Homework #4

Sunday, September 22, 2024

Sect 4.1 - #41

$$S = \text{span}\left\{ (1,1,1,1,1), (1,1,1,1,1) \right\}$$

$\therefore$ let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$\therefore S = C(A)$ ← row space of A

$S^\perp = N(A)$ ← Nullspace of A ← Fundamental Theorem.

$\therefore$ Solve $Ax = 0$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

x_1, x_2 $\quad \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ x_1 & x_2 & 0 & 0 & 0 \end{bmatrix}$

free variable: x_3, x_4

Pivot variable: x_1, x_2

$\Rightarrow$ $x_1 + x_2 + x_3 = 0$

$x_1 + x_2 + x_4 = 0$

$\Rightarrow$ $x_1 + (x_2 + x_3) + 3x_4 = 0$

$x_1 + 3x_4 - 3x_4 - 3x_4$

$+ (x_2 + x_3 + x_4) - 3x_4 = 0$

$3x_4 - 3x_4 - 3x_4 - 3x_4$

$= -3x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_4 \\ -3x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Ie $x_3 = 1$, $x_4 = 0$

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in S^\perp$$

Ie $x_3 = 0$, $x_4 = 1$

$$x_2 = \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in S^\perp$$

Sect 4.1 - #45

Ie S subspace, T is contained in a subspace V, then that $S^\perp$ contain $V^\perp$

Proof

$S \subset V$ ← S is a subspace of V

let $w \in V^\perp$

$\therefore$ ALS $\quad (w, s) = 0$, for all $s \in S$

$\therefore$ ALS T, $(w, t) = 0$, for all $\forall t \in T$ ←

$\Rightarrow w \in S^\perp$

$\Rightarrow V^\perp \subset S^\perp$

QED

Sect 4.4 - #18

$a = (1,-1,-1,0)$, $b = (0,1,-1,-1)$, $c = (0,0,1,-1)$

let $p_a = ax = (1,-1,-1,0)$ $\quad \|a\|^2 = 3$

$p_b = bx = \frac{ax \cdot b}{a \cdot a} a$

$= (0,1,-1,-1)$

$+ \frac{1}{3}(1,-1,-1,0)$

$= (0,1,-1,-1) + \frac{1}{3}(1,-1,-1,0)$

$+ \frac{1}{3}(1,-1,-1,0)$

$\|a\|^2 = \frac{2}{3}$

$p_c = bx = \frac{ax \cdot b}{a \cdot a} a - \frac{bx \cdot c}{b \cdot b} b$

$= \frac{1}{3}(1,-1,-1,0) - \frac{1}{3}(1,-1,-1,0)$

$+ \frac{1}{3}(1,-1,-1,0)$

$+ (0,1,-1,-1)$

$+ \frac{1}{3}(1,-1,-1,0)$

$\|b\|^2 = 3$

$A = a = (1,-1,-1,0)$ $\Rightarrow$ $x_1 = \frac{1}{3}(1,-1,-1,0)$

$B = b = (0,1,-1,-1)$ $\Rightarrow$ $x_2 = \frac{2}{3}(1,-1,-1,0)$

$C = c = (0,0,1,-1)$ $\Rightarrow$ $x_3 = \frac{2}{3}(1,1,1,-1)$

Sect 4.4 - #22

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\quad c = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}$

$p_a = a_x = \frac{ax \cdot b}{a \cdot a} a$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \|a\|^2 = 4$

$p_b = a_x = \frac{ax \cdot b}{a \cdot a} a$

$= \frac{10}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \\ 2.5 \\ 2.5 \end{bmatrix}$ $\quad \|a\|^2 = 2.5$

$p_c = a_x = \frac{ax \cdot b}{a \cdot a} a - \frac{ax \cdot b}{a \cdot a} a$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.5 \\ -1.5 \\ -1.5 \end{bmatrix}$

$\|a\|^2 = 2$

$P_1 = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad P_2 = \frac{1}{4} \begin{bmatrix} -3 \\ -3 \\ 3 \\ 3 \end{bmatrix}$ $\quad P_3 = \frac{1}{4} \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$

