HW 02 — Q1

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(1) Book 2.3 #3

$$A=\begin{bmatrix}1&1&0\\4&6&1\\-2&2&0\end{bmatrix}\xrightarrow{R_2=R_2-4R_1}\begin{bmatrix}1&1&0\\0&2&1\\-2&2&0\end{bmatrix}\xrightarrow{R_3=R_3+2R_1}\begin{bmatrix}1&1&0\\0&2&1\\0&4&0\end{bmatrix}\xrightarrow{R_3=R_3-2R_2}\begin{bmatrix}1&1&0\\0&2&1\\0&0&-2\end{bmatrix}$$

$$\therefore E_{21}=\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}\quad E_{31}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\quad E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\quad U=\begin{bmatrix}1&1&0\\0&2&1\\0&0&-2\end{bmatrix}\Rightarrow E_{32}E_{31}E_{21}A=U$$

$$\therefore M=E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\2&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-4&1&0\\10&-2&1\end{bmatrix}\qquad MA=U$$

(2) Book 2.3 #25

$$[A\,\vdots\,b]=\left[\begin{array}{ccc|c}1&2&3&1\\2&3&4&2\\3&5&7&6\end{array}\right]\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1}\left[\begin{array}{ccc|c}1&2&3&1\\0&-1&-2&0\\0&-1&-2&3\end{array}\right]\xrightarrow{R_3=R_3-R_2}\left[\begin{array}{ccc|c}1&2&3&1\\0&-1&-2&0\\0&0&0&3\end{array}\right]$$

$\because [A\,\vdots\,b]\in\mathbb{R}^{3\times4}$, but only have 2 pivot variables

$\therefore$ this system has no solution

If we change "6" to "3", then

$$[A\,\vdots\,b']=\left[\begin{array}{ccc|c}1&2&3&1\\2&3&4&2\\3&5&7&3\end{array}\right]\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1}\left[\begin{array}{ccc|c}1&2&3&1\\0&-1&-2&0\\0&-1&-2&0\end{array}\right]\xrightarrow[R_2=-R_2]{R_3=R_3-R_2}\left[\begin{array}{ccc|c}1&2&3&1\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$$\xrightarrow{R_1=R_1-2R_2}\left[\begin{array}{ccc|c}1&0&-1&1\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$$\therefore\begin{cases}x-z=1\\y+2z=0\end{cases}\Rightarrow\begin{cases}x=z+1\\y=-2z\\z\end{cases}\quad\text{has solution}$$

(3) Book 2.5 #25

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}]{R_1 = \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow[R_3 = 2R_3]{R_2 = \frac{2}{3}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 3 & -1 & 0 & 2 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{8}{3} & -\frac{2}{3} & -\frac{2}{3} & 2 \end{array}\right] \xrightarrow{R_3 = \frac{3}{8}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \xrightarrow[R_2 = R_2 - \frac{1}{3}R_3]{R_1 = R_1 - \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & -\frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \xrightarrow{R_1 = R_1 - \frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$[B \mid I] = \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_3 = R_3 - R_2 \\ R_2 = -R_2}]{R_1 = R_1 + R_2 + R_3} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \end{array}\right] \xrightarrow[R_2 = R_3]{R_1 = R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$\because$ B has a free variables

$\therefore$ B doesn't have inverse, B^{-1} doesn't exist

(4) Book 2.5 #30

① $$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

when $a \neq 0$ and $a \neq b \Rightarrow a \neq 0$ and $a - b \neq 0$

$\therefore$ A has three pivot variable , $3=n$

$\therefore$ A is invertible

② $C=\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow[R_2=R_2-\frac{c}{2}R_1]{R_3=R_3-4R_1} \begin{bmatrix} 2 & c & c \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} \\ 0 & 7-4c & -3c \end{bmatrix}$

$\therefore$ if $7-4c=-3c$ and $c-\frac{c^2}{2}=0$, then C will have free variable which means C is not invertible

