4.39. Find the mean and variance of X in Problem 4.17.

4.17. A random variable X has pdf:

$$f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

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$\because \int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

$$\therefore \int_{-1}^{1} c(1-x^2)\,dx = c\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \frac{4}{3}c = 1$$

$\therefore c = \frac{3}{4}$

$$\therefore f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & -1 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\,dx$$

$$= \frac{3}{4}\int_{-1}^{1}(x - x^3)\,dx$$

$$= \frac{3}{4}\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_{-1}^{1}$$

$$= 0$$

$$\because E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx = \int_{-1}^{1} x^2 \cdot \frac{3}{4}(1-x^2)\,dx$$

$$= \frac{3}{4}\int_{-1}^{1}(x^2 - x^4)\,dx$$

$$= \frac{3}{4}\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_{-1}^{1}$$

$$= \frac{3}{4} \times \frac{4}{15}$$

$$= \frac{1}{5}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = \frac{1}{5} - 0^2 = \frac{1}{5}$$

4.40. Find the mean and variance of X in Problem 4.18.

4.18. A random variable X has pdf:

$$f_X(x) = \begin{cases} cx(1-x^2) & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find c and plot the pdf.
(b) Plot the cdf of X.
(c) Find $P[0 < X < 0.5]$, $P[X = 1]$, $P[.25 < X < 0.5]$.

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$\because \int_{-\infty}^{+\infty} f_X(x) = 1$

$\therefore \int_0^1 cx(1-x^2)\,dx = \int_0^1 c(x-x^3)\,dx$

$= c\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^1$

$= \frac{c}{4} = 1$

$\therefore c = 4$

$\therefore f_X(x) = \begin{cases} 4x(1-x^2) & , 0 \le x \le 1 \\ 0 & , \text{elsewhere} \end{cases}$

$\therefore E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_0^1 4x^2(1-x^2)\,dx$

$= 4\int_0^1 (x^2 - x^4)\,dx$

$= 4\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$

$= 4 \times \frac{2}{15} = \frac{8}{15}$

$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx = \int_0^1 4x^3(1-x^2)\,dx$

$= 4\int_0^1 (x^3 - x^5)\,dx$

$= 4\left(\frac{1}{4}x^4 - \frac{1}{6}x^6\right)\Big|_0^1$

$= 4 \times \frac{1}{12} = \frac{1}{3}$

$\therefore Var(X) = E[X^2] - (E[X])^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}$

4.53. Let $Y = A\cos(\omega t) + c$ where A has mean m and variance σ^2 and ω and c are constants. Find the mean and variance of Y. Compare the results to those obtained in Example 4.15.

Example 4.15 Expected Values of a Sinusoid with Random Phase

Let $Y = a\cos(\omega t + \Theta)$ where a, ω, and t are constants, and Θ is a uniform random variable in the interval $(0, 2\pi)$. The random variable Y results from sampling the amplitude of a sinusoid with random phase Θ. Find the expected value of Y and expected value of the power of Y, Y^2.

$$E[Y] = E[a\cos(\omega t + \Theta)]$$

$$= \int_0^{2\pi} a\cos(\omega t + \theta)\frac{d\theta}{2\pi} = -a\sin(\omega t + \theta)\Big|_0^{2\pi}$$

$$= -a\sin(\omega t + 2\pi) + a\sin(\omega t) = 0.$$

The average power is

$$E[Y^2] = E[a^2\cos^2(\omega t + \Theta)] = E\left[\frac{a^2}{2} + \frac{a^2}{2}\cos(2\omega t + 2\Theta)\right]$$

$$= \frac{a^2}{2} + \frac{a^2}{2}\int_0^{2\pi}\cos(2\omega t + \theta)\frac{d\theta}{2\pi} = \frac{a^2}{2}.$$

Note that these answers are in agreement with the time averages of sinusoids: the time average ("dc" value) of the sinusoid is zero; the time-average power is $a^2/2$.

