HW02 — Book

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Corollary 7: If $A \subset B$, the $P[A] \leq P[B]$

(a) proof: $P[A \cup B \cup C] = P[A] + P[B] + P[C] \underline{- P[A \cap B] - P[B \cap C] - P[C \cap A] + P[A \cap B \cap C]}$

pick $a \in A \cap B \cap C$

$\therefore a \in A \cap B$ AND $a \in C$ (def of $\cap$)

$\therefore A \cap B \cap C \subset A \cap B$ (def of $\subset$)

$\therefore P[A \cap B \cap C] \leq P[A \cap B]$ (Corollary 7)

$\therefore P[A \cap B \cap C] - P[A \cap B] \leq 0$

$\because P[B \cap C] \in [0,1]$, $P[C \cap A] \in [0,1]$ (def of P)

$\therefore - P[A \cap B] - P[B \cap C] - P[C \cap A] + P[A \cap B \cap C] < 0$

$\therefore P[A \cup B \cup C] \leq P[A] + P[B] + P[C]$

QED

(b) proof: Basic: when $n=1$, $P[\bigcup_{k=1}^{1} A_k] = P[A_1]$

when $n=2$, $P[\bigcup_{k=1}^{2} A_k] = P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$

$= \sum_{k=1}^{2} P[A_k] - P[A_1 \cap A_2]$

$\because P[A_1 \cap A_2] \in [0,1]$

$\therefore P[\bigcup_{k=1}^{2} A_k] \leq \sum_{k=1}^{2} P[A_k]$

Induction Hypothesis: when $n=r$, $P[\bigcup_{k=1}^{r} A_k] \leq \sum_{k=1}^{r} P[A_k]$

then, when $n=r+1$: $P[\bigcup_{k=1}^{r+1} A_k] = P[(\bigcup_{k=1}^{r} A_k) \cup A_{r+1}] = P[\bigcup_{k=1}^{r} A_k] + P[A_{r+1}] - P[(\bigcup_{k=1}^{r} A_k) \cap A_{k+1}]$

$\leq \sum_{k=1}^{r} P[A_k] + P[A_{r+1}] - P[(\bigcup_{k=1}^{r} A_k) \cap A_{k+1}]$

$= \sum_{k=1}^{r+1} P[A_k] - P[(\bigcup_{k=1}^{r} A_k) \cap A_{k+1}]$

$\because P[(\bigcup_{k=1}^{r} A_k) \cap A_{k+1}] \in [0,1]$

$\therefore P[\bigcup_{k=1}^{r+1} A_k] \leq \sum_{k=1}^{r+1} P[A_k]$

QED

(c) proof: $P[\bigcap_{k=1}^{n} A_k]$

$= P[(\bigcup_{k=1}^{n} A_k^c)^c]$ (De Morgan's)

$= 1 - P[\bigcup_{k=1}^{n} A_k^c]$ (def of ^{c})

$\because P[\bigcup_{k=1}^{n} A_k^c] \leq \sum_{k=1}^{n} P[A_k^c]$ (Question (b))

$\therefore P[\bigcap_{k=1}^{n} A_k] \geq 1 - \sum_{k=1}^{n} P[A_k^c]$

QED

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(a) ① If $A \cap B = \phi$:

$\therefore P[A|B] = \frac{P(A\cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$

② If $A \subset B \Rightarrow A \cap B = A$

$\therefore P[A|B] = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}$

$\because$ when $A \subset B$, $P[A] \leq P[B]$

$\therefore P[A|B] = \frac{P(A)}{P(B)} \in [0,1]$

③ If $A \supset B \Rightarrow A \cap B = B$

$\therefore P[A|B] = \frac{P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(b) proof: If $P[A|B] = \frac{P(A\cap B)}{P(B)} > P(A)$

$\because P(A) \in [0,1]$, $P(B) \in [0,1]$

$\therefore P(B) < \frac{P(A\cap B)}{P(A)} = P[B|A]$

QED

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(i) proof: $P[A|B] = \frac{P[A\cap B]}{P[B]}$

① $\because P[A\cap B] \geq 0$ and $P[B] \geq 0$ (def of P)

$\therefore P[A|B] = \frac{P[A\cap B]}{P[B]} \geq 0$

② $\because A\cap B \subset B$

$\therefore P[A\cap B] \leq P[B]$

$\therefore P[A|B] = \frac{P[A\cap B]}{P[B]} \leq 1$

$\therefore P[A|B] \in [0,1]$

QED

(ii) proof: $P[S|B] = \frac{P[S\cap B]}{P[B]}$

$\because S\cap B = B$

$\therefore P[S|B] = \frac{P[B]}{P[B]} = 1$

QED

(iii) proof: $P[A\cup C|B] = \frac{P[(A\cup C)\cap B]}{P[B]}$

$= \frac{P[(A\cap B)\cup(C\cap B)]}{P[B]}$ (distributive)

$= \frac{P[A\cap B] + P[C\cap B] - P[(A\cap B)\cap(C\cap B)]}{P[B]}$

$= \frac{P[A\cap B]}{P[B]} + \frac{P[C\cap B]}{P[B]} - \frac{P[A\cap B\cap C]}{P[B]}$

$= P[A|B] + P[C|B] - \frac{P[A\cap C\cap B]}{P[B]}$

$\because A\cap C = \phi \Rightarrow A\cap C\cap B = \phi\cap B = \phi$

$\therefore P[A\cap C\cap B] = P[\phi] = 0$

$\therefore P[A\cup C|B] = P[A|B] + P[C|B]$

QED

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proof: $P[A\cap B\cap C] = P[A\cap(B\cap C)]$ (associative)

$= P[A|B\cap C]\cdot P[B\cap C]$ (conditional pro.)

