A \cup B = B \cup A$ QED − ①

② $A \cap B = B \cap A$

Claim 1: $A \cap B \subseteq B \cap A$

prof: pick $x \in A \cap B$

∴ $x \in A$ AND $x \in B$ (def of $\cap$)

∴ $x \in B$ AND $x \in A$ (def of AND)

∴ $x \in B \cap A$ (def of $\cap$)

∴ $A \cap B \subseteq B \cap A$ (def of $\subseteq$)

Claim 2: $B \cap A \subseteq A \cap B$

prof: pick $x \in B \cap A$

∴ $x \in B$ AND $x \in A$

∴ $x \in A$ AND $x \in B$

∴ $x \in A \cap B$

∴ $B \cap A \subseteq A \cap B$

∴ $A \cap B = B \cap A$ QED − ②

QED − Communitive Properties

* Associative Properties

① $(A \cup B) \cup C = A \cup (B \cup C)$

Claim 1: $(A \cup B) \cup C \subset A \cup (B \cup C)$

prof: pick $x \in (A \cup B) \cup C$

$\therefore x \in A \cup B$ OR $x \in C$ (def of $\cup$)

$\therefore x \in A$ OR $\underline{x \in B \text{ OR } x \in C}$ (def of $\cup$)

$\therefore x \in A$ OR $x \in B \cup C$ (def of $\cup$)

$\therefore x \in A \cup (B \cup C)$ (def of $\cup$)

$\therefore (A \cup B) \cup C \subset A \cup (B \cup C)$ (def of $\subset$)

Claim 2: $A \cup (B \cup C) \subset (A \cup B) \cup C$

prof: pick $x \in A \cup (B \cup C)$

$\therefore x \in A$ OR $x \in B \cup C$

$\therefore \underline{x \in A \text{ OR } x \in B}$ OR $x \in C$

$\therefore x \in A \cup B$ OR $x \in C$

$\therefore x \in (A \cup B) \cup C$

$\therefore A \cup (B \cup C) \subset (A \cup B) \cup C$

$\therefore (A \cup B) \cup C = A \cup (B \cup C)$ QED - ①

② $(A \cap B) \cap C = A \cap (B \cap C)$

Claim 1: $(A \cap B) \cap C \subset A \cap (B \cap C)$

prof: pick $x \in (A \cap B) \cap C$

$\therefore x \in A \cap B$ AND $x \in C$ (def of $\cap$)

$\therefore x \in A$ AND $\underline{x \in B \text{ AND } x \in C}$ (def of $\cap$)

$\therefore x \in A$ AND $x \in B \cap C$ (def of $\cap$)

$\therefore x \in A \cap (B \cap C)$ (def of $\cap$)

$\therefore (A \cap B) \cap C \subset A \cap (B \cap C)$ (def of $\subset$)

Claim 2: $A \cap (B \cap C) \subset (A \cap B) \cap C$

prof: pick $x \in A \cap (B \cap C)$

$\therefore x \in A$ AND $x \in B \cap C$

$\therefore x \in A$ AND $x \in B$ AND $x \in C$

$\therefore x \in A \cap B$ AND $x \in C$

$\therefore x \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) \subset (A \cap B) \cap C$

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$ QED – ②

QED – Associative Properties

* Distributive Properties

Lemma 3 : $(P \vee Q) \,\&\, (P \vee R) = P \vee (Q \,\&\, R)$

prof :

P	Q	R	(P	∨	Q)	&	(P	∨	R)	=	P	∨	(Q	&	R)
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	0	1	1	1	0	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1	1	1	0	0	1
0	0	1	0	0	0	0	0	1	1	1	0	0	0	0	1
1	1	0	1	1	1	1	1	1	0	1	1	1	1	0	0
0	1	0	0	1	1	0	0	0	0	1	0	0	1	0	0
1	0	0	1	1	0	1	1	1	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0

$\therefore$ valid

$\therefore$ QED

Lemma 3 : $(P \,\&\, Q) \vee (P \,\&\, R) = P \,\&\, (Q \vee R)$

prof :

P	Q	R	(P	&	Q)	∨	(P	&	R)	=	P	&	(Q	∨	R)
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	0	0	0	1	1	0	0	1	1	1
1	0	1	1	0	0	1	1	1	1	1	1	1	0	1	1
0	0	1	0	0	0	0	0	0	1	1	0	0	0	1	1
1	1	0	1	1	1	1	1	0	0	1	1	1	1	1	0
0	1	0	0	0	1	0	0	0	0	1	0	0	1	1	0
1	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0

