HW03 - Q1

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(1) Book 3.1 #10

(a) $\because$ the plane of vectors $[b_1, b_2, b_3]$, with $b_1 = b_2$

① zero vector: $0 = (0,0,0)$

$\because b_1 = 0 = b_2$

$\therefore 0$ is in the plane

② addition: Suppose $u = (u_1, u_2, u_3)$. $V = (V_1, V_1, V_3)$, $u_1 = u_2, V_1 = V_2$

$\therefore u + V = (u_1, u_2, u_3) + (V_1, V_2, V_3)$

$= (u_1 + V_1, u_2 + V_2, u_3 + V_3)$

$\because u_1 = u_2, V_1 = V_2$

$\therefore u_1 + V_1 = u_2 + V_2$

$\therefore S$ is closed under vector addition

③ multiplication: Suppose $u = (u_1, u_2, u_3)$, $u_1 = u_2$, c is a scalar

$\therefore cu = c(u_1, u_2, u_3) = (cu_1, cu_2, cu_3)$

$\because u_1 = u_2$

$\therefore cu_1 = cu_2$

$\therefore S$ is closed under scalar multiplication

$\therefore$ the plane of (b_1, b_2, b_3) is a subspace of $\mathbb{R}^3$

(b) the plane of vector with $b_1 = 1$

① zero vector: $\because 0 = [0,0,0]$ $b_1 = 0 \neq 1$

$\therefore 0$ is not in the plane

$\therefore$ the plane of vector with $b_1 = 1$ is not a subspace of $\mathbb{R}^3$

(c) the vectors with $b_1 b_2 b_3 = 0$

$\therefore \exists i \in [1,3]$, $b_i = 0$

① zero vector: $0 = (0,0,0)$

$\therefore b_1 = b_2 = b_3 = 0$, $b_1 b_2 b_3 = 0$

$\therefore 0$ is in the vectors

② addition: Suppose $u = (0, u_2, u_3)$ $v = (v_1, 0, 0)$

$u_2 \neq 0, u_3 \neq 0, v_1 \neq 0$

$\therefore u + v = (0, u_2, u_3) + (v_1, 0, 0)$

$= (v_1, u_2, u_3)$

$\because v_1 u_2 u_3 \neq 0$

$\therefore$ is not closed under vector addition

$\therefore$ the vectors with $b_1 b_2 b_3 = 0$ is not a subspace of $\mathbb{R}^3$

(d) All linear combinations of v and w

$u = \alpha v + \beta w = \alpha(1,4,0) + \beta(2,2,2)$

$= (\alpha + 2\beta,\ 4\alpha + 2\beta,\ 2\beta) \qquad \alpha, \beta \in \mathbb{R}$

① zero vector: when $\alpha = \beta = 0$

$u = 0v + 0w = 0$

$\therefore 0$ is a linear combination of v and w

② addition: Suppose $u_1 = a_1 v + a_2 w$, $u_2 = b_1 v + b_2 w$

$\therefore u_1 + u_2 = a_1 v + a_2 w + b_1 v + b_2 w$

$= (a_1 + b_1)v + (a_2 + b_2)w$

$\because a_1 + b_1 \in \mathbb{R}$, $a_2 + b_2 \in \mathbb{R}$

$\therefore$ is closed under vector addition

③ multiplication: Suppose $u = \alpha v + \beta w$, c is a scalar

$\therefore cu = c(\alpha v + \beta w)$

$= c\alpha v + c\beta w$

$\because c\alpha \in \mathbb{R}$, $c\beta \in \mathbb{R}$

$\therefore$ is closed under scalar multiplication

$\therefore$ All linear combinations of v and w is a subspace of $\mathbb{R}^3$

(e) All vectors that satisfy $b_1+b_2+b_3=0$

① zero vector: $0=(0,0,0)$

$\therefore b_1=b_2=b_3=0 \Rightarrow b_1+b_2+b_3=0$

$\therefore 0$ is satisfy

② addition: Suppose $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$

$u_1+u_2+u_3=0$, $v_1+v_2+v_3=0$

$\therefore u+v=(u_1,u_2,u_3)+(v_1,v_2,v_3)$

$=(u_1+v_1,\ u_2+v_2,\ u_3+v_3)$

$\because (u_1+v_1)+(u_2+v_2)+(u_3+v_3)$

$=(u_1+u_2+u_3)+(v_1+v_2+v_3)=0+0=0$

$\therefore$ is closed under vector addition

③ multiplication: Suppose $u=(u_1,u_2,u_3)$, c is a scalar

$u_1+u_2+u_3=0$

$\therefore cu=c(u_1,u_2,u_3)$

$=(cu_1,cu_2,cu_3)$

$\because cu_1+cu_2+cu_3=c(u_1+u_2+u_3)=c\cdot 0=0$

$\therefore$ is closed under scalar multiplication

$\therefore$ All vectors that satisfy $b_1+b_2+b_3=0$ is a subspace of $\mathbb{R}^3$

