HW5 — Book

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Leon Chapter #4 4.5

(a) define H: # of heads in the 3 tosses of a fair coin

$H \sim B(3, \frac{1}{2})$

$\therefore P(Y=-3) = P(H=0) = C_3^0 (\frac{1}{2})^0 (\frac{1}{2})^3 = \frac{1}{8}$

$P(Y=-1) = P(H=1) = C_3^1 (\frac{1}{2})^1 (\frac{1}{2})^{3-1} = \frac{3}{8}$

$P(Y=1) = P(H=2) = C_3^2 (\frac{1}{2})^2 (\frac{1}{2})^{3-2} = \frac{3}{8}$

$P(Y=3) = P(H=3) = C_3^3 (\frac{1}{2})^3 (\frac{1}{2})^0 = \frac{1}{8}$

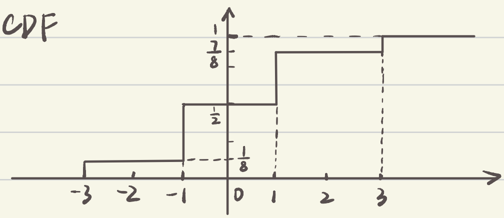
$\therefore F_Y(-3) = P(Y \le -3) = \frac{1}{8}$

$F_Y(-1) = P(Y \le -1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$F_Y(1) = P(Y \le 1) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

$F_Y(3) = P(Y \le 3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

$\therefore$ CDF



(b) $P(|Y| < y) = P(-y < Y < y) = F_Y(y) - F_Y(-y)$

4.5. Let Y be the difference between the number of heads and the number of tails in the 3 tosses of a fair coin.

(a) Plot the cdf of the random variable Y.

(b) Express $P[|Y| < y]$ in terms of the cdf of Y.

Leon Chapter #4 4.25

4.25. Find and plot the pdf of the Weibull random variable in Problem 4.15a.

4.15. For $\beta > 0$ and $\lambda > 0$, the Weibull random variable Y has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-(x/\lambda)^\beta} & \text{for } x \geq 0. \end{cases}$$

(a) Plot the cdf of Y for $\beta = 0.5$, 1, and 2.

① $\beta = 0.5$: $F_X(x) = \begin{cases} 0 & , \text{for } x < 0 \\ 1 - e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}} & , \text{for } x \geq 0 \end{cases}$

$\therefore$ when $x \geq 0$, $f_X(x) = \frac{d}{dx}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}}\right)$

$$= -e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}} \cdot \frac{d}{dx}\left(-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}\right)$$

$$= -e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}} \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{x\lambda}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{4x\lambda}\right)^{\frac{1}{2}} e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{2}}}$$

② $\beta = 1$: $F_X(x) = \begin{cases} 0 & , \text{for } x < 0 \\ 1 - e^{-\frac{x}{\lambda}} & , \text{for } x \geq 0 \end{cases}$

$\therefore$ when $x \geq 0$, $f_X(x) = \frac{d}{dx}\left(1 - e^{-\frac{x}{\lambda}}\right) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$

③ $\beta = 2$: $F_X(x) = \begin{cases} 0 & , \text{for } x < 0 \\ 1 - e^{-\left(\frac{x}{\lambda}\right)^2} & , \text{for } x \geq 0 \end{cases}$

$\therefore$ when $x \geq 0$, $f_X(x) = \frac{d}{dx}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)$

$$= -e^{-\left(\frac{x}{\lambda}\right)^2} \frac{d}{dx}\left(-\frac{x^2}{\lambda^2}\right)$$

$$= \frac{2x}{\lambda^2} e^{-\left(\frac{x}{\lambda}\right)^2}$$

4.26. Find the cdf of the Cauchy random variable which has pdf:

$$f_X(x) = \frac{\alpha/\pi}{x^2 + \alpha^2} \qquad -\infty < x < \infty.$$

Leon Chapter #4 4.26

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{\frac{\alpha}{\pi}}{t^2+\alpha^2}\, dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{x} \frac{\alpha}{t^2+\alpha^2}\, dt$$

let $t = \alpha \tan\theta \Rightarrow dt = d\alpha\tan\theta = \alpha(1+\tan^2\theta)\, d\theta$

