HW07 – Q1

Book Set 6.1 #19

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B
(b) the determinant of $B^{\mathrm{T}}B$
(c) the eigenvalues of $B^{\mathrm{T}}B$
(d) the eigenvalues of $(B^2 + I)^{-1}$.

(a) $\text{rank}(B) = n - (\# \text{ of } \lambda = 0)$
$= 3 - 1$
$= 2$

(b) $\det(B^T B) = \det(B^T)\det(B)$ — $\det(AB) = \det(A)\det(B)$
$= \det^2(B)$ — $\det(A^T) = \det(A)$
$= \left(\prod_{i=1}^{3} \lambda_i\right)^2$ — $\det(A) = \prod \lambda_i$
$= (0 \times 1 \times 2)^2$
$= 0$

(c) I don't know how to get the eigenvalues of $B^T B$

(d) for B^2 : $B^2 x = BBx = B\lambda x = \lambda Bx = \lambda^2 x$
$\Rightarrow \lambda_1' = 0^2 = 0 \quad \lambda_2' = 1^2 = 1 \quad \lambda_3' = 2^2 = 4$

for $B^2 + I$: $(B^2 + I)x = B^2 x + Ix = \lambda^2 x + x = (\lambda^2 + 1)x$
$\Rightarrow \lambda_1'' = 0 + 1 = 1 \quad \lambda_2'' = 1 + 1 = 2 \quad \lambda_3'' = 4 + 1 = 5$

for $(B^2 + I)^{-1}$: $\because Ix = A^{-1}Ax = A^{-1}\lambda x = \lambda A^{-1} x$
$\therefore A^{-1} x = \frac{1}{\lambda} x$
$\Rightarrow \lambda_1 = \frac{1}{1} = 1 \quad \lambda_2 = \frac{1}{2} \quad \lambda_3 = \frac{1}{5}$

24 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

Book Set 6.1 #24

$$\because P_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 2 \\ 4 & 2-\lambda & 4 \\ 2 & 1 & 2-\lambda \end{bmatrix} \xlongequal{R_2 = R_2 - 2R_3} \det\begin{bmatrix} 2-\lambda & 1 & 2 \\ 0 & -\lambda & 2\lambda \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\det\begin{bmatrix} -\lambda & 2\lambda \\ 1 & 2-\lambda \end{bmatrix} + 2\det\begin{bmatrix} 1 & 2 \\ -\lambda & 2\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[-\lambda(2-\lambda) - 2\lambda\right] + 2\left[2\lambda - 2(-\lambda)\right]$$

$$= (2-\lambda)(\lambda^2 - 4\lambda) + 8\lambda$$

$$= \lambda(-\lambda^2 + 6\lambda)$$

$$= \lambda^2(6-\lambda)$$

Let $P_A(\lambda) = 0$

$\therefore \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 6$

① For $\lambda_1 = \lambda_2 = 0$: $(A - \lambda I)x = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} x = 0$

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore 2x_1 + x_2 + 2x_3 = 0$

$\therefore x_1 = -\frac{1}{2}x_2 - x_3$

$$\therefore x = \begin{bmatrix} -\frac{1}{2}x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad x_2, x_3 \in \mathbb{C} \ \& \neq 0$$

$$\therefore \text{let } x_1 = x_2 = 0, \quad e_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

② For $\lambda_3=6$: $(A-\lambda I)x=\begin{bmatrix}-4&1&2\\4&-4&4\\2&1&-4\end{bmatrix}x=0$

$$\begin{bmatrix}-4&1&2\\4&-4&4\\2&1&-4\end{bmatrix}\longrightarrow\begin{bmatrix}2&1&-4\\0&-6&12\\0&3&-6\end{bmatrix}\longrightarrow\begin{bmatrix}2&1&-4\\0&1&-2\\0&0&0\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}2&0&-2\\0&1&-2\\0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}\quad\therefore\begin{cases}x_1-x_3=0\\x_2-2x_3=0\end{cases}\Rightarrow\begin{cases}x_1=x_3\\x_2=2x_3\end{cases}$$

$$\therefore x=\begin{bmatrix}x_3\\2x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\2\\1\end{bmatrix}\qquad x_3\in\mathbb{C}\ \&\ \neq 0$$

$\therefore$ let $x_3=0$, $e_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Book Set 6.1 #27