(f) All vectors with $b_1 \le b_2 \le b_3$

① zero vector: $0 = (0, 0, 0)$

$\therefore b_1 \le b_2 \le b_3 = 0$

$\therefore 0$ is satisfy

② addition: Suppose $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

$u_1 \le u_2 \le u_3$, $v_1 \le v_2 \le v_3$

$\therefore u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$\because u_1 \le u_2 \le u_3$, $v_1 \le v_2 \le v_3$

$\therefore u_1 + v_1 \le u_2 + v_2 \le u_3 + v_3$

$\therefore$ is closed under vector addition

③ multiplication: Suppose $u = (u_1, u_2, u_3)$, c is a scalar

$u_1 \le u_2 \le u_3$

$\therefore cu = c(u_1, u_2, u_3)$

$= (cu_1, cu_2, cu_3)$

if $c < 0 \Rightarrow cu_1 \ge cu_2 \ge cu_3$

$\therefore$ is not closed under scalar multiplication

$\therefore$ All vectors with $b_1 \le b_2 \le b_3$ is not a subspace of $\mathbb{R}^3$

(2) Book 3.1 #20

(a) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & 2 & b_3 \end{array}\right] \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3+b_1 \end{array}\right] \Rightarrow \begin{cases} b_1 = x_1+4x_2+2x_3 \\ b_2-2b_1=0 \\ b_3+b_1=0 \end{cases}$

$\therefore$ when $\begin{cases} b_2=2b_1 \\ b_3=-b_1 \end{cases}$, the system is solvable

(b) $\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_3+b_1 \end{array}\right] \Rightarrow \begin{cases} b_1 = x_1+4x_2 \\ b_2-2b_1=x_2 \\ b_3+b_1=0 \end{cases}$

$\therefore$ when $b_3=-b_1$, the system is solvable

(3) Book 3.3 #18

① $A=\begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{bmatrix} \xrightarrow{R_3=R_3-2R_2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore A$ has 2 pivot and 1 free variable

$\therefore \text{rank}(A)=2$

$A^T=\begin{bmatrix} 1 & 2 & -1 \\ 4 & 11 & 2 \\ 0 & 5 & 10 \end{bmatrix} \xrightarrow{R_2=R_2-4R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix} \xrightarrow[R_3=\frac{1}{5}R_3]{R_2=\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore A^T$ has 2 pivot and 1 free variable

$\therefore \text{rank}(A^T)=2$

② $A=\begin{bmatrix}1&0&1\\1&1&2\\1&1&q\end{bmatrix}\xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1}\begin{bmatrix}1&0&1\\0&1&1\\0&1&q-1\end{bmatrix}\xrightarrow{R_3=R_3-R_2}\begin{bmatrix}1&0&1\\0&1&1\\0&0&q-2\end{bmatrix}$

$\therefore$ Case 1: when $q-2=0 \Rightarrow q=2$

A has 2 pivot and 1 free variable

$\therefore \operatorname{rank}(A)=2$

Case 2: when $q-2\neq 0 \Rightarrow q\neq 2$

A has 3 pivot variables

$\therefore \operatorname{rank}(A)=3$

$A^T=\begin{bmatrix}1&1&1\\0&1&1\\1&2&q\end{bmatrix}\xrightarrow{R_3=R_3-R_1}\begin{bmatrix}1&1&1\\0&1&1\\0&1&q-1\end{bmatrix}\xrightarrow{R_3=R_3-R_2}\begin{bmatrix}1&1&1\\0&1&1\\0&0&q-2\end{bmatrix}$

