HW09 — Book

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5.1. Let X be the maximum and let Y be the minimum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the probabilities for all values of (X, Y).

(c) Find $P[X = Y]$.

(d) Repeat parts b and c if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

Section 5.7: Conditional Probability and Conditional Expectation

5.75. **(a)** Find $p_Y(y \mid x)$ and $p_X(x \mid y)$ in Problem 5.1 assuming fair coins are used.

(b) Find $p_Y(y \mid x)$ and $p_X(x \mid y)$ in Problem 5.1 assuming Carlos uses a coin with $p = 3/4$.

(c) What is the effect on $p_X(x \mid y)$ of Carlos using a biased coin?

(d) Find $E[Y \mid X = x]$ and $E[X \mid Y = y]$ in part a; then find $E[X]$ and $E[Y]$.

(e) Find $E[Y \mid X = x]$ and $E[X \mid Y = y]$ in part b; then find $E[X]$ and $E[Y]$.

$$S = \{(TT,TT),(TH,TT),(HT,TT),(HH,TT),$$
$$(TT,TH),(TH,TH),(HT,TH),(HH,TH),$$
$$(TT,HT),(TH,HT),(HT,HT),(HH,HT),$$
$$(TT,HH),(TH,HH),(HT,HH),(HH,HH),\}$$

$$\therefore S_{XY} = \{(0,0),(1,0),(2,0),(1,1),(2,1),(2,2)\}$$

(a) $\because$ fair coin

$$\therefore P[(X,Y)=(0,0)] = \frac{1}{16}$$
$$P[(X,Y)=(1,0)] = \frac{4}{16} = \frac{1}{4}$$
$$P[(X,Y)=(2,0)] = \frac{2}{16} = \frac{1}{8}$$
$$P[(X,Y)=(1,1)] = \frac{4}{16} = \frac{1}{4}$$
$$P[(X,Y)=(2,1)] = \frac{4}{16} = \frac{1}{4}$$
$$P[(X,Y)=(2,2)] = \frac{1}{16}$$

$\Rightarrow$

$$\begin{cases} P(x=0) = \frac{1}{16} \\ P(x=1) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \\ P(x=2) = \frac{1}{8}+\frac{1}{4}+\frac{1}{16} = \frac{7}{16} \end{cases}$$

$$\begin{cases} P(y=0) = \frac{1}{16}+\frac{1}{4}+\frac{1}{8} = \frac{7}{16} \\ P(y=1) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \\ P(y=2) = \frac{1}{16} \end{cases}$$

Y \ X	0	1	2	
0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{7}{16}$
1		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2			$\frac{1}{16}$	$\frac{1}{16}$
	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{16}$	

$$\therefore P_Y(y=0 \mid x=0) = \frac{P(x=0,y=0)}{P(x=0)} = \frac{1/16}{1/16} = 1$$
$$P_Y(y=0 \mid x=1) = \frac{P(x=1,y=0)}{P(x=1)} = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P_Y(y=1 \mid x=1) = \frac{P(x=1,y=1)}{P(x=1)} = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P_Y(y=0 \mid x=2) = \frac{P(x=2,y=0)}{P(x=2)} = \frac{1/8}{7/16} = \frac{2}{7}$$
$$P_Y(y=1 \mid x=2) = \frac{P(x=2,y=1)}{P(x=2)} = \frac{1/4}{7/16} = \frac{4}{7}$$
$$P_Y(y=2 \mid x=2) = \frac{P(x=2,y=2)}{P(x=2)} = \frac{1/16}{7/16} = \frac{1}{7}$$

$$\therefore\ P_X(x=0\mid y=0)=\frac{P(x=0,y=0)}{P(y=0)}=\frac{1/16}{7/16}=\frac{1}{7}$$

$$P_X(x=1\mid y=0)=\frac{P(x=1,y=0)}{P(y=0)}=\frac{1/4}{7/16}=\frac{4}{7}$$

$$P_X(x=2\mid y=0)=\frac{P(x=2,y=0)}{P(y=0)}=\frac{1/8}{7/16}=\frac{2}{7}$$

$$P_X(x=1\mid y=1)=\frac{P(x=1,y=1)}{P(y=1)}=\frac{1/4}{1/2}=\frac{1}{2}$$

$$P_X(x=2\mid y=1)=\frac{P(x=2,y=1)}{P(y=1)}=\frac{1/4}{1/2}=\frac{1}{2}$$

$$P_X(x=2\mid y=2)=\frac{P(x=2,y=2)}{P(y=2)}=\frac{1/16}{1/16}=1$$

(b) $\because\ p=3/4$

$$\therefore\ P[(X,Y)=(0,0)]=\left(\tfrac{1}{4}\right)^4=\frac{1}{256}$$

$$P[(X,Y)=(1,0)]=\binom{4}{1}\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^1=\frac{3}{64}$$