$\therefore \begin{cases} 7-4c=-3c \Rightarrow c=7 \\ c-\frac{c^2}{2}=0 \Rightarrow c(1-\frac{c}{2})=0 \rightarrow c=0 \\ \qquad\qquad\qquad\qquad\quad \searrow 1-\frac{c}{2}=0 \Rightarrow c=2 \end{cases}$

$\therefore$ when $c=7$ or $c=0$ or $c=2$, then C is not invertible

俞昊柳

(5) Book 2.6 #5

$\because A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

$\therefore E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad U=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow EA=U$

$\therefore E^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}=L \Rightarrow A=LU=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$

(6) Book 2.6 #13

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow[\substack{R_3=R_3-R_1 \\ R_4=R_4-R_1}]{\substack{R_1=R_1 \\ R_2=R_2-R_1}} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \xrightarrow[R_4=R_4-R_2]{R_3=R_3-R_2}$$

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \xrightarrow{R_4=R_4-R_3} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$\therefore E_1=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad E_2=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad E_3=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$U=\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \Rightarrow E_3E_2E_1A=U$$

$$\therefore A=E_1^{-1}E_2^{-1}E_3^{-1}U=LU \quad , L=E_1^{-1}E_2^{-1}E_3^{-1}$$

$$\therefore E_1^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad E_3^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1 \end{bmatrix} = \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1 \end{bmatrix}$$

when U has 4 pivots, $A = LU$ has 4 pivots

$\therefore a \neq 0$ and $b-a \neq 0$ and $c-b \neq 0$ and $d-c \neq 0$

$$\therefore \text{4 conditions: } \begin{cases} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{cases}$$

(7) Book 2.6 #22

$$A = \begin{bmatrix} 5&3&1\\3&3&1\\1&1&1 \end{bmatrix} \xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - R_3} \begin{bmatrix} 4&2&0\\2&2&0\\1&1&1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 2&0&0\\2&2&0\\1&1&1 \end{bmatrix} \xrightarrow[R_2 = \frac{1}{2}R_2]{R_1 = \frac{1}{2}R_1} \begin{bmatrix} 1&0&0\\1&1&0\\1&1&1 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1&0&0\\1&1&0\\1&1&1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1&0&-1\\0&1&-1\\0&0&1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1&-1&0\\0&1&0\\0&0&1 \end{bmatrix} \quad E_3 = \begin{bmatrix} \frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&1 \end{bmatrix} \Rightarrow E_3E_2E_1A = L$$

$$\therefore A = E_1^{-1}E_2^{-1}E_3^{-1}L = UL, \quad U = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$\therefore E_1^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore U = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = UL = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

HW 02 — Q2

$$[A \vdots b] = \left[\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ -3 & 1 & 2 & -2 & b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right] \xrightarrow[\substack{R_3 = R_3 + R_2 - 5R_1 \\ R_4 = R_4 - 6R_1}]{R_2 = 2R_2 + 3R_1} \left[\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & -14 & -7 & -5 & -5a+b+c \\ 0 & -14 & -7 & -5 & -6a+d \end{array}\right]$$

$$\xrightarrow[R_4 = R_4 + R_2]{R_3 = R_3 + R_2} \left[\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 0 & 0 & 0 & -2a+3b+c \\ 0 & 0 & 0 & 0 & -3a+2b+d \end{array}\right]$$

$\therefore$ if the linear system is consistent

$$\begin{cases} -2a+3b+c = 0 \\ -3a+2b+d = 0 \end{cases}$$

HW 02 — Q3

prof: $A_{n\times n} \equiv [a_{ij}]$, $a_{ij} \begin{cases} = 0, & i<j \\ \neq 0, & i \geq j \end{cases}$

$$\begin{bmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

suppose $A^{-1}_{n\times n} = B_{n\times n} \equiv [b_{ij}]$, $AB = I \equiv [h_{ij}]$

pick l^{th} column of I,

Case 1: when $i<l$, $h_{il}=0$

- $h_{1l} = \sum_{k=1}^{n} a_{1k} \cdot b_{kl} = 0$

$\because$ when $k>1$, $a_{1k}=0$, and $a_{11} \neq 0$

$\therefore h_{1l} = a_{11} \cdot b_{1l} = 0 \Rightarrow b_{1l} = 0$

- $h_{2l} = \sum_{k=1}^{n} a_{2k} \cdot b_{kl} = 0$

$\because$ Similarly when $k>2$, $a_{2k}=0$, and $a_{22} \neq 0$

$\therefore h_{2l} = a_{21} \cdot \underline{b_{1l}}^{=0} + a_{22} \cdot b_{2l} = \underline{a_{22}}^{\neq 0} \cdot b_{2l} = 0$