$\Rightarrow \theta = \arctan\frac{t}{\alpha} \Rightarrow \begin{cases} t \to x \Rightarrow \theta \to \arctan\frac{x}{\alpha} \\ t \to -\infty \Rightarrow \theta \to -\frac{\pi}{2} \end{cases}$

$$\therefore = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\arctan\frac{x}{\alpha}} \frac{\alpha^2(1+\tan^2\theta)}{\alpha^2\tan^2\theta + \alpha^2}\, d\theta$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\arctan\frac{x}{\alpha}} 1\, d\theta$$

$$= \frac{1}{\pi} \left(\arctan\frac{x}{\alpha} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{\pi} \arctan\frac{x}{\alpha} + \frac{1}{2}$$

4.34. The Pareto random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & x < x_m \\ 1 - \dfrac{x_m^\alpha}{x^\alpha} & x \geq x_m. \end{cases}$$

(a) Find and plot the pdf of X.

(b) Repeat Problem 4.33 parts a and b for the Pareto random variable.

(c) What happens to $P[X > t + x \mid X > t]$ as t becomes large? Interpret this result.

Leon Chapter #4 4.34

(a) when $x \geq x_m$, $f_X(x) = \frac{d}{dx}\left(1 - \frac{x_m^\alpha}{x^\alpha}\right) = \frac{d}{dx}\left(1 - x_m^\alpha \cdot \frac{1}{x^\alpha}\right)$

$$= (-x_m^\alpha)\,\frac{-\alpha x^{\alpha-1}}{x^{2\alpha}}$$

$$= \frac{\alpha \cdot x_m^\alpha}{x^{\alpha+1}}$$

$$\therefore f_X(x) = \begin{cases} 0, & x < x_m \\ \dfrac{\alpha x_m^\alpha}{x^{\alpha+1}}, & x \geq x_m \end{cases}$$

(b) ① $F_X(x \mid X > t) = \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} = \dfrac{\left(1 - \left(\frac{x_m}{x}\right)^\alpha\right) - \left(1 - \left(\frac{x_m}{t}\right)^\alpha\right)}{1 - \left(1 - \left(\frac{x_m}{t}\right)^\alpha\right)}$

$$= \frac{\left(\frac{x_m}{t}\right)^\alpha - \left(\frac{x_m}{x}\right)^\alpha}{\left(\frac{x_m}{t}\right)^\alpha} = 1 - \left(\frac{x}{t}\right)^\alpha \quad (x > t)$$

② $f_X(x \mid X > t) = \dfrac{f_X(x)}{1 - F_X(t)} = \dfrac{\alpha x_m^\alpha / x^{\alpha+1}}{1 - \left(1 - \left(\frac{x_m}{t}\right)^\alpha\right)} = \dfrac{\alpha t^\alpha}{x^{\alpha+1}} \quad (x > t)$

(c) $\because P[X > t+x \mid X > t] = \dfrac{P[X > t+x]}{P[X > t]}$

$$= \frac{1 - F_X(t+x)}{1 - F_X(t)}$$

$$= \frac{(x_m / t+x)^\alpha}{(x_m / t)^\alpha}$$

$$= \left(\frac{t}{t+x}\right)^\alpha$$

$$\therefore \lim_{t \to +\infty} \left(\frac{t}{t+x}\right)^\alpha = 1$$

4.63. Let X be a Gaussian random variable with $m = 5$ and $\sigma^2 = 16$.

(a) Find $P[X > 4]$, $P[X \geq 7]$, $P[6.72 < X < 10.16]$, $P[2 < X < 7]$, $P[6 \leq X \leq 8]$.

(b) $P[X < a] = 0.8869$, find a.

(c) $P[X > b] = 0.11131$, find b.

(d) $P[13 < X \leq c] = 0.0123$, find c.