$\therefore$ valid

$\therefore$ QED

① $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Claim 1: $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

prof: pick $x \in A \cup (B \cap C)$

$\therefore x \in A$ OR $x \in B \cap C$ (def of $\cup$)

$\therefore \underbrace{x \in A}_{P}$ OR $\underbrace{(x \in B \text{ AND } x \in C)}_{Q \ \& \ R}$ (def of $\cap$)

$\therefore (x \in A$ OR $x \in B)$ AND $(x \in A$ OR $x \in C)$ (Lemma 3: $(P \vee Q) \& (P \vee R) = P \vee (Q \& R)$)

$\therefore x \in A \cup B$ AND $x \in A \cup C$ (def of $\cup$)

$\therefore x \in (A \cup B) \cap (A \cup C)$ (def of $\cap$)

$\therefore A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ (def of $\subset$)

Claim 2: $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

prof: pick $x \in (A \cup B) \cap (A \cup C)$

$\therefore x \in A \cup B$ AND $x \in A \cup C$ (def of $\cap$)

$\therefore (x \in A$ OR $x \in B)$ AND $(x \in A$ OR $x \in C)$ (def of $\cup$)

$\therefore x \in A$ OR $(x \in B$ AND $x \in C)$ (Lemma 3: $(P \vee Q) \& (P \vee R) = P \vee (Q \& R)$)

$\therefore x \in A$ OR $x \in B \cap C$ (def of $\cap$)

$\therefore x \in A \cup (B \cap C)$ (def of $\cup$)

$\therefore (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ (def of $\subset$)

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ QED – ①

② $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Claim 1: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

prof: pick $x \in A \cap (B \cup C)$

$\therefore x \in A$ AND $x \in B \cup C$ (def of $\cap$)

$\therefore x \in A$ AND $(x \in B$ OR $x \in C)$ (def of $\cup$)

$\therefore (x \in A$ AND $x \in B)$ OR $(x \in A$ AND $x \in C)$ (Lemma 4: $(P \& Q) \vee (P \& R) = P \& (Q \vee R)$)

$\therefore x \in A \cap B$ OR $x \in A \cap C$ (def of $\cap$)

$\therefore x \in (A \cap B) \cup (A \cap C)$ (def of $\cup$)

$\therefore A\cap(B\cup C) \subset (A\cap B)\cup(A\cap C)$ (def of $\subset$)

Claim 2: $(A\cap B)\cup(A\cap C) \subset A\cap(B\cup C)$

prof: pick $x\in(A\cap B)\cup(A\cap C)$

$\therefore x\in A\cap B$ OR $x\in A\cap C$ (def of $\cup$)

$\therefore (x\in A$ AND $x\in B)$ OR $(x\in A$ AND $x\in C)$ (def of $\cap$)

$\therefore x\in A$ AND $(x\in B$ OR $x\in C)$ (Lemma 4: $(P\&Q)\vee(P\&R) = P\&(Q\vee C)$)

$\therefore x\in A$ AND $x\in B\cup C$ (def of $\cup$)

$\therefore x\in A\cap(B\cup C)$ (def of $\cap$)

$\therefore (A\cap B)\cup(A\cap C) \subset A\cap(B\cup C)$ (def of $\subset$)

$\therefore A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ QED — ②

QED — Distributive Porperties

* De Morgan's Law

① $(A\cup B)^c = A^c\cap B^c$

Claim 1: $(A\cup B)^c \subset A^c\cap B^c$

prof: pick $x\in(A\cup B)^c$

$\therefore x\notin A\cup B$ (def of ^{c})

$\therefore x\notin A$ AND $x\notin B$ (def of $\cup$)

$\therefore x\in A^c$ AND $x\in B^c$ (def of ^{c})

$\therefore x\in A^c\cap B^c$ (def of $\cap$)

$\therefore (A\cup B)^c \subset A^c\cap B^c$ (def of $\subset$)

Claim 2: $A^c\cap B^c \subset (A\cup B)^c$

prof: pick $x\in A^c\cap B^c$

$\therefore x\in A^c$ AND $x\in B^c$ (def of $\cap$)

$\therefore x\notin A$ AND $x\notin B$ (def of ^{c})

$\therefore x\notin A\cup B$ (def of $\cup$)

$\therefore x\in(A\cup B)^c$ (def of ^{c})

$\therefore A^c\cap B^c \subset (A\cup B)^c$ (def of $\subset$)

