Problem Set 3

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1. **LDE, LTI, DTFT. (32pts)** Consider the following LDE corresponding to an LTI system with the initial condition and $y[n] = 0$ for all $n < 0$.

$$y[n] - \left(\frac{1}{3}\right) y[n-1] = x[n]$$

(a) (10pts) Compute the impulse response $h[n]$ by solving the LDE.

(b) (2pts) Is the system causal? Briefly explain.

(c) (2pts) Is the system BIBO stable? Briefly explain.

(d) (2pts) Find $H(e^{j\omega})$, the DTFT of $h[n]$.

(e) Find the output of the system for the following input signals, using convolution ($y[n] = h[n] \circledast x[n]$). Simplify your answer.
(i) (5pts) $x[n] = 2^n u[n]$
(ii) (5pts) $x[n] = \delta[n] + 2\delta[n-1] - \delta[n+1]$.

(f) (6pts, 3pts each answer) Can you use the DTFT and its properties to compute the answers for (e). If yes, show the computations. If not, give a brief explanation.

(a) Let $x[n] = \delta[n]$,

$h[n] - \frac{1}{3} h[n-1] = \delta[n]$

$\because$ LTI system with the initial condition and $y[n]=0$ for all $n<0$

$\therefore$ $h[n]=0$ for all $n<0$

$\therefore$ when $n=0$: $h[0] - \frac{1}{3} h[-1] = \delta[0]$

$h[0] - 0 = 1$

$\therefore h[0] = 1 = \left(\frac{1}{3}\right)^0$

$n=1$: $h[1] - \frac{1}{3} h[0] = \delta[1]$

$h[1] - \frac{1}{3} = 0$

$\therefore h[1] = \frac{1}{3}$

$n=2$: $h[2] - \frac{1}{3} h[1] = \delta[2]$

$h[2] - \frac{1}{3} \times \frac{1}{3} = 0$

$\therefore h[2] = \left(\frac{1}{3}\right)^2$

$\vdots$

$\therefore h[n] = \left(\frac{1}{3}\right)^n u[n]$

(b) $\because$ for all $n<0$, $h[n] = (\frac{1}{3})^n u[n] = 0$

$\therefore$ casual

(c) $\because \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=0}^{+\infty} (\frac{1}{3})^n = \frac{3}{2} < \infty$

$\therefore$ BIBO stable

(d) $H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-jwn}$

$= \sum_{n=-\infty}^{+\infty} (\frac{1}{3})^n u[n] e^{-jwn}$

$= \sum_{n=0}^{+\infty} (\frac{1}{3} e^{-jw})^n$

$= \dfrac{1}{1-\frac{1}{3}e^{-jw}}$

(e) (i) $y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$

$= \sum_{k=-\infty}^{+\infty} (\frac{1}{3})^k u[k] \cdot 2^{n-k} u[n-k]$

$= \sum_{k=0}^{n} (\frac{1}{3})^k \cdot 2^{n-k}$

$= 2^n \sum_{k=0}^{n} (\frac{1}{6})^k$

$= 2^n \cdot \dfrac{1-(\frac{1}{6})^{n+1}}{1-\frac{1}{6}}$

$= \frac{6}{5} \cdot 2^n (1-(\frac{1}{6})^{n+1}) u[n]$

(ii) $y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$

$= \sum_{k=-\infty}^{+\infty} (\delta[k] + 2\delta[k-1] - \delta[k+1]) (\frac{1}{3})^{n-k} u[n-k]$

$= \delta[0] \cdot (\frac{1}{3})^n u[n] + 2\delta[0] \cdot (\frac{1}{3})^{n-1} u[n-1] - \delta[0] (\frac{1}{3})^{n+1} u[n+1]$

$= (\frac{1}{3})^n u[n] + 2(\frac{1}{3})^{n-1} u[n-1] - (\frac{1}{3})^{n+1} u[n+1]$

$\therefore$ when $n<-1$, $y[n] = 0+0+0 = 0$

$-1 \le n < 0$, $y[n] = 0+0-(\frac{1}{3})^{n+1} = -\frac{1}{3}(\frac{1}{3})^n$

$0 \le n < 1$, $y[n] = (\frac{1}{3})^n - (\frac{1}{3})^{n+1} = \frac{2}{3}(\frac{1}{3})^n$