① $\because$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - R_1 \\ R_4 = R_4 - R_1}]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ only has 1 pivot

$\therefore \text{rank}(A) = 1$

$$\therefore P_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix}$$

$$\underline{\underline{R_1 = R_1 + R_2 + R_3 + R_4}} \det \begin{bmatrix} 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix}$$

$$\underline{\underline{\substack{C_2 = C_2 - C_1 \\ C_3 = C_3 - C_1 \\ C_4 = C_4 - C_1}}} \det \begin{bmatrix} 4-\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$$

$$= (4-\lambda)(-\lambda)(-\lambda)(-\lambda)$$

$$= \lambda^3(\lambda - 4)$$

Let $P_A(\lambda) = 0$

$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 4$

② $\because$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 = R_4 - R_2]{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ has 2 pivot}$$

$\therefore rank(C) = 2$

$$\therefore P_C(\lambda) = \det(C - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} + \det \begin{bmatrix} 0 & 1-\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} + (1-\lambda)\det \begin{bmatrix} 0 & 1-\lambda \\ 1 & 0 \end{bmatrix} - \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^4 - (1-\lambda)^2 - (1-\lambda)^2 + 1$$

$$= (1-\lambda)^4 - 2(1-\lambda)^2 + 1$$

$$= [(1-\lambda)^2 - 1]^2$$

Let $P_C(\lambda) = 0$

$\therefore [(1-\lambda)^2 - 1]^2 = 0 \Rightarrow (1-\lambda)^2 = 1 \Rightarrow 1-\lambda = \pm 1$

$\therefore \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = 2$

29 (Review) Find the eigenvalues of A, B, and C:

$$A=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix} \text{ and } B=\begin{bmatrix}0&0&1\\0&2&0\\3&0&0\end{bmatrix} \text{ and } C=\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}.$$

Book Set 6.1 #29

① $\because$ A is a upper triangular matrix

$\therefore \lambda_A = a_{ii} \quad i\in[1,n]$

$\therefore \lambda_1=1,\ \lambda_2=4,\ \lambda_3=6$

② $P_B(\lambda)=\det(B-\lambda I)=\det\begin{bmatrix}-\lambda&0&1\\0&2-\lambda&0\\3&0&-\lambda\end{bmatrix}$

$=-\lambda\det\begin{bmatrix}2-\lambda&0\\0&-\lambda\end{bmatrix}+\det\begin{bmatrix}0&2-\lambda\\3&0\end{bmatrix}$

$=(-\lambda)(-\lambda)(2-\lambda)-3(2-\lambda)$

$=(2-\lambda)(\lambda^2-3)$

Let $P_B(\lambda)=0$

$\therefore \lambda_1=2,\ \lambda_2=\sqrt{3},\ \lambda_3=-\sqrt{3}$

③ $P_C(\lambda)=\det(C-\lambda I)=\det\begin{bmatrix}2-\lambda&2&2\\2&2-\lambda&2\\2&2&2-\lambda\end{bmatrix}$

$\overset{R_1=R_1+R_2+R_3}{=\!=\!=} \det\begin{bmatrix}6-\lambda&6-\lambda&6-\lambda\\2&2-\lambda&2\\2&2&2-\lambda\end{bmatrix}$

$\overset{C_2=C_2-C_1}{\underset{C_3=C_3-C_1}{=\!=\!=}} \det\begin{bmatrix}6-\lambda&0&0\\2&-\lambda&0\\2&0&-\lambda\end{bmatrix}$

$=(6-\lambda)(-\lambda)(-\lambda)$

$=\lambda^2(6-\lambda)$

Let $P_c(\lambda) = 0$

$\therefore \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 6$

HW07 — Q2

Book Set 6.2 #9

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$G_{k+2} = \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$
$$G_{k+1} = G_{k+1}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X\Lambda^n X^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

(a) $[A]\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \\ G_{k+1} \end{bmatrix}$

$\therefore A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

$\therefore P_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ 1 & -\lambda \end{bmatrix} = \lambda(\lambda-\frac{1}{2}) - \frac{1}{2} = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}$

$= (\lambda-1)(\lambda+\frac{1}{2})$

let $P_A(\lambda) = 0$

$\therefore \lambda_1 = 1 \quad \lambda_2 = -\frac{1}{2}$

① For $\lambda_1 = 1$: $(A-\lambda_1 I)x = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ z \end{bmatrix} \quad z \in \mathbb{C} \ \& \neq 0$

