1. (a)

#5 $\quad v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Let $u_1 = \dfrac{v}{\|v\|} = \dfrac{v}{\sqrt{\sum_{i=1}^{2} v_i^2}}$; $\|v\| = \sqrt{1^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$

$u_2 = \dfrac{w}{\|w\|} = \dfrac{w}{\sqrt{\sum_{i=1}^{3} w_i^2}}$; $\|w\| = \sqrt{2^2+1^2+2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

r is perpendicular to u_1 means $r \cdot u_1 = 0$

Let $r = \begin{bmatrix} a \\ b \end{bmatrix} \qquad r \cdot u_1 = r^T u_1 = r^T \dfrac{v}{\|v\|} = \dfrac{1}{\|v\|} r^T v$

$$= \frac{1}{\sqrt{10}}(a + 3b) = 0$$

So $\quad a + 3b = 0 \Rightarrow a = -3b$.

Then $\quad r = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3b \\ b \end{bmatrix}$ and $\|r\| = \sqrt{a^2+b^2} = \sqrt{(-3b)^2+b^2} = \sqrt{10b^2}$
$= b\sqrt{10}$

$\therefore \quad \dfrac{r}{\|r\|} = \dfrac{1}{b\sqrt{10}} \begin{bmatrix} -3b \\ b \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$

s is perpendicular to u_2 means $s \cdot u_2 = 0$

Let $s = \begin{bmatrix} c \\ d \\ e \end{bmatrix}$, $\quad s \cdot u_2 = s^T u_2 = s^T \dfrac{w}{\|w\|} = \dfrac{1}{\|w\|} s^T w = \dfrac{1}{3} s^T w$

$$= \frac{1}{3}(2c + d + 2e)$$

Let $d = 2c$; $\quad 2e = -2c - d = -2c - 2c = -4c$.
$\Rightarrow e = -2c$

$\therefore \quad s = \begin{bmatrix} c \\ 2c \\ -2c \end{bmatrix}$ and $\|s\| = \sqrt{c^2 + d^2 + e^2} = \sqrt{c^2 + (2c)^2 + (-2c)^2}$
$= \sqrt{c^2 + 4c^2 + 4c^2} = \sqrt{9c^2} = 3c$

$\dfrac{s}{\|s\|} = \dfrac{1}{3c} \begin{bmatrix} c \\ 2c \\ -2c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

<u>CHECK</u> : $\quad v \cdot \dfrac{r}{\|r\|} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}^T \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = -\dfrac{3}{\sqrt{10}} + \dfrac{3}{\sqrt{10}} = 0$

$w \cdot \dfrac{s}{\|s\|} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \dfrac{2}{3} + \dfrac{2}{3} - \dfrac{4}{3} = 0.$

#6

(a) $V = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$V \cdot w = V^T w = 2w_1 - w_2 = 0 \Rightarrow 2w_1 = w_2$.

$\therefore$ All $w = \begin{bmatrix} c \\ 2c \end{bmatrix}$ such that $c \in \mathbb{R}$.

(b) $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$V \cdot u = V^T u = u_1 + u_2 + u_3 = 0 \Rightarrow u_1 + u_2 + u_3 = 0$

$$\therefore \underbrace{u_1 + u_2 + u_3 = 0}_{\text{Hyperplane in } \mathbb{R}^3 : \text{ Plane.}}$$

(c)

$V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $W = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$\underbrace{V \cdot x = 0}_{V \perp x}$ and $\underbrace{W \cdot x = 0}_{W \perp x}$

$\therefore x_1 + x_2 + x_3 = 0$ and $x_1 + 2x_2 + 3x_3 = 0$

$$\begin{array}{rl} & x_1 + x_2 + x_3 = 0 \\ - & x_1 + 2x_2 + 3x_3 = 0 \\ \hline & -x_2 - 2x_3 = 0 \end{array}$$

Let $x_2 = -2x_3$

$\therefore x_1 = -x_2 - x_3 = 2x_3 - x_3 = x_3$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ where $x_3 \in \mathbb{R}$.