$$P[(X,Y)=(2,0)]=\binom{2}{1}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^2=\frac{9}{128}$$

$$P[(X,Y)=(1,1)]=\binom{2}{1}\binom{2}{1}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^2=\frac{9}{64}$$

$$P[(X,Y)=(2,1)]=\binom{4}{1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^3=\frac{27}{64}$$

$$P[(X,Y)=(2,2)]=\left(\tfrac{3}{4}\right)^4=\frac{81}{256}$$

$\Rightarrow$

Y \ X	0	1	2	
0	$\frac{1}{256}$	$\frac{3}{64}$	$\frac{9}{128}$	$\frac{31}{256}$
1		$\frac{9}{64}$	$\frac{27}{64}$	$\frac{144}{256}=\frac{36}{64}$
2			$\frac{81}{256}$	$\frac{81}{256}$
	$\frac{1}{256}$	$\frac{48}{256}$ $=\frac{12}{64}$	$\frac{207}{256}$	

$$\therefore\ P_Y(y=0\mid x=0)=\frac{P(x=0,y=0)}{P(x=0)}=\frac{1/256}{1/256}=1$$

$$P_Y(y=0\mid x=1)=\frac{P(x=1,y=0)}{P(x=1)}=\frac{3/64}{12/64}=\frac{1}{4}$$

$$P_Y(y=1\mid x=1)=\frac{P(x=1,y=1)}{P(x=1)}=\frac{9/64}{12/64}=\frac{3}{4}$$

$$P_Y(y=0\mid x=2)=\frac{P(x=2,y=0)}{P(x=2)}=\frac{9/128}{207/256}=\frac{18}{207}$$

$$P_Y(y=1\mid x=2)=\frac{P(x=2,y=1)}{P(x=2)}=\frac{27/64}{207/256}=\frac{108}{207}$$

$$P_Y(y=2\mid x=2)=\frac{P(x=2,y=2)}{P(x=2)}=\frac{81/256}{207/256}=\frac{81}{207}$$

$$\therefore\ p_X(x=0\mid y=0)=\frac{P(x=0,y=0)}{P(y=0)}=\frac{1/256}{31/256}=\frac{1}{31}$$

$$p_X(x=1\mid y=0)=\frac{P(x=1,y=0)}{P(y=0)}=\frac{3/64}{31/256}=\frac{12}{31}$$

$$p_X(x=2\mid y=0)=\frac{P(x=2,y=0)}{P(y=0)}=\frac{9/128}{31/256}=\frac{18}{31}$$

$$p_X(x=1\mid y=1)=\frac{P(x=1,y=1)}{P(y=1)}=\frac{9/64}{36/64}=\frac{1}{4}$$

$$p_X(x=2\mid y=1)=\frac{P(x=2,y=1)}{P(y=1)}=\frac{27/64}{36/64}=\frac{3}{4}$$

$$p_X(x=2\mid y=2)=\frac{P(x=2,y=2)}{P(y=2)}=\frac{81/256}{81/256}=1$$

(c) When Carlos uses a biased coin with $p=3/4$, the probability distribution of X shifts towards higher values because Carlos is more likely to get heads.
This change increases the likelihood of $X=2$ and affects $p_X(x,y)$, making higher values of X more probable given any $Y=y$

(d) $E[Y\mid X=0]=0\cdot 1=0$

$E[Y\mid X=1]=0\cdot\frac{1}{2}+1\cdot\frac{1}{2}=\frac{1}{2}$

$E[Y\mid X=2]=0\cdot\frac{2}{7}+1\cdot\frac{4}{7}+2\cdot\frac{1}{7}=\frac{6}{7}$

$E[X\mid Y=0]=0\cdot\frac{1}{7}+1\cdot\frac{4}{7}+2\cdot\frac{2}{7}=\frac{8}{7}$

$E[X\mid Y=1]=1\cdot\frac{1}{2}+2\cdot\frac{1}{2}=\frac{3}{2}$

$E[X\mid Y=2]=2\cdot 1=2$

$E[X]=0\cdot\frac{1}{16}+1\cdot\frac{1}{2}+2\cdot\frac{7}{16}=\frac{11}{8}$

$E[Y]=0\cdot\frac{7}{16}+1\cdot\frac{1}{2}+2\cdot\frac{1}{16}=\frac{5}{8}$

(e) $E[Y|X=0] = 0 \cdot 1 = 0$

$E[Y|X=1] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}$

$E[Y|X=2] = 0 \cdot \frac{18}{207} + 1 \cdot \frac{108}{207} + 2 \cdot \frac{81}{207} = \frac{270}{207}$

$E[X|Y=0] = 0 \cdot \frac{1}{31} + 1 \cdot \frac{12}{31} + 2 \cdot \frac{18}{31} = \frac{48}{31}$