$\therefore b_{2l} = 0$

$\vdots$

- $h_{(l-1)l} = \sum_{k=1}^{n} a_{(l-1)k} \cdot b_{kl} = 0$

Similarly: $h_{(l-1)l} = a_{(l-1)1} \cdot \underline{b_{1l}}^{=0} + a_{(l-1)2} \cdot \underline{b_{2l}}^{=0} + \cdots + a_{(l-1)(l-1)} \cdot b_{(l-1)l}$

$= \underline{a_{(l-1)(l-1)}}^{\neq 0} \cdot b_{(l-1)l} = 0$

$\therefore b_{(l-1)l} = 0$

$\therefore$ when $k<l$, $b_{kl}=0$

$\therefore$ So the elements above the diagonal of B are all 0

Case 2: when $i=l$, $h_{ll}=1$

$h_{ll}=\sum_{k=1}^{n} a_{lk}\cdot b_{kl}=1$

$\because$ when $k>l$, $a_{lk}=0$, and $a_{ll}\neq 0$

$\therefore h_{ll}=\sum_{k=1}^{l} a_{lk}\cdot b_{kl}=a_{l1}\cdot \underline{b_{1l}}+a_{l2}\cdot \underline{b_{2l}}+\cdots+a_{ll}\cdot b_{ll}=1$

$\because$ accroding to Case 1, when $k<l$, $b_{kl}=0$

$\therefore h_{ll}=a_{ll}\cdot b_{ll}=1$

$\therefore b_{ll}=\frac{1}{a_{ll}}\neq 0$

$\therefore$ So the diagonal elements of B are the reciprocals of the corresponding diagonal elements of A

Case 3: when $i>l$, $h_{il}=0$

• $h_{(l+1)l}=\sum_{k=1}^{n} a_{(l+1)k}\cdot b_{kl}=0$

$\because$ when $k>l+1$, $a_{(l+1)k}=0$

$\therefore h_{(l+1)l}=a_{(l+1)1}\cdot b_{1l}+\cdots+a_{(l+1)\cdot l}\cdot b_{ll}+a_{(l+1)\cdot(l+1)}\cdot b_{(l+1)\cdot l}=0$

$\because$ according to Case 1 and Case 2, $b_{ll}\neq 0$, $a_{(l+1)(l+1)}\neq 0$

$\therefore h_{(l+1)l}=a_{(l+1)\cdot l}\cdot \overset{\neq 0}{\underline{b_{ll}}}+\overset{\neq 0}{\underline{a_{(l+1)(l+1)}}}\cdot b_{(l+1)\cdot l}=0$

$\therefore$ So $b_{(l+1)\cdot l}$ is not necessarily 0

⋮

$\therefore$ Similary, the elements below the diagonal of B are uncertain

In conclusion: the inverse of a lower triangular matrix A with nonzero diagonal elements is itself lower triangular, and this inverse also has nonzero diagonal elements.

QED

HW 02 — Q4

suppose $x\begin{bmatrix}1\\3\\2\end{bmatrix}+y\begin{bmatrix}2\\8\\-1\end{bmatrix}+z\begin{bmatrix}-1\\9\\2\end{bmatrix}=\begin{bmatrix}6\\10\\2\end{bmatrix} \Rightarrow \begin{bmatrix}1&2&-1\\3&8&9\\2&-1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\2\end{bmatrix}$

$$\therefore \left[\begin{array}{ccc|c}1&2&-1&6\\3&8&9&10\\2&-1&2&2\end{array}\right] \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-3R_1} \left[\begin{array}{ccc|c}1&2&-1&6\\0&2&12&-8\\0&-5&4&-10\end{array}\right] \xrightarrow[\substack{R_1=R_1-2R_2\\R_3=R_3+5R_2}]{R_2=\frac{1}{2}R_2} \left[\begin{array}{ccc|c}1&0&-13&14\\0&1&6&-4\\0&0&34&-30\end{array}\right]$$

