Assigned: 19 November

Homework #12

EE 503: Fall 2024

Instructions: Write your solutions to these homework problems on separate sheets of paper. Submit your work to Canvas by the due date. Show all work and box answers where appropriate. Do not guess.

Due: Tuesday, 26 November at 12:00.

1. The present value of a bond is the sum of its discounted semi-annual coupon payments and its discounted "par" value at maturity $P = \sum_{k=1}^{n} \frac{C}{(1+r)^k} + \frac{M}{(1+r)^n}$. Here C is the semi-annual (twice-a-year) coupon payment in dollars, P is the price of the bond in dollars, n is the number of payout periods (twice the number of years), r is the interest rate (one half the required annual yield), M is the bounds maturity value, and k is the payment time period. Prove that the first term (present value of coupon payments) obeys the annuity formula

$$C \cdot \left(\frac{1 - \frac{1}{(1+r)^n}}{r} \right)$$

(Note: A "zero-coupon" bond is one where $C = 0$ and so an investor receives indirect interest as the difference between the bond's maturity value and its purchase price.) Suppose first that you want to find the price of a 20-year 10% coupon bound with a par value of \$1,000. Suppose that the required yield is 11% per year. So there will be 40 semi-annual coupon payments of \$50 and you will receive \$1,000 in 40 6-month periods from now. What is P? Then find P if the required yield is 6.8%. What happens if the required yield is 10%?

2. Suppose a random stream of future dividends can shrink as well as grow: $D_{k+1} = D_k(1+g)$ with probability p, $D_{k+1} = D_k(1-g)$ with probability p_D, and $D_{k+1} = D_k$ with probability $1-p-p_D$, for constant profit growth rate $g > 0$. Then what is the expected value $E[P]$ for a summable infinite stream of such dividends? What random constraint on g does summability impose? Now consider the probability p_B that the firm goes bankrupt in any given year and so $D_{k+1} = 0$ with probability p_B. Then what is the expected value $E[P]$ for a summable infinite stream of such dividends and what constraint does summability impose?

3. A low-g stock has an average annual profit growth rate less than the current discount rate: $g < r$. Then the rational-asset price is $P = D_0 \sum_{k=1}^{\infty} \left(\frac{1+g}{1+r}\right)^k$ where D_0 is the current profit. Locate such a company on either the New York Stock Exchange or NASDAQ. Use $r = 0.08$ and base your estimate of g on at least the past six-years of profit data. Compare your estimate of the firm's rational share price with its market price. (Note: The two-stage RAP model $P = D_0 \frac{a - a^{N+1}}{1-a} + D_0 a^N \frac{1+g_2}{r-g_2}$ applies in the more general case where stage-1 growth is $g_1 > r$ for the first N years and stage-2 growth is $g_2 < r$ from then on where $a = \frac{1+g_1}{1+r}$.)

4. Let $X(t)$ be the x component of a random particle in a Brownian motion. Let x_0 be the initial position of the particle: $X(t_0) = x_0$. Let $p(x, t|x_0)$ be the conditional probability density function of $X(t+t_0)$ given the starting point $X(t_0) = x_0$. The Brownian motion defines a one-dimensional diffusion process if $\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}$ for some diffusion coefficient D. Albert Einstein first showed in in 1905 that the time-varying Gaussian conditional density $P(x, t|x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x - x_0)^2\right)$ satisfies this partial differential equation. Show that Einstein was right. What is the value of the diffusion coefficient D?

5. A *financial "derivative"* is a financial asset that derives its value from an underlying asset such as a stock or bond or mortgage. Call options on stocks are the simplest derivatives. A call lets you buy a fixed number of stock shares at a fixed price for a fixed length of time (usually less than 6 months). You need not exercise the call if you choose not to and it would make no sense to do so if the call's "exercise" price were more than the market share price. (A put option lets you sell a bundle of shares at a fixed price for a fixed length of time.) The popular Black-Scholes model for options pricing assumes that in the short term the underlying stock obeys a Brownian diffusion. The equilibrium condition of this Nobel-prize winning result is $C(S, t) = SN(d_1) - Fe^{-r\tau}N(d_2)$ where $d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S}{F} + \left(r + \frac{\sigma^2}{2}\right)\tau\right)$ and $d_2 = d_1 - \sigma\sqrt{t}$. Here $C(S, t)$ is the value (rational price) of the call option at time t of a stock with share price S at t (and that pays no dividend), $\tau = t - T$ is the remaining time of the option before it expires at time T, σ^2 is the dispersion or volatility of S, F is the exercise price of the call option, N is the CDF of a standard normal random variable, and r is the risk-free interest rate. Compute the B-S value of a call options if S=\$47, F=\$45, $\tau = 0.5$ (183/365 rounded-off), $r = 0.1$, and $\sigma = 0.25$. Find C if $\sigma = 0.4$. Then find C for two different interest rates $r > 0$ that you choose.