2. $V = \mathbb{R}^{n \times n}$

$\langle A, B \rangle = \text{Tr}(B^T A)$

- $\langle A+A', B \rangle$
  - $= \text{Tr}(B^T (A + A'))$
  - $= \text{Tr}(B^T A + B^T A')$
  - $= \text{Tr}(B^T A) + \text{Tr}(B^T A')$
  - $= \langle A, B \rangle + \langle A', B \rangle$
- $\langle A, B \rangle$
  - $= \text{Tr}(B^T A)$
  - $= \text{Tr}((B^T A)^T)$
  - $= \text{Tr}(A^T B)$
  - $= \langle B, A \rangle$
- $\langle A, A \rangle$
  - $= \text{Tr}(A^T A)$
  - $= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji}^2$
  - ≥ 0

$\langle A, A \rangle = 0 \iff A$ is a zero matrix.

(3) let $x = [x_1, x_2, x_3, x_4]$

$\rightarrow x_1 - x_2 = 0$

$x_3 - x_4 = 0$

$\Rightarrow x = [x_1, -x_1, x_3, -x_3]$

$x_1 - 2x_2 + 3x_3 + 4x_4 = 0$

$3x_1 - 5x_2 + 7x_3 + 9x_4 = 0$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 3 & -5 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

A

$R_1 : R_1$ $\quad \begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}$

$R_2 : R_2 - 3R_1$ $\quad \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}$

$\Rightarrow$ free variables: x_2, x_4

Pivot variables: x_1, x_3

$\therefore$ $\boxed{x_1 = 2x_2 + 3x_4}$

$x_1 = 2x_2 + 3x_4$

$= 2(x_2 + 3x_4) - 3x_2 - 9x_4$

$= 2x_2 + 3x_4 - 3x_2 - 9x_4$

$= -x_2 - 6x_4$

$\boxed{x_1 = -x_2 - 6x_4}$

$[-x_2 - 6x_4, x_2, x_3, -x_3] = [-x_2, x_2, 3x_3 + 6x_4, -x_3, x_4]$

$= [-x_2, x_2, x_3, -x_3] + [-6x_4, 0, 6x_4, x_4]$

$= x_2 [-1, 1, 0, 0] + x_3 [0, 0, 3, -1]$

let $x_2 = 0$, and $x_4 = 1$ $\quad \rightarrow [-6, 0, 6, 1]$

$x_3 = 0$, and $x_4 = 0$ $\quad \rightarrow [-1, 1, 0, 0]$

$\therefore$

Basis : $\{ [-1, 1, 0, 0], [-6, 0, 6, 1] \}$

4. $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ $\quad u_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$u_1 \cdot u_2 = 1 + 1 + -1 + -1 = 0$

$u_1 \cdot u_3 = 1 + -1 + 1 + -1 = 0$

$u_2 \cdot u_3 = 1 + 1 + -1 + -1 = 0$

$\therefore \{u_1, u_2, u_3\}$ is an orthogonal subset

Find $x \in \mathbb{R}^4$ $\quad u_1^T x = u_2^T x = u_3^T x = 0$

let $A = \begin{bmatrix} \text{---} & u_1 & \text{---} \\ \text{---} & u_2 & \text{---} \\ \text{---} & u_3 & \text{---} \end{bmatrix}$

Find x such that $Ax = 0$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Apply elementary row operations

$R_1 : R_1$ $\quad \begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}$

$R_2 : R_2 - R_1$ $\quad \begin{array}{cccc} 0 & 0 & -2 & -2 \end{array}$

$R_3 : R_3 - R_1$ $\quad \begin{array}{cccc} 0 & -2 & 0 & -2 \end{array}$

$\therefore$ x_4 is a free variable. let $x_4 = t \in \mathbb{R}$

R_2 $\quad -2x_3 = 2t \quad \Rightarrow \quad x_3 = -t$

R_3 $\quad -2x_2 - 2x_4 = 0 \quad \Rightarrow \quad x_2 + x_4 = 0$

$x_2 + x_4 = 0$

$x_1 + x_2 + x_3 + x_4 = 0$

$x_1 - x_2 - x_3 - x_4 = 0 \Rightarrow x_1 = -t$

$\therefore$ $x = \begin{bmatrix} -t \\ -t \\ -t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$.

let $t = 1$:

$\therefore$ $x = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ can be the fourth vector

Check.

