HW02 — Handout — Q1

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(a)

P	Q	P	⇔	[(P	&	Q)	V	(P	&	~	Q)]
1	1	1	1	1	1	1	1	1	0	0	1
0	1	0	1	0	0	1	0	0	0	0	1
1	0	1	1	1	0	0	1	1	1	1	0
0	0	0	1	0	0	0	0	0	0	1	0

∴ valid

(b)

P	Q	~	(P	→	Q)	⇔	(P	&	Q)	V	(~	P	&	~	Q)
1	1	0	1	1	1	0	1	1	1	1	0	1	0	0	1
0	1	0	0	1	1	1	0	0	1	0	1	0	0	0	1
1	0	1	1	0	0	0	1	0	0	0	0	1	0	1	0
0	0	0	0	1	0	0	0	0	0	1	1	0	1	1	0

∴ invalid

(c)

P	Q	R	P	&	(Q	V	R)	⇔	[(P	&	Q)	V	(P	&	R)]
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	0	0	1	0	0	0	1
1	0	1	1	1	0	1	1	1	1	0	0	1	1	1	1
0	0	1	0	0	0	1	1	1	0	0	0	0	0	0	1
1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	0
0	1	0	0	0	1	1	0	1	0	0	1	0	0	0	0
1	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

∴ valid

(d)

P	Q	R	{(P	→	Q)	&	[P	V	(~	P	&	R)]}	⟺	[P	⟺	(P	&	Q)]
1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1
0	1	1	0	1	1	1	0	1	1	0	1	1	1	0	1	0	0	1
1	0	1	1	0	0	0	1	1	0	1	0	1	1	1	0	1	0	0
0	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1	0	0	0
1	1	0	1	1	1	1	1	1	0	1	0	0	1	1	1	1	1	1
0	1	0	0	1	1	0	0	0	1	0	0	0	0	0	1	0	0	1
1	0	0	1	0	0	0	1	1	0	1	0	0	1	1	0	1	0	0
0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0

∴ invalid

HW02 — Handout — Q2

(a) prof: Basic: when $n=1$

$$\frac{1}{1\cdot 2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Induction Hypothesis: when $n=k$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$$

then when $n=k+1$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{(k+2)-1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+1}{(k+1)(k+2)}$$

$$= 1 - \frac{1}{k+2}$$

$$\therefore \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

QED

(b) prof: Basic: when $n=2$, $10^2-1 = 99 = 11\times 9$

Induction Hypothesis: when $n=2k$, $10^{2k}-1 = 11c$, c is a scalar

then when $n=2(k+1)$

$$10^{2(k+1)}-1 = 10^{2k}\times 10^2 - 1$$

$$= (10^{2k}-1)\times 10^2 + 10^2 - 1$$

$$= (10^{2k}-1)\times 100 + 99$$

$$= 11c\times 100 + 11\times 9$$

$$= 11(100c+9)$$

$\therefore 10^n - 1$ is divisible by 11 for every even $n \geq 1$

QED

(c) prof: Basic: when $n=1$, $x^1+\frac{1}{x^1}=x+\frac{1}{x}$ is an integer

Induction Hypothesis: when $n\le k$, $x^k+\frac{1}{x^k}$ is an integer

then when $n=k+1$

$$x^{k+1}+\frac{1}{x^{k+1}}=\left(x^{k+1}+\frac{1}{x^{k-1}}+x^{k-1}+\frac{1}{x^{k+1}}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right)$$

$$=\underline{\left(x+\frac{1}{x}\right)}\,\underline{\left(x^k+\frac{1}{x^k}\right)}-\underline{\left(x^{k-1}+\frac{1}{x^{k-1}}\right)}$$

$\because \left(x+\frac{1}{x}\right), \left(x^k+\frac{1}{x^k}\right), \left(x^{k-1}+\frac{1}{x^{k-1}}\right)$ are all integers

$\therefore x^{k+1}+\frac{1}{x^{k+1}}$ is an integer

$\therefore$ if $x+\frac{1}{x}$ is an integer, then $x^n+\frac{1}{x^n}$ is an integer

QED

HW02 — Handout — Q3

① Proof: $2^{A\cup B} = 2^A \cup 2^B$

Claim 1: $2^{A\cup B} \subset 2^A \cup 2^B$

prof: pick $S \in 2^{A\cup B}$

$\therefore S \subset A\cup B$ (def of 2^X)