$1 \le n$, $y[n] = (\frac{1}{3})^n + 2(\frac{1}{3})^{n-1} - (\frac{1}{3})^{n+1} = \frac{20}{3}(\frac{1}{3})^n$

(f) (i) $X(e^{jw}) = \sum_{n=-\infty}^{+\infty} 2^n u[n] e^{-jwn}$

$= \sum_{n=0}^{+\infty} 2^n e^{-jwn}$

$= \sum_{n=0}^{+\infty} (2e^{-jw})^n$

$\because \sum_{n=-\infty}^{+\infty} |x[n]| = \sum_{n=0}^{\infty} 2^n = \infty$ is not summable

$\therefore$ DTFT does not exist

$\therefore$ cannot compute

(ii) $\because \delta[n] \xrightarrow{DTFT} 1$

$\delta[n-1] \xrightarrow{DTFT} e^{-jw}$

$\delta[n+1] \xrightarrow{DTFT} e^{jw}$

$\therefore X(e^{jw}) = 1 + 2e^{-jw} - e^{jw}$

$\therefore Y(e^{jw}) = H(e^{jw}) X(e^{jw})$

$= \frac{1}{1-\frac{1}{3}e^{-jw}} (1 + 2e^{-jw} - e^{jw})$

$= \frac{1}{1-\frac{1}{3}e^{-jw}} + \frac{2e^{-jw}}{1-\frac{1}{3}e^{-jw}} - \frac{e^{jw}}{1-\frac{1}{3}e^{-jw}}$

$\because \frac{1}{1-\frac{1}{3}e^{-jw}} \xrightarrow{IDTFT} (\frac{1}{3})^n u[n]$

$\frac{2e^{-jw}}{1-\frac{1}{3}e^{-jw}} \xrightarrow{IDTFT} 2 \cdot (\frac{1}{3})^{n-1} u[n-1]$

$\frac{e^{jw}}{1-\frac{1}{3}e^{-jw}} \xrightarrow{IDTFT} (\frac{1}{3})^{n+1} u[n+1]$

$\therefore y[n] = (\frac{1}{3})^n u[n] + 2 \cdot (\frac{1}{3})^{n-1} u[n-1] - (\frac{1}{3})^{n+1} u[n+1]$

same as (e) (ii)

2. **DTFT practice.(28pts, 4pts each)** Compute the DTFT and the inverse DTFT for the following sequences. It should be obvious which one to compute for each case.

(a) $x[n] = (\frac{2}{3})^n u[n+1]$

(b) $x[n] = (\frac{1}{2})^{2|n|}$

(c) $x[n] = \cos\left(\frac{\pi n}{4}\right) + \sin(n)$

(d) $x[n] = (\frac{1}{3})^n \cos\left(\frac{n}{3}\right) u[n]$

(e) $X(e^{j\omega}) = \sin(4\omega)$ for $-\pi \le \omega \le \pi$

(f) $X(e^{j\omega}) = \sin\left(\frac{\omega}{4}\right)$ for $-\pi \le \omega \le \pi$

(g) $X(e^{j\omega}) = \sin^2(\omega) + \cos^2(3\omega)$ for $-\pi \le \omega \le \pi$

(a) $x[n] = (\frac{2}{3})^n u[n+1]$

$$\therefore X(e^{jw}) = \sum_{n=-\infty}^{+\infty} (\tfrac{2}{3})^n u[n+1] e^{-jwn}$$
$$= \sum_{n=-1}^{+\infty} (\tfrac{2}{3})^n e^{-jwn}$$
$$= \sum_{n=-1}^{+\infty} (\tfrac{2}{3} e^{-jw})^n$$

Let $m = n+1$, $n = m-1$

$$\therefore X(e^{jw}) = \sum_{m=0}^{+\infty} (\tfrac{2}{3} e^{-jw})^{m-1}$$
$$= (\tfrac{2}{3} e^{-jw})^{-1} \sum_{m=0}^{+\infty} (\tfrac{2}{3} e^{-jw})^m$$
$$= \frac{\frac{3}{2} e^{jw}}{1 - \frac{2}{3} e^{-jw}}$$