$\therefore$ let $z = 1 \quad e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② For $\lambda_2 = -\frac{1}{2}$: $(A-\lambda_2 I)x = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ -2z \end{bmatrix} \quad z \in \mathbb{C} \ \& \neq 0$

$\therefore$ let $z = 1 \quad e_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $X = [e_1, e_2] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ $\quad X^{-1} = \frac{1}{-2-1} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ and $A = X \Lambda X^{-1}$

$\because \Lambda^2 = \Lambda\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^2 \end{bmatrix}$

$\Lambda^3 = \Lambda^2 \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^3 \end{bmatrix}$

$\therefore \Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix}$

$$
\begin{aligned}
\therefore A^n &= X \Lambda^n X^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & (-\frac{1}{2})^n \\ 1 & (-\frac{1}{2})^{n-1} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}+\frac{1}{3}(-\frac{1}{2})^n & \frac{1}{3}-\frac{1}{3}(-\frac{1}{2})^n \\ \frac{2}{3}+\frac{1}{3}(-\frac{1}{2})^n & \frac{1}{3}-\frac{1}{3}(-\frac{1}{2})^n \end{bmatrix}
\end{aligned}
$$

when $n \to \infty$, $A^n = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(C) $\because G_0 = 0,\ G_1 = 1$

$$\therefore \begin{bmatrix} G_2 \\ G_1 \end{bmatrix} = A \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$$

$$\begin{bmatrix} G_3 \\ G_2 \end{bmatrix} = A \begin{bmatrix} G_2 \\ G_1 \end{bmatrix} = AA \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$$

$$\begin{bmatrix} G_4 \\ G_3 \end{bmatrix} = A \begin{bmatrix} G_3 \\ G_2 \end{bmatrix} = AA^2 \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} \xlongequal{\text{when } n\to\infty} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$\therefore$ when $n \to \infty$, Gibnacci numbers approach $\frac{2}{3}$

19 Diagonalize B and compute $X\Lambda^k X^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

Book Set 6.2 #19

$\because B$ is upper triangular matrix

$\therefore \lambda_1 = b_{11} = 5 \quad \lambda_2 = b_{22} = 4$

① For $\lambda_1 = 5$: $(B - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ 0 \end{bmatrix}$, $z \in \mathbb{C}$ & $\neq 0$

Let $z = 1$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② For $\lambda_2 = 4$: $(B - \lambda I)x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ -z \end{bmatrix}$, $z \in \mathbb{C}$ & $\neq 0$

Let $z = 1$, $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore X = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad X^{-1} = \frac{1}{-1-0} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$

$\therefore B = X \Lambda X^{-1}$

$B^k = X \Lambda^k X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 4^k \\ 0 & -4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = X\sqrt{\Lambda}X^{-1}$. Why is there no real matrix square root of B?

Book Set 6.2 #27

① For $\lambda_1 = 1$, $(A-\lambda I)x = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ -z \end{bmatrix}$, $z \in \mathbb{C}$ & $\neq 0$

Let $z = 1$, $e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 9$, $(A-\lambda I)x = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} z \\ z \end{bmatrix}$, $z \in \mathbb{C}$ & $\neq 0$

Let $z = 1$, $e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore X = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad X^{-1} = \frac{1}{1-(-1)} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$A = X\Lambda X^{-1}$

$\therefore R = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{9} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

② $\because$ eigenvalues of B is $\lambda_1 = -1$, $\lambda_2 = 9$

$\therefore B = X\Lambda X^{-1}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix}$

$\because \lambda_1 = -1 < 0$

$\therefore \sqrt{\Lambda} = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{9} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & 3 \end{bmatrix}$ is not a real matrix

$\therefore R = X\sqrt{\Lambda}X^{-1}$ is not a real matrix

3. Let A be a 4×4 matrix where

$$A = \begin{bmatrix} 2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Find a real matrix B such that $A = B^3$.

