Problem Set 6

Due: April 2nd, 11.59pm PST on Gradescope — EE483 Spring 2025

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1. **(50pts) Relation between circular and standard convolution**.

   (a) (5 pts) Consider the causal signals $x[n]$ and $h[n]$ of length N_1 and N_2 respectively. Show that $y[n] = x[n] \circledast h[n]$ is also a causal sequence of length $N_1 + N_2 - 1$.

   (b) (15pts) Let $\tilde{x}[n]$ and $\tilde{h}[n]$ be the zero-padded N length versions of $x[n]$ and $h[n]$, respectively. Prove that if $N \geq N_1 + N_2 - 1$, then

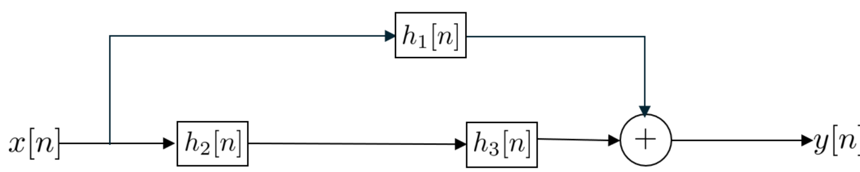
$$y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{1}$$

   is equal to

$$y_c[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[\langle n-k \rangle_N] = \tilde{x}[n] \textcircled{\scriptsize N} \tilde{h}[n] \tag{2}$$

   for $0 \leq n \leq N-1$. Explain why this equivalence is not true when $N < N_1 + N_2 - 1$.

   (c) Now consider four causal signals $x[n]$, $h_1[n]$, $h_2[n]$ and $h_3[n]$ of lengths N_1, N_2, N_3 and N_4, respectively. We want to compute the output $y[n]$ shown in the figure below using forward and inverse FFT (FFT and IFFT).



   i. (5pts) Express the input-output relation of the above system using standard convolution.

   ii. (10pts) We want to replace *all* the standard convolutions in (i) with M-modulo circular convolution. Using (a) and (b), find the minimum value of M that gives the same output as in (i).

   iii. (15pts)Derive an efficient algorithm to compute $y[n]$ using FFT and IFFT. The algorithm should use the IFFT function only once. (Assume the FFT and IFFT functions are available.) Indicate the complexity of this algorithm using $\mathcal{O}(\cdot)$ notation.

2. **(20 pts) Fast DCT using FFT** Let $x[n]$ be a N-length sequence. The N-length DCT-II coefficients $C[k]$ of the sequence $x[n]$ are given by

$$C[k] = \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)k\right] \qquad \text{for } k = 0, \ldots N-1.$$

   A naive implementation of the above formula requires $\mathcal{O}(N^2)$ computations. We are interested in finding a computationally efficient way to compute $C[k]$ using FFT, which has a computational complexity of $\mathcal{O}(N \log N)$ for signals of length N. Consider the sequence $y[n]$ of length $2N$ defined as

$$y[n] = \begin{cases} x[n] & 0 \leq n \leq N-1 \\ x[2N-n-1] & N \leq n \leq 2N-1 \end{cases}.$$

   Let $Y[k]$ be the $2N$-length DFT of $y[n]$.

(a) (10 pts)Prove that

$$Y[k] = 2e^{\frac{j\pi k}{2N}} \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)k\right]$$

for $k = 0, \ldots, N-1$.

(b) (10 pts)Give an algorithm that takes $x[n]$ as input and computes $C[k]$ using $Y[k]$. What is the computational complexity of your algorithm? Express your answer using $\mathcal{O}(\cdot)$ notation. (You don't need to explain FFT implementation in your algorithm. Assume the FFT function is available.)

3. **(30 pts)DFT and inverse DFT** If we denote the DFT operation by $\mathcal{F}$, and its inverse by $\mathcal{F}^{-1}$. That is, for any signal $x[n]$ of length N its DFT is $X[k] = \mathcal{F}\{x[n]\}$ and $x[n] = \mathcal{F}^{-1}\{X[k]\}$.

   (a) (15 pts) Prove that

$$\mathcal{F}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{F}\{X^*[k]\}\right)^*, \tag{3}$$

   and

$$x[n] = \frac{1}{N}\mathcal{F}\{X[\langle -k\rangle_N]\} \tag{4}$$

   We saw in lecture that the complexity of computing $\mathcal{F}\{x[n]\}$ is $\mathcal{O}(N\log(N))$ by using FFT algorithms. What is the complexity of computing the inverse DFT, i.e., $\mathcal{F}^{-1}\{X[k]\}$?

   (b) (15 pts) Compute the N length signal corresponding to applying the DFT operation $L \geq 1$ times to a signal $x[n]$. That is, compute DFT of $x[n]$ then apply the DFT to the result, and then apply DFT to the result and so on. Your result should not have any unresolved sums, and should only depend on the signal $x[n]$, its DFT $X[k]$ and N Hint: first compute for the case $L = 2$, i.e., $\mathcal{F}\{\{\mathcal{F}\{x[n]\}\}$, and then generalize your result for larger L.