Leon Chapter #4 4.63

$X \sim G(5,16)$, $m=5$, $\sigma=4$

(a) ① $P[X>4] = 1-P[X\leq 4]$

$= 1-P[Z\leq \frac{4-m}{\sigma}]$

$= 1-P[Z\leq -0.25]$

$= 1-0.5987$

$= 0.4013$

② $P[X\geq 7] = 1-P[X<7]$

$= 1-P[Z< \frac{7-m}{\sigma}]$

$= 1-P[Z<0.5]$

$= 1-0.6915$

$= 0.3085$

③ $P[6.72<X<10.16] = P[X<10.16]-P[X\leq 6.72]$

$= P[Z<1.29]-P[Z\leq 0.43]$

$= 0.9015-0.6664$

$= 0.2351$

④ $P[2<X<7] = P[X<7]-P[X\leq 2]$

$= P[Z<0.5]-P[X\leq -0.75]$

$= 0.6915-0.2266$

$= 0.4649$

⑤ $P[6\leq X\leq 8] = P[X\leq 8]-P[X<6]$

$= P[Z\leq 0.75]-P[X<0.25]$

$= 0.7734-0.5987$

$= 0.1747$

(b) $\because 0.8869 = P[Z<1.21]$

$\therefore P[X<a] = P[Z<\frac{a-m}{\sigma}] = P[Z<1.21]$

$\therefore \frac{a-5}{4} = 1.21 \Rightarrow a = 9.84$

(c) $\because P[X > b] = 1 - P[X \le b] = 0.11131$

$\therefore P[X \le b] = 1 - 0.11131 = 0.88869 = P[Z < 1.22]$

$\therefore P[X \le b] = P[Z \le \frac{b-m}{\sigma}] = P[Z < 1.22]$

$\therefore \frac{b-5}{4} = 1.22 \Rightarrow b = 9.88$

(d) $\because P[13 < X \le c] = P[X \le c] - P[X \le 13]$

$= P[Z \le \frac{c-m}{\sigma}] - P[Z \le 2]$

$= P[Z \le \frac{c-m}{\sigma}] - 0.9772$

$= 0.0123$

$\therefore P[Z \le \frac{c-m}{\sigma}] = 0.9895 = P[Z \le 2.31]$

$\therefore \frac{c-5}{4} = 2.31 \Rightarrow c = 14.24$

4.64. Show that the Q-function for the Gaussian random variable satisfies $Q(-x) = 1 - Q(x)$.

Leon Chapter #4 4.64

$\because$ Q-function: $Q(x) = 1 - \Phi(x) = P(Z > x) \quad Z \sim N(0,1)$

$$= \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$\therefore$ for Gaussian distribution: $Q(-x) = P(Z > -x) = P(Z < x) \quad (\because \text{symmetry})$

$$\therefore Q(-x) = \Phi(x)$$

$\therefore Q(x) = 1 - \Phi(x) = 1 - Q(-x)$

QED

4.67. A binary transmission system transmits a signal X (-1 to send a "0" bit; $+1$ to send a "1" bit). The received signal is $Y = X + N$ where noise N has a zero-mean Gaussian distribution with variance σ^2. Assume that "0" bits are three times as likely as "1" bits.

(a) Find the conditional pdf of Y given the input value: $f_Y(y|X=+1)$ and $f_Y(y|X=-1)$.

(b) The receiver decides a "0" was transmitted if the observed value of y satisfies

$$f_Y(y|X=-1)P[X=-1] > f_Y(y|X=+1)P[X=+1]$$

and it decides a "1" was transmitted otherwise. Use the results from part a to show that this decision rule is equivalent to: If $y < T$ decide "0"; if $y \geq T$ decide "1".

(c) What is the probability that the receiver makes an error given that a $+1$ was transmitted? a -1 was transmitted? Assume $\sigma^2 = 1/16$.

(d) What is the overall probability of error?