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$\because Y = A\cos(wt) + c$

$\therefore E[Y] = E[A\cos(wt) + c]$

$= \cos(wt)\,E[A] + c$

$= \cos(wt)\cdot m + c$

$= m\cos(wt) + c$

$E[Y^2] = E[(A\cos(wt) + c)^2]$

$= E[A^2\cos^2(wt) + 2cA\cos(wt) + c^2]$

$= \cos^2(wt)\,E[A^2] + 2c\cos(wt)E[A] + c^2$

$\because Var(A) = E[A^2] - (E[A])^2$

$\therefore E[A^2] = \sigma^2 + m$

$\therefore E[Y^2] = (\sigma^2 + m)\cos^2(wt) + 2cm\cos(wt) + c^2$

$\therefore Var(Y) = E[Y^2] - (E[Y])^2$

$= (\sigma^2 + m)\cos^2(wt) + \cancel{2cm\cos(wt)} + \cancel{c^2} - m^2\cos^2(wt) - \cancel{2cm\cos(wt)} - \cancel{c^2}$

$= \sigma^2\cos^2(wt)$

4.54. A limiter is shown in Fig. P4.2.

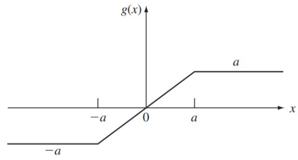

FIGURE P4.2

(a) Find an expression for the mean and variance of $Y = g(X)$ for an arbitrary continuous random variable X.

(b) Evaluate the mean and variance if X is a Laplacian random variable with $\lambda = a = 1$.

(c) Repeat part (b) if X is from Problem 4.17 with $a = 1/2$.

(d) Evaluate the mean and variance if $X = U^3$ where U is a uniform random variable in the unit interval, $[-1, 1]$ and $a = 1/2$.

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(a) $Y = g(X) = \begin{cases} -a, & X < -a \\ X, & -a \le X < a \\ a, & a \le X \end{cases}$

$$\therefore E[Y] = E[g(X)]$$
$$= \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$
$$= \int_{-\infty}^{-a} (-a) f_X(x)\,dx + \int_{-a}^{+a} x f_X(x)\,dx + \int_{a}^{+\infty} a f_X(x)\,dx$$
$$= -aF_X(-a) + E[X \mid -a \le X \le a] + a(1 - F_X(a))$$

$$E[Y^2] = \int_{-\infty}^{-a} (-a)^2 f_X(x)\,dx + \int_{-a}^{+a} x^2 f_X(x)\,dx + \int_{a}^{+\infty} a^2 f_X(x)\,dx$$
$$= a^2 F_X(-a) + E[X^2 \mid -a \le x \le a] + a^2(1 - F_X(a))$$

$$\therefore Var(Y) = E[Y^2] - (E[Y])^2$$

(b) $f_X(x) = \frac{1}{2} e^{-|x|}$, $a = 1$

$$\therefore E[Y] = \int_{-\infty}^{-a} -\frac{1}{2} e^{-|x|}\,dx + \int_{-a}^{a} \frac{1}{2} x e^{-|x|}\,dx + \int_{a}^{+\infty} \frac{1}{2} e^{-|x|}\,dx$$
$$= \frac{1}{2}\left(\int_{-\infty}^{-1} -e^{x}\,dx + \int_{-1}^{0} x e^{x}\,dx + \int_{0}^{1} x e^{-x}\,dx + \int_{1}^{+\infty} e^{-x}\,dx\right)$$

$$\because \int_{-\infty}^{-1} -e^{x}\,dx = (-e^{x})\Big|_{-\infty}^{-1} = -\frac{1}{e}$$