$\therefore (A\cup B)^c = A^c\cap B^c$ QED — ①

② $(A\cap B)^c = A^c \cup B^c$

Claim 1: $(A\cap B)^c \subset A^c \cup B^c$

prof: pick $x \in (A\cap B)^c$

$\therefore x \notin A\cap B$ (def of ^{c})

$\therefore x \notin A$ OR $x \notin B$ (def of $\cap$)

$\therefore x \in A^c$ OR $x \in B^c$ (def of ^{c})

$\therefore x \in A^c \cup B^c$ (def of $\cup$)

$\therefore (A\cap B)^c \subset A^c \cup B^c$ (def of $\subset$)

Claim 2: $A^c \cup B^c \subset (A\cap B)^c$

prof: pick $x \in A^c \cup B^c$

$\therefore x \in A^c$ OR $x \in B^c$ (def of $\cup$)

$\therefore x \notin A$ OR $x \notin B$ (def of ^{c})

$\therefore x \notin A\cap B$ (def of $\cap$)

$\therefore x \in (A\cap B)^c$ (def of ^{c})

$\therefore A^c \cup B^c \subset (A\cap B)^c$ (def of $\subset$)

$\therefore (A\cap B)^c = A^c \cup B^c$ QED – ②

QED – De Morgan's Law

Proof: $A\cap(B\triangle C) = (A\cap B)\triangle(A\cap C)$

Left side $A\cap(B\triangle C)$

$= A\cap((B-C)\cup(C-B))$ (def of $\triangle$)

$= A\cap((B\cap C^c)\cup(C\cap B^c))$ (def of $-$)

$= (A\cap(B\cap C^c))\cup(A\cap(C\cap B^c))$ (Distributive Properties)

Right side $(A\cap B)\triangle(A\cap C)$

$= ((A\cap B)-(A\cap C))\cup((A\cap C)-(A\cap B))$ (def of $\triangle$)

$= (\underline{(A\cap B)\cap(A\cap C)^c})\cup((A\cap C)\cap(A\cap B)^c)$ (def of $-$)

Left part $= (A\cap B)\cap(A^c\cup C^c)$ (De Morgan's Law)

$= ((A\cap B)\cap A^c)\cup((A\cap B)\cap C^c)$ (Distributive Properties)

$= (A \cap A^c \cap B) \cup (A \cap (B \cap C^c))$ (Communitive and Associative Pro.)

$= (\phi \cap B) \cup (A \cap (B \cap C^c))$ (Properties of $\cap$)

$= \phi \cup (A \cap (B \cap C^c))$ (Properties of $\cap$)

$= A \cap (B \cap C^c)$ (Properties of $\cup$)

Similarly, Right part $= A \cap (C \cap B^c)$

$\therefore$ Right side $= (A \cap (B \cap C^c)) \cup (A \cap (C \cap B^c))$

$\therefore$ Left side = Right side

$\therefore A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ QED

Q3:

(a) Proof: $f(A \cup B) = f(A) \cup f(B)$

Claim 1: $f(A \cup B) \subset f(A) \cup f(B)$

prof: pick $y = f(x) \in f(A \cup B)$

$\therefore x \in A \cup B$ (def of f)

$\therefore x \in A$ OR $x \in B$ (def of $\cup$)

$\therefore f(x) \in f(A)$ OR $f(x) \in f(B)$ (def of f)

$\therefore f(x) \in f(A) \cup f(B)$ (def of $\cup$)

$\therefore f(A \cup B) \subset f(A) \cup f(B)$ (def of $\subset$)

Claim 2: $f(A) \cup f(B) \subset f(A \cup B)$

prof: pick $y = f(x) \in f(A) \cup f(B)$

$\therefore f(x) \in f(A)$ OR $f(x) \in f(B)$ (def of $\cup$)

$\therefore x \in A$ OR $x \in B$ (def of f)

$\therefore x \in A \cup B$ (def of $\cup$)

$\therefore f(x) \in f(A \cup B)$ (def of f)

$\therefore f(A) \cup f(B) \subset f(A \cup B)$ (def of $\subset$)

$\therefore f(A \cup B) = f(A) \cup f(B)$ QED

(b) Proof $f^{-1}(A) \subset f^{-1}(B)$ if $A \subset B$

prof: Suppose $A \subset B$, pick $x \in f^{-1}(A)$

$\therefore f(x) \in A$ (def of f^{-1})

$\because A \subset B$

$\therefore f(x) \in B$ (def of $\subset$)

$\therefore x \in f^{-1}(B)$ (def of f^{-1})