$\therefore S \subset A$ OR $S \subset B$ (def of $\cup$)

$\therefore S \in 2^A$ OR $S \in 2^B$ (def of 2^X)

$\therefore S \in 2^A \cup 2^B$ (def of $\cup$)

$\therefore 2^{A\cup B} \subset 2^A \cup 2^B$ (def of $\subset$)

Claim 2: $2^A \cup 2^B \subset 2^{A\cup B}$

prof: pick $S \in 2^A \cup 2^B$

$\therefore S \in 2^A$ OR $S \in 2^B$ (def of $\cup$)

$\therefore S \subset A$ OR $S \subset B$ (def of 2^X)

$\therefore S \subset A\cup B$ (def of $\cup$)

$\therefore S \in 2^{A\cup B}$ (def of 2^X)

$\therefore 2^A \cup 2^B \subset 2^{A\cup B}$ (def of $\subset$)

$\therefore 2^{A\cup B} = 2^A \cup 2^B$ (def of $=$)

QED

② Proof $2^A \subset 2^B$ IFF $A \subset B$

Claim 1: $2^A \subset 2^B \longrightarrow A \subset B$

prof: pick a subset $S = A$

$\therefore S \in 2^A$ (def of 2^X)

Suppose $2^A \subset 2^B$

$\therefore S \in 2^B$ (def of $\subset$)

$\therefore S \subset B$ (def of 2^X)

$\therefore A \subset B$ (def of $=$)

$\therefore 2^A \subset 2^B \longrightarrow A \subset B$

Claim 2: $A \subset B \longrightarrow 2^A \subset 2^B$

prof: pick $S \in 2^A$

$\therefore S \subset A$ (def of 2^X)

Suppose $A \subset B$

$\therefore S \subset B$ (transitive)

$\therefore S \in 2^B$ (def of 2^X)

$\therefore 2^A \subset 2^B$ (def of $\subset$)

$\therefore A \subset B \rightarrow 2^A \subset 2^B$

$\therefore 2^A \subset 2^B$ IFF $A \subset B$ QED

③ Proof $2^{f^{-1}(A)} \cap 2^{f^{-1}(B)} = 2^{f^{-1}(A \cap B)}$

Claim 1: $2^{f^{-1}(A)} \cap 2^{f^{-1}(B)} \subset 2^{f^{-1}(A \cap B)}$

prof: pick $S \in 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)}$

$\therefore S \in 2^{f^{-1}(A)}$ AND $S \in 2^{f^{-1}(B)}$ (def of $\cap$)

$\therefore S \subset f^{-1}(A)$ AND $S \subset f^{-1}(B)$ (def of 2^X)

$\therefore f(S) \subset A$ AND $f(S) \subset B$ (def of f^{-1})

$\therefore f(S) \subset A \cap B$ (def of $\cap$)

$\therefore S \subset f^{-1}(A \cap B)$ (def of f^{-1})

$\therefore S \in 2^{f^{-1}(A \cap B)}$ (def of 2^X)

$\therefore 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)} \subset 2^{f^{-1}(A \cap B)}$ (def of $\subset$)

Claim 2: $2^{f^{-1}(A \cap B)} \subset 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)}$

prof: pick $S \in 2^{f^{-1}(A \cap B)}$

$\therefore S \subset f^{-1}(A \cap B)$ (def of 2^X)

$\therefore f(S) \subset A \cap B$ (def of f^{-1})

$\therefore f(S) \subset A$ AND $f(S) \subset B$ (def of $\cap$)

$\therefore S \subset f^{-1}(A)$ AND $S \subset f^{-1}(B)$ (def of f^{-1})

$\therefore S \in 2^{f^{-1}(A)}$ AND $S \in 2^{f^{-1}(B)}$ (def of 2^X)

$\therefore S \in 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)}$ (def of $\cap$)

$\therefore 2^{f^{-1}(A \cap B)} \subseteq 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)}$ (def of $\subseteq$)

$\therefore 2^{f^{-1}(A)} \cap 2^{f^{-1}(B)} = 2^{f^{-1}(A \cap B)}$ (def of $=$) QED