$(\int u\,dv = uv - \int v\,du)$

$$\int_{-1}^{0} x e^{x}\,dx = xe^{x}\Big|_{-1}^{0} - \int_{-1}^{0} e^{x}\,dx = \frac{1}{e} - e^{x}\Big|_{-1}^{0} = \frac{1}{e} - \left(1 - \frac{1}{e}\right) = \frac{2}{e} - 1$$

$$\int_{0}^{1} x e^{-x}\,dx = \int_{0}^{1} (-x)(-e^{-x})\,dx = -xe^{-x}\Big|_{0}^{1} - \int_{0}^{1} -e^{-x}\,dx$$
$$= -\frac{1}{e} - (e^{-x})\Big|_{0}^{1} = -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = 1 - \frac{2}{e}$$

$$\int_{1}^{\infty} e^{-x}\,dx = (-e^{-x})\Big|_{1}^{\infty} = \frac{1}{e}$$

$$\therefore E[Y] = \frac{1}{2}\left(-\frac{1}{e} + \left(\frac{2}{e} - 1\right) + \left(1 - \frac{2}{e}\right) + \frac{1}{e}\right) = 0$$

$$E[Y^2] = \int_{-\infty}^{-a} (-a)^2 f_X(x)dx + \int_{-a}^{+a} x^2 f_X(x)dx + \int_{a}^{+\infty} a^2 f_X(x)dx$$

$$= \int_{-\infty}^{1} \frac{1}{2}e^{-|x|}dx + \int_{-1}^{1} x^2 \cdot \frac{1}{2}e^{-|x|}dx + \int_{1}^{\infty} \frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\left(\int_{-\infty}^{-1} e^{x}dx + \int_{-1}^{0} x^2e^{x}dx + \int_{0}^{1} x^2e^{-x}dx + \int_{1}^{\infty} e^{-x}dx\right)$$

$$\because \int_{-1}^{0} x^2e^{x}dx = \int_{-1}^{0} x^2 de^{x} = x^2e^{x}\Big|_{-1}^{0} - \int_{-1}^{0} 2xe^{x}dx$$

$$= -\frac{1}{e} - \int_{-1}^{0} 2x de^{x} = -\frac{1}{e} - \left(2xe^{x}\Big|_{-1}^{0} - \int_{-1}^{0} 2e^{x}dx\right)$$

$$= -\frac{1}{e} - \frac{2}{e} + (2e^{x})\Big|_{-1}^{0}$$

$$= -\frac{3}{e} + 2 - \frac{2}{e} = 2 - \frac{5}{e}$$

$$\int_{0}^{1} x^2e^{-x}dx = -\int_{0}^{1} x^2 de^{-x} = -x^2 \cdot e^{-x}\Big|_{0}^{1} + \int_{0}^{1} 2xe^{-x}dx$$

$$= -\frac{1}{e} - 2\int_{0}^{1} x de^{-x} = -\frac{1}{e} - 2\left(xe^{-x}\Big|_{0}^{1} - \int_{0}^{1} e^{-x}dx\right)$$

$$= -\frac{1}{e} - \frac{2}{e} + (-2e^{-x})\Big|_{0}^{1}$$

$$= -\frac{3}{e} - \frac{2}{e} + 2 = 2 - \frac{5}{e}$$

$$\therefore E[Y^2] = \frac{1}{2}\left(\frac{1}{e} + 2 - \frac{5}{e} + 2 - \frac{5}{e} + \frac{1}{e}\right) = \frac{1}{2}\left(4 - \frac{8}{e}\right) = 2 - \frac{4}{e}$$

$$\therefore V[Y] = E[Y^2] - (E[Y])^2 = 2 - \frac{4}{e} - 0 = 2 - \frac{4}{e}$$

(c) $\because E[Y] = \int_{-\infty}^{-a} -\frac{1}{2}e^{-|x|}dx + \int_{-a}^{a} \frac{1}{2}xe^{-|x|}dx + \int_{a}^{+\infty} \frac{1}{2}e^{-|x|}dx$

$$= \frac{1}{2}\left(\int_{-\infty}^{-\frac{1}{2}} -e^{x}dx + \int_{-\frac{1}{2}}^{0} xe^{x}dx + \int_{0}^{\frac{1}{2}} xe^{-x}dx + \int_{\frac{1}{2}}^{+\infty} e^{-x}dx\right)$$

$$E[Y^2] = \int_{-\infty}^{-a} (-a)^2 f_X(x)dx + \int_{-a}^{+a} x^2 f_X(x)dx + \int_{a}^{+\infty} a^2 f_X(x)dx$$