$\because P[B\cap C] = P[B|C]\cdot P[C]$ (conditional pro.)

$\therefore P[A\cap B\cap C] = P[A|B\cap C]\cdot P[B|C]\cdot P[C]$

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(a) define A: a lot with k defective items is accepted

$$P(A) = \frac{100-k}{100}\cdot\frac{99-k}{99} = \frac{(100-k)(99-k)}{9900}$$

(b) define: D: this item is defective

D^c: this item is not defective

F: test n items are all not defective

F^c: test n items at least one is defective

$\therefore P(D) = \frac{50}{100} = \frac{1}{2}$, $P(D^c) = 1 - P(D) = \frac{1}{2}$

$\therefore P(F) = P(D)^n = (\frac{1}{2})^n$, $P(F^c) = 1 - (\frac{1}{2})^n$

$\therefore 1 - (\frac{1}{2})^n \geqslant 99\%$

$(\frac{1}{2})^n \leq \frac{1}{100}$

$n \lg\frac{1}{2} \leq \lg\frac{1}{100} = -2$

$\therefore n \geqslant \frac{\lg\frac{1}{100}}{\lg\frac{1}{2}} \approx \frac{-2}{-0.301} \approx 6.64$

$\because$ n is a integer

$\therefore n = 7$

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(a) $\mathcal{F} = \sigma(\{A, B\}) = \sigma(\{1,2,3\}, \{3,4,5\})$

$= \sigma(\{\phi, \{1,2,3\}, \{3,4,5\}, \{1,2,3\}^c, \{3,4,5\}^c, \Omega\})$

$= \sigma(\{\phi, \{1,2\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}, \Omega\})$

$= \sigma(\{\phi, \{1,2\}, \{4,5\}, \{1,2\} \cup \{4,5\}, \{1,2,3\}, \{3,4,5\}, \Omega\})$

$= \sigma(\{\phi, \{1,2\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}, \{1,2,4,5\}, \{1,2,4,5\}^c, \Omega\})$

$= \{\phi, \{3\}, \{1,2\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}, \{1,2,4,5\}, \Omega\}$

(b) $P(\phi) = 0$, $P(\Omega) = 1$

$P(A) = P(\{1,2,3\}) = \frac{5}{8}$, $P(B) = P(\{3,4,5\}) = \frac{7}{8}$

$P(\{4,5\}) = 1 - P(A) = 1 - \frac{5}{8} = \frac{3}{8}$

$P(\{1,2\}) = 1 - P(B) = 1 - \frac{7}{8} = \frac{1}{8}$

$P(\{1,2,4,5\}) = P(\{4,5\}) + P(\{1,2\}) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

$P(\{3\}) = 1 - P(\{1,2,4,5\}) = 1 - \frac{1}{2} = \frac{1}{2}$

(c) we cannot compute $P(\{1\})$

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(a) $\underline{I}$ $\underline{M}$

$\underline{A}$: $\underline{\underline{C}}$: pick $A \in \bigcap_\alpha \mathcal{A}_\alpha$, $\quad \therefore A \in \mathcal{A}_\alpha$, $\forall \alpha$

$\therefore A^c \in \mathcal{A}_\alpha$, $\forall \alpha$

$\therefore A^c \in \bigcap_\alpha A_\alpha$, $\forall \alpha$

$\therefore \bigcap_\alpha \mathcal{A}_\alpha$ is closed under complement

$\underline{\underline{U}}$: pick $A_1, A_2 \in \bigcap_\alpha \mathcal{A}_\alpha$, $\quad \therefore A_1, A_2 \in \mathcal{A}_\alpha$, $\forall \alpha$

$\therefore A_1 \cup A_2 \in \mathcal{A}_\alpha$, $\forall \alpha$

$\therefore A_1 \cup A_2 \in \bigcap_\alpha \mathcal{A}_\alpha$

$\therefore \bigcap_\alpha \mathcal{A}_\alpha$ is closed under countable union

$\underline{\underline{I}}$: $\because$ total set $\Omega \in \mathcal{A}_\alpha$, $\forall \alpha$

$\therefore \Omega \in \bigcap_\alpha \mathcal{A}_\alpha$

$\underline{C}$: $\therefore \bigcap_\alpha \mathcal{A}_\alpha$ is CUT

$\therefore \bigcap_\alpha \mathcal{A}_\alpha$ is S.A.