$\therefore$ Similarly, when $q=2$, $\operatorname{rank}(A^T)=2$

when $q\neq 2$, $\operatorname{rank}(A^T)=3$

$(\therefore \operatorname{rank}(A)=\operatorname{rank}(A^T))$

(4) Book 3.3 #19

① $A=\begin{bmatrix}1&1&5\\1&0&1\end{bmatrix}\xrightarrow{R_2=R_2-R_1}\begin{bmatrix}1&1&5\\0&-1&-4\end{bmatrix}$ $\therefore \operatorname{rank}(A)=$ number of pivot $=2$

$AA^T=\begin{bmatrix}1&1&5\\1&0&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\\5&1\end{bmatrix}=\begin{bmatrix}27&6\\6&2\end{bmatrix}\xrightarrow[R_2=R_2-\frac{2}{3}R_1]{R_1=\frac{1}{3}R_1}\begin{bmatrix}9&2\\0&\frac{2}{3}\end{bmatrix}$ $\therefore \operatorname{rank}(AA^T)=2$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 6 & 5 & 26 \end{bmatrix} \xrightarrow[R_2 = R_1]{R_1 = R_2} \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 6 \\ 6 & 5 & 26 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - 6R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2]{R_2 = -R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \operatorname{rank}(A^TA) = 2$$

② $$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow[R_3 = R_3 - \frac{1}{2}R_1]{R_2 = R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \therefore \operatorname{rank}(A) = 2$$

$$AA^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}]{R_1 = \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \operatorname{rank}(AA^T) = 2$$

$$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{1}{2}R_1} \begin{bmatrix} 6 & 3 \\ 0 & \frac{7}{2} \end{bmatrix} \quad \therefore \operatorname{rank}(A^TA) = 2$$

$$(\therefore \operatorname{rank}(A) = \operatorname{rank}(AA^T) = \operatorname{rank}(A^TA))$$

(5) Book 3.4 #11

(a) $\because V_1 = (1, 1, -1)$, $V_2 = (-1, -1, 1)$

$\because$ when $a_1 = 1$, $a_2 = 1$

$a_1V_1 + a_2V_2 = (1, 1, -1) + (-1, -1, 1)$

$= (0, 0, 0)$

$\therefore$ v_1 and v_2 are linear dependent

and Span(v_1, v_2) only has 1 basic vector

$\therefore$ dimension is 1, is a line

(b) Suppose $A = [v_1^T, v_2^T, v_3^T] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ rank$(A) = 2$

$\therefore$ Span(v_1, v_2, v_3) only has 2 basis vector, dimension is 2

$\therefore$ is a plane

(c) Suppose $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$

they belongs to all vectors in $\mathbb{R}^3$ with whole number components

$\because$ $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ IFF $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ v_1, v_2, v_3 are linear independent

$\therefore$ they are a basis of $\mathbb{R}^3$

$\therefore$ dimension is 3

$\therefore$ is $\mathbb{R}^3$

(d) Suppose $v_1 = (2,1,1)$, $v_2 = (1,2,1)$, $v_3 = (1,1,2)$

they belongs to all vector with positive components

Suppose $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix}$

$\therefore \operatorname{rank}(A) = 3 = n$

$\therefore v_1, v_2, v_3$ are linear independent

$\therefore$ they are a basis of $\mathbb{R}^3$

$\therefore$ dimension is 3

$\therefore$ is $\mathbb{R}^3$

(6) Book 3.4 #26

(a) Suppose diagnal matrix $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$, and $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$

$\therefore$ one basis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ the dimension is 3

(b) Suppose symmetric matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. $A^T = A$

$\therefore$ one basis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ the dimension is 6

(c) Suppose skew-symmetric matrix $A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$, $A^T = -A$

$\therefore$ one basis is:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$\therefore$ the dimension is 3

HW03 − Q2

Proof: Claim 1: u and v are linear dependent $\longrightarrow ad-cd=0$

Suppose u and v are linear dependent

$\therefore A=\begin{bmatrix} u \\ v \end{bmatrix}$, $\text{rank}(A)<2$

$\therefore A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2=R_2-\frac{c}{a}R_1} \begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix}$

$\therefore d-\frac{c}{a}b=0 \Rightarrow \frac{c}{a}b=d$

$\therefore ad-bc=0$

Claim 2: $ad-bc=0 \longrightarrow u$ and v are linear dependent

Suppose $ad-bc=0$

$\therefore A=\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2=R_2-\frac{c}{a}R_1} \begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix}=\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\therefore \text{rank}(A)=1 \neq n=2$