$$= \int_{-\infty}^{-\frac{1}{2}} \frac{1}{2}e^{-|x|}dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cdot \frac{1}{2}e^{-|x|}dx + \int_{\frac{1}{2}}^{\infty} \frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\left(\int_{-\infty}^{-\frac{1}{2}} e^{x}dx + \int_{-\frac{1}{2}}^{0} x^2e^{x}dx + \int_{0}^{\frac{1}{2}} x^2e^{-x}dx + \int_{\frac{1}{2}}^{\infty} e^{-x}dx\right)$$

$$V[Y] = E[Y^2] - (E[Y])^2$$

(d) $\because u \sim U(-1,1)$

$\therefore f_U(u) = \frac{1}{2} \quad -1 \le u \le 1$

$\because X = U^3$

$\therefore u = X^{\frac{1}{3}} \quad f_X(x) = f_U(u)\left|\frac{du}{dx}\right| = \frac{1}{2} \cdot \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{6} x^{-\frac{2}{3}} \quad -1 \le x \le 1$

$\because a = \frac{1}{2}$

$\therefore E[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$

$$= \int_{-1}^{-\frac{1}{2}} -\frac{1}{2} \cdot \frac{1}{6} x^{-\frac{2}{3}} dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x \cdot \frac{1}{6} x^{-\frac{2}{3}} dx + \int_{\frac{1}{2}}^{1} \frac{1}{2} \cdot \frac{1}{6} x^{-\frac{2}{3}} dx$$

$E[Y^2] = \int_{-\infty}^{+\infty} g^2(x) f_X(x)\, dx$

$$= \int_{-1}^{-\frac{1}{2}} \frac{1}{4} \cdot \frac{1}{6} x^{-\frac{2}{3}} dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cdot \frac{1}{6} x^{-\frac{2}{3}} dx + \int_{\frac{1}{2}}^{1} \frac{1}{4} \cdot \frac{1}{6} x^{-\frac{2}{3}} dx$$

$V[Y] = E[Y^2] - (E[Y])^2$

4.56. Let $Y = 3X + 2$.

(a) Find the mean and variance of Y in terms of the mean and variance of X.

(b) Evaluate the mean and variance of Y if X is Laplacian.

(c) Evaluate the mean and variance of Y if X is an arbitrary Gaussian random variable.

(d) Evaluate the mean and variance of Y if $X = b\cos(2\pi U)$ where U is a uniform random variable in the unit interval.

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(a) $E[Y] = E[3X+2] = 3E[X]+2$

$V[Y] = V[3X+2] = 9V[X]$

(b) $X \sim \text{Laplacian}(\mu, b)$

$\therefore E[X] = \mu \quad V[X] = 2b^2$

$\therefore E[Y] = 3\mu + 2$

$V[Y] = 18b^2$

(c) $X \sim N(\mu, \sigma^2)$

$\therefore E[X] = \mu \quad V[X] = \sigma^2$

$\therefore E[Y] = 3\mu + 2$

$V[Y] = 9\sigma^2$

(d) Suppose $u \sim U(a, b)$

$\therefore E[X] = E[b\cos(2\pi U)] = bE[\cos(2\pi U)]$

$$= b\int_{-\infty}^{+\infty} \cos(2\pi u)\frac{1}{b-a}\,du$$

$$= \frac{b}{b-a}\int_a^b \cos(2\pi u)\,du$$

$$= \frac{b}{b-a}\left(\frac{\sin(2\pi u)}{2\pi}\right)\Bigg|_a^b$$

$$= \frac{0-0}{2\pi(b-a)} = 0$$

$E[X^2] = E[b^2\cos^2(2\pi U)] = b^2E[\cos^2(2\pi U)]$

$$= b^2\int_{-\infty}^{+\infty} \cos^2(2\pi u)\frac{1}{b-a}\,du$$

$$= \frac{b^2}{b-a}\int_a^b \frac{\cos(4\pi u)+1}{2}\,du$$

$$= \frac{b^2}{2(b-a)}\int_a^b \cos(4\pi u)+1\,du$$

$$= \frac{b^2}{2(b-a)} \left[(b-a) + \int_a^b \cos(4\pi u)\, du \right]$$

$$= \frac{b^2}{2} + \frac{b^2}{2(b-a)} \left(\frac{\sin(4\pi u)}{4\pi} \right) \Big|_a^b$$