$\therefore f^{-1}(A) \subset f^{-1}(B)$ (def of $\subset$) QED

(C) Proof: $f^{-1}(A \triangle B) = f^{-1}(A) \triangle f^{-1}(B)$

* Lemma: $(P \& Q) \& (Q \vee P) = P \& Q$

prof: by Truth-Table, propositional calculus

P	Q	(P	&	Q)	&	(Q	V	P)	=	P	&	Q
1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	0	0	1	0	1	1	0	1	0	0	1
1	0	1	0	0	0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0

$\therefore$ valid

$\therefore$ QED

Claim 1: $f^{-1}(A \triangle B) \subset f^{-1}(A) \triangle f^{-1}(B)$

prof: pick $x \in f^{-1}(A \triangle B)$

$\therefore f(x) \in A \triangle B$ (def of f^{-1})

$\therefore f(x) \in (A-B) \cup (B-A)$ (def of $\triangle$)

$\therefore f(x) \in (A-B)$ OR $f(x) \in (B-A)$ (def of $\cup$)

$\therefore f(x) \in A \cap B^c$ OR $f(x) \in B \cap A^c$ (def of ^{c})

Case 1: $f(x) \in A \cap B^c$ AND $f(x) \notin B \cap A^c$

$\therefore f(x) \in A \cap B^c$ AND $f(x) \in (B \cap A^c)^c$ (def of ^{c})

$\therefore f(x) \in A \cap B^c$ AND $f(x) \in B^c \cup A$ (De Morgan's Law)

$\therefore \underbrace{(f(x) \in A \text{ AND } f(x) \in B^c)}_{P \,\&\, Q}$ AND $\underbrace{(f(x) \in B^c \text{ OR } f(x) \in A)}_{Q \,\vee\, P}$ (def of $\cap, \cup$)

&

$\therefore f(x) \in A$ AND $f(x) \in B^c$ (Lemma: $(P \& Q) \& (Q \vee P) = P \& Q$)

$\therefore f(x) \in A$ AND $f(x) \notin B$ (def of ^{c})

$\therefore x \in f^{-1}(A)$ AND $x \notin f^{-1}(B)$ (def of f^{-1})

$\therefore x \in f^{-1}(A)$ AND $x \in (f^{-1}(B))^c$ (def of ^{c})

$\therefore x \in f^{-1}(A) \cap (f^{-1}(B))^c$ (def of $\cap$)

$\therefore x \in f^{-1}(A) - f^{-1}(B)$ (def of $-$)

$\therefore x \in (f^{-1}(A) - f^{-1}(B)) \cup (f^{-1}(B) - f^{-1}(A))$ (Lemma: $P \to P \vee Q$)

$\therefore x \in f^{-1}(A) \triangle f^{-1}(B)$ (def of $\triangle$)

Case 2: $f(x) \in B \cap A^c$ AND $f(x) \notin A \cap B^c$

$\therefore f(x) \in B \cap A^c$ AND $f(x) \in (A \cap B^c)^c$ (def of ^{c})

$\therefore f(x) \in B \cap A^c$ AND $f(x) \in A^c \cup B$ (De Morgan's Law)

$\therefore (\underbrace{f(x) \in B \text{ AND } f(x) \in A^c}_{P \ \& \ Q})$ AND $(\underbrace{f(x) \in A^c \text{ OR } f(x) \in B}_{Q \vee P})$ (def of $\cap, \cup$)

&

$\therefore f(x) \in B$ AND $f(x) \in A^c$ (Lemma: $(P \& Q) \& (Q \vee P) = P \& Q$

$\therefore f(x) \in B$ AND $f(x) \notin A$ (def of ^{c})

$\therefore x \in f^{-1}(B)$ AND $x \notin f^{-1}(A)$ (def of f^{-1})

$\therefore x \in f^{-1}(B)$ AND $x \in (f^{-1}(A))^c$ (def of ^{c})

$\therefore x \in f^{-1}(B) \cap (f^{-1}(A))^c$ (def of $\cap$)

$\therefore x \in f^{-1}(B) - f^{-1}(A)$ (def of $-$)

$\therefore x \in (f^{-1}(B) - f^{-1}(A)) \cup (f^{-1}(A) - f^{-1}(B))$ (Lemma: $P \to P \vee Q$)

$\therefore x \in (f^{-1}(A) - f^{-1}(B)) \cup (f^{-1}(B) - f^{-1}(A))$ (Communitive Properties)

$\therefore x \in f^{-1}(A) \triangle f^{-1}(B)$ (def of $\triangle$)

Case 3: $f(x) \in A \cap B^c$ AND $f(x) \in B \cap A^c$

$\therefore f(x) \in A$ AND $f(x) \in B^c$ AND $f(x) \in B$ AND $f(x) \in A^c$ (def of $\cap$)

