HW 01 — Book

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Leon-Garcia Chapter #2, 117(d)

① Proof $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$

Claim 1: $f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$

prof: pick $x\in f^{-1}(A\cup B)$

$\therefore f(x)\in A\cup B$ (def of f^{-1})

$\therefore f(x)\in A$ OR $f(x)\in B$ (def of $\cup$)

$\therefore x\in f^{-1}(A)$ OR $x\in f^{-1}(B)$ (def of f^{-1})

$\therefore x\in f^{-1}(A)\cup f^{-1}(B)$ (def of $\cup$)

$\therefore f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$ (def of $\subset$)

Claim 2: $f^{-1}(A)\cup f^{-1}(B)\subset f^{-1}(A\cup B)$

prof: pick $x\in f^{-1}(A)\cup f^{-1}(B)$

$\therefore x\in f^{-1}(A)$ OR $x\in f^{-1}(B)$ (def of $\cup$)

$\therefore f(x)\in A$ OR $f(x)\in B$ (def of f^{-1})

$\therefore f(x)\in A\cup B$ (def of $\cup$)

$\therefore x\in f^{-1}(A\cup B)$ (def of f^{-1})

$\therefore f^{-1}(A)\cup f^{-1}(B)\subset f^{-1}(A\cup B)$ (def of $\subset$)

$\therefore f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$ QED

② Proof $f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B)$

Claim 1: $f^{-1}(A\cap B)\subset f^{-1}(A)\cap f^{-1}(B)$

prof: pick $x\in f^{-1}(A\cap B)$

$\therefore f(x)\in A\cap B$ (def of f^{-1})

$\therefore f(x)\in A$ AND $f(x)\in B$ (def of $\cap$)

$\therefore x\in f^{-1}(A)$ AND $x\in f^{-1}(B)$ (def of f^{-1})

$\therefore x\in f^{-1}(A)\cap f^{-1}(B)$ (def of $\cap$)

$\therefore f^{-1}(A\cap B)\subset f^{-1}(A)\cap f^{-1}(B)$ (def of $\subset$)

Claim 2: $f^{-1}(A)\cap f^{-1}(B)\subset f^{-1}(A\cap B)$

prof: pick $x\in f^{-1}(A)\cap f^{-1}(B)$

$\therefore x \in f^{-1}(A)$ AND $x \in f^{-1}(B)$ (def of $\cap$)

$\therefore f(x) \in A$ AND $f(x) \in B$ (def of f^{-1})

$\therefore f(x) \in A \cap B$ (def of $\cap$)

$\therefore x \in f^{-1}(A \cap B)$ (def of f^{-1})

$\therefore f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$ (def of $\subset$)

$\therefore f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ QED

③ Proof $f^{-1}(A^c) = f^{-1}(A)^c$

Claim 1: $f^{-1}(A^c) \subset f^{-1}(A)^c$

prof: pick $x \in f^{-1}(A^c)$

$\therefore f(x) \in A^c$ (def of f^{-1})

$\therefore f(x) \notin A$ (def of ^{c})

$\therefore x \notin f^{-1}(A)$ (def of f^{-1})

$\therefore x \in f^{-1}(A)^c$ (def of ^{c})

$\therefore f^{-1}(A^c) \subset f^{-1}(A)^c$ (def of $\subset$)

Claim 2: $f^{-1}(A)^c \subset f^{-1}(A^c)$

prof: pick $x \in f^{-1}(A)^c$

$\therefore x \notin f^{-1}(A)$ (def of ^{c})

$\therefore f(x) \notin A$ (def of f^{-1})

$\therefore f(x) \in A^c$ (def of ^{c})

$\therefore x \in f^{-1}(A^c)$ (def of f^{-1})

$\therefore f^{-1}(A)^c \subset f^{-1}(A^c)$ (def of $\subset$)

$\therefore f^{-1}(A^c) = f^{-1}(A)^c$ QED

Gubner Chapter #1, 15 (page 50)

(a) Proof $B \subset Y \longrightarrow f^{-1}(B^c) = f^{-1}(B)^c$

$\because f: X \to Y$ & $B \subset Y$

$\therefore f^{-1}(B) = \{x \in X : f(x) \in B\}$

Claim 1: $f^{-1}(B^c) \subset f^{-1}(B)^c$

prof: pick $x \in f^{-1}(B^c)$

$\therefore f(x) \in B^c$ (def of f^{-1})

$\therefore f(x) \notin B$ (def of ^{c})

$\therefore x \notin f^{-1}(B)$ (def of f^{-1})

$\therefore x \in f^{-1}(B)^c$ (def of ^{c})

$\therefore f^{-1}(B^c) \subset f^{-1}(B)^c$ (def of $\subset$)

Claim 2: $f^{-1}(B)^c \subset f^{-1}(B^c)$

prof: pick $x \in f^{-1}(B)^c$

$\therefore x \notin f^{-1}(B)$ (def of ^{c})