$$= \frac{b^2}{2} + \frac{b^2}{2(b-a)} \frac{0-0}{4\pi}$$

$$= \frac{b^2}{2}$$

$\therefore V[X] = E[X^2] - (E[X])^2 = \frac{b^2}{2} - 0 = \frac{b^2}{2}$

$\therefore E[Y] = 3E[X] + 2 = 3 \times 0 + 2 = 2$

$V[Y] = 9V[X] = \frac{9}{2} b^2$

4.57. Find the nth moment of U, the uniform random variable in the unit interval. Repeat for X uniform in $[a, b]$.

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$\because X \sim U(a,b)$

$$\therefore f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{eles} \end{cases}$$

$$\therefore E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx = \int_a^b x^n \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{1}{n+1}x^{n+1}\right)\Big|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^{n+1}-a^{n+1}}{n+1}$$

4.70. **(a)** Show that the gamma random variable has mean:

$$E[X] = \alpha/\lambda.$$

(b) Show that the gamma random variable has second moment, and variance given by:

$$E[X^2] = \alpha(\alpha+1)/\lambda^2 \text{ and } \mathrm{VAR}[X] = \alpha/\lambda^2.$$

(c) Use parts a and b to obtain the mean and variance of an m-Erlang random variable.

(d) Use parts a and b to obtain the mean and variance of a chi-square random variable.

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(a) $X \sim \text{Gamma}(\alpha, \beta) \sim \text{Gamma}(\alpha, \frac{1}{\lambda})$

$$\therefore f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad , x>0$$

$$\therefore E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^{+\infty} x \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x}\,dx$$

Let $u = \lambda x$, $x = \frac{u}{\lambda}$, $du = \lambda dx$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{u}{\lambda}\right)^\alpha e^{-u}\,\lambda du$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)\,\lambda^{\alpha+1}} \int_0^{+\infty} u^\alpha e^{-u}\,du$$

$$= \frac{1}{\Gamma(\alpha)\lambda} \cdot \Gamma(\alpha+1)$$

$$= \frac{1}{\Gamma(\alpha)\lambda} \cdot \alpha\Gamma(\alpha)$$

$$= \alpha/\lambda$$

(b) $$\therefore E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx = \int_0^{+\infty} x^n \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+n-1} e^{-\lambda x}\,dx$$

Let $u = \lambda x$, $x = \frac{u}{\lambda}$, $du = \lambda dx$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{u}{\lambda}\right)^{\alpha+n-1} e^{-u}\,\lambda du$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)\,\lambda^{\alpha+n}} \int_0^{+\infty} u^{\alpha+n-1} e^{-u}\,du$$

$$= \frac{1}{\Gamma(\alpha)\lambda^n}\Gamma(\alpha+n)$$

$$= \left(\frac{1}{\lambda}\right)^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$\therefore E[X^2] = \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} = \frac{(\alpha+1)!}{\lambda^2(\alpha-1)!} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\therefore V[X] = E[X^2] - (E[X])^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha^2+\alpha-\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

(c) $X \sim \text{Erlang} \sim \text{Gamma}(\alpha, \frac{1}{\lambda})$ $\quad \alpha \in \mathbb{Z}^+$

$$\therefore \begin{cases} E[X] = \frac{\alpha}{\lambda} \\ V[X] = \frac{\alpha}{\lambda^2} \end{cases} \quad \alpha \in \mathbb{Z}^+$$

(d) $X \sim \chi^2(k) \sim \text{Gamma}(\frac{k}{2}, \frac{1}{2})$ $\quad k \in \mathbb{Z}^+$

$$\therefore \begin{cases} E[X] = \frac{k}{2} / \frac{1}{2} = k \\ V[X] = \frac{k}{2} / (\frac{1}{2})^2 = 2k \end{cases}$$