6.71. Let N_1 be the number of Web page requests arriving at a server in the period (0, 100) ms and let N_2 be the *total* combined number of Web page requests arriving at a server in the period (0, 200) ms. Assume page requests occur every 1-ms interval according to independent Bernoulli trials with probability of success p.

(a) Find the minimum linear mean square estimator for N_2 given N_1 and the associated mean square error.

(b) Find the minimum mean square error estimator for N_2 given N_1 and the associated mean square error.

(c) Find the maximum a posteriori estimator for N_2 given N_1.

(d) Repeat parts a, b, and c for the estimation of N_1 given N_2.

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$\because N_1 \sim \text{Binomial}(100, p)$

$N' \sim \text{Binomial}(100, p)$

$N_2 = N_1 + N'$

N_1 and N' are independent

$\therefore E[N_1] = E[N'] = 100p$

$E[N_2] = E[N_1 + N'] = E[N_1] + E[N'] = 200p$

$V[N_1] = V[N'] = 100p(1-p)$

$V[N_2] = V[N_1 + N'] \overset{id}{=} V[N_1] + V[N'] = 200p(1-p)$

$Cov(N_1, N_2) = Cov(N_1, N_1 + N') \overset{id}{=} V[N_1] = 100p(1-p)$

(a) $$\hat{N_2}^{LMMSE} = E[N_2] + \frac{Cov(N_1, N_2)}{V[N_1]}(N_1 - E[N_1])$$

$$= 200p + \frac{100p(1-p)}{100p(1-p)}(N_1 - 100p)$$

$$= N_1 + 100p$$

$MSE = V[N_2 | N_1] = V[N_1 + N' | N_1] = V[N'] = 100p(1-p)$

(b) $\hat{N_2}^{MMSE} = E[N_2 | N_1] = E[N_1 + N' | N_1] \overset{id}{=} N_1 + E[N'] = N_1 + 100p$

$MSE = V[N'] = 100p(1-p)$

(c) $\hat{N_2}^{MAP} = \underset{n_2}{\arg\max}\, P(N_2 = n_2 | N_1 = n_1)$

$\because P(N_2 = n_2) = \binom{200}{n_2} p^{n_2} (1-p)^{200-n_2}$

$P(N_1 = n_1 | N_2 = n_2) = \binom{100}{n_1} \left(\frac{n_2}{200}\right)^{n_1} \left(1 - \frac{n_2}{200}\right)^{100-n_1}$

$\therefore P(N_2 = n_2 | N_1 = n_1) \propto \binom{100}{n_1} \left(\frac{n_2}{200}\right)^{n_1} \left(1 - \frac{n_2}{200}\right)^{100-n_1} \cdot \binom{200}{n_2} p^{n_2} (1-p)^{200-n_2}$

$$\therefore \widehat{N_2}^{MAP} = \arg\max_{n_2} \left[\left(\frac{n_2}{200}\right)^{n_1} \left(1-\frac{n_2}{200}\right)^{100-n_1} \cdot \binom{200}{n_2} p^{n_2} (1-p)^{200-n_2} \right]$$

if $p \ll 1$, $\widehat{N_2}^{MAP} \approx n_1 + 100p = E[N_2 \mid N_1 = n_1]$

(d) Similarly: $\widehat{N_1}^{LMMSE} = \widehat{N_1}^{MMSE} = N_2 - 100p$

$MSE_{LMMSE} = MSE_{MMSE} = 100p(1-p)$

6.72. Let $Y = X + N$ where X and N are independent Gaussian random variables with different variances and N is zero mean.

(a) Plot the correlation coefficient between the "observed signal" Y and the "desired signal" X as a function of the signal-to-noise ratio σ_X/σ_N.

(b) Find the minimum mean square error estimator for X given Y.

(c) Find the MAP and ML estimators for X given Y.

(d) Compare the mean square error of the estimators in parts a, b and c.

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(a) $X \sim N(0, \sigma_x^2)$ $N \sim N(0, \sigma_N^2)$ they are independent

$\therefore Var(Y) = Var(X+N) = Var(X) + Var(N) = \sigma_x^2 + \sigma_N^2$

$Cov(X,Y) = Cov(X,X) + Cov(X,N) = Var(X) = \sigma_x^2$

$$\therefore \rho_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\sigma_x^2}{\sqrt{\sigma_x^2(\sigma_x^2+\sigma_N^2)}} = \sqrt{\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}}$$

Suppose $SNR = \sigma_x^2/\sigma_N^2$

$$\therefore \rho_{xy} = \sqrt{\frac{SNR}{1+SNR}}$$

(b) $$\widehat{X^{MMSE}} = E[X|Y] = \frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2} Y$$

(c) $\because P(Y|X) \sim N(X, \sigma_N^2)$

$$\therefore P(X|Y) \propto P(Y|X)p(X) \sim N\left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}Y, \frac{\sigma_x^2\sigma_N^2}{\sigma_x^2+\sigma_N^2}\right)$$

$$\widehat{X^{MAP}} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2} Y$$

$\because p(Y|X) \sim N(X, \sigma_N^2)$

$$\therefore \widehat{X^{ML}} = \arg\max_x p(Y|X) = Y$$

(d) $$MSE_{MMSE} = MSE_{MAP} = E\left[\left(X - \frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}Y\right)^2\right] = \frac{\sigma_x^2\sigma_N^2}{\sigma_x^2+\sigma_N^2}$$

$$MSE_{ML} = E[(X-Y)^2] = \sigma_x^2 + \sigma_N^2$$

$\therefore$ MMSE and MAP are better than ML