$\therefore f(x) \notin B$ (def of f^{-1})

$\therefore f(x) \in B^c$ (def of ^{c})

$\therefore x \in f^{-1}(B^c)$ (def of f^{-1})

$\therefore f^{-1}(B)^c \subset f^{-1}(B^c)$ (def of $\subset$)

$\therefore f^{-1}(B^c) = f^{-1}(B)^c$

QED

(b) Proof $f^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$

Claim 1: $f^{-1}(\bigcup_{n=1}^{\infty} B_n) \subset \bigcup_{n=1}^{\infty} f^{-1}(B_n)$

prof: pick $x \in f^{-1}(\bigcup_{n=1}^{\infty} B_n)$

$\therefore f(x) \in \bigcup_{n=1}^{\infty} B_n$ (def of f^{-1})

$\therefore \exists n^*, f(x) \in B_{n^*}$ (def of $\cup$)

$\therefore x \in f^{-1}(B_{n^*})$ (def of f^{-1})

$\therefore x \in f^{-1}(B_{n^*}) \cup [\bigcup_{n=1, \& n \neq n^*}^{\infty} f^{-1}(B_n)]$ (Lemma: $P \to P \vee Q$)

$\therefore x \in \bigcup_{n=1}^{\infty} f^{-1}(B_n)$ (def of $\cup$)

Lemma: $P \to P \vee Q$

prof:

P	Q	P	→	P	∨	Q
1	1	1	1	1	1	1
0	1	0	1	0	1	1
1	0	1	1	1	1	0
0	0	0	1	0	0	0

$\therefore$ valid

$\therefore$ QED

$\therefore f^{-1}(\bigcup_{n=1}^{\infty} B_n) \subset \bigcup_{n=1}^{\infty} f^{-1}(B_n)$ (def of $\subset$)

Claim2: $\bigcup_{n=1}^{\infty} f^{-1}(B_n) \subset f^{-1}(\bigcup_{n=1}^{\infty} B_n)$

prof: pick $x \in \bigcup_{n=1}^{\infty} f^{-1}(B_n)$

$\therefore \exists n^*, x \in f^{-1}(B_{n^*})$ (def of $\cup$)

$\therefore f(x) \in B_{n^*}$ (def of f^{-1})

$\therefore f(x) \in B_{n^*} \cup [\bigcup_{n=1, n \neq n^*}^{\infty} B_n]$ (Lemma: $P \rightarrow P \vee Q$)

$\therefore f(x) \in \bigcup_{n=1}^{\infty} B_n$ (def of $\cup$)

$\therefore x \in f^{-1}(\bigcup_{n=1}^{\infty} B_n)$ (def of f^{-1})

$\therefore \bigcup_{n=1}^{\infty} f^{-1}(B_n) \subset f^{-1}(\bigcup_{n=1}^{\infty} B_n)$ (def of $\subset$)

$\therefore \bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}(\bigcup_{n=1}^{\infty} B_n)$ (def of $=$) QED

(C) Proof $f^{-1}(\bigcap_{n=1}^{\infty} B_n) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$

Claim1: $f^{-1}(\bigcap_{n=1}^{\infty} B_n) \subset \bigcap_{n=1}^{\infty} f^{-1}(B_n)$

prof: pick $x \in f^{-1}(\bigcap_{n=1}^{\infty} B_n)$

$\therefore f(x) \in \bigcap_{n=1}^{\infty} B_n$ (def of f^{-1})

$\therefore \forall n^*, f(x) \in B_{n^*}$ (def of $\cap$)

$\therefore \forall n^*, x \in f^{-1}(B_{n^*})$ (def of f^{-1})

$\therefore x \in \bigcap_{n=1}^{\infty} f^{-1}(B_n)$ (def of $\cap$)

$\therefore f^{-1}(\bigcap_{n=1}^{\infty} B_n) \subset \bigcap_{n=1}^{\infty} f^{-1}(B_n)$ (def of $\subset$)

Claim2: $\bigcap_{n=1}^{\infty} f^{-1}(B_n) \subset f^{-1}(\bigcap_{n=1}^{\infty} B_n)$

prof: pick $x \in \bigcap_{n=1}^{\infty} f^{-1}(B_n)$

$\therefore \forall n^*, x \in f^{-1}(B_{n^*})$ (def of $\cap$)

$\therefore \forall n^*, f(x) \in B_{n^*}$ (def of f^{-1})

$\therefore f(x) \in \bigcap_{n=1}^{\infty} B_n$ (def of $\cap$)

$\therefore x \in f^{-1}(\bigcap_{n=1}^{\infty} B_n)$ (def of f^{-1})

$\therefore \bigcap_{n=1}^{\infty} f^{-1}(B_n) \subset f^{-1}(\bigcap_{n=1}^{\infty} B_n)$ (def of $\subset$)

$\therefore f^{-1}(\bigcap_{n=1}^{\infty} B_n) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$ (def of $=$) QED