(b) $\mathcal{A}_1 = \sigma(\{1\}, \{2\}, \{3,4\})$

$= \sigma(\phi, \{1\}, \{1\}^c, \{2\}, \{2\}^c, \{3,4\}, \{3,4\}^c, \Omega)$

$= \{\phi, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \Omega\}$

$\mathcal{A}_2 = \sigma(\{1\}, \{3\}, \{2,4\})$

$= \sigma(\phi, \{1\}, \{1\}^c, \{3\}, \{3\}^c, \{2,4\}, \{2,4\}^c, \Omega)$

$= \{\phi, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{2,3,4\}, \{1,2,4\}, \Omega\}$

$\mathcal{A}_1 \cap \mathcal{A}_2 = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \Omega\}$

(c) proof: Claim 1: $\sigma(C) \subset \bigcap_{C \subset \mathcal{A}} \mathcal{A}$

prof: pick $A \in \sigma(C)$

$\because \mathcal{A}$ is S.A.

$\therefore A \in \mathcal{A} \quad \forall \mathcal{A} : C \subset \mathcal{A}$

$\therefore A \in \bigcap_{C \subset \mathcal{A}} \mathcal{A}$

$\therefore \sigma(C) \subset \bigcap_{C \subset \mathcal{A}} \mathcal{A}$

Claim 2: $\bigcap_{C \subset \mathcal{A}} \mathcal{A} \subset \sigma(C)$

proof: pick $A \in \bigcap_{C \subset \mathcal{A}} \mathcal{A}$

$\therefore A \in \mathcal{A} \quad \forall \mathcal{A}: C \subset \mathcal{A}$

$\therefore A \in C$

$\therefore A \in \sigma(C)$

$\therefore \bigcap_{C \subset \mathcal{A}} \mathcal{A} \subset \sigma(C)$

$\therefore \sigma(C) = \bigcap_{\mathcal{A}: C \subset \mathcal{A}} \mathcal{A}$

QED

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no.

counterexample: $\Omega = \{1, 2, 3\}$

$\mathcal{F} = \{\phi, \{1\}, \{2,3\}, \Omega\}$ is S.A.

$\mathcal{G} = \{\phi, \{2\}, \{1,3\}, \Omega\}$ is S.A.

$\therefore \mathcal{H} = \mathcal{F} \cup \mathcal{G} = \{\phi, \{1\}, \{2\}, \{2,3\}, \{1,3\}, \Omega\}$

$\because \{1\} \cup \{2\} = \{1,2\} \notin \mathcal{H}$

$\therefore \mathcal{H}$ is not a S.A.

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① proof: $P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

$$= \frac{P(A \cap (B \cap C))}{P(C)}$$

$$= \frac{P(A | B \cap C) \cdot P(B \cap C)}{P(C)}$$

$$= P(A | B \cap C) P(B | C)$$

② proof: $P(A \cap B \cap C) = P(A \cap (B \cap C))$

$$= P(A | B \cap C) \cdot P(B \cap C)$$

$$= P(A | B \cap C) \cdot P(B | C) \cdot P(C)$$

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(a) define : M : workstation is from M

H : workstation is from H

$P(M) = \frac{140}{140+60} = 70\%$

$P(H) = \frac{60}{140+60} = 30\%$

(b) I : total probability

M : $P(E) = \sum_{k=1}^{n} P(E|H_k) \cdot P(H_k) = P(E|H) \cdot P(H) + P(E|H^c) \cdot P(H^c)$

A : define D : workstation is defective

$P(D) = P(D|M) \cdot P(M) + P(D|H) \cdot P(H)$

$= 10\% \times 70\% + 20\% \times 30\%$

$= 7\% + 6\%$

$= 13\%$

C : the probability that your workstation is defective is 13%

(c) I : Bayes theorem

M : $P(H_k|E) = \frac{P(H_k) P(E|H_k)}{\sum_{j=1}^{n} P(E|H_j) \cdot P(H_j)}$ for partition $\{H_k\}$

A : $P(M|D) = \frac{P(M) P(D|M)}{P(M)P(D|M) + P(H)P(D|H)}$

$= \frac{70\% \times 10\%}{70\% \times 10\% + 30\% \times 20\%}$

$= \frac{7}{13}$

C : when your workstation is defective, the probability that it came from MM is $\frac{7}{13}$.

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(a) Define: D: this chip is defective

$\therefore P(D) = \frac{2}{5}$

(b) I: conditional propability

M: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

A: Define: F: friends reports that one is defective and one is not

$P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{1}{3}$

(c) ① I: total probalility

M: $P(E) = \sum_{k=1}^{n} P(E|H_k) \cdot P(H_k)$

A: Define: F_0: friend takes away two good chips

F_1: friend takes away one is defective and one is not

F_2: friend takes away two defective chips

$P(F_0) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

$P(F_1) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$

$P(F_2) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

$P(D) = P(D|F_0)P(F_0) + P(D|F_1)P(F_1) + P(D|F_2)P(F_2)$

$= \frac{2}{3} \times \frac{3}{10} + \frac{1}{3} \times \frac{3}{5} + 0 \times \frac{1}{10}$

$= \frac{1}{5} + \frac{1}{5}$

$= \frac{2}{5}$

C: ...

② Yes, $\frac{2}{5}$