$\therefore f(x) \in \phi$

$\therefore$ Case 3 doesn't exist

$\therefore x \in f^{-1}(A) \triangle f^{-1}(B)$

$\therefore f^{-1}(A \triangle B) \subset f^{-1}(A) \triangle f^{-1}(B)$ (def of $\subset$)

Claim 2: $f^{-1}(A) \triangle f^{-1}(B) \subset f^{-1}(A \triangle B)$

prof: pick $x \in f^{-1}(A) \triangle f^{-1}(B)$

$\therefore x \in (f^{-1}(A) - f^{-1}(B)) \cup (f^{-1}(B) - f^{-1}(A))$ (def of $\triangle$)

$\therefore x \in f^{-1}(A) - f^{-1}(B)$ OR $x \in f^{-1}(B) - f^{-1}(A)$ (def of $\cup$)

$\therefore x \in f^{-1}(A) \cap (f^{-1}(B))^c$ OR $x \in f^{-1}(B) \cap (f^{-1}(A))^c$ (def of $-$)

Case 1: $x \in f^{-1}(A) \cap (f^{-1}(B))^c$ AND $x \notin f^{-1}(B) \cap (f^{-1}(A))^c$

$\therefore (x \in f^{-1}(A)$ AND $x \in (f^{-1}(B))^c)$ AND $(x \notin f^{-1}(B)$ OR $x \in (f^{-1}(A))^c)$

(def of $\cap$)

$\therefore (x \in f^{-1}(A)$ AND $x \in (f^{-1}(B))^c)$ AND $(x \in (f^{-1}(B))^c$ OR $x \in f^{-1}(A))$

(def of ^{c})

$\therefore x \in f^{-1}(A)$ AND $x \in (f^{-1}(B))^c$ (Lemma: $(P \& Q) \& (Q \vee P) = P \& Q$)

$\therefore x \in f^{-1}(A)$ AND $x \notin f^{-1}(B)$ (def of ^{c})

$\therefore f(x) \in A$ AND $f(x) \notin B$ (def of f^{-1})

$\therefore f(x) \in A$ AND $f(x) \in B^c$ (def of ^{c})

$\therefore f(x) \in A \cap B^c$ (def of $\cap$)

$\therefore f(x) \in A - B$ (def of $-$)

$\therefore f(x) \in (A-B) \cup (B-A)$ (Lemma: $P \to P \vee Q$)

$\therefore f(x) \in A \triangle B$ (def of $\triangle$)

$\therefore x \in f^{-1}(A \triangle B)$ (def of f^{-1})

Case 2: $x \in f^{-1}(B) \cap (f^{-1}(A))^c$ AND $x \notin f^{-1}(A) \cap (f^{-1}(B))^c$

$\therefore (x \in f^{-1}(B)$ AND $x \in (f^{-1}(A))^c)$ AND $(x \notin f^{-1}(A)$ OR $x \notin (f^{-1}(B))^c)$

$\therefore (x \in f^{-1}(B)$ AND $x \in (f^{-1}(A))^c)$ AND $(x \in (f^{-1}(A))^c$ OR $x \in f^{-1}(B))$

$\therefore x \in f^{-1}(B)$ AND $x \in (f^{-1}(A))^c$

$\therefore x \in f^{-1}(B)$ AND $x \notin f^{-1}(A)$

$\therefore f(x) \in B$ AND $f(x) \notin A$

$\therefore f(x) \in B$ AND $f(x) \in A^c$

$\therefore f(x) \in B \cap A^c$

$\therefore f(x) \in B - A$

$\therefore f(x) \in (B-A) \cup (A-B)$

$\therefore f(x) \in (A-B) \cup (B-A)$ (Communitive Properties)

$\therefore f(x) \in A \triangle B$

$\therefore x \in f^{-1}(A \triangle B)$

Case 3: $x \in f^{-1}(A) \cap (f^{-1}(B))^c$ AND $x \in f^{-1}(B) \cap (f^{-1}(A))^c$

$\therefore x \in f^{-1}(A)$ AND $x \in (f^{-1}(B))^c$ AND $x \in f^{-1}(B)$ AND $x \in (f^{-1}(A))^c$

$\therefore x \in \phi$

$\therefore$ Case 3 doesn't exist

$\therefore x \in f^{-1}(A \triangle B)$

$\therefore f^{-1}(A) \triangle f^{-1}(B) \subset f^{-1}(A \triangle B)$ (def of $\subset$)