(b) $x[n] = (\frac{1}{2})^{2|n|} = (\frac{1}{4})^{|n|} = (\frac{1}{4})^n u[n] + (\frac{1}{4})^{-n} u[-n-1]$

$$\therefore X(e^{jw}) = \sum_{n=-\infty}^{+\infty} \left((\tfrac{1}{4})^n u[n] + (\tfrac{1}{4})^{-n} u[-n-1]\right) e^{-jwn}$$
$$= \sum_{n=0}^{+\infty} (\tfrac{1}{4})^n e^{-jwn} + \sum_{n=-\infty}^{-1} (\tfrac{1}{4})^{-n} e^{-jwn} \quad (\text{Let } m=-n)$$
$$= \sum_{n=0}^{+\infty} (\tfrac{1}{4} e^{-jw})^n + \sum_{m=1}^{+\infty} (\tfrac{1}{4} e^{jw})^m$$
$$= \sum_{n=0}^{+\infty} (\tfrac{1}{4} e^{-jw})^n + \sum_{m=0}^{+\infty} (\tfrac{1}{4} e^{jw})^m - (\tfrac{1}{4} e^{jw})^0$$
$$= \frac{1}{1 - \frac{1}{4} e^{-jw}} + \frac{1}{1 - \frac{1}{4} e^{jw}} - 1$$

(c) $x[n] = \cos(\frac{\pi n}{4}) + \sin(n)$

$$= \frac{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}}{2} + \frac{e^{jn} - e^{-jn}}{2j}$$

$$= \frac{1}{2} e^{j\frac{\pi n}{4}} + \frac{1}{2} e^{-j\frac{\pi n}{4}} + \frac{1}{2j} e^{jn} - \frac{1}{2j} e^{-jn}$$

$$\therefore X(e^{jw}) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2} e^{j\frac{\pi n}{4}} + \frac{1}{2} e^{-j\frac{\pi n}{4}} + \frac{1}{2j} e^{jn} - \frac{1}{2j} e^{-jn}\right) e^{-jwn}$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} 2\pi \delta(w - \frac{\pi}{4} + 2\pi k) + \frac{1}{2} \sum_{k=-\infty}^{+\infty} 2\pi \delta(w + \frac{\pi}{4} + 2\pi k)$$

$$+ \frac{1}{2j} \sum_{k=-\infty}^{+\infty} 2\pi \delta(w - 1 + 2\pi k) - \frac{1}{2j} \sum_{k=-\infty}^{+\infty} 2\pi \delta(w + 1 + 2\pi k)$$

$$= \pi \sum_{k=-\infty}^{+\infty} \left(\delta(w - \frac{\pi}{4} + 2\pi k) + \delta(w + \frac{\pi}{4} + 2\pi k)\right)$$

$$+ \frac{\pi}{j} \sum_{k=-\infty}^{+\infty} \left(\delta(w - 1 + 2\pi k) - \delta(w + 1 + 2\pi k)\right)$$

$$= \pi\left(\delta(w - \frac{\pi}{4}) + \delta(w + \frac{\pi}{4})\right) + \frac{\pi}{j}\left(\delta(w-1) - \delta(w+1)\right)$$

(d) $x[n] = (\frac{1}{3})^n \cos(\frac{n}{3}) u[n]$

$$= (\frac{1}{3})^n \frac{e^{j\frac{1}{3}n} + e^{-j\frac{1}{3}n}}{2} u[n]$$

$$\therefore X(e^{jw}) = \sum_{n=+\infty}^{-\infty} (\frac{1}{3})^n \cdot \frac{1}{2} \cdot \left(e^{j\frac{1}{3}n} + e^{-j\frac{1}{3}n}\right) u[n]\, e^{-jwn}$$

$$= \frac{1}{2} \sum_{n=0}^{-\infty} \left(\frac{1}{3} e^{-j(w-\frac{1}{3})}\right)^n + \left(\frac{1}{3} e^{-j(w+\frac{1}{3})}\right)^n$$

$$= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{3} e^{-j(w-\frac{1}{3})}} + \frac{1}{1 - \frac{1}{3} e^{-j(w+\frac{1}{3})}}\right)$$

