HW06 — Q1

14 By applying row operations to produce an upper triangular U, compute

$$\det\begin{bmatrix}1&2&3&0\\2&6&6&1\\-1&0&0&3\\0&2&0&7\end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}.$$

(1) Book Set 5.1 #14

(a)

$$\det\begin{bmatrix}1&2&3&0\\2&6&6&1\\-1&0&0&3\\0&2&0&7\end{bmatrix} \underset{R_3=R_3+R_1}{\overset{R_2=R_2-2R_1}{=\!=\!=}} \det\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&2&3&3\\0&2&0&7\end{bmatrix}$$

$$\underset{R_4=R_4-R_2}{\overset{R_3=R_3-R_2}{=\!=\!=}} \det\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&0&3&2\\0&0&0&6\end{bmatrix} = 1\times2\times3\times6=36$$

(b)

$$\det\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix} \overset{R_2=R_2+\frac{1}{2}R_1}{=\!=\!=} \det\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

$$\overset{R_3=R_3+\frac{2}{3}R_2}{=\!=\!=} \det\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&-1&2\end{bmatrix} \overset{R_4=R_4+\frac{3}{4}R_3}{=\!=\!=} \det\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&0&\frac{5}{4}\end{bmatrix}$$

$$= 2\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}$$

$$=5$$

16 Find the determinants of a rank one matrix and a skew-symmetric matrix:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}.$$

(2) Book Set 5.1 #16

① $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 5 \\ 2 & -8 & 10 \\ 3 & -12 & 15 \end{bmatrix}$

$\therefore \det A = \det \begin{bmatrix} 1 & -4 & 5 \\ 2 & -8 & 10 \\ 3 & -12 & 15 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \det \begin{bmatrix} 1 & -4 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

② $\det A = \det \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = (-1) \times 1 \times \det \begin{bmatrix} -1 & 4 \\ -3 & 0 \end{bmatrix} + 3 \times \det \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix}$

$= (-1) \times 12 + 3 \times 4$

$= 0$

18 Use row operations to show that the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{bmatrix} = (b-a)(c-a)(c-b).$$

(3) Book Set 5.1 #18

$$\det\begin{bmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{bmatrix} \underset{R_3=R_3-R_1}{\overset{R_2=R_2-R_1}{=\!=\!=}} \det\begin{bmatrix}1 & a & a^2\\0 & b-a & b^2-a^2\\0 & c-a & c^2-a^2\end{bmatrix}$$

$$\overset{R_3=R_3-\frac{c-a}{b-a}R_2}{=\!=\!=} \det\begin{bmatrix}1 & a & a^2\\0 & b-a & b^2-a^2\\0 & 0 & c^2-a^2-\frac{(b^2-a^2)(c-a)}{(b-a)}\end{bmatrix}$$

$$= 1\times(b-a)\times\left[\underline{(c+a)}(c-a)-\underline{(b+a)}(c-a)\right]$$

$$= (b-a)(c-a)(c-b)$$

QED

24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

(4) Book Set 5.1 #24

① $\det L = \det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = 1 \times 1 \times 1 = 1$

② $\det U = \det \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = 3 \times 2 \times (-1) = -6$

③ $\det A = \det \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = 3 \det \begin{bmatrix} 8 & 7 \\ 5 & -9 \end{bmatrix} - 3 \det \begin{bmatrix} 6 & 7 \\ -3 & -9 \end{bmatrix} + 4 \det \begin{bmatrix} 6 & 8 \\ -3 & 5 \end{bmatrix}$

$= 3(-72-35) - 3(-54+21) + 4(30+24)$

$= -321 + 99 + 216$

$= -6$

④ $\because U^{-1}L^{-1} = (LU)^{-1} = A^{-1}$

$\therefore \det(U^{-1}L^{-1}) = \det A^{-1} = (\det A)^{-1} = -\frac{1}{6}$

⑤ $\det(U^{-1}L^{-1}A) = \det(U^{-1}L^{-1}) \cdot \det A = (-\frac{1}{6}) \times (-6) = 1$

HW06 — Q2

2 Compute the determinants of A, B, C, D. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

(1) Book Set 5.2 #2

① $\det A = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = -1-1 = -2$

$\because \det A \neq 0$

$\therefore$ the colums of A are independent

② $\det B = \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$

$= 1(45-48) - 2(36-42) + 3(32-35)$

$= -3 + 12 - 9 = 0$

$\because \det B = 0$

$\therefore$ the colums of B are dependent

③ $\det C = \det \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = \det A \cdot \det A = (-2)^2 = 4$

$\because \det C \neq 0$

$\therefore$ the colums of C are independent

④ $\det D = \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det A \cdot \det B = (-2) \times 0 = 0$

$\because \det D = 0$

$\therefore$ the colums of D are dependent

3 Show that $\det A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1?
What is the rank of A?
What are the 6 terms in $\det A$?