$u_1 \cdot x = 1 + 1 + 1 + 1 = 0$

$u_2 \cdot x = 1 + 1 + 1 + 1 = 0$

$u_3 \cdot x = 1 + 1 + 1 + 1 = 0$

(5) $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ $\quad u_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$u_1 \cdot x = \frac{u_1 \cdot x}{u_1 \cdot u_1} u_1$

$u_1 \cdot u_1 = 1 + 1 + 1 + 1 = 4$

$u_1 \cdot x = x_1 + x_2 + x_3 + x_4$

$\Rightarrow u_1 \cdot x = \frac{x_1 + x_2 + x_3 + x_4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2 + x_3 + x_4}{4} \\ \frac{x_1 + x_2 + x_3 + x_4}{4} \\ \frac{x_1 + x_2 + x_3 + x_4}{4} \\ \frac{x_1 + x_2 + x_3 + x_4}{4} \end{bmatrix}$

$u_2 \cdot x = \frac{u_2 \cdot x}{u_2 \cdot u_2} u_2$

$u_2 \cdot u_2 = 1 + 1 + 1 + 1 = 4$

$u_2 \cdot x = x_1 + x_2 - x_3 - x_4$

$\Rightarrow u_2 \cdot x = \frac{x_1 + x_2 - x_3 - x_4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2 - x_3 - x_4}{4} \\ \frac{x_1 + x_2 - x_3 - x_4}{4} \\ \frac{-x_1 - x_2 + x_3 + x_4}{4} \\ \frac{-x_1 - x_2 + x_3 + x_4}{4} \end{bmatrix}$

$u_3 \cdot x = \frac{u_3 \cdot x}{u_3 \cdot u_3} u_3$

$u_3 \cdot u_3 = 1 + 1 + 1 + 1 = 4$

$u_3 \cdot x = x_1 - x_2 + x_3 - x_4$

$\Rightarrow u_3 \cdot x = \frac{x_1 - x_2 + x_3 - x_4}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{x_1 - x_2 + x_3 - x_4}{4} \\ \frac{-x_1 + x_2 - x_3 + x_4}{4} \\ \frac{x_1 - x_2 + x_3 - x_4}{4} \\ \frac{-x_1 + x_2 - x_3 + x_4}{4} \end{bmatrix}$

$u_1 \cdot x = \frac{x_1 + x_2 + x_3 + x_4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$u_2 \cdot x = \frac{x_1 + x_2 - x_3 - x_4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

$u_3 \cdot x = \frac{x_1 - x_2 + x_3 - x_4}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ $P_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$P_2 = \frac{u_2}{\|u_2\|} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$P_3 = \frac{u_3}{\|u_3\|} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$\therefore \{P_1, P_2, P_3\}$ is an orthonormal basis

6. let $Z \subset \mathbb{R}^n$

$Z^\perp = \{ x \in \mathbb{R}^n : x \cdot y = 0 \ \forall y \in Z \}$

Claim 1 : $0 \in Z^\perp$

Proof

$0 \cdot y = \sum_{i=1}^{n} 0 \cdot y_i = 0 \quad \forall y \in Z$

$\therefore 0 \in Z^\perp$

QED Claim 1

Claim 2 : If $x, y \in Z^\perp$ and $x, y \in Z$, then $x + y \in Z^\perp$

Proof

$x \in Z^\perp$, $y \in Z^\perp \Rightarrow x \cdot y = 0 \ \forall y \in Z$

$(x + y) \cdot y = x(y) + y(y)$

$= x \cdot y + y \cdot y = 0 \quad \forall y \in Z$

$\therefore x + y \in Z^\perp$

QED Claim 2