Leon Chapter #4 4.67

(a) $\because N \sim N(0,\sigma^2)$

① when $X=+1$, $Y=X+N \sim N(1,\sigma^2)$

$\therefore f_Y(y|X=+1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-1}{\sigma}\right)^2}$

② when $X=-1$, $Y=X+N \sim N(-1,\sigma^2)$

$\therefore f_Y(y|X=-1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y+1}{\sigma}\right)^2}$

(b) $\because P[X=-1] = 3P[X=+1]$

$\therefore 3 f_Y(y|X=-1) > f_Y(y|X=+1)$

$\therefore \frac{3}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y+1}{\sigma}\right)^2} > \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-1}{\sigma}\right)^2}$

$\therefore 3e^{-\frac{1}{2}\left(\frac{y+1}{\sigma}\right)^2} > e^{-\frac{1}{2}\left(\frac{y-1}{\sigma}\right)^2}$

$\therefore \ln 3 - \frac{1}{2}\left(\frac{y+1}{\sigma}\right)^2 > -\frac{1}{2}\left(\frac{y-1}{\sigma}\right)^2$

$\therefore (y+1)^2 - (y-1)^2 < 2\sigma^2 \ln 3$

$\therefore 4y < 2\sigma^2 \ln 3$

$\therefore y < \frac{\sigma^2}{2} \ln 3$

$\therefore T = \frac{\sigma^2}{2} \ln 3 \Rightarrow$ if $y<T$, decide "0"
if $y \geq T$, decide "1"

(c) $\sigma^2 = 1/16 \Rightarrow \sigma = \frac{1}{4}$

$P(\text{error}|X=+1) = P(y<T|X=+1) = P\left(Z < \frac{T-1}{\sigma}\right) = P\left(Z < \frac{\frac{\sigma^2}{2}\ln 3 - 1}{\sigma}\right)$

$= P\left(Z < \frac{1}{8}\ln 3 - 4\right)$

$= P(Z < -3.8626)$

$= 5.61 \times 10^{-5}$

$$P(error \mid x=-1) = P(y \geq T \mid x=-1)$$
$$= 1 - P(y < T \mid x=-1)$$
$$= 1 - P\left(Z < \frac{T+1}{\sigma}\right)$$
$$= 1 - P\left(Z < \frac{\frac{\sigma^2}{2}\ln 3 + 1}{\sigma}\right)$$
$$= 1 - P\left(Z < \frac{1}{8}\ln 3 + 4\right)$$
$$= 1 - P(Z < 4.1373)$$
$$= 2.11 \times 10^{-5}$$

(d) total probability:

$$P(error) = P(error \mid x=+1)P(x=+1) + P(error \mid x=-1)P(x=-1)$$
$$= 5.61 \times 10^{-5} \times \frac{1}{4} + 2.11 \times 10^{-5} \times \frac{3}{4}$$
$$= 2.985 \times 10^{-5}$$

4.68. Two chips are being considered for use in a certain system. The lifetime of chip 1 is modeled by a Gaussian random variable with mean 20,000 hours and standard deviation 5000 hours. (The probability of negative lifetime is negligible.) The lifetime of chip 2 is also a Gaussian random variable but with mean 22,000 hours and standard deviation 1000 hours. Which chip is preferred if the target lifetime of the system is 20,000 hours? 24,000 hours?

Leon Chapter #4 4.68

$C_1 \sim N(20000, 5000^2)$

$C_2 \sim N(22000, 1000^2)$

① 20000 hours

$$P(C_1 \geq 20000) = 1 - P(C_1 < 20000) = 1 - P\left(Z < \frac{20000-20000}{5000}\right) = 1 - P(Z<0) = 0.5$$

$<$

$$P(C_2 \geq 20000) = 1 - P(C_2 < 20000) = 1 - P\left(Z < \frac{20000-22000}{1000}\right) = 1 - P(Z<-2) = 1 - 0.0228 = 0.9772$$

$\therefore C_2$ is preferred

② 24000 hours

$$P(C_1 \geq 24000) = 1 - P(C_1 < 24000) = 1 - P\left(Z < \frac{24000-20000}{5000}\right) = 1 - P(Z<0.8) = 1 - 0.7881 = 0.2119$$

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$$P(C_2 \geq 24000) = 1 - P(C_2 < 24000) = 1 - P\left(Z < \frac{24000-22000}{1000}\right) = 1 - P(Z<2) = 1 - 0.9772 = 0.0228$$

$\therefore C_1$ is preferred