(2) Book Set 5.2 #3

$$\det A = \det \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix} = \prod_{i=1}^{3} a_{ii} = x \cdot 0 \cdot x = 0$$

QED

① $M_{11} = \det \begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix} = 0$

$M_{12} = (-1) \det \begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix} = 0$

$M_{13} = \det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

② $\because A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ has two pivot variables

$\therefore \operatorname{rank} A = 2$

③ $\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma^1} \cdots a_{n\sigma^n}$

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

$\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$= x \cdot 0 \cdot x - x \cdot x \cdot 0 - x \cdot 0 \cdot x - x \cdot 0 \cdot 0 + x \cdot x \cdot 0 + x \cdot 0 \cdot 0$

$= 0 - 0 - 0 - 0 + 0 + 0$

$= 0$

Problems 1–5 are about Cramer's Rule for $x = A^{-1}b$.

1 Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

(3) Book Set 5.3 #1

(a) $$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \det A = \det\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = 8-5=3$$

$$\therefore x_1 = \frac{\det\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}}{\det A} = \frac{-6}{3} = -2$$

$$x_2 = \frac{\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}{\det A} = \frac{3}{3} = 1$$

(b) $$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \det A = \det\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 2\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= 6-2=4$$

$$\therefore x_1 = \frac{\det\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}}{\det A} = \frac{\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}{\det A} = \frac{3}{4}$$

$$x_2 = \frac{\det\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}}{\det A} = \frac{-\det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}{\det A} = -\frac{2}{4} = -\frac{1}{2}$$

$$x_3 = \frac{\det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}}{\det A} = \frac{\det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}{\det A} = \frac{1}{4}$$

Problems 6–15 are about $A^{-1} = C^{\mathrm{T}} / \det A$. Remember to transpose C.

6 Find A^{-1} from the cofactor formula $C^{\mathrm{T}} / \det A$. Use symmetry in part (b).

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(4) Book Set 5.3 #6

(a) $$\det A = \det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 3 & 0 \\ 7 & 1 \end{bmatrix} = 3$$

$$C \equiv [C_{ij}] = (-1)^{i+j} [m_{ij}] = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{C^T}{\det A} = \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$$

(b) $$\det A = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4-1) + (-2) = 4$$

$$C \equiv [C_{ij}] = (-1)^{i+j} [m_{ij}] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$\Rightarrow C = C^T$, C is symmetry

$$\therefore A^{-1} = \frac{C^T}{\det A} = \frac{C}{\det A} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

HW06 — Q3

3. Let A be a matrix in $\mathbb{R}^{n\times n}$. Show that the determinant of kA is $k^n Det(A)$.

Suppose $A_{n\times n} \equiv [a_{ij}]$

$\therefore kA_{n\times n} \equiv [ka_{ij}]$

$$\therefore \det(kA) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, ka_{1\sigma^1} \cdots ka_{n\sigma^n}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, k^n (a_{1\sigma^1} \cdots a_{n\sigma^n})$$

$$= k^n \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma^1} \cdots a_{n\sigma^n}$$

$$= k^n \det(A)$$

QED

HW06 — Q4

4. Suppose A is an orthogonal matrix in $\mathbb{R}^{n\times n}$. Show that $Det(A) = \pm 1$.

proof: ① $\because$ A is a orthogonal matrix

$\therefore$ the rows of A are linearly independent

$\therefore \det(A) \neq 0$

② $\because$ A is a orthogonal matrix

$\therefore AA^T = A^TA = I$

$\therefore A^T = A^{-1}$

$\because \begin{cases} \det(A^T) = \det(A) \\ \det(A^{-1}) = (\det(A))^{-1}, \text{ if } \det(A) \neq 0 \end{cases}$

$\therefore \det(A) = (\det(A))^{-1}$

$\therefore (\det(A))^2 = 1$

$\therefore \det(A) = \pm 1$

QED

HW06 — Q5

5. Show that if A is triangular then $Adj(A)$ is triangular.

proof: $\because$ A is triangular

$\therefore$ A is upper triangular or lower triangular

Suppose A is upper triangular (WLOG)