$\therefore f^{-1}(A \triangle B) = f^{-1}(A) \triangle f^{-1}(B)$ QED

Q4:

① the power set $2^X = \{\phi, \{N\}, \{W\}, \{S\}, \{E\}, \{N,W\}, \{N,S\}, \{N,E\}, \{W,S\}, \{W,E\}, \{S,E\}, \{N,W,S\}, \{N,W,E\}, \{N,S,E\}, \{W,S,E\}, X\}$

② $\because$ Sample Space X contains four elements : $|X| = 4$

$\therefore$ power set 2^X contains 2^4 elements (subsets) : $|2^X| = 2^4 = 16$

$\therefore$ there are 2^{16} set collections $\mathcal{A} \in 2^X$

③ $\mathcal{Q}_1 = \{\phi, X\}$

$\mathcal{Q}_2 = \{\phi, \{N\}, \{W,S,E\}, X\}$

$\mathcal{Q}_3 = \{\phi, \{W\}, \{N,S,E\}, X\}$

$\mathcal{Q}_4 = \{\phi, \{S\}, \{N,W,E\}, X\}$

$\mathcal{Q}_5 = \{\phi, \{E\}, \{N,W,S\}, X\}$

$\mathcal{Q}_6 = \{\phi, \{N,W\}, \{S,E\}, X\}$

$\mathcal{Q}_7 = \{\phi, \{N,S\}, \{W,E\}, X\}$

$\mathcal{Q}_8 = \{\phi, \{N,E\}, \{W,S\}, X\}$

$\mathcal{A}_9 = \{\phi, \{N\}, \{W\}, \{N,W\}, \{S,E\}, \{N,S,E\}, \{W,S,E\}, X\}$

$\mathcal{A}_{10} = \{\phi, \{N\}, \{S\}, \{N,S\}, \{W,E\}, \{N,W,E\}, \{W,S,E\}, X\}$

$\mathcal{A}_{11} = \{\phi, \{N\}, \{E\}, \{N,E\}, \{W,S\}, \{N,W,S\}, \{W,S,E\}, X\}$

$\mathcal{A}_{12} = \{\phi, \{W\}, \{S\}, \{W,S\}, \{N,E\}, \{N,W,E\}, \{N,S,E\}, X\}$

$\mathcal{A}_{13} = \{\phi, \{W\}, \{E\}, \{W,E\}, \{N,S\}, \{N,W,S\}, \{N,S,E\}, X\}$

$\mathcal{A}_{14} = \{\phi, \{S\}, \{E\}, \{S,E\}, \{N,W\}, \{N,W,S\}, \{N,W,E\}, X\}$

$\mathcal{A}_{15} = \{\phi, \{N\}, \{W\}, \{S\}, \{E\}, \{N,W\}, \{N,S\}, \{N,E\}, \{W,S\}, \{W,E\},$
$\{S,E\}, \{N,W,S\}, \{N,W,E\}, \{N,S,E\}, \{W,S,E\}, X\} = 2^X$

$\therefore$ there are 15 S.A. we can define on X

④ e.g. $\mathcal{A} = \{\phi, \{S\}, X\}$

$\underline{I}$: Sigma-Algebra

$\underline{M}$: $\mathcal{A} \subset 2^X$ is a S.A. IFF $\mathcal{A}$ is CUT:

C: complement : $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

U: countable union: $A_1 \in \mathcal{A}, A_2 \in \mathcal{A}, \cdots A_n \in \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

T: total set : $X \in \mathcal{A}$

$\underline{A}$: $\underline{\underline{C}}$: $\phi^c = X \in \mathcal{A}$

$\{S\}^c = \{N, W, T\} \notin \mathcal{A}$

$\therefore \mathcal{A}$ is not under closed complement, because $\{S\}^c \notin \mathcal{A}$

$\underline{\underline{U}}$: $\phi \cup A = A \in \mathcal{A}$

$X \cup A = X \in \mathcal{A}$

$\therefore \mathcal{A}$ is under countable union

$\underline{\underline{T}}$: $X \in \mathcal{A}$

$\underline{C}$: $\therefore \mathcal{A}$ is $\cancel{C}UT$

$\therefore \mathcal{A}$ is not a Sigma-Algebra

* If we add $\{S\}^c = \{N, W, T\}$

$\mathcal{A}' = \{\phi, \{S\}, \{N, W, T\}, X\}$

$\underline{A}$: $\underline{\underline{C}}$: $\phi^c = X \in \mathcal{Q}'$

$\{s\}^c = \{N, W, T\} \in \mathcal{Q}'$

$\therefore \mathcal{Q}'$ is under closed complement

$\underline{\underline{U}}$: $\phi \cup A = A \in \mathcal{Q}'$

$\{s\} \cup \{N, W, T\} = \{N, W, S, T\} \in \mathcal{Q}'$

$X \cup A = X \in \mathcal{Q}'$

$\therefore \mathcal{Q}'$ is under closed countable union

$\underline{\underline{T}}$: total set $X \in \mathcal{Q}'$

$\underline{C}$: $\therefore \mathcal{Q}'$ is CUT

$\therefore \mathcal{Q}'$ is a S.A.

