HW 01 — Q1

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(1) Book #5

unit vector $u_1 = \frac{v}{\|v\|} = \frac{(1,3)}{\sqrt{1^2+3^2}} = \frac{(1,3)}{\sqrt{10}} = (\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10})$

$u_2 = \frac{w}{\|w\|} = \frac{(2,1,2)}{\sqrt{2^2+1^2+2^2}} = \frac{(2,1,2)}{\sqrt{9}} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$

Suppose $U_1' = (x_1, y_1)$, $U_1' \perp u_1 \Rightarrow U_1' \perp v \Rightarrow U_1' \cdot v = 0$

$\therefore U_1' \cdot v = x_1 + 3y_1 = 0 \Rightarrow x_1 = -3y_1$

$\therefore$ Suppose $y_1 = 1$, $x_1 = -3y_1 = -3 \Rightarrow U_1' = (-3, 1)$

$\therefore$ unit vector $U_1 = \frac{U_1'}{\|U_1'\|} = \frac{(-3,1)}{\sqrt{(-3)^2+1^2}} = (-\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10})$

Similarly, suppose $U_2' = (x_2, y_2, z_2)$, $U_2' \perp w$

$\therefore U_2' \cdot w = 2x_2 + y_2 + 2z_2 = 0$

$\therefore$ Suppose $x_2 = 1$, $z_2 = 1 \Rightarrow y_2 = -4 \Rightarrow U_2' = (1, -4, 1)$

$\therefore$ unit vector $U_2 = \frac{U_2'}{\|U_2'\|} = \frac{(1,-4,1)}{\sqrt{1^2+(-4)^2+1^2}} = (\frac{\sqrt{2}}{6}, -\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6})$

(2) Book #6

(a) $\because w \perp v$

$\therefore w \cdot v = 0 \Rightarrow 2w_1 - w_2 = 0 \Rightarrow 2w_1 = w_2$

$\therefore$ vector $w = (w_1, 2w_1)$ $w_1 \in \mathbb{R}$

(b) Suppose $u = (x, y, z) \perp V$

$\therefore u \cdot V = 0 \Rightarrow x + y + z = 0$

$\therefore$ is a plane

(C) Suppose $u=(x,y,z)$, $u \perp v_1=(1,1,1)$ & $u \perp v_2=(1,2,3)$

$$\therefore \begin{cases} u \cdot v_1 = 0 \\ u \cdot v_2 = 0 \end{cases} \Rightarrow \begin{array}{l} x+y+z=0 \\ x+2y+3z=0 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x-z=0 \\ y+2z=0 \end{cases} \Rightarrow \text{Suppose } z=1 \Rightarrow \begin{cases} x=z=1 \\ y=-2z=-2 \end{cases}$$

$\therefore$ Suppose $z=1$, $u=(1,-2,1)$

$\therefore$ $\forall u \perp v_1$ & $u \perp v_2$, u lie on a line (with the same direction of $(1,-2,1)$

(3) Book #13

Suppose $v=(x_1,y_1,z_1)$, $w=(x_2,y_2,z_2)$, and $u=(1,0,1)$

$$\therefore \begin{cases} v \perp u \\ w \perp u \\ v \perp w \end{cases} \Rightarrow \begin{cases} v \cdot u = 0 \\ w \cdot u = 0 \\ v \cdot w = 0 \end{cases} \Rightarrow \begin{cases} x_1+z_1=0 \\ x_2+z_2=0 \\ x_1x_2+y_1y_2+z_1z_2=0 \end{cases}$$

$$\therefore \begin{cases} x_1=-z_1 \\ x_2=-z_2 \end{cases} \Rightarrow x_1x_2=(-z_1)(-z_2)=z_1z_2 \Rightarrow y_1y_2=-2x_1x_2$$

$\therefore$ Suppose $x_1=x_2=1$, $y_1=1$

$\therefore$ $v=(1,1,-1)$ and $w=(1,-2,-1)$ that are perpendicular to $(1,0,1)$ and to each other

(4) Book #16

$\because$ $v=(1,1,\cdots,1)$ in 9 dimensions

$\therefore \|v\|=\sqrt{1^2+1^2+\cdots+1^2}=\sqrt{9\times 1^2}=\sqrt{9}=3$

$\therefore u=\frac{v}{\|v\|}=\frac{(1,1,\cdots,1)}{3}=(\frac{1}{3},\frac{1}{3},\cdots,\frac{1}{3})$ in 9 dimensions