(e) $X(e^{jw}) = \sin(4w)$ for $-\pi \le w \le \pi$

$$= \frac{e^{j4w} - e^{-j4w}}{2j}$$

$$= -\frac{j}{2}\left(e^{j4w} - e^{-j4w}\right)$$

$$\because \delta[n - n_0] \xrightarrow{DTFT} e^{-jwn_0}$$

$$\therefore x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} -\frac{j}{2}\left(e^{j4w} - e^{-j4w}\right) e^{jwn}\, dw$$

$$= -\frac{j}{2}\left(\delta[n+4] - \delta[n-4]\right)$$

$$= \frac{j}{2}\left(\delta[n-4] - \delta[n+4]\right)$$

(f) $X(e^{jw}) = \sin\left(\frac{w}{4}\right)$ for $-\pi \le w \le \pi$

$$= \frac{e^{j\frac{w}{4}} - e^{-j\frac{w}{4}}}{2j}$$

$$= -\frac{j}{2}\left(e^{j\frac{w}{4}} - e^{-j\frac{w}{4}}\right)$$

$$\therefore x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} -\frac{j}{2}\left(e^{j\frac{w}{4}} - e^{-j\frac{w}{4}}\right) e^{jwn}\, dw$$

$$= -\frac{j}{2}\left(\delta[n+\tfrac{w}{4}] - \delta[n-\tfrac{w}{4}]\right)$$

$$= \frac{j}{2}\left(\delta[n-\tfrac{w}{4}] - \delta[n+\tfrac{w}{4}]\right)$$

(g) $X(e^{jw}) = \sin^2(w) + \cos^2(3w)$ for $-\pi \le w \le \pi$

$$= \frac{1-\cos(2w)}{2} + \frac{\cos(6w)+1}{2}$$

$$= 1 - \frac{1}{2}\cos(2w) + \frac{1}{2}\cos(6w)$$

Suppose $X_1(e^{jw}) = 1$,

$$X_2(e^{jw}) = \frac{1}{2}\cos(2w) = \frac{1}{2}\,\frac{e^{j2w} + e^{-j2w}}{2} = \frac{1}{4}\left(e^{j2w} + e^{-j2w}\right)$$

$$X_3(e^{jw}) = \frac{1}{2}\cos(6w) = \frac{1}{2}\,\frac{e^{j6w} + e^{-j6w}}{2} = \frac{1}{4}\left(e^{j6w} + e^{-j6w}\right)$$

$$\therefore x_1[n] = \delta[n]$$

$$x_2[n] = \frac{1}{4}\left(\delta[n+2] + \delta[n-2]\right)$$

$$x_3[n] = \frac{1}{4}\left(\delta[n+6] + \delta[n-6]\right)$$

$$\therefore x[n] = \delta[n] + \frac{1}{4}\left(\delta[n+2] + \delta[n-2] + \delta[n+6] + \delta[n-6]\right)$$

3. **DTFT Properties (14 pts)**

(a) (2 pts) Compute $X(e^{j\omega})$, the DTFT of the signal $x[n] = \frac{\sin(\frac{\pi n}{8})}{\pi n}$, and sketch the magnitude of the the DTFT, $|X(e^{j\omega})|$, for $\omega \in [-\pi, \pi]$ by hand.

(b) For each of the following signals, use the properties of DTFT to find their corresponding DTFT and sketch their magnitude for $\omega \in [-\pi, \pi]$ by hand.

i. (3pts)$y_1[n] = x[n] \cos\left(\frac{\pi n}{4}\right)$

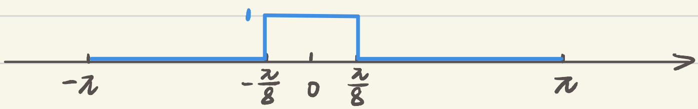
ii. (3pts) $y_2[n] = x[n] \circledast \cos\left(\frac{\pi n}{12}\right)$

iii. (3pts) $y_3[n] = x[n] \circledast \sin\left(\frac{\pi n}{10}\right) \circledast \cos\left(\frac{\pi n}{4}\right)$

iv. (3pts) $y_4[n] = x[n+1] + x[n-1]$

(a) $\because \frac{\sin w_c n}{\pi n} \xrightarrow{DTFT} \begin{cases} 1, & 0 \le |w| \le w_c \\ 0, & w_c < |w| \le \pi \end{cases}$