HW02 — Handout — Q4

$\Omega = \{v, w, x, y, z\}$

① $\sigma(\phi) = \{\phi, \Omega\}$

② $\sigma(\{w,x\}) = \{\phi, \{w,x\}, \{w,x\}^c, \Omega\}$

$= \{\phi, \{w,x\}, \{x,y,z\}, \Omega\}$

③ $\sigma(\{w,x,y\}) = \{\phi, \{w,x,y\}, \{w,x,y\}^c, \Omega\}$

$= \{\phi, \{w,x,y\}, \{v,z\}, \Omega\}$

④ $\sigma(\{w\}) = \{\phi, \{w\}, \{v,x,y,z\}, \Omega\}$

$\sigma(\{z\}) = \{\phi, \{z\}, \{v,w,x,y\}, \Omega\}$

$\sigma(\{w\}) \cup \sigma(\{z\}) = \{\phi, \{w\}, \{z\}, \{v,x,y,z\}, \{v,w,x,y\}, \Omega\}$

$\therefore \sigma(\sigma(\{w\}) \cup \sigma(\{z\})) = \{\phi, \{w\}, \{z\}, \{w\}\cup\{z\}, (\{w\}\cup\{z\})^c, \{v,x,y,z\}, \{v,w,x,y\}, \Omega\}$

$= \{\phi, \{w\}, \{z\}, \{w,z\}, \{v,x,y\}, \{v,x,y,z\}, \{v,w,x,y\}, \Omega\}$

HW02 — Handout — Q5

proof: (contradiction)

Suppose $\sqrt{3}$ is a rational number

we must have $\sqrt{3} = \frac{a}{b}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$, a and b are coprime

$\therefore 3 = \frac{a^2}{b^2} \Rightarrow a^2 = 3b^2$

$\therefore$ a^2 is divisible by 3

$\because$ 3 is a prime number

$\therefore$ a must be divisible by 3

Suppose $a = 3k$, $k \in \mathbb{Z}$

$\therefore a^2 = 3b^2 \Rightarrow 9k^2 = 3b^2$

$\therefore b^2 = 3k^2$

Similary b is divisible by 3

$\therefore$ a and b have same factor 3

$\therefore$ controdiction

$\therefore$ $\sqrt{3}$ is an irrational number

HW02 — Handout — Q6

proof: Basic: when $n=1$, $Ae_1=\lambda_1 e_1$

only one vector e_1 is linear independent

Induction Hypothesis: when $n=k$, $Ae_r=\lambda_r e_r$, $r\in[1,k]$

n corresponding eigenvalues $\lambda_1,\cdots\lambda_k$ are distinct and n eigenvector

$e_1,\cdots,e_k$ are linear independent,

$\sum_{r=1}^{k} c_r e_r=0$ IFF $c_1=\cdots=c_k=0$

then when $n=k+1$, $Ae_r=\lambda_r e_r$, $r\in[1,k]$, and $\lambda_1,\cdots,\lambda_{k+1}$ are distinct

Suppose $\sum_{r=1}^{k+1} c_r e_r = c_1e_1+\cdots+c_ke_k+c_{k+1}e_{k+1}=0$ ①

$\therefore A\cdot\sum_{r=1}^{k+1} c_r e_r = A0=0$

$\therefore c_1Ae_1+\cdots+c_kAe_k+c_{k+1}Ae_{k+1}=0$

$\therefore c_1\lambda_1e_1+\cdots+c_k\lambda_ke_k+c_{k+1}\lambda_{k+1}e_{k+1}=0$ ②

$\therefore$ ② − ①: $c_1(\lambda_1-1)e_1+\cdots+c_k(\lambda_k-1)e_k+c_{k+1}(\lambda_{k+1}-1)e_{k+1}=0$

$\because \lambda_1,\cdots,\lambda_{k+1}$ are distinct

and $e_1\cdots e_k$ are linear independent

$\therefore c_1(\lambda_1-1)e_1+\cdots+c_k(\lambda_k-1)e_k=0$, $c_1=\cdots=c_k=0$

$\therefore c_{k+1}(\lambda_{k+1}-1)e_{k+1}=0$

$\because e_{k+1}$ is not null vector

$\therefore$ we must have $c_{k+1}=0$

$\therefore c_1=\cdots=c_k=c_{k+1}=0$

$\therefore e_1,\cdots,e_{k+1}$ are linear independent

QED