$$\therefore \begin{cases}x-13z=14\\y+6z=-4\\34z=-30\end{cases} \Rightarrow \begin{cases}x=\frac{43}{17}\\y=\frac{22}{17}\\z=-\frac{15}{17}\end{cases}$$

$$\therefore [6,10,2]=\frac{43}{17}[1,3,2]+\frac{22}{17}[2,8,-1]-\frac{15}{17}[-1,9,2]$$

is a linear combination

HW 02 — Q5

① zero vector: If $b=0$, then $Ax=0$, $x=0$ is the solution

Else if $b\neq 0$, then $x=0$ is not the solution of $Ax=b$

∴ zero vector might not in the solution set

② addition: Suppose $Ax_1=b$ and $Ax_2=b$

then $A(x_1+x_2)=Ax_1+Ax_2=2b$

∴ solution set is not closed under vector addition

③ multiplication: Suppose $Ax=b$, c is a scalar

then $A(cx)=cAx=cb$

∴ solution set is not closed under scalar multiplication

∴ the solution set is not a subspace of $\mathbb{R}^{n\times 1}$

HW 02 — Q6

Suppose $W_1, W_2, \cdots, W_n$ are all subspace of a vector space V

① zero vector: $\because \forall k$, W_k is a subspace of V

$\therefore \forall k, \vec{0} \in W_k$

$\therefore \vec{0} \in \bigcap_{k=1}^{n} W_k$

$\therefore$ zero vector belongs to the intersection of subspaces

② addition: pick vectors $\vec{x}_1, \vec{x}_2 \in \bigcap_{k=1}^{n} W_k$

$\therefore \forall k, \vec{x}_1 \in W_k$ and $\vec{x}_2 \in W_k$

$\because W_k$ is a subspace of V

$\therefore W_k$ is closed under vector addition

$\therefore \forall k, \vec{x}_1 + \vec{x}_2 \in W_k$

$\therefore \vec{x}_1 + \vec{x}_2 \in \bigcap_{k=1}^{n} W_k$

$\therefore$ the intersection is closed under vector addition

③ multiplication: pick vector $\vec{x} \in \bigcap_{k=1}^{n} W_k$, c is a scalar

$\therefore \forall k, \vec{x} \in W_k$

$\because W_k$ is a subspace of V

$\therefore W_k$ is closed under scalar multiplication

$\therefore \forall k, c\vec{x} \in W_k$

$\therefore c\vec{x} \in \bigcap_{k=1}^{n} W_k$

$\therefore$ the intersection is closed under scalar multiplication

$\therefore$ the intersection of any number of subspaces of V is a subspace of V

QED

HW 02 — Q7

$L(S) = \{ w : w = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n, \quad a_1, a_2, \cdots a_n \in F \}$

① zero vector: Suppose $a_1 = a_2 = \cdots = a_n = 0$

then $w = 0\alpha_1 + 0\alpha_2 + \cdots + 0\alpha_n = 0$

$\therefore$ zero vector belongs to $L(S)$

② addition: Suppose $\begin{cases} w_1 = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \\ w_2 = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n \end{cases}$

$\therefore w_1 + w_2 = (a_1+b_1)\alpha_1 + (a_2+b_2)\alpha_2 + \cdots + (a_n+b_n)\alpha_n$

$\because \forall i,\ a_i + b_i \in F$

$\therefore w_1 + w_2 \in L(S)$

$\therefore L(S)$ is closed under vector addition

③ multiplication: Suppose $w = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

pick a scalar c

$\therefore cw = (ca_1)\alpha_1 + (ca_2)\alpha_2 + \cdots + (ca_n)\alpha_n$

$\because \forall i,\ ca_i \in F$

$\therefore cw \in L(S)$

$\therefore L(S)$ is closed under scalar multiplication

$\therefore L(S)$ is a subspace of V

QED