$\therefore X(e^{jw}) = \begin{cases} 1, & 0 \le |w| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} \le |w| \le \pi \end{cases}$



(b)(i) Let $g[n] = \cos(\frac{\pi n}{4})$

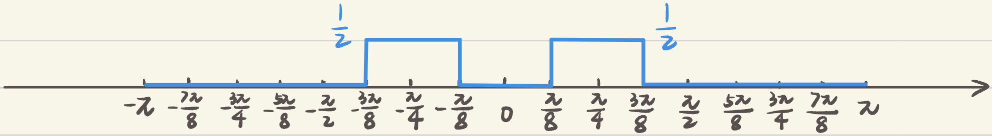
$\therefore y_1[n] = x[n]\, g[n]$

$\therefore Y_1(e^{jw}) = \frac{1}{2\pi} X(e^{jw}) \circledast G(e^{jw})$

$\because g[n] = \cos(\frac{\pi n}{4}) = \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}$

$\therefore G(e^{jw}) = \pi(\delta[w + \frac{\pi}{4}] + \delta[w - \frac{\pi}{4}])$

$\therefore Y_1(e^{jw}) = \frac{1}{2}(X(e^{j(w-\frac{\pi}{4})}) + X(e^{j(w+\frac{\pi}{4})}))$



(ii) Let $g[n] = \cos(\frac{\pi n}{12}) = \frac{e^{j\frac{\pi}{12}n} + e^{-j\frac{\pi}{12}n}}{2}$

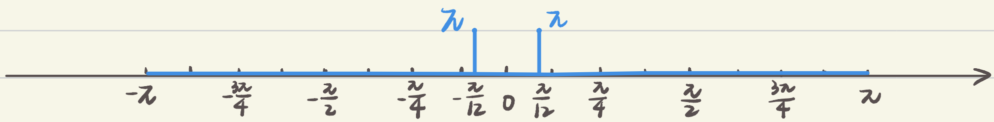
$\therefore G(e^{jw}) = \pi(\delta[w+\frac{\pi}{12}] + \delta[w-\frac{\pi}{12}])$

$\therefore y_2[n] = x[n] \circledast g[n]$

$Y_2(e^{jw}) = X(e^{jw}) \cdot G(e^{jw})$

$= \pi X(e^{jw})\delta[w+\frac{\pi}{12}] + \pi X(e^{jw})\delta[w-\frac{\pi}{12}]$

$= \begin{cases} \pi, & w = \pm\frac{\pi}{12} \\ 0, & \text{else} \end{cases}$



(iii) Let $g_1[n] = \sin(\frac{\pi n}{10}) = \frac{e^{j\frac{\pi}{10}n} - e^{-j\frac{\pi}{10}n}}{2j}$

$g_2[n] = \cos(\frac{\pi n}{4}) = \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}$

$\therefore G_1(e^{jw}) = \frac{\pi}{j}(\delta[w-\frac{\pi}{10}] - \delta[w+\frac{\pi}{10}])$

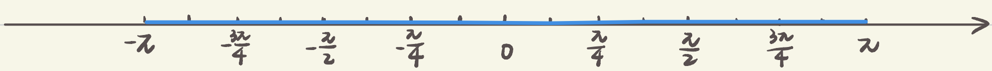
$G_2(e^{jw}) = \pi(\delta[w+\frac{\pi}{4}] + \delta[w-\frac{\pi}{4}])$

$\because y_3[n] = x[n] \circledast g_1[n] \circledast g_2[n]$

$\therefore Y_3(e^{jw}) = X(e^{jw}) \cdot G_1(e^{jw}) \cdot G_2(e^{jw})$

$= \frac{\pi^2}{j} X(e^{jw}) \cdot (\delta[w-\frac{\pi}{10}] - \delta[w+\frac{\pi}{10}])(\delta[w+\frac{\pi}{4}] + \delta[w-\frac{\pi}{4}])$

