HW04 — Book

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(a) $N \sim G(p)$

① if $k > m$: $P[N=k \mid N \le m] = 0$

② if $k \le m$: $P[N \le m] = 1 - P[N > m] = 1 - (1-p)^m$

$P[(N=k) \cap (N \le m)] = P[N=k] = p(1-p)^{k-1}$

$\therefore P[N=k \mid N \le m] = \dfrac{P[(N=k)\cap(N\le m)]}{P[N\le m]} = \dfrac{p(1-p)^{k-1}}{1-(1-p)^m}$

(b) define H : N is odd

$$\therefore P[H] = P[N=1] + P[N=3] + P[N=5] + \cdots$$
$$= p(1-p)^{1-1} + p(1-p)^{3-1} + p(1-p)^{5-1} + \cdots$$
$$= p \times 1 + p(1-p)^2 + p(1-p)^4 + \cdots$$
$$= p\left[1 + (1-p)^2 + (1-p)^4 + \cdots\right]$$
$$= p \sum_{n=0}^{\infty} (1-p)^{2n}$$
$$= p \frac{1}{1-(1-p)^2}$$
$$= \frac{1}{2-p}$$

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Proof: $\because M \sim G(p)$

$\therefore P[M \geq j+1] = P[M > j] = (1-p)^{j}$

$P[M \geq k+j] = P[M > k+j-1] = (1-p)^{k+j-1}$

$P[M \geq k] = P[M > k-1] = (1-p)^{k-1}$

$\because k > 1$

$\therefore k+j > 1+j$

$\therefore P[(M \geq k+j) \cap (M \geq j+1)] = P[M \geq k+j]$

$$\therefore P[M \geq k+j \mid M \geq j+1] = \frac{P[(M \geq k+j) \cap (M \geq j+1)]}{P[M \geq j+1]}$$

$$= \frac{P[M \geq k+j]}{P[M \geq j+1]}$$

$$= \frac{(1-p)^{k+j-1}}{(1-p)^{j}}$$

$$= (1-p)^{k-1} = P[M \geq k]$$

QED

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I: Poisson Law

M: $b \longrightarrow P$ if: $np = \lambda$ & $n >> 1$ & $p << 1$

$X \sim P(\lambda)$ $P(x=k) = \dfrac{e^{-\lambda}\lambda^{k}}{k!}$, $k = 0, 1, 2, \cdots$

A: define n: 1/12 of a slice of the cake has n bits

$\therefore n = 1/12\,(48+24+12) = 7$

Green : $p_G = 12/(48+24+12) = \frac{1}{7}$ $\lambda_G = np_G = 7 \times \frac{1}{7} = 1$

Red : $p_R = 24/(48+24+12) = \frac{2}{7}$ $\lambda_R = np_R = 7 \times \frac{2}{7} = 2$

Raisins: $p_S = 48/(48+24+12) = \frac{4}{7}$ $\lambda_S = np_S = 7 \times \frac{4}{7} = 4$

(a) $P(G=0) = \dfrac{e^{-1} 1^{0}}{0!} = e^{-1} \approx 0.368$

(b) $P(R \le 2) = P(R=0) + P(R=1) + P(R=2)$

$= \dfrac{e^{-2} 2^{0}}{0!} + \dfrac{e^{-2} 2^{1}}{1!} + \dfrac{e^{-2} 2^{2}}{2!}$

$= e^{-2}(1+2+2)$

$= 5e^{-2}$

≈ 0.677

$\therefore P[B] = P(G=0) \cdot P(R \le 2) \approx 0.249$

(c) $P(R=0) = \dfrac{e^{-2} \cdot 2^{0}}{0!} = e^{-2} \approx 0.135$

$P(S>5) = 1 - P(S \le 5)$

$= 1 - (P(S=0) + P(S=1) + P(S=2) + P(S=3) + P(S=4) + P(S=5))$

$= 1 - \left(\dfrac{e^{-4} 4^{0}}{0!} + \dfrac{e^{-4} 4^{1}}{1!} + \dfrac{e^{-4} 4^{2}}{2!} + \dfrac{e^{-4} 4^{3}}{3!} + \dfrac{e^{-4} 4^{4}}{4!} + \dfrac{e^{-4} 4^{5}}{5!} \right)$

≈ 0.215

$\therefore P[C] = P(G=0) \cdot P(R=0) \cdot P(S>5) \approx 0.0107$

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I: Poisson Law

M: $b \longrightarrow P$ if: $np=\lambda$ & $n>>1$ & $p<<1$

$X \sim P(\lambda)$ $P(x=k)=\dfrac{e^{-\lambda}\lambda^{k}}{k!}$, $k=0,1,2,\cdots$

A: ① $\because \alpha=\dfrac{\lambda}{n\mu}=\dfrac{5}{n\cdot 1}=\dfrac{5}{n}$, $X \sim P(\alpha)$

$\therefore P(X>4)=1-P(X\le 4)<10\%$

$\therefore P(X\le 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)>90\%$

$$=\frac{e^{-\frac{5}{n}}\cdot(\frac{5}{n})^{0}}{0!}+\frac{e^{-\frac{5}{n}}(\frac{5}{n})^{1}}{1!}+\frac{e^{-\frac{5}{n}}(\frac{5}{n})^{2}}{2!}+\frac{e^{-\frac{5}{n}}(\frac{5}{n})^{3}}{3!}+\frac{e^{-\frac{5}{n}}(\frac{5}{n})^{4}}{4!}$$

$$=e^{-\frac{5}{n}}\left(1+\frac{5}{n}+\frac{(\frac{5}{n})^{2}}{2}+\frac{(\frac{5}{n})^{3}}{6}+\frac{(\frac{5}{n})^{4}}{24}\right)>0.9$$

$\because n$ is integer

$\therefore n=3$

② $P[X=0]=\dfrac{e^{-\frac{5}{n}}\cdot(\frac{5}{n})^{0}}{0!}=e^{-\frac{5}{n}}=e^{-\frac{5}{3}}\approx 0.189$

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I: Poisson Law

M: $b \longrightarrow P$. if: $np=\lambda$, $n \gg 1$, $p \ll 1$

$X \sim P(\lambda)$ $\quad P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$ $\quad k=0,1,2,\cdots$

A: $P[\text{request}] = \dfrac{6000}{60 \times 1000} = 0.1 \ll 1$, $n = 100 \gg 1$

$\therefore \lambda = np = 10$

$\therefore$ we can use poisson approximation

(a) $P[X=0] = \dfrac{e^{-10}\,10^0}{0!} = e^{-10} \approx 4.54 \times 10^{-5}$

(b) $P[B] = P[X=5] + P[X=6] + P[X=7] + P[X=8] + P[X=9] + P[X=10]$

$$= \frac{e^{-10}10^5}{5!} + \frac{e^{-10}10^6}{6!} + \frac{e^{-10}10^7}{7!} + \frac{e^{-10}10^8}{8!} + \frac{e^{-10}10^9}{9!} + \frac{e^{-10}10^{10}}{10!}$$

$= e^{-10} \times 12197.97178$

≈ 0.554