$\therefore \mathcal{Q}'$ is the minimal augment of $\mathcal{Q}$

Q5:

① the power set $2^X = \{\phi, \{w\}, \{x\}, \{y\}, \{z\}, \{w,x\}, \{w,y\}, \{w,z\}, \{x,y\}, \{x,z\},$
$\{y,z\}, \{w,x,y\}, \{w,x,z\}, \{w,y,z\}, \{x,y,z\}, X\}$

② $\because$ Sample Space X contains four elements : $|X| = 4$

$\therefore$ power set 2^X contains 2^4 elements (subsets) : $|2^X| = 2^4 = 16$

$\therefore$ there are 2^{16} set collections $\mathcal{A} \in 2^X$

③ e.g.1: $\mathcal{Q}_1 = \{\phi, \{w\}, \{x\}, X\}$

$\underline{I}$: Sigma-Algebra

$\underline{M}$: $\mathcal{Q} \subset 2^X$ is a S.A. IFF $\mathcal{Q}$ is CUT :

C : complement : $A \in \mathcal{Q} \Rightarrow A^c \in \mathcal{Q}$

U : countable union : $A_1 \in \mathcal{Q}, A_2 \in \mathcal{Q}, \cdots A_n \in \mathcal{Q} \Rightarrow \bigcup_{k=1}^{n} A_k \in \mathcal{Q}$

T : total set $X \in \mathcal{Q}$

$\underline{A}$: $\underline{\underline{C}}$: $\phi^c = X \in \mathcal{Q}_1$

$\{w\}^c = \{x, y, z\} \notin \mathcal{Q}_1$

$\{x\}^c = \{w, y, z\} \notin \mathcal{Q}_1$

$\therefore \mathcal{Q}_1$ is not under closed complement,
because $\{w\}^c \notin \mathcal{Q}_1$ and $\{x\}^c \notin \mathcal{Q}_1$

$\underline{\underline{U}}$: $\phi \cup A = A \in \mathcal{Q}_1$
$\{w\} \cup \{x\} = \{w, x\} \notin \mathcal{Q}_1$
$X \cup A = A \in \mathcal{Q}_1$
$\therefore \mathcal{Q}_1$ is not under closed countable union
because $\{w\} \cup \{x\} \notin \mathcal{Q}_1$

$\underline{I}$: total set $X \in \mathcal{Q}_1$

$\underline{C}$: $\therefore \mathcal{Q}_1$ is $\cancel{C}\cancel{U}T$
$\therefore \mathcal{Q}_1$ is not a Sigma-Algebra

* If we add $\{w, x\}, \{y, z\}, \{w, y, z\}, \{x, y, z\}$
$\mathcal{Q}_1' = \{\phi, \{w\}, \{x\}, \{w, x\}, \{y, z\}, \{w, y, z\}, \{x, y, z\}, X\}$

$\underline{A}$: $\underline{\underline{C}}$: $\phi^c = X \in \mathcal{Q}_1'$
$\{w\}^c = \{x, y, z\} \in \mathcal{Q}_1'$
$\{x\}^c = \{w, y, z\} \in \mathcal{Q}_1'$
$\{w, x\}^c = \{y, z\} \in \mathcal{Q}_1'$
$\therefore \mathcal{Q}_1'$ is under closed complement

$\underline{\underline{U}}$: $\phi \cup A = A \in \mathcal{Q}_1'$
$\{w\} \cup \{x\} = \{w, x\} \in \mathcal{Q}_1'$
$\{w\} \cup \{y, z\} = \{w, y, z\} \in \mathcal{Q}_1'$
$\{w\} \cup \{x, y, z\} = \{w, x, y, z\} = X \in \mathcal{Q}_1'$
$\{x\} \cup \{y, z\} = \{x, y, z\} \in \mathcal{Q}_1'$
$\{x\} \cup \{w, y, z\} = \{w, x, y, z\} = X \in \mathcal{Q}_1'$
$\{w, x\} \cup \{y, z\} = \{w, x, y, z\} = X \in \mathcal{Q}_1'$
$\{w, x\} \cup \{w, y, z\} = \{w, x, y, z\} = X \in \mathcal{Q}_1'$
$\{w, x\} \cup \{x, y, z\} = \{w, x, y, z\} = X \in \mathcal{Q}_1'$

