HOMEWORK SET #10

EE 510: Linear Algebra for Engineering

Assigned: 2 November 2024
Due: 10 November 2024

Directions: Please include all the necessary plots and clearly derive any pertinent mathematical expression.

1. Let random vector X be a 2D-Gaussian vector with mean $\underline{\mu}_X$ and covariance matrix K_{XX}. We have

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \underline{\mu}_X = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad K_{XX} = \begin{bmatrix} {\sigma_1}^2 & \sigma_{12} \\ \sigma_{21} & {\sigma_2}^2 \end{bmatrix}$$

where μ_1 and μ_2 are the respective means of X_1 and X_2, ${\sigma_1}^2$ and ${\sigma_2}^2$ are the respective variances of X_1 and X_2, and σ_{12} is the covariance between X_1 and X_2. Note that $\sigma_{12} = \sigma_{21}$ so K_{XX} is a symmetric matrix ($K_{XX} = {K_{XX}}^T$).

Let X be a Gaussian vector such that:

$$\underline{\mu}_X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad K_{XX} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

a) Randomly select 5,000 samples from the 2D-Gaussian distribution with mean $\underline{\mu}_X$ and covariance K_{XX} and plot them. (You can use the code provided in Figure 1 below or any other software).

b) Define the ellipsoid $\mathcal{E}_2$ where

$$\mathcal{E}_2 = \left\{ \mathbf{x} = [x_1, x_2]^T : \ (\mathbf{x} - \underline{\mu}_{\mathbf{X}})^T K_{xx}^{-1} (\mathbf{x} - \underline{\mu}_{\mathbf{X}}), \ x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R} \right\}. \tag{1}$$

c) Derive the principal axes and their respective lengths of $\mathcal{E}_2$.

d) Sketch the ellipsoid $\mathcal{E}_2$. Compare this plot with the sample plot.

e) Repeat 1(a)-1(d) with

$$\underline{\mu}_X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad K_{XX} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

f) Repeat 1(a)-1(d) with

$$\underline{\mu}_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad K_{XX} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

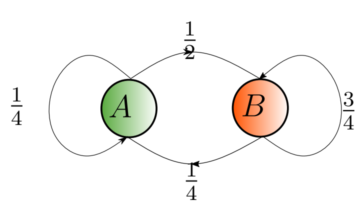
```python
# Importing the necessary modules
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import multivariate_normal

plt.rcParams['figure.figsize']=8,8

def generate_and_plot(kx, mu):

  distr = multivariate_normal(
    cov = Kx, mean = mu,
    seed = 1000)

  data = distr.rvs(size = 5000)
  plt.grid()

  plt.plot(data[:,0],data[:,1], 'o', c='lime',
           markeredgewidth = 0.5,
           markeredgecolor = 'black')

  plt.title(r'Random samples from a 2D-Gaussain distribution')
  plt.xlabel(r'$x_1$')
  plt.ylabel(r'$x_2$')
  plt.axis('equal')


# Define the mean and covariance matrix
Kx = np.array([[2.0, 1.0],
    [1.0, 4.0]])
mu = np.array([0,0])
random_seed = 10


generate_and_plot(Kx, mu)
```

Figure 1: Python code for number 1(a).

2. Each year, 50% of the population of the city A migrates to the city B, while only 25% of the population in B will move to A. Let a_k and b_k denote the respective proportions of the total population living in A and B at the end of year k, assuming $a_k + b_k = 1$. The system of linear equations that describe the population proportion at year $k+1$ is

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \end{bmatrix} = A \begin{bmatrix} a_k \\ b_k \end{bmatrix} \qquad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

We called the transpose of A the transition matrix.



(a) Compute A^k for any k integer, and show that

$$\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Hint: Use diagonalization to find $A = QDQ^{-1}$ where D is the diagonal matrix formed by the eigenevalues,and then find $A^k = QD^kQ^{-1}$.
(b) Assuming initially a distribution $[a_0, b_0]^T = [\frac{1}{2}, \frac{1}{2}]^T$ If this migration pattern continues, compute the long-run population distribution. Will city A be desserted?

3. Find the principal axes, their lengths, and sketch the following ellipsoids:

a) $2x_1^2 + 4x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3 = 1$

b) $5x_1^2 + 2x_2^2 + 4x_3^2 - 2\sqrt{2}x_1x_3 = 1.$