$\therefore$ the rows of A (u and v) are linear dependent

HW03 - Q3

Suppose $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -4 & 1 \\ 2 & 3 & 0 & -1 & -1 \\ 1 & -6 & 3 & -8 & 7 \end{bmatrix}$

$$\therefore A \xrightarrow[R_2=R_2-2R_3]{R_1=R_1-3R_3} \begin{bmatrix} 0 & 20 & -8 & 20 & -20 \\ 0 & 15 & -6 & 15 & -15 \\ 1 & -6 & 3 & -8 & 7 \end{bmatrix} \xrightarrow[R_3=\frac{1}{4}R_1]{\substack{R_1=R_3 \\ R_2=\frac{1}{3}R_2}} \begin{bmatrix} 1 & -6 & 3 & -8 & 7 \\ 0 & 5 & -2 & 5 & -5 \\ 0 & 5 & -2 & 5 & -5 \end{bmatrix}$$

$$\xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 1 & -6 & 3 & -8 & 7 \\ 0 & 5 & -2 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A has 2 pivot $\Rightarrow \text{rank}(A) = 2 < 3$

$\therefore$ the rows of A are linear dependent

$\therefore$ set $\{[3,2,1,-4,1], [2,3,0,-1,1], [1,-6,3,-8,7]\}$ are <u>not</u> linear independent

HW03 - Q4

Suppose $Span(\{u_1, \cdots, u_r\}) \cap Span(\{w_1, \cdots, w_s\}) = V \neq \{0\}$

$\therefore v \in Span(\{u_1, \cdots, u_r\}) \Rightarrow v = a_1u_1 + \cdots + a_ru_r \quad , \exists a_i \neq 0 , i \in [1, r]$

and

$v \in Span(\{w_1, \cdots, w_s\}) \Rightarrow v = b_1w_1 + \cdots + b_sw_s \quad , \exists b_j \neq 0 , j \in [1, r]$

$\therefore a_1u_1 + \cdots + a_ru_r = b_1w_1 + \cdots + b_sw_s$

$\therefore (a_1u_1 + \cdots + a_ru_r) - (b_1w_1 + \cdots + b_sw_s) = 0$

$\therefore a_1u_1 + \cdots + a_ru_r + (-b_1)w_1 + \cdots + (-b_s)w_s = 0 \quad , a_i \neq 0 , (-b_j) \neq 0$

$\because \{u_1, \cdots, u_r, w_1, \cdots, w_s\}$ is a linear independent subset

$\therefore \alpha_1u_1 + \cdots + \alpha_ru_r + \beta_1w_1 + \cdots + \beta_sw_s = 0$ IFF $\alpha_1 = \cdots = \alpha_r = \beta_1 = \cdots = \beta_s = 0$

$\therefore$ CONTRADICTION

$\therefore Span(\{u_1, \cdots, u_r\}) \cap Span(\{w_1, \cdots, w_s\}) = \{0\}$

QED

HW03 — Q5

①

$$[A^T \mid B] = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 4 \\ 1 & 1 & 1 & 3 & 1 \\ 3 & 3 & 2 & 8 & 2 \\ 1 & 0 & 1 & 2 & 5 \end{array}\right] \xrightarrow[\substack{R_3 = R_3 - 3R_4 \\ R_4 = R_1 - 2R_4}]{\substack{R_1 = R_4 \\ R_2 = R_2 - R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 3 & -1 & 2 & -13 \\ 0 & 1 & -2 & -1 & -6 \end{array}\right]$$

$$\xrightarrow[R_4 = R_4 - R_2]{R_3 = R_3 - 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -2 \end{array}\right] \xrightarrow[R_4 = R_4 - 2R_3]{R_3 = -R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ have infinite solution

$\therefore$ B belows to the row space $C(A^T)$

②

$$[A^T \mid C] = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 3 & 2 \\ 3 & 3 & 2 & 8 & 3 \\ 1 & 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow[\substack{R_3 = R_3 - 3R_4 \\ R_4 = R_1 - 2R_4}]{\substack{R_1 = R_4 \\ R_2 = R_2 - R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 3 & -1 & 2 & -9 \\ 0 & 1 & -2 & -1 & -7 \end{array}\right]$$

$$\xrightarrow[R_4 = R_4 - R_2]{R_3 = R_3 - 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & -2 & -2 & -5 \end{array}\right] \xrightarrow[R_4 = R_4 - 2R_3]{R_3 = -R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ don't have solution

$\therefore$ C doesn't belong to the row space $C(A^T)$