Problem Set 8

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1. **Transfer function (30 pts, 5 each)** For the following transfer functions

   (a) $\frac{z-4}{(z-2)^2}$

   (b) $\frac{(z-1/4)^4}{(z+1/2)^5}$

   (c) $\frac{2-z^{-1}}{z-2}$

   (d) $\frac{(z-2)^2}{z+2}$

   (e) $\frac{(z-1/4)^5}{(z+1/2)^4}$

   (f) $\frac{z^2-z-2}{(z-4)^2(z-2)}$

   Sketch the zero/pole locations and identify their possible ROCs, and for each of them indicate whether the system is stable, unstable, causal or non causal. You do not have to compute the impulse response to answer this question.

(a) $\frac{z-4}{(z-2)^2} \Rightarrow \begin{cases} \text{zeros: } z_1 = 4,\ z_2 = \infty \\ \text{poles: } \lambda_1 = \lambda_2 = 2 \end{cases}$

$\therefore$ possible ROCs:

① $\{z: |z| < 2\}$

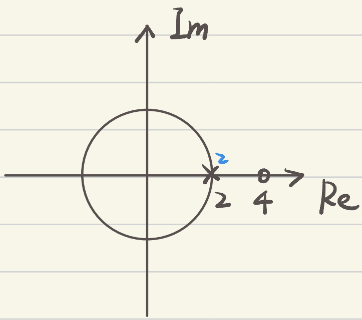
$\because$ it contains the unit circle

$\therefore$ it's not causal but stable

② $\{z: |z| > 2\}$

$\because$ it doesn't contain the unit circle

$\therefore$ it's causal but unstable



(b) $\frac{(z-\frac{1}{4})^4}{(z+\frac{1}{2})^5} \Rightarrow \begin{cases} \text{zeros: } z_1 = z_2 = z_3 = z_4 = \frac{1}{4},\ z_5 = \infty \\ \text{poles: } \lambda = -\frac{1}{2} \text{ with multiplicity } 5 \end{cases}$

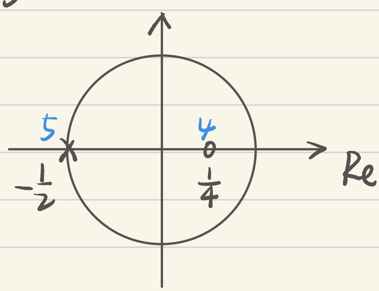
$\therefore$ possible ROCs:

① $\{z: |z| < \frac{1}{2}\}$

not causal and unstable

② $\{z: |z| > \frac{1}{2}\}$

causal and stable



(c) $\frac{z - z^{-1}}{z - 2} = \frac{z^{-1}(z^2 - z^{-1})}{1 - 2z^{-1}} \Rightarrow \begin{cases} \text{zeros: } z_1 = \infty,\ z_2 = \frac{1}{2} \\ \text{poles: } \lambda_1 = 2,\ \lambda_2 = 0 \end{cases}$

$\therefore$ possible ROCs:

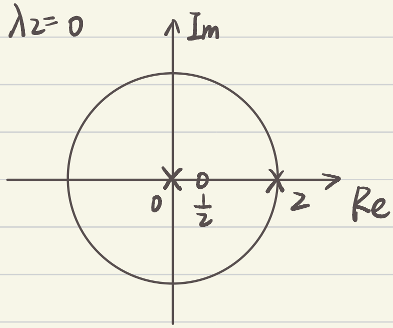
① $\{z: 0 < |z| < 2\}$

$\because |z| \neq 0 \Rightarrow$ non-anticausal signal

$\therefore$ not causal but stable

② $\{z: |z| > 2\}$

causal but unstable



(d) $\frac{(z-2)^2}{z+2} = \frac{(1-2z^{-1})^2}{z^{-1}(1+2z^{-1})} \Rightarrow \begin{cases} \text{zeros: } z_1 = z_2 = 2 \\ \text{poles: } \lambda_1 = \infty,\ \lambda_2 = -2 \end{cases}$

$\therefore$ possible ROCs:

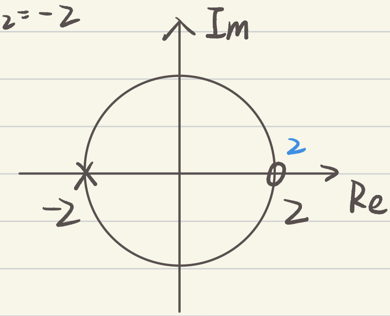
① $\{z: |z| < 2\}$

not causal but stable

② $\{z: 2 < |z| < \infty\}$

$\because |z| \neq \infty \Rightarrow$ not causal

$\therefore$ not causal and unstable



(e) $\frac{(z-\frac{1}{4})^5}{(z+\frac{1}{2})^4} = \frac{(1-\frac{1}{4}z^{-1})^5}{z^{-1}(1+\frac{1}{2}z^{-1})^4} \Rightarrow \begin{cases} \text{zeros: } z = \frac{1}{4} \text{ with multiplicity } 5 \\ \text{poles: } \lambda_1 = \infty,\ \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -\frac{1}{2} \end{cases}$

$\therefore$ possible ROCs:

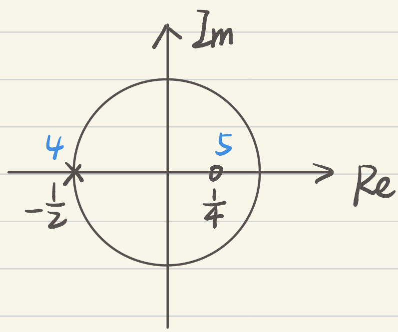
① $\{z: |z| < \frac{1}{2}\}$

not causal and unstable

② $\{z: \frac{1}{2} < |z| < \infty\}$

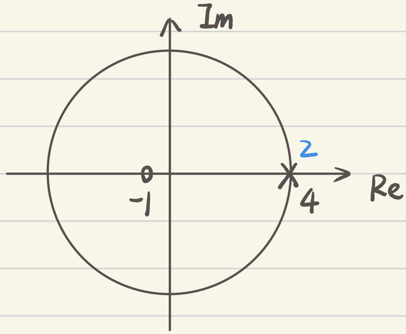
$\because |z| \neq \infty$

$\therefore$ not causal but stable



1f) $\dfrac{z^2-z-2}{(z-4)^2(z-2)} = \dfrac{\cancel{(z-2)}(z+1)}{(z-4)^2\cancel{(z-2)}} = \dfrac{(z+1)}{(z-4)^2} = \dfrac{z^{-1}(1+z^{-1})}{(1-4z^{-1})^2}$

$\Rightarrow \begin{cases} \text{zeros: } z_1=\infty,\ z_2=-1 \\ \text{poles: } \lambda_1=\lambda_2=4 \end{cases}$



$\therefore$ possible ROCs:

① $\{z: |z|<4\}$

not causal but stable

② $\{z: |z|>4\}$

causal but unstable

2. **Filter type (30 pts, 5 each)** For the systems of the previous problem, indicate when does the DTFT converges uniformly. If the DTFT exists, use the pole/zero locations to indicate whether the system is low-pass, high-pass, or something else. Plot the magnitude response using MATLAB to verify your results.

(a) $\frac{z-4}{(z-2)^2} \Rightarrow \begin{cases} \text{zeros: } z_1 = 4, \ (z_2 = \infty) \\ \text{poles: } \lambda_1 = \lambda_2 = 2 \end{cases}$

$\therefore$ when ROC is $\{z : |z| < 2\}$, the DTFT converges uniformly

($\because$ contain the unit circle)

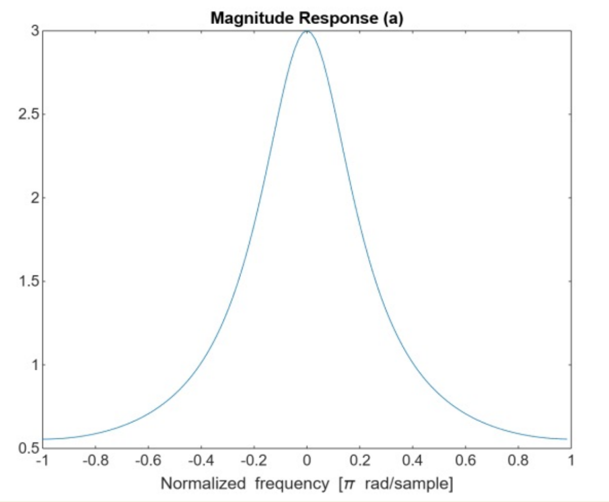
$\because$ # of poles > # of zeros

and poles are closer to the unit circle ($z=1$) than the zeros

$\therefore$ the effect of the poles dominates

$\therefore$ is a low-pass filter

```
%(a)
b = [0 1 -4];
a = [1 -4 4];
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (a)');
```