$E[X|Y=1] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} = \frac{7}{4}$

$E[X|Y=2] = 2 \cdot 1 = 2$

$E[X] = 0 \cdot \frac{1}{256} + 1 \cdot \frac{12}{64} + 2 \cdot \frac{207}{256} = \frac{462}{256} = \frac{231}{128}$

$E[Y] = 0 \cdot \frac{31}{256} + 1 \cdot \frac{36}{64} + 2 \cdot \frac{81}{256} = \frac{306}{256} = \frac{153}{128}$

5.26. Let X and Y have joint pdf:

$$f_{X,Y}(x,y) = k(x+y) \qquad \text{for } 0 \le x \le 1, 0 \le y \le 1.$$

(a) Find k.
(b) Find the joint cdf of (X, Y).
(c) Find the marginal pdf of X and of Y.
(d) Find $P[X < Y]$, $P[Y < X^2]$, $P[X + Y > 0.5]$.

5.80. (a) Find $f_Y(y|x)$ in Problem 5.26.
(b) Find $P[Y > X | x]$.
(c) Find $P[Y > X]$ using part b.
(d) Find $E[Y | X = x]$.

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$$\because \int_{x=0}^{x=1}\int_{y=0}^{y=1} k(x+y)\,dy\,dx$$

$$= k\int_{x=0}^{x=1} (xy+\tfrac{1}{2}y^2)\Big|_0^1 dx$$

$$= k\int_{x=0}^{x=1} x+\tfrac{1}{2}\,dx$$

$$= k\,(\tfrac{1}{2}x^2+\tfrac{1}{2}x)\Big|_0^1 dx$$

$$= k = 1$$

$$\therefore f_{X,Y}(x,y) = x+y$$

(a) $f_X(x) = \int_0^1 x+y\,dy = (xy+\frac{1}{2}y^2)\Big|_0^1 = x+\frac{1}{2}$

$$\therefore f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2x+2y}{2x+1}$$

(b) $P[Y>X|x] = \int_x^1 \frac{2x+2y}{2x+1}\,dy = \int_x^1 \frac{2x}{2x+1} + \frac{2}{2x+1}y\,dy$

$$= \left(\frac{2x}{2x+1}y + \frac{1}{2x+1}y^2\right)\Big|_x^1$$

$$= 1 - \frac{2x^2}{2x+1} - \frac{x^2}{2x+1}$$

$$= 1 - \frac{3x^2}{2x+1}$$

$$= \frac{-3x^2+2x+1}{2x+1}$$

(c) $P[Y>X] = \int_0^1 P[Y>X|x]\cdot f_X(x)\,dx$

$$= \int_0^1 \frac{-3x^2+2x+1}{2x+1}\cdot(x+\tfrac{1}{2})\,dx$$

$$= \frac{1}{2}\int_0^1 -3x^2+2x+1\,dx$$

$$= \frac{1}{2}(-x^3+x^2+x)\Big|_0^1 = \frac{1}{2}$$

(d) $E[Y|X=x] = \int_0^1 y \, f_Y(y|x) \, dy$

$$= \int_0^1 y \, \frac{2x+2y}{2x+1} \, dy$$

$$= \frac{1}{2x+1} \int_0^1 2xy + 2y^2 \, dy$$

$$= \frac{1}{2x+1} \left(xy^2 + \frac{2}{3}y^3 \right) \Big|_0^1$$

$$= \frac{x+\frac{2}{3}}{2x+1}$$

$$= \frac{3x+2}{6x+3}$$

6.50. Let $\mathbf{X} = (X_1, X_2)$ have covariance matrix:

$$\mathbf{K_X} = \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{K_X}$.

(b) Find the orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{K_X}$. Verify that $\mathbf{P}$ is orthogonal and that $\mathbf{P}^T\mathbf{K_X}\mathbf{P} = \Lambda$.

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(a) $P_{K_X}(\lambda) = \det(K_X - \lambda I) = \begin{vmatrix} 1-\lambda & 1/4 \\ 1/4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (\frac{1}{4})^2 = (1-\lambda+\frac{1}{4})(1-\lambda-\frac{1}{4}) = (\frac{5}{4}-\lambda)(\frac{3}{4}-\lambda)$

$\therefore$ Let $P_{K_X}(\lambda) = 0$, $\lambda_1 = \frac{5}{4}$, $\lambda_2 = \frac{3}{4}$

For $\lambda_1 = \frac{5}{4}$: $(K_X - \lambda_1 I)x = \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} x = 0$ $\quad\therefore x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = \frac{3}{4}$: $(K_X - \lambda_2 I)x = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} x = 0$ $\quad\therefore x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $e_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad e_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore P = [e_1\ e_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$P^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\because PP^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ P is orthogonal matrix

$\therefore P^T K_X P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5/4 & 3/4 \\ 5/4 & -3/4 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 5/2 & 0 \\ 0 & 3/2 \end{bmatrix}$

$= \begin{bmatrix} 5/4 & 0 \\ 0 & 3/4 \end{bmatrix} = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$