HW07 — Q3

$$P_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 4 & 2 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)\det\begin{bmatrix} -(1+\lambda) & 0 & 0 \\ 1 & -\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$= -(2-\lambda)(1+\lambda)\det\begin{bmatrix} -\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)(1+\lambda)(\lambda)(1-\lambda)$$

Let $P_A(\lambda) = 0$

$\therefore \lambda_1 = 2,\ \lambda_2 = -1,\ \lambda_3 = 0,\ \lambda_4 = 1$

① For $\lambda_1 = 2$:

$$(A-\lambda I)x = \begin{bmatrix} 0 & 4 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 4 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 10 & -4 \\ 0 & 0 & -6 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \therefore \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases} \Rightarrow x = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_1 \in \mathbb{C} \ \& \neq C$$

$$\therefore \text{Let } x_1 = 1,\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

② For $\lambda_2=-1$,

$$(A-\lambda I)x=\begin{bmatrix}3&4&2&0\\0&0&0&0\\0&1&1&1\\0&1&0&2\end{bmatrix}x=0$$

$$\therefore \begin{bmatrix}3&4&2&0\\0&0&0&0\\0&1&1&1\\0&1&0&2\end{bmatrix}\longrightarrow\begin{bmatrix}3&0&2&-8\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{bmatrix}\quad \therefore \begin{cases}3x_1+2x_3-8x_4=0\\x_2+2x_4=0\\x_3-x_4=0\end{cases}\Rightarrow x=\begin{bmatrix}2x_4\\-2x_4\\x_4\\x_4\end{bmatrix}\quad x_4\in\mathbb{C}\ \&\ x_4\neq 0$$

$\therefore$ Let $x_4=1$, $e_2=\begin{bmatrix}2\\-2\\1\\1\end{bmatrix}$

③ For $\lambda_3=0$,

$$(A-\lambda I)x=\begin{bmatrix}2&4&2&0\\0&-1&0&0\\0&1&0&1\\0&1&0&1\end{bmatrix}x=0$$

$$\therefore \begin{bmatrix}2&4&2&0\\0&-1&0&0\\0&1&0&1\\0&1&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}2&0&2&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

$$\therefore \begin{cases}x_1+x_3=0\\x_2=0\\x_4=0\end{cases}\Rightarrow x=\begin{bmatrix}-x_3\\0\\x_3\\0\end{bmatrix}\quad x_3\in\mathbb{C}\ \&\ \neq 0$$

$\therefore$ Let $x_3=-1$, $e_3=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$

④ For $\lambda_4 = 1$:

$$(A-\lambda I)x = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} x = 0$$

$$\therefore \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{cases} x_1 + 2x_4 = 0 \\ x_2 = 0 \\ x_3 - x_4 = 0 \end{cases} \Rightarrow x = \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \end{bmatrix} \quad x_4 \in \mathbb{K} \ \& \neq 0$$

$$\therefore \text{Let } x_4 = -1, \quad e_4 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$\therefore$

$$E = [e_1\ e_2\ e_3\ e_4] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[E|I] = \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & -1 \end{array}\right]$$

$$\therefore E^{-1} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -\frac{1}{2} & 0 & -1 \end{bmatrix}$$

$$\because B^3 = A = E\Lambda E^{-1}$$

$$\therefore B = \sqrt[3]{A} = E\sqrt[3]{\Lambda}E^{-1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt[3]{2} & 0 & 0 & 0 \\ 0 & \sqrt[3]{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt[3]{1} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -\frac{1}{2} & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[3]{2} & -2 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -\frac{1}{2} & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[3]{2} & 2\sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{2}-2 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

HW07 - Q4

4. Show that if λ is an eigenvalue of A with the corresponding eigenvector x, then for any scalar c, x is an eigenvector of $A - cI$ corresponding to the eigenvalue $\lambda - c$.

$\because Ax = \lambda x \quad , x \neq 0$

$$\therefore (A - cI)x = Ax - cIx$$
$$= \lambda x - cx$$
$$= (\lambda - c)x$$

QED

HW07 — Q5

5. Show that if T is invertible then the trace of $T^{-1}AT$ equals the trace of A.

Let $B = T^{-1}AT$

$\therefore A = TBT^{-1}$ and T is invertible

$\therefore A \sim B$

$\therefore$ A and B have same eigenvalues

$\because \mathrm{Tr}(A) = \prod_{i=1}^{n} \lambda_i = \mathrm{Tr}(B)$

$\therefore \mathrm{Tr}(A) = \mathrm{Tr}(T^{-1}AT)$

QED