(b) $\dfrac{(z-\frac{1}{4})^4}{(z+\frac{1}{2})^5} \Rightarrow \begin{cases} \text{zeros: } z=\frac{1}{4} \text{ with multiplicity } 4,\ (z_5=\infty) \\ \text{poles: } \lambda=-\frac{1}{2} \text{ with multiplicity } 5 \end{cases}$

$\therefore$ when $\{z: |z|>\frac{1}{2}\}$, the DTFT converges uniformly

$\because$ poles $\lambda=-\frac{1}{2}$ next to $z=-1$ where is high frequencies

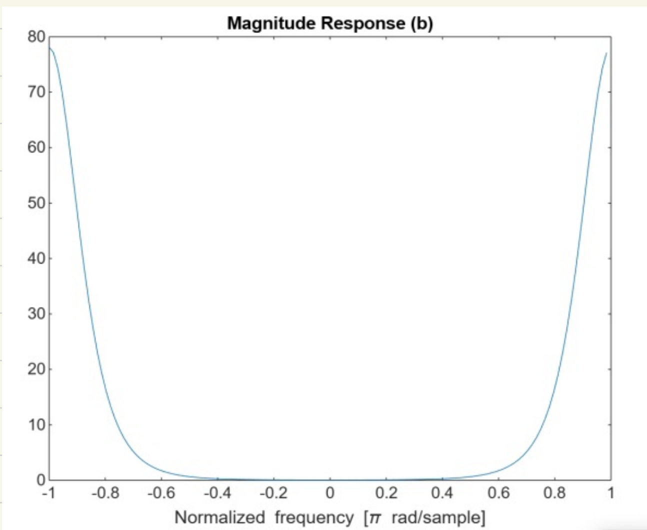
$\Rightarrow$ poles boost the high frequencies

zeros $z=\frac{1}{4}$ next to $z=1$ where is low frequencies

$\Rightarrow$ zeros attenuate the low frequencies

$\therefore$ is a high-pass filter

```
%(b)
syms z;
b = double(fliplr(coeffs(expand((z - 1/4)^4))));
a = double(fliplr(coeffs(expand((z + 1/2)^5))));
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (b)');
```



(c) $\frac{2-z^{-1}}{z-2} = \frac{z^{-1}(2-z^{-1})}{1-2z^{-1}} \Rightarrow \begin{cases} \text{zeros: } z_1 = \frac{1}{2}, \ (z_2 = \infty) \\ \text{poles: } \lambda_1 = 2, \ \lambda_2 = 0 \end{cases}$

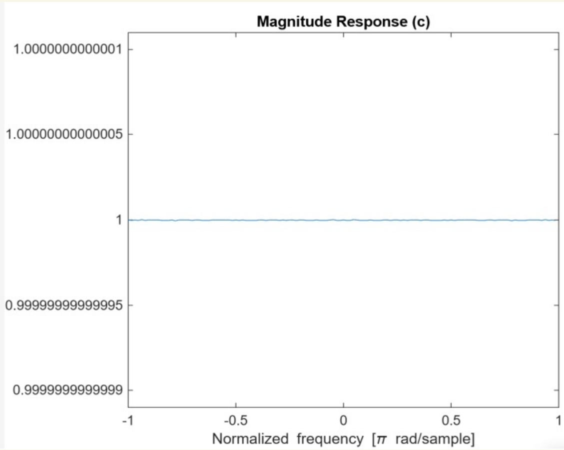
$\therefore$ when $\{z: 0<|z|<2\}$, the DTFT converges uniformly

$\because$ poles and zeros appear in conjugate reciprocal locations

$\lambda_1 = \frac{1}{z_1}$, $\lambda_2 = \frac{1}{z_2}$

$\therefore$ is an all pass filter

```
%(c)
b = [0 2 -1];
a = [1 -2];
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (c)');
```



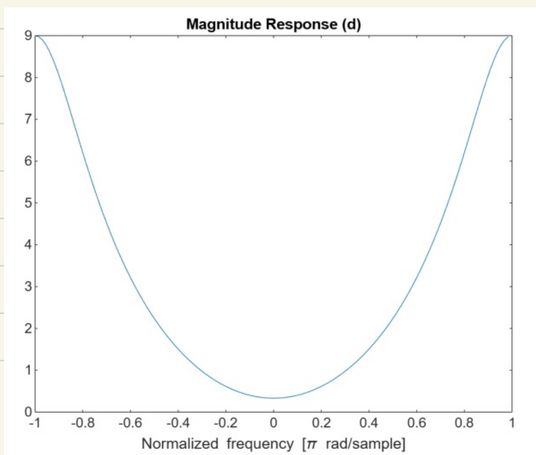
(d) $\frac{(z-2)^2}{z+2} = \frac{(1-2z^{-1})^2}{z^{-1}(1+2z^{-1})} \Rightarrow \begin{cases} \text{zeros: } z_1 = z_2 = 2 \\ \text{poles: } \lambda_1 = \infty, \ \lambda_2 = -2 \end{cases}$