Points on a line : $c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ where $c \in \mathbb{R}$.

#13. $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$u \cdot v = 0 \Rightarrow v_1 + 0v_2 + v_3 = 0 \quad \therefore \quad v_1 = -v_3$

$\therefore \quad v = \begin{bmatrix} v_1 \\ 0 \\ -v_1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ where $c \in \mathbb{R}$.

$v \cdot u = 0, \quad v \cdot w = 0, \quad u \cdot w = 0$

#16. $\|v\| = \sqrt{\sum_{j=1}^{9} v_j^2} = \sqrt{\sum_{j=1}^{9} 1^2} = \sqrt{9} = 3.$

$u = \dfrac{v}{\|v\|} = \dfrac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w = \dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad v \cdot w = v^T w = 0$

2-ⓐ $u, v, w \in \mathbb{C}^n$ and $k \in \mathbb{C}$

$$(u+v)\cdot w = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \sum_{i=1}^{n} (u_i+v_i)\, w_i^*$$

$$= \sum_{i=1}^{n} (u_i w_i^* + v_i w_i^*)$$

$$= \sum_{i=1}^{n} u_i w_i^* + \sum_{i=1}^{n} v_i w_i^*$$

$$= (u \cdot w) + (v \cdot w)$$

$\therefore$ QED.

ⓑ $\|u\| \geq 0$; $\|u\| = 0$ if and only if $u = \underline{0}$.

Claim 1: If $\|u\| = 0$, then $u = \underline{0}$

Proof 1: Assume that $u \neq \underline{0}$:

$\therefore \quad \|u\|^2 = \left(\sum_{j=1}^{n} u_j \cdot u_j^*\right)^2 > 0$ — because $u_k \neq 0$ for some k

$\therefore \quad \|u\| = \sqrt{\|u\|^2} > 0$

$\therefore \quad \|u\| = 0$ implies $u = \underline{0}$ QED Claim 1

Claim 2: If $u = \underline{0}$, then $\|u\| = 0$

Proof: Let $u = \underline{0}$

$$\|u\| = \sqrt{\sum_{j=1}^{n} 0^2} = 0 \qquad \text{QED Claim 2.}$$

$\therefore$ QED

ⓒ $$k(u+v) = k\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = k\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} ku_1 + kv_1 \\ \vdots \\ ku_n + kv_n \end{bmatrix}$$

$$= \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix} + \begin{bmatrix} kv_1 \\ \vdots \\ kv_n \end{bmatrix} = k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + k\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = ku + kv$$

---

3 ⓐ $A \in \mathbb{R}^{m \times n}$ $\qquad$ $B \in \mathbb{R}^{n \times m}$

A has a zero row:

$\therefore \exists k : 1 \le k \le m, \quad a_{kl} = 0 \quad \forall l : 1 \le l \le n$ —— ①

$$(AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

NOTE: $\underset{m \times n}{A} \; \underset{n \times m}{B} = \underset{m \times m}{C}$

$$(AB)_{kj} = \sum_{l=1}^{n} a_{kl} b_{lj} = \sum_{l=1}^{n} 0 \, b_{lj} = 0$$ — For all j because of ①

$\therefore \forall j : 1 \le j \le m, \quad (AB)_{kj} = 0$

QED.

ⓑ B has a zero column:

$\exists k : 1 \le k \le m$ and $b_{lk} = 0 \quad \forall l : 1 \le l \le n.$ —— ②

$$(AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$\therefore \forall i : 1 \le i \le m, \quad (AB)_{ik} = \sum_{l=1}^{n} a_{il} b_{lk} = \sum_{l=1}^{n} a_{il} \, 0 = 0$ because of ②

So $\forall i : 1 \le i \le m, \quad (AB)_{ik} = 0$

$\therefore \underline{\exists k : 1 \le k \le m}$ such that $\underline{\forall i : 1 \le i \le m}, \quad (AB)_{ik} = 0$

QED

4 ⓐ $A_1, A_2, \ldots, A_n \in \mathbb{R}$

Proof: By induction

Basis Step: Let $n = 2$

Claim: $(A_1 A_2)^T = A_2^T A_1^T$

Let $A_1 = [a_{ij}]$, $A_2 = [b_{ij}]$ for $1 \le i \le n$, $1 \le j \le n$.