$= 0$



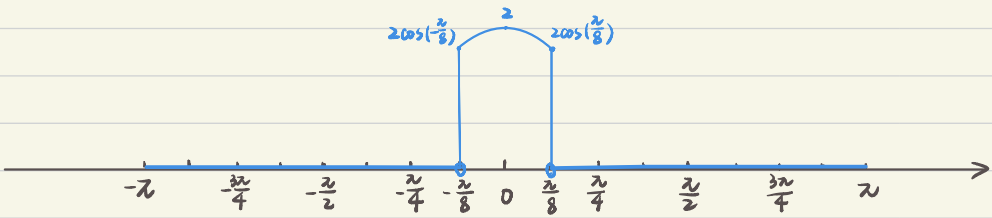
(iv) $\because y_4[n] = x[n+1] + x[n-1]$

$\therefore Y_4(e^{jw}) = e^{jw} X(e^{jw}) + e^{-jw} X(e^{jw})$

$= X(e^{jw})(e^{jw} + e^{-jw})$

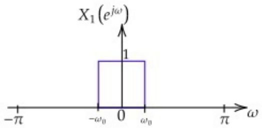
$= X(e^{jw}) \cdot 2\cos(w)$

$= \begin{cases} 2\cos(w), & 0 \le |w| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} \le |w| \le 0 \end{cases}$

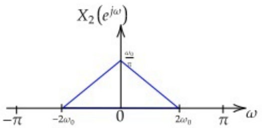


4. **Inverse DTFT and Properties. 36 pts, a)8pts, b)8pts, c)12pts, d)8pts** For each of the sketches of $X(e^{jw})$ below, find the corresponding signals $x[n]$ in the simplified form. (Hint: Express each of these as sums of sacled, shifted rectangles ($X_1(e^{j\omega})$) and triangles ($X_2(e^{j\omega})$) and use their inverse DTFTs)

DTFT Pairs:

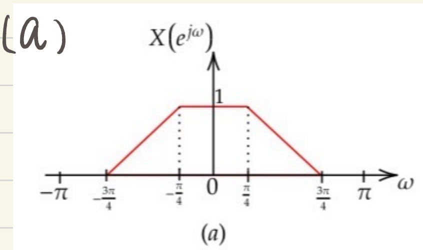


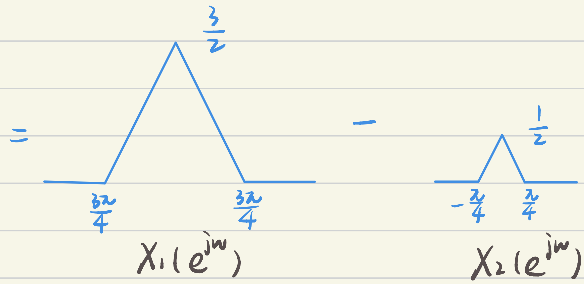
$$X_1(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases} \rightleftharpoons x_1[n] = \begin{cases} \dfrac{\sin(\omega_0 n)}{\pi n} & n \neq 0 \\ \dfrac{\omega_0}{\pi} & n = 0 \end{cases}$$



$$X_2(e^{j\omega}) = \begin{cases} \dfrac{1}{2\pi}(2\omega_0 - |\omega|) & |\omega| \le 2\omega_0 \\ 0 & 2\omega_0 < |\omega| \le \pi \end{cases} \rightleftharpoons x_2[n] = \begin{cases} \left(\dfrac{\sin(\omega_0 n)}{\pi n}\right)^2 & n \neq 0 \\ \left(\dfrac{\omega_0}{\pi}\right)^2 & n = 0 \end{cases}$$

(a)





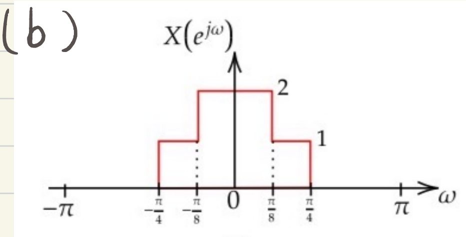
$$X_1(e^{jw}) = \begin{cases} \frac{2}{\pi}\left(\frac{3\pi}{4} - |w|\right), & |w| \le \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} 4\left(\frac{\sin(\frac{3\pi}{8}n)}{\pi n}\right)^2, & n \neq 0 \\ 4\left(\frac{3}{8}\right)^2, & n = 0 \end{cases}$$

$$X_2(e^{jw}) = \begin{cases} \frac{2}{\pi}\left(\frac{\pi}{4} - |w|\right), & |w| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} 4\left(\frac{\sin(\frac{\pi}{8}n)}{\pi n}\right)^2, & n \neq 0 \\ 4\left(\frac{1}{8}\right)^2, & n = 0 \end{cases}$$