$\therefore$ when $\{z: |z|<2\}$, the DTFT converges uniformly

$\because$ poles are next to $z=-1$ and zeros are next to $z=1$

$\therefore$ is a high-pass filter

```
%(d)
b = [1 -4 4];
a = [0 1 2];
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (d)');
```



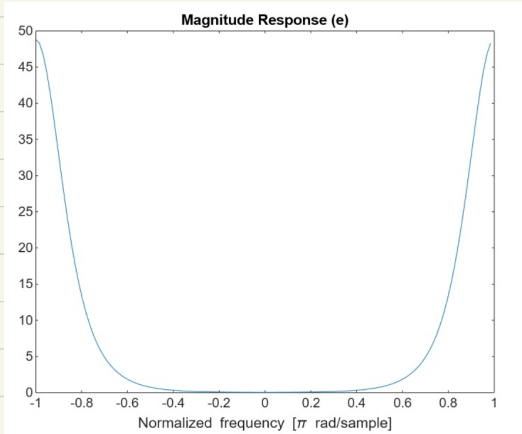
(e) $\dfrac{(z-\frac{1}{4})^5}{(z+\frac{1}{2})^4} = \dfrac{(1-\frac{1}{4}z^{-1})^5}{z^{-1}(1+\frac{1}{2}z^{-1})^4} \Rightarrow \begin{cases} \text{zeros: } z=\frac{1}{4} \text{ with multiplicity } 5 \\ \text{poles: } \lambda_1=\infty,\ \lambda_2=\lambda_3=\lambda_4=\lambda_5=-\frac{1}{2} \end{cases}$

$\therefore$ when $\{z: \frac{1}{2}<|z|<\infty\}$, the DTFT converges uniformly

$\because$ poles are next to $z=-1$ and zeros are next to $z=1$

$\therefore$ is a high-pass filter

```
%(e)
b = double(fliplr(coeffs(expand((z - 1/4)^5))));
a = double(fliplr(coeffs(expand((z + 1/2)^4))));
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (e)');
```

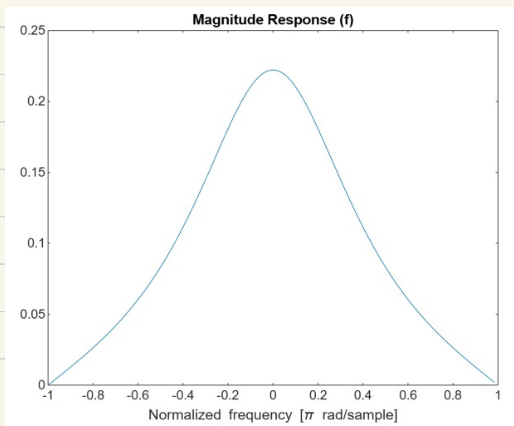


(f) $\dfrac{z^2-z-2}{(z-4)^2(z-2)} = \dfrac{z^{-1}(1+z^{-1})}{(1-4z^{-1})^2} \Rightarrow \begin{cases} \text{zeros: } z_1=-1,\ (z_2=\infty) \\ \text{poles: } \lambda_1=\lambda_2=4 \end{cases}$

$\therefore$ when $\{z: |z|<4\}$, the DTFT converges uniformly

$\because$ poles are next to $z=1$ and zeros are $z=-1$

$\therefore$ is a low-pass filter




```
%(f)
b = [0 1 1];
a = [1 -8 16];
[h, w] = freqz(b, a, 128, 'whole');
h = fftshift(h);
w = (w - pi) / pi;

figure;
plot(w, abs(h));
xlabel('Normalized frequency [\pi rad/sample]');
title('Magnitude Response (f)');
```

3. **GLP filters (35 pts)**

A) (5 pts) Prove that $(1 - az^{-1})(1 - a^*z^{-1}) = 1 - 2\text{Re}(a)z^{-1} + |a|^2 z^{-2}$.

B) You are provided with the following information about the filter $h[n]$.