Let $C = A_1 A_2 = [c_{ij}]$

$$\therefore \quad C = \begin{bmatrix} \text{---}\ a_{1:}\ \text{---} \\ \text{---}\ a_{i:}\ \text{---} \\ \text{---}\ a_{n:}\ \text{---} \end{bmatrix} \begin{bmatrix} | & & | & & | \\ b_{:1} & \cdots & b_{:j} & \cdots & b_{:n} \\ | & & | & & | \end{bmatrix} = [c_{ij}]$$

NOTE: $a_{i:}$ is the i^{th} row of A.

$b_{:j}$ is the j^{th} column of B.

$$\therefore \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$\Rightarrow C^T = (A_1 A_2)^T = [\tilde{c}_{ij}]$ where $\tilde{c}_{ij} = c_{ji}$ — definition of transpose.

Let $D = A_2^T A_1^T = [d_{ij}]$. So what is d_{ji}?

$$D = \begin{bmatrix} \text{---}\ b_{:1}\ \text{---} \\ \text{---}\ b_{:j}\ \text{---} \\ \text{---}\ b_{:n}\ \text{---} \end{bmatrix} \begin{bmatrix} | & & | & & | \\ a_{1:} & \cdots & a_{i:} & \cdots & a_{n:} \\ | & & | & & | \end{bmatrix}$$

$$d_{ji} = \sum_{k=1}^{n} b_{kj} a_{ik} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\therefore \quad c_{ij} = d_{ji} \Rightarrow C = D^T \Rightarrow A_1 A_2 = (A_1^T A_2^T)^T$$

$$\Rightarrow (A_1 A_2)^T = A_1^T A_2^T \quad \text{—— QED Basis Step.}$$

Induction Step:

Claim: If $(A_1 A_2 \ldots A_n)^T = A_n^T \ldots A_2^T A_1^T$ then

$(A_1 A_2 \ldots A_n A_{n+1})^T = A_{n+1}^T \ldots A_2^T A_1^T$

Induction hypothesis (IH):

Assume $(A_1 A_2 \ldots A_n)^T = A_n^T \ldots A_2^T A_1^T$:

Check for $n+1$:

$$(A_1 A_2 \ldots A_n A_{n+1})^T = \Big(\underbrace{(A_1 A_2 \ldots A_n)}_{A} \underbrace{A_{n+1}}_{B} \Big)^T \quad \text{——— Basis Step}$$

$$= A_{n+1}^T (A_1 A_2 \ldots A_n)^T$$

$$= A_{n+1}^T (A_n^T \ldots A_2^T A_1^T).$$

$\therefore$ QED Induction Step.

) $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$

roof : By induction

asis Step : $n = 2$

laim : $(A_1 A_2)^{-1} = (A_2^{-1} A_1^{-1})$

$$A_1 A_2 A_2^{-1} A_1^{-1} = A_1 I A_1^{-1} = A_1 A_1^{-1} = I$$

$$A_2^{-1} A_1^{-1} A_1 A_2 = A_2^{-1} I A_2 = A_2^{-1} A_2 = I$$

$\therefore$ $A_2^{-1} A_1^{-1}$ is the inverse of $A_1 A_2$.

$\therefore$ QED Basis Step.

Induction Step :

Claim : If $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$ then

$$(A_1 A_2 \dots A_n A_{n+1})^{-1} = A_{n+1}^{-1} A_n^{-1} \dots A_2^{-1} A_1^{-1}.$$

roof : Induction Hypothesis (IH) :

Assume $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$ —— Induction Hypothesis

Check $(n+1)$:

$$(A_1 A_2 \dots A_n A_{n+1})^{-1} = (\underbrace{A_1}_{A} \underbrace{(A_2 \dots A_n A_{n+1})}_{B})^{-1}$$

$$= \underbrace{(A_2 \dots A_n A_{n+1})^{-1}}_{B^{-1}} \underbrace{A_1^{-1}}_{A^{-1}}$$ —— Basis Step.

$$= (A_{n+1}^{-1} A_n^{-1} \dots A_2^{-1}) A_1^{-1}$$ —— Induction Hypothesis

$$= A_{n+1}^{-1} A_n^{-1} \dots A_2^{-1} A_1^{-1}$$ $\therefore$ QED Induction Step

$\therefore$ $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$ —— Basis Step and Induction Step

$\therefore$ QED.