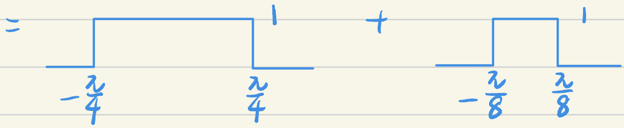
$$\because X(e^{jw}) = X_1(e^{jw}) - X_2(e^{jw})$$

$$\therefore x[n] = x_1[n] - x_2[n] = \begin{cases} 4\left(\frac{\sin(\frac{3\pi}{8}n)}{\pi n}\right)^2 - 4\left(\frac{\sin(\frac{\pi}{8}n)}{\pi n}\right)^2, & n \neq 0 \\ 4\left(\frac{3}{8}\right)^2 - 4\left(\frac{1}{8}\right)^2 = \frac{1}{2}, & n = 0 \end{cases}$$

(b)



(b)



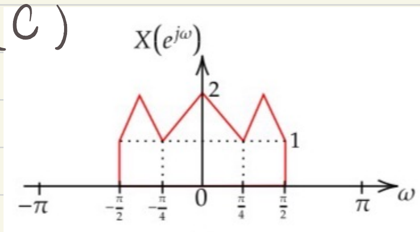
$$X_1(e^{jw}) = \begin{cases} 1, & |w| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} \frac{\sin(\frac{\pi}{4}n)}{\pi n}, & n \neq 0 \\ \frac{1}{4}, & n = 0 \end{cases}$$

$$X_2(e^{jw}) = \begin{cases} 1, & |w| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |w| \le \pi \end{cases} \longleftrightarrow x_2[n] = \begin{cases} \frac{\sin(\frac{\pi}{8}n)}{\pi n}, & n \neq 0 \\ \frac{1}{8}, & n = 0 \end{cases}$$

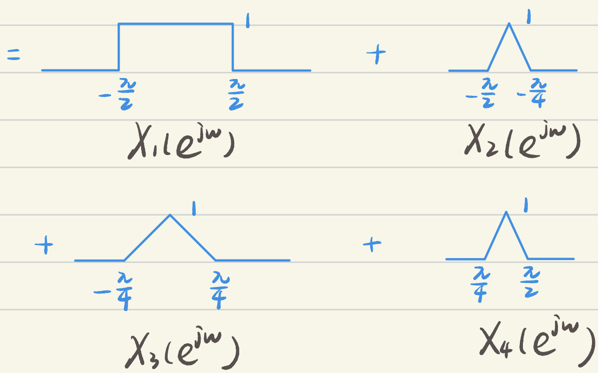
$$\because X(e^{jw}) = X_1(e^{jw}) + X_2(e^{jw})$$

$$\therefore x[n] = x_1[n] + x_2[n] = \begin{cases} \frac{\sin(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n)}{\pi n}, & n \neq 0 \\ \frac{1}{4} + \frac{1}{8} = \frac{3}{8}, & n = 0 \end{cases}$$

(c)



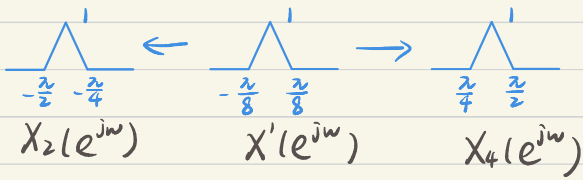
(c)



$$X_1(e^{jw}) = \begin{cases} 1, & |w| \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} \frac{\sin(\frac{\pi}{2}n)}{\pi n}, & n \ne 0 \\ \frac{1}{2}, & n = 0 \end{cases}$$

$$X_3(e^{jw}) = \begin{cases} \frac{4}{\pi}(\frac{\pi}{4} - |w|), & |w| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |w| \le \pi \end{cases} \longleftrightarrow x_3[n] = \begin{cases} 8\left(\frac{\sin(\frac{\pi}{8}n)}{\pi n}\right)^2, & n \ne 0 \\ 8(\frac{1}{8})^2, & n = 0 \end{cases}$$