Suppose $w'=(a_1,a_2,\cdots,a_9)$, $w'\perp v$

$\therefore w'\cdot v=0 \Rightarrow a_1+a_2+\cdots+a_9=0$

$\therefore$ suppose $a_1=1$, $a_2=-1$, $a_3=a_4=\cdots=a_9=0$

$\therefore w'=(1,-1,0,0,\cdots 0)$, $w'\perp v$

$\therefore w=\frac{w'}{\|w'\|}=\frac{(1,-1,0,0,\cdots 0)}{\sqrt{1^2+(-1)^2+0^2+\cdots+0^2}}=\frac{(1,-1,0,0,\cdots 0)}{\sqrt{2}}=(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0,0,\cdots,0)$

HW 01 — Q2

(a) Proof $(u+v)\cdot w = u\cdot w + v\cdot w$

prof: $(u+v)\cdot w$

$$= \sum_{k=1}^{n} (u+v)_k \cdot w_k^*$$

$$= \sum_{k=1}^{n} (u_k \cdot w_k^* + v_k \cdot w_k^*)$$

$$= \sum_{k=1}^{n} u_k \cdot w_k^* + \sum_{k=1}^{n} v_k \cdot w_k^*$$

$$= u\cdot w + v\cdot w$$

QED

(b) ① Proof $\|u\| \geqslant 0$

prof: $\|u\| = \sqrt{\|u\|^2} = \sqrt{u\cdot u^*} = \sqrt{\sum_{k=1}^{n} |u_k|^2}$

$\because \forall k \in [1,n],\ |u_k|^2 \geqslant 0$

$\therefore \sum_{k=1}^{n} |u_k|^2 \geqslant 0$

$\therefore \sqrt{\sum_{k=1}^{n} |u_k|^2} \geqslant 0$

$\therefore \|u\| \geqslant 0$

② Proof $\|u\| = 0$ if and only if $u = 0$

Claim 1: $\|u\| = 0 \longrightarrow u = 0$

prof: $\because \|u\| = \sqrt{\sum_{k=1}^{n} |u_k|^2} = 0$

$\therefore \sum_{k=1}^{n} |u_k|^2 = 0$

$\therefore \forall k \in [1,n],\ |u_k|^2 = 0 \Rightarrow u_k = 0$

$\therefore u = 0$

Claim2: $u=0 \longrightarrow \|u\|=0$

$\because u=0$

$\therefore \forall k\in[1,n],\ u_k=0$

$\therefore \|u\|^2=u\cdot u^*=\sum_{k=1}^{n}|u_k|^2=0$

$\therefore \|u\|=\sqrt{\|u\|^2}=\sqrt{0}=0$

$\therefore \|u\|=0$ IFF $u=0$ QED

(c) Proof $k(u+v)=ku+kv$

prof: $k(u+v)=\begin{bmatrix} k(u_1+v_1) \\ k(u_2+v_2) \\ \vdots \\ k(u_n+v_n) \end{bmatrix}=\begin{bmatrix} ku_1+kv_1 \\ ku_2+kv_2 \\ \vdots \\ ku_n+kv_n \end{bmatrix}=\begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}+\begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}=ku+kv$

QED

HW 01 — Q3

$\because A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$

$\therefore$ Suppose $AB = C$, $A \equiv [a_{ij}]$, $B \equiv [b_{ij}]$, $C \equiv [c_{ij}]$

$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

(a) Proof if A has a zero row, then AB has a zero row

prof: Suppose l^{th} row of A is a zero row

$\therefore \forall k \in [1,n], a_{lk} = 0$

$\therefore \forall k \in [1,n], a_{lk} \cdot b_{kj} = 0$

$\therefore \forall j \in [1,n], c_{lj} = \sum_{k=1}^{n} a_{lk} \cdot b_{kj} = 0$

$\therefore l^{th}$ row of C is a zero row

$\rightarrow l^{th}$ row of AB is a zero row QED

(b) Proof if B has a zero column, then AB has a zero column

prof: Suppose l^{th} column of B is a zero column

$\therefore \forall k \in [1,n], b_{kl} = 0$

$\therefore \forall k \in [1,n], a_{ik} \cdot b_{kl} = 0$

$\therefore \forall i \in [1,n], c_{il} = \sum_{k=1}^{n} a_{ik} \cdot b_{kl} = 0$

$\therefore l^{th}$ column of C is a zero column

$\Rightarrow l^{th}$ column of AB is a zero column QED

HW 01 — Q4

(a) Proof $(A_1A_2\cdots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \cdots A_2^T A_1^T$

prof: $\because (AB)^T = B^T A^T$

$\therefore (A_1A_2\cdots A_{n-1}A_n)^T = ((A_1A_2\cdots A_{n-1})A_n)^T = A_n^T(A_1A_2\cdots A_{n-1})^T$