- $h[n]$ is a causal FIR filter and has a generalized linear phase (GLP).
- All the zeros of $H(z)$ are located on the unit circle and have multiplicity 1 (simple).
- For $\omega \in [0, \pi]$, $H(e^{j\omega}) = 0$ only at the frequencies $\omega = \pi/3$, $\omega = 2\pi/3$, and $\omega = \pi$.
- $h[0] = 1$.

(a) (10 pts) Find all the zeros of $H(z)$ and indicate the order of the filter $h[n]$

(c) (5pts) Roughly sketch by hand the magnitude response of the filter $h[n]$ and state whether the filter is low-pass, high-pass, band-pass, or any other?

(d) (10 pts) Find an expression for $H(z)$ and hence determine $h[n]$.

(e) (5 pts) (i) Is $h[n]$ real? (ii) Does $h[n]$ satisfy the symmetric/antisymmetric property for GLP learned in the class?

A) $(1-az^{-1})(1-a^*z^{-1}) = 1 - a^*z^{-1} - az^{-1} + a \cdot a^* z^{-2}$

$= 1 - (a^* + a)z^{-1} + |a|^2 z^{-2}$

$= 1 - 2\text{Re}(a) z^{-1} + |a|^2 z^{-2}$

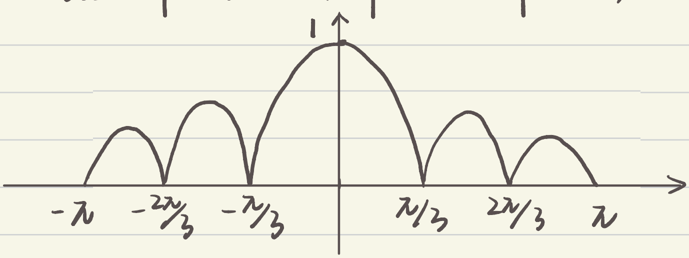
B) (a) $\because$ only when $w = \pi/3$, $w = 2\pi/3$, $w = \pi$, $H(e^{jw}) = 0$

$\therefore$ zeros of $H(z)$ are $z_1 = e^{j\frac{\pi}{3}}$, $z_2 = e^{-j\frac{\pi}{3}}$,

$z_3 = e^{j\frac{2\pi}{3}}$, $z_4 = e^{-j\frac{2\pi}{3}}$, $z_5 = 1$

$\therefore$ order of $h[n]$ = # of zeros of $H(z)$ is 5

(c)



it's a band-pass

(d) $H(z) = (1 - e^{j\frac{\pi}{3}}z^{-1})(1 - e^{-j\frac{\pi}{3}}z^{-1})(1 - e^{j\frac{2\pi}{3}}z^{-1})(1 - e^{-j\frac{2\pi}{3}}z^{-1})(1 + z^{-1})$

$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$\therefore h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$

$= \sum_{k=0}^{5} \delta[n-k]$

(e) $\because$ zeros of $H(z)$ are conjegate pairs

$\therefore$ $h[n]$ is real

$\because$ $h[0]=1$ and $h[5]=1$

$\therefore$ $h[n]=h[5-n]$

$\therefore$ $h[n]$ satisfies the symmetric property

4. **Transfer function (30 pts)**. Consider an LTI system with impulse response $h[n]$ whose DTFT $H(e^{j\omega})$ converges uniformly. The transfer function is

$$H(z) = \frac{2z-3}{z-\frac{1}{3}} \tag{1}$$

(a) (5 pts) Is this system (i) causal, (ii) BIBO stable?

(b) (10 pts)Let $x[n] = \left(\frac{3}{2}\right)^n u[n]$. Find $X(z)$ and then find the output of the LTI system when $x[n]$ is given as input using $z-$transform approach.

(c) (10 pts) Derive a new transfer function $G(z)$ that has the same magnitude response as $H(z)$, but with minimum group delay. Derive the impulse response $g[n]$ of such system.

(d) (5 pts)Plot the magnitude and phase response of $H(z)$ and $G(z)$ using MATLAB.