$\therefore A_{n\times n} \equiv [a_{ij}]$, $a_{ij} = \begin{cases} 0 , & i>j \\ \text{else} , & i \le j \end{cases}$

$\therefore C_{n\times n} \equiv [C_{ij}] = (-1)^{i+j} [m_{ij}] = (-1)^{i+j} \det A'_{ij}$

$\because$ a upper triangular matrix (A) remove one row and one column (A'_{ij}) are still upper triangular matrix

$\therefore \det A'_{ij} = \prod_{k=1}^{n-1} a'_{kk}$

$\because$ when $i<j$, remove i^{th} row and j^{th} colume

$\exists\, t \in [1, n-1]$, $a'_{tt} = 0$

$\therefore \det A'_{ij} = 0$

$\therefore m_{ij} = \begin{cases} 0 , & i<j \\ \text{else} , & i \ge j \end{cases}$

$\therefore$ C is lower triangular matrix

$\therefore Adj(A) = C^T$ is upper triangular matrix

QED

6. Suppose $A = [a_{ij}]$ is triangular. Show that

a) A is invertible if and only if each diagonal element $a_{ii} \neq 0$.

b) The diagonal elements of A^{-1} (if it exists) are a_{ii}^{-1}, the reciprocals of the diagonal elements of A.

HW06 — Q6

(a) Claim1: A is invertible $\longrightarrow$ each diagonal element $a_{ii} \neq 0$

proof: $\because$ A is invertible

$\therefore \det(A) \neq 0$

$\therefore \det(A) = \prod_{i=1}^{n} a_{ii} \neq 0$ — A is triangular

$\therefore \forall i \in [1,n]\ a_{ii} \neq 0$

QED

Claim2: each diagonal element $a_{ii} \neq 0 \longrightarrow A$ is invertible

$\because \forall i \in [1,n],\ a_{ii} \neq 0$

$\therefore \det(A) = \prod_{i=1}^{n} a_{ii} \neq 0$ — A is triangular

$\therefore$ A is invertible

QED

(b) Suppose $A \equiv [a_{ij}]$, $A^{-1} \equiv [a'_{ij}]$, $\mathrm{Adj}(A) = \tilde{A}^{T} = C^{T} \equiv [c_{ij}]^{T}$

$\because A^{-1} = \dfrac{\mathrm{Adj}(A)}{\det(A)}$

$\therefore a'_{ii} = \dfrac{C_{ii}}{\det(A)}$

$\because$ A is triangular matrix

$\therefore \det(A) = \prod_{t=1}^{n} a_{tt}$

$C_{ii} = \det(A \text{ remove } i^{th} \text{ row and } i^{th} \text{ column}) = \prod_{\substack{k=1 \\ k \neq i}}^{n} a_{kk}$

$\therefore a'_{ii} = \dfrac{C_{ii}}{\det(A)} = \dfrac{\prod_{\substack{k=1 \\ k\neq i}}^{n} a_{kk}}{\prod_{t=1}^{n} a_{tt}} = \dfrac{1}{a_{ii}} = a_{ii}^{-1}$

QED

7. Find the volume of $V(S)$ of the parallelopiped S in $\mathbb{R}^4$ bounded by the following vectors:

$$\alpha_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 5 \\ 3 \\ 7 \\ 9 \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 7 \end{bmatrix}.$$

HW06 — Q7

let $A_{4\times 4} = \begin{bmatrix} | & | & | & | \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ | & | & | & | \end{bmatrix}$

$$\therefore \det(A) = \det\begin{bmatrix} 2 & 2 & 5 & 3 \\ 2 & 3 & 3 & 2 \\ 3 & 3 & 7 & 4 \\ 3 & 2 & 9 & 7 \end{bmatrix} \underset{R_4 = R_4 - R_3}{\overset{R_2 = R_2 - R_1}{=\!=\!=}} \det\begin{bmatrix} 2 & 2 & 5 & 3 \\ 0 & 1 & -2 & -1 \\ 3 & 3 & 7 & 4 \\ 0 & -1 & 2 & 3 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & -2 & -1 \\ 3 & 7 & 4 \\ -1 & 2 & 3 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 5 & 3 \\ 1 & -2 & -1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & -2 & -1 \\ 3 & 7 & 4 \\ 0 & 0 & 2 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 5 & 3 \\ 1 & -2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 2\times 2\det\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix} + 3\times 2\det\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$= 4(7+6) + 6(-4-5)$$

$$= 52 - 54$$

$$= -2$$

$$\therefore V(S) = |\det(A)| = 2$$