. Ass. : A is unitary $\Rightarrow$ $A^H A = A A^H = I$

B is unitary $\Rightarrow$ $B^H B = B B^H = I$

a) $A^H (A^H)^H = A^H A$ —— def. of H

$= I$ —— A is unitary.

$(A^H)^H A^H = A A^H$ —— def. of H

$= I$ —— A is unitary.

$\therefore$ A^H is unitary.

) $A^{-1} (A^{-1})^H = A^{-1} (A^H)^H$ —— A is unitary

$= A^{-1} A$

$= I$

$(A^{-1})^H A^{-1} = (A^H)^H A^{-1}$ —— A is unitary

$= A A^{-1}$

$= I$

$\therefore$ A^{-1} is unitary.

(c) $A^H B^{-1} (A^H B^{-1})^H = A^H B^{-1} (B^{-1})^H (A^H)^H$ —— $(XY)^H = Y^H X^H$

$= A^H B^{-1} (B^H)^H (A^H)^H$ —— B is unitary

$= A^H B^{-1} B A$

$= A^H A$

$= I$

$(A^H B^{-1})^H (A^H B^{-1}) = (B^{-1})^H (A^H)^H A^H B^{-1}$

$= (B^H)^H (A^H)^H A^H B^{-1}$

$= B \underbrace{A A^H}_{I} B^{-1}$

$= B I B^{-1}$ —— A is unitary

$= B B^{-1}$

$= I$

$\therefore A^H B^{-1}$ is a unitary matrix.

---

6 (a) $(A + A^H)^H = A^H + (A^H)^H$

$= A^H + A$ —— $(A^H)^H = A$

$= A + A^H$ —— Commutativity

$\therefore (A + A^H)$ is Hermitian.

(b) $(A - A^H)^H = A^H - (A^H)^H$

$= A^H - A$ —— $(A^H)^H = A$

$= -(A - A^H)$ —— commutativity

$\therefore (A - A^H)$ is Skew Hermitian.

(c)

Let $B = \frac{1}{2}(A + A^H)$ and $C = \frac{1}{2}(A - A^H)$

B is Hermitian —— 6(a)

C is Skew Hermitian —— 6(b)

$\therefore B + C = \frac{1}{2}(A + A^H) + \frac{1}{2}(A - A^H)$

$= \frac{1}{2}A + \cancel{\frac{1}{2}A^H} + \frac{1}{2}A - \cancel{\frac{1}{2}A^H}$

$= A$

QED.

7) $A \in \mathbb{R}^{\cdots}$ and $A' = A^{-1}$.

$\therefore \; A A^T = I_n$

Vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ are mutually orthogonal iff $u_i \cdot u_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

$$A = \begin{bmatrix} \text{---} & A_{1:} & \text{---} \\ \text{---} & A_{2:} & \text{---} \\ & \vdots & \\ \text{---} & A_{n:} & \text{---} \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} | & | & & | \\ A_{1:} & A_{2:} & \cdots & A_{n:} \\ | & | & & | \end{bmatrix}$$

$A A^T = I_n$ —— because A is orthogonal

$$\therefore \; (A A^T)_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad \text{——} \quad (1)$$

$$(A A^T)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = A_{i:} \cdot A_{j:} \quad \text{——} \quad (2)$$

From (1) & (2)

$$\therefore \; \underbrace{A_{i:}}_{i^{th}\text{ row}} \cdot \underbrace{A_{j:}}_{j^{th}\text{ row}} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$\therefore$ The rows of A are mutually orthogonal.