(a) $\because H(z) = \frac{2z-3}{z-\frac{1}{3}} \Rightarrow \begin{cases} \text{zero}: z = \frac{3}{2} \\ \text{pole}: \lambda = \frac{1}{3} \end{cases}$

and DTFT converges uniformly

$\therefore$ ROC : $\{z : |z| > \frac{1}{3}\}$

$\therefore$ causal and stable

(b) $\because x[n] = (\frac{3}{2})^n u[n]$

$\therefore X(z) = \frac{1}{1-\frac{3}{2}z^{-1}}, \quad |z| > \frac{3}{2}$

$\therefore$ Let $Y(z) = H(z)X(z) = \frac{2z-3}{z-\frac{1}{3}} \cdot \frac{1}{1-\frac{3}{2}z^{-1}}$

$= \frac{\cancel{2-3z^{-1}}\ 2}{(1-\frac{1}{3}z^{-1})\cancel{(1-\frac{3}{2}z^{-1})}}$

$= \frac{2}{1-\frac{1}{3}z^{-1}}$

$\because \text{ROC}_{H(z)} = \{z : |z| > \frac{1}{3}\}$

$\text{ROC}_{X(z)} = \{z : |z| > \frac{3}{2}\}$

$\therefore \text{ROC}_{Y(z)} = \text{ROC}_{H(z)} \cap \text{ROC}_{X(z)} = \{z : |z| > \frac{3}{2}\}$

$\therefore y[n] = 2(\frac{1}{3})^n u[n]$

(c) $\because$ for $H(z)$ $\begin{cases} \text{zero: } z = \frac{3}{2} \\ \text{pole: } \lambda = \frac{1}{3} \end{cases}$

and $G(z)$ has the same magnitude response and with minimum group delay

$\therefore G(z)$ must have zero $z = 1/(\frac{3}{2}) = \frac{2}{3}$

$$\because H(z) = \frac{2z-3}{z-\frac{1}{3}} = \frac{2-3z^{-1}}{1-\frac{1}{3}z^{-1}} = \frac{2(1-\frac{3}{2}z^{-1})}{1-\frac{1}{3}z^{-1}}$$

$$\therefore \text{Let } G(z) = \frac{2\times\frac{3}{2}(1-\frac{2}{3}z^{-1})}{1-\frac{1}{3}z^{-1}} = \frac{3(1-\frac{2}{3}z^{-1})}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{3}{1-\frac{1}{3}z^{-1}} - \frac{2z^{-1}}{1-\frac{1}{3}z^{-1}} \qquad \text{ROC: } \{z : |z| > \tfrac{1}{3}\}$$

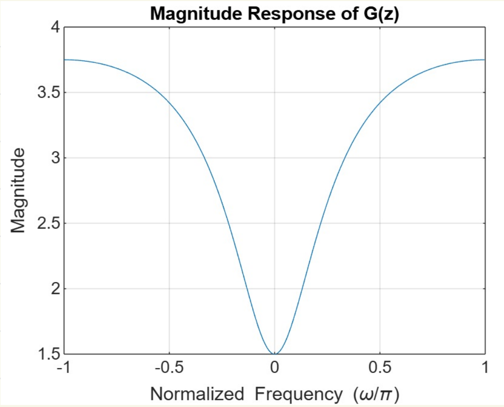
$$\therefore g[n] = 3(\tfrac{1}{3})^n u[n] - 2(\tfrac{1}{3})^{n-1} u[n-1]$$
$$= (\tfrac{1}{3})^{n-1} u[n] - 2(\tfrac{1}{3})^{n-1} u[n-1]$$

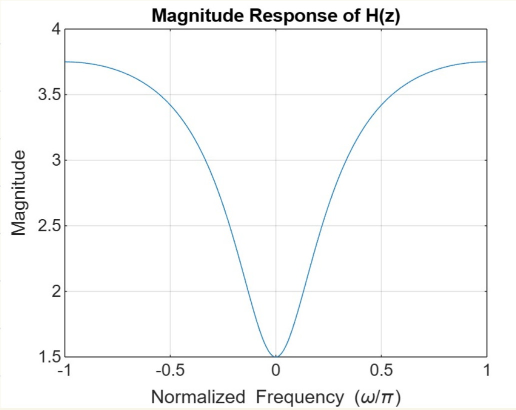
(d)

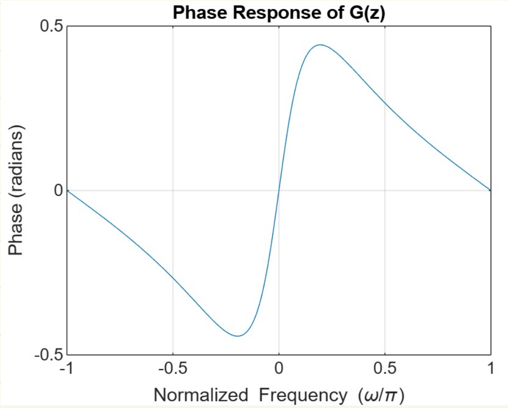
```
b_H = [2 -3];
a_H = [1 -1/3];

b_G = [3 -2];
a_G = [1 -1/3];

[h_H, w] = freqz(b_H, a_H, 512, 'whole');
[h_G, ~] = freqz(b_G, a_G, 512, 'whole');

h_H = fftshift(h_H);
h_G = fftshift(h_G);
w = (w - pi) / pi;
```