$\{w,y,z\} \cup \{x,y,z\} = \{w,x,y,z\} = X \in \mathcal{Q}_1'$

$X \cup A = X \in \mathcal{Q}_1'$

$\therefore \mathcal{Q}_1'$ is under closed countable union

$\underline{I}$: total set $X \in \mathcal{Q}_1'$

$\underline{C}$: $\therefore \mathcal{Q}_1'$ is CUT

$\therefore \mathcal{Q}_1'$ is a Sigma-Algebra

$\therefore \mathcal{Q}_1'$ is the minimal augment of $\mathcal{Q}_1$

e.g.2: $\mathcal{Q}_2 = \{\phi, \{w,x\}, \{y,z\}\}$

Similarly, $\mathcal{Q}_2$ is not a S.A.

Because: $\underline{C}$: $\phi^c = X \notin \mathcal{Q}_2$ X

$\underline{U}$: $\{w,x\} \cup \{y,z\} = \{w,x,y,z\} = X \notin \mathcal{Q}_2$ X

$\underline{I}$: total set $X \notin \mathcal{Q}_2$ X

$\therefore \mathcal{Q}_2$ is $\cancel{CUT}$, $\mathcal{Q}_2$ is not a S.A.

If we add $X = \{w,x,y,z\}$

$\therefore \mathcal{Q}_2' = \{\phi, \{w,x\}, \{y,z\}, X\}$

$\therefore \mathcal{Q}_2'$ is CUT, $\mathcal{Q}_2'$ is a S.A.

and $\mathcal{Q}_2'$ is the minimal augment of $\mathcal{Q}_2$

e.g.3: $\mathcal{Q}_3 = \{\{w\}, \{x,y,z\}, X\}$

Similarly, $\mathcal{Q}_3$ is not a S.A.

Because: $\underline{C}$: $X^c = \phi \notin \mathcal{Q}_3$ X

$\underline{U}$: $\mathcal{Q}_3$ is under closed countable union V

$\underline{I}$: total set $X \in \mathcal{Q}_3$ V

$\therefore \mathcal{Q}_3$ is $\cancel{C}$UT, $\mathcal{Q}_3$ is not a S.A.

If we add ϕ

$\therefore \mathcal{Q}_3' = \{\phi, \{w\}, \{x,y,z\}, X\}$

$\therefore \mathcal{Q}_3'$ is CUT, $\mathcal{Q}_3'$ is a S.A.

and $\mathcal{Q}_3'$ is the minimal agument of $\mathcal{Q}_3$

e.g.4: $\mathcal{Q}_4 = \{\phi, \{w\}, \{z\}, \{x,y\}, \{w,x,y\}, \{x,y,z\}, X\}$

Similarly, $\mathcal{Q}_4$ is not a S.A.

Because: $\underline{\underline{C}}$: $\{x\}^c = \{w,y,z\} \notin \mathcal{Q}_4$

$\{x,y\}^c = \{w,z\} \notin \mathcal{Q}_4$ ×

$\{w,x,y\}^c = \{z\} \notin \mathcal{Q}_4$

$\underline{\underline{U}}$: $\{w\} \cup \{x\} = \{w,x\} \notin \mathcal{Q}_4$ ×

$\underline{\underline{T}}$: total set $X \in \mathcal{Q}_4$

$\therefore \mathcal{Q}_4$ is $\cancel{C}UT$, $\mathcal{Q}_4$ is not a S.A.

If we add $\{z\}, \{y\}, \{w,x\}, \{w,y\}, \{w,z\}, \{x,z\}, \{y,z\}, \{w,y,z\}, \{w,x,z\}$

$\therefore \mathcal{Q}_4' = \{\phi, \{w\}, \{x\}, \{y\}, \{z\}, \{w,x\}, \{w,y\}, \{w,z\}, \{x,y\}, \{x,z\}, \{y,z\},$
$\{w,x,y\}, \{w,x,z\}, \{w,y,z\}, \{x,y,z\}, X\} = 2^X$

$\therefore \mathcal{Q}_4'$ is CUT, $\mathcal{Q}_4'$ is a S.A.

and $\mathcal{Q}_4'$ is the minimal agument of $\mathcal{Q}_4$