$\because$ $X_2(e^{jw})$ and $X_4(e^{jw})$ shifting from the same DTFT $X'(e^{jw})$



$$X'(e^{jw}) = \begin{cases} \frac{8}{\pi}(\frac{\pi}{8} - |w|), & |w| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |w| \le \pi \end{cases} \longleftrightarrow x'[n] = \begin{cases} 16\left(\frac{\sin(\frac{\pi}{16}n)}{\pi n}\right)^2, & n \ne 0 \\ 16(\frac{1}{16})^2, & n = 0 \end{cases}$$

$$\therefore X_2(e^{jw}) = X'(e^{j(w+\frac{3\pi}{8})}) \longleftrightarrow x_2[n] = \begin{cases} 16\left(\frac{\sin(\frac{\pi}{16}n)}{\pi n}\right)^2 e^{-j\frac{3\pi}{8}n}, & n \ne 0 \\ \frac{1}{16}, & n = 0 \end{cases}$$

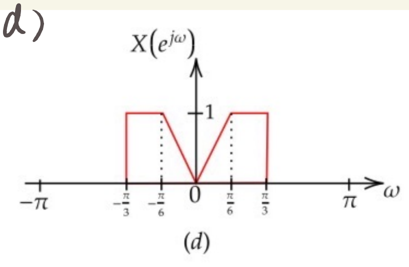
$$X_4(e^{jw}) = X'(e^{j(w-\frac{3\pi}{8})}) \longleftrightarrow x_4[n] = \begin{cases} 16\left(\frac{\sin(\frac{\pi}{16}n)}{\pi n}\right)^2 e^{j\frac{3\pi}{8}n}, & n \ne 0 \\ \frac{1}{16}, & n = 0 \end{cases}$$

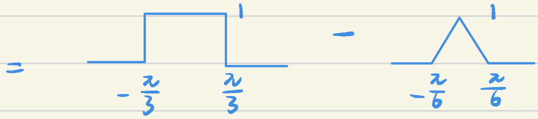
$$\because X(e^{jw}) = X_1(e^{jw}) + X_2(e^{jw}) + X_3(e^{jw}) + X_4(e^{jw})$$

$$\therefore x[n] = x_1[n] + x_2[n] + x_3[n] + x_4[n]$$

$$= \begin{cases} \frac{\sin(\frac{\pi}{2}n)}{\pi n} + 16\left(\frac{\sin(\frac{\pi}{16}n)}{\pi n}\right)^2 e^{-j\frac{3\pi}{8}n} + 8\cdot\left(\frac{\sin(\frac{\pi}{8}n)}{\pi n}\right)^2 + 16\left(\frac{\sin(\frac{\pi}{16}n)}{\pi n}\right)^2 e^{j\frac{3\pi}{8}n}, & n \ne 0 \\ \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} = \frac{3}{4}, & n = 0 \end{cases}$$

(d)





$X_1(e^{jw})$ $\quad$ $X_2(e^{jw})$

$$X_1(e^{jw}) = \begin{cases} 1, & |w| \le \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} \frac{\sin(\frac{\pi}{3}n)}{\pi n}, & n \neq 0 \\ \frac{1}{3}, & n = 0 \end{cases}$$

$$X_2(e^{jw}) = \begin{cases} \frac{6}{\pi}\left(\frac{\pi}{6} - |w|\right), & |w| \le \frac{\pi}{6} \\ 0, & \frac{\pi}{6} < |w| \le \pi \end{cases} \longleftrightarrow x_1[n] = \begin{cases} 12\left(\frac{\sin(\frac{\pi}{12}n)}{\pi n}\right)^2, & n \neq 0 \\ 12\left(\frac{1}{12}\right)^2, & n = 0 \end{cases}$$

$$\because X(e^{jw}) = X_1(e^{jw}) - X_2(e^{jw})$$

$$\therefore x[n] = x_1[n] - x_2[n] = \begin{cases} \frac{\sin(\frac{\pi}{3}n)}{\pi n} - 12\left(\frac{\sin(\frac{\pi}{12}n)}{\pi n}\right)^2, & n \neq 0 \\ \frac{1}{3} - 12\left(\frac{1}{12}\right)^2 = \frac{1}{4}, & n = 0 \end{cases}$$