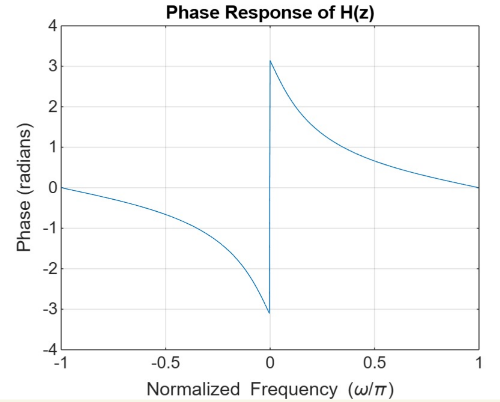
```
figure;
plot(w, abs(h_G));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Magnitude');
title('Magnitude Response of G(z)');
grid on;

figure;
plot(w, angle(h_G));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Phase (radians)');
title('Phase Response of G(z)');
grid on;
```

```
figure;
plot(w, abs(h_H));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Magnitude');
title('Magnitude Response of H(z)');
grid on;

figure;
plot(w, angle(h_H));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Phase (radians)');
title('Phase Response of H(z)');
grid on;
```


Magnitude Response of G(z)
Magnitude
Normalized Frequency (ω/π)


Magnitude Response of H(z)
Magnitude
Normalized Frequency (ω/π)


Phase Response of G(z)
Phase (radians)
Normalized Frequency (ω/π)


Phase Response of H(z)
Phase (radians)
Normalized Frequency (ω/π)

5. **Causality and minimum phase (25 pts)**. The pole-zero plot and the ROC of a transfer function $H(z)$ are shown in Figure 1.

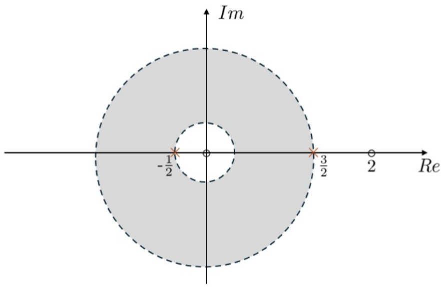


Figure 1: Pole-zero plot and ROC of $H(z)$.

(a) (10 pts)If $\lim_{z\to\infty} H(z) = 1$, (i) find $H(z)$, (ii) find $h[n]$, and (iii) plot the magnitude and phase response of $H(z)$ using MATLAB.

(b) (15 pts)Does this system have minimum group delay? If not, (i) find a new transfer function $G(z)$ that has the same magnitude response as $H(z)$ with minimum group delay, (ii) plot the pole-zero plot of such system and show the ROC, (iii) find $g[n]$, and (iv) plot the magnitude and phase response of $G(z)$ using MATLAB.

(a) according to Figure 1 $\Rightarrow \begin{cases} \text{zeros: } z_1 = 0,\ z_2 = 2 \\ \text{poles: } \lambda_1 = -\frac{1}{2},\ \lambda_2 = \frac{3}{2} \end{cases}$

$\because \lim_{z\to\infty} H(z) = 1$

$\therefore H(z) = \dfrac{z(z-2)}{(z+\frac{1}{2})(z-\frac{3}{2})} = \dfrac{1-2z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{2}z^{-1})}$

$\therefore$ Let $H(z) = \dfrac{A}{1+\frac{1}{2}z^{-1}} + \dfrac{B}{1-\frac{3}{2}z^{-1}}$

$\therefore (1-\frac{3}{2}z^{-1})A + (1+\frac{1}{2}z^{-1})B = 1-2z^{-1}$

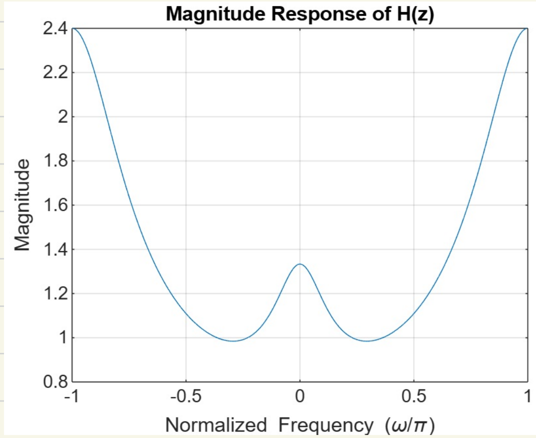
$(-\frac{3}{2}A + \frac{1}{2}B)z^{-1} + (A+B) = -2z^{-1} + 1$