Similary $(A_1A_2\cdots A_{n-2}A_{n-1})^T = A_{n-1}^T(A_1A_2\cdots A_{n-2})^T$

$\therefore (A_1A_2\cdots A_{n-1}A_n)^T = A_n^T A_{n-1}^T (A_1A_2\cdots A_{n-2})^T$

$\therefore$ repeat the process

$(A_1A_2\cdots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \cdots A_4^T A_3^T (A_1A_2)^T$

$= A_n^T A_{n-1}^T \cdots A_4^T A_3^T (A_2^T \cdot A_1^T)$

$\therefore (A_1A_2\cdots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \cdots A_2^T A_1^T$ QED

(b) Proof $(A_1A_2\cdots A_{n-1}A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1}$ if the A_k's are invertible

prof: $(A_1A_2\cdots A_{n-1}A_n)(A_n^{-1}A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1})$

$= A_1A_2\cdots A_{n-1}(A_n \cdot A_n^{-1})A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1}$

$= A_1A_2\cdots A_{n-1}\cdot I \cdot A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1}$

$= A_1A_2\cdots(A_{n-1}A_{n-1}^{-1})\cdots A_2^{-1}A_1^{-1}$

$\therefore$ repeat the process

$(A_1A_2\cdots A_{n-1}A_n)(A_n^{-1}A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1})$

$= A_1 \cdot I \cdot A_1^{-1}$

$= A_1 \cdot A_1^{-1}$

$= I$

$\therefore (A_n^{-1}A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1}) = (A_1A_2\cdots A_{n-1}A_n)^{-1}$

QED

HW 01 — Q5

$\because$ A and B are unitary

$\therefore A^{-1} = A^H,\ B^{-1} = B^H$

① prof: $\because A^{-1} = A^H$

$\therefore (A^H)^{-1} = (A^{-1})^{-1} = A$

$\because (A^H)^H = A$

$\therefore (A^H)^{-1} = (A^H)^H$

$\therefore A^H$ is unitary

② proof: $\because A^{-1} = A^H$

$\therefore (A^{-1})^H = (A^H)^H = A$

$\because (A^{-1})^{-1} = A$

$\therefore (A^{-1})^{-1} = (A^{-1})^H$

$\therefore A^{-1}$ is unitary

③ prof: $\because A^{-1} = A^H,\ B^{-1} = B^H$

$\therefore (A^H B^{-1})^{-1} = (B^{-1})^{-1}(A^H)^{-1} = B(A^{-1})^{-1} = BA$

$(A^H B^{-1})^H = (B^{-1})^H (A^H)^H = (B^H)^H (A^H)^H = BA$

$\therefore (A^H B^{-1})^{-1} = (A^H B^{-1})^H$

$\therefore A^H B^{-1}$ is unitary

HW 01 — Q6

(a) prof: $(A+A^H)^H = (A^H + (A^H)^H) = A^H + A = A + A^H$

$\therefore (A+A^H)$ is Hermitian

(b) prof: $(A-A^H)^H = (A^H - (A^H)^H) = A^H - A = -(A-A^H)$

$\therefore (A-A^H)$ is skew-Hermitian

(c) prof: Suppose $B = \frac{1}{2}(M+M^H)$, $C = \frac{1}{2}(M-M^H)$, M is a square matrix

$$\therefore B^H = \left(\frac{1}{2}(M+M^H)\right)^H = \frac{1}{2}(M^H + (M^H)^H) = \frac{1}{2}(M^H + M) = B$$

$$C^H = \left(\frac{1}{2}(M-M^H)\right)^H = \frac{1}{2}(M^H - (M^H)^H) = \frac{1}{2}(M^H - M) = -C$$

$\therefore B$ is a Hermitian $\quad \therefore C$ is a skew-Hermitian

$\therefore A = B + C = \frac{1}{2}(M+M^H) + \frac{1}{2}(M-M^H) = \frac{1}{2}\cdot 2M = M$

$\therefore A$ is a square matrix

and A can be presented as a sum of a Hermitian and a skew-Hermitian matrix

HW 01 — Q7

prof: $\because$ A is a orthogonal matrix

$\therefore A^{-1} = A^T$, $AA^T = A^TA = I$

$\therefore AA^T = [m_{ij}]$ $\begin{cases} m_{ij} = 1 \text{ , only if } i=j \\ m_{ij} = 0 \text{ , } i \neq j \end{cases}$

$\therefore$ when $i \neq j$, $m_{ij} = \sum_{k=1}^{n} a_{ik} \cdot \underbrace{a^T_{kj}} = \sum_{k=1}^{n} a_{ik} \cdot a_{jk} = 0$

$\therefore$ when $i \neq j$, i^{th} row of A $\cdot$ j^{th} row of $A = 0$

$\therefore$ the row of A are mutually orthogonal