Assigned: 27 August

Homework #1

EE 503: Fall 2024

Instructions: Write your solutions to these homework problems. Submit your work to Brightspace by the due date. Show all work and box answers where appropriate. Do not guess.

Due: Tuesday, 03 September at 12:00.

1. Consider the integer function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = n^2 + 5$ for any integer n in the set of integers $\mathbb{Z}$. Define the range subsets $A = \{5, 6, 14, 21\} \subset \mathbb{Z}$ and $B = \{9, 14\} \subset \mathbb{Z}$. Define the pullback or *inverse image set* $f^{-1}(S)$ as the set of pre-images $z \in \mathbb{Z}$ of S under the mapping $f : f^{-1}(S) = \{z \in \mathbb{Z} : f(z) \in S\}$.

   (a) Find $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(A \cup B)$, and $f^{-1}(A \cap B)$.

   (b) Verify the commutations $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

   (c) Is it also true that $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) = f(A) \cap f(B)$?

2. Prove or disprove the following statements. You must prove *any* set theoretic theorems that you use. That includes set associativity, commutativity, distributivity, De Morgan's law, *etc.*

   (a) $A \subset B$ if and only if $A - B = \emptyset$.

   (b) $A \cap (B \; \Delta \; C) = (A \cap B) \; \Delta \; (A \cap C)$, where the *symmetric difference* $A \; \Delta \; B = (A - B) \cup (B - A)$.

3. For all functions f and all sets A and B prove or disprove the following statements.

   (a) $f(A \cup B) = f(A) \cup f(B)$.

   (b) $f^{-1}(A) \subset f^{-1}(B)$ if $A \subset B$.

   (c) $f^{-1}(A \; \Delta \; B) = f^{-1}(A) \; \Delta \; f^{-1}(B)$.

4. Let $X = \{N, W, S, E\}$. Find the power set 2^X. How many possible set collections $\mathcal{A} \in 2^X$ are there? How many sigma-algebras can we define on X? Produce all of them. Produce a set collection that is not a sigma-algebra and then show how to minimally augment it to make it a sigma-algebra.

5. Let $X = \{w, x, y, z\}$. Find the power set 2^X. How many possible set collections $\mathcal{A} \in 2^X$ are there? Produce four set collections $\mathcal{A} \subset 2^X$ that are *not* sigma-algebras. Show how to minimally augment these four set collections so that they are sigma-algebras.