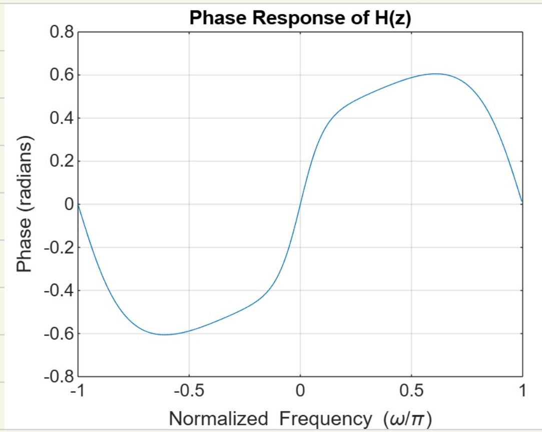
$\therefore \begin{cases} -\frac{3}{2}A + \frac{1}{2}B = -2 \\ A+B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{5}{4} \\ B = -\frac{1}{4} \end{cases}$

$\therefore H(z) = \dfrac{\frac{5}{4}}{1+\frac{1}{2}z^{-1}} - \dfrac{\frac{1}{4}}{1-\frac{3}{2}z^{-1}}$ ROC: $\{z: \frac{1}{2} < |z| < \frac{3}{2}\}$

$\therefore h[n] = \frac{5}{4}(-\frac{1}{2})^n u[n] + \frac{1}{4}(\frac{3}{2})^n u[-n-1]$

```
b_H = [1 -2];
a_H = [1 -1 -3/4];
[h_H, w] = freqz(b_H, a_H, 512, 'whole');
h_H = fftshift(h_H);
w = (w - pi) / pi;

figure;
plot(w, abs(h_H));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Magnitude');
title('Magnitude Response of H(z)');
grid on;

figure;
plot(w, angle(h_H));
xlabel('Normalized Frequency (\omega/\pi)');
ylabel('Phase (radians)');
title('Phase Response of H(z)');
grid on;
```





(b) $\because$ zero $z_2 = 2 > 1$

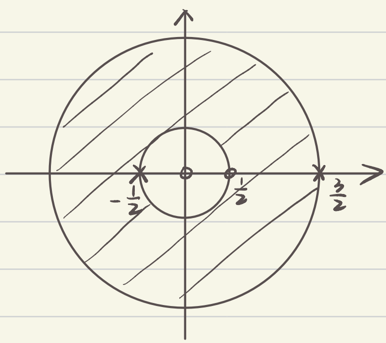
$\therefore$ doesn't have minimum group delay

(i) Let new zero of $G(z)$ $z_2 = \frac{1}{2}$

$$\therefore G(z) = \frac{2(1-\frac{1}{2}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{2}z^{-1})}$$

(ii) for $G(z) \Rightarrow \begin{cases} \text{zeros: } z_1 = 0,\ z_2 = \frac{1}{2} \\ \text{poles: } \lambda_1 = -\frac{1}{2},\ \lambda_2 = \frac{3}{2} \end{cases}$

$\text{ROC}_{G(z)} = \text{ROC}_{H(z)} = \{ z : \frac{1}{2} < |z| < \frac{3}{2} \}$



(iii) Let $G(z) = \dfrac{A}{1+\frac{1}{2}z^{-1}} + \dfrac{B}{1-\frac{3}{2}z^{-1}}$

$\therefore (1-\frac{3}{2}z^{-1})A + (1+\frac{1}{2}z^{-1})B = 2(1-\frac{1}{2}z^{-1})$

$(-\frac{3}{2}A + \frac{1}{2}B)z^{-1} + (A+B) = -z^{-1} + 2$

$$\therefore \begin{cases} -\frac{3}{2}A + \frac{1}{2}B = -1 \\ A+B = 2 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$$

$\therefore G(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}} + \dfrac{1}{1-\frac{3}{2}z^{-1}} \quad \{z: \frac{1}{2} < |z| < \frac{3}{2}\}$

$\therefore g[n] = (\frac{1}{2})^n u[n] - (-\frac{3}{2})^n u[-n-1]$

(iv)

```
b_H = [2 -1];
a_H = [1 -1 -3/4];
[h_H, w] = freqz(b_H, a_H, 512, 'whole');
h_H = fftshift(h_H);
w = (w - pi) / pi;
```