HW08 — Book

Leon Chapter #7 7.15

7.15. Suppose that the number of particle emissions by a radioactive mass in t seconds is a Poisson random variable with mean λt. Use the Chebyshev inequality to obtain a bound for the probability that $|N(t)/t - \lambda|$ exceeds ε.

$\because E[N(t)] = \lambda t$

$\therefore N(t) \sim \text{Poisson}(\lambda t)$

$V[N(t)] = \lambda t$

$\therefore E[N(t)/t] = \lambda t / t = \lambda$

$V[N(t)/t] = \lambda t / t^2 = \lambda / t$

$$\therefore P[|N(t)/t - \lambda| > \varepsilon] = P[|N(t)/t - E[N(t)/t]| > \varepsilon]$$

$$\overset{\text{C.I.}}{\leq} \frac{V[N(t)/t]}{\varepsilon^2}$$

$$= \frac{\lambda/t}{\varepsilon^2}$$

$$= \frac{\lambda}{t\varepsilon^2}$$

7.16. Suppose that 20% of voters are in favor of certain legislation. A large number n of voters are polled and a relative frequency estimate $f_A(n)$ for the above proportion is obtained.

Use Eq. (7.20) to determine how many voters should be polled in order that the probability is at least .95 that $f_A(n)$ differs from 0.20 by less than 0.02.

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Suppose X voters of n are in favor of certain legislation

$\therefore X \sim \text{Bernoulli}(0.2)$

$E[X] = 0.2$, $V[X] = 0.2 \times 0.8 = 0.16$

$\therefore E[f_A(n)] = \mu = E[X] = 0.2$

$V[f_A(n)] = \sigma^2/n = V[X]/n = 0.16/n$

$\because P[|f_A(n) - \mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$ (7.20)

$\therefore P[|f_A(n) - 0.2| < 0.02] \geq 1 - \frac{0.16}{n(0.02)^2}$

$\because P[|f_A(n) - 0.2| < 0.02] \geq 0.95$

$\therefore 1 - \frac{0.16}{n(0.02)^2} \geq 0.95$

$\therefore \frac{0.16}{n(0.02)^2} \leq 0.05$

$\therefore n \geq \frac{0.16}{(0.02)^2 \, 0.05} = 8000$

$\therefore$ at least 8000 voters

7.17. A fair die is tossed 20 times. Use Eq. (7.20) to bound the probability that the total number of dots is between 60 and 80.

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Suppose the number of dots in 1 time is X

$\therefore X \sim \text{uniform}(6)$

$\therefore E[X] = \frac{6+1}{2} = \frac{7}{2}$, $V[X] = \frac{6^2-1}{12} = \frac{35}{12}$

Suppose the total number of dots in 20 times is N_n

$\therefore E[N_{20}] = E[X_1 + X_2 + \cdots + X_{20}] = E[X_1] + E[X_2] \cdots + E[X_{20}] = 20\,E[X] = 20 \times \frac{7}{2} = 70$

$V[N_{20}] = V[X_1 + X_2 + \cdots + X_{20}] \overset{iid}{=} V[X_1] + V[X_2] + \cdots + V[X_{20}] = 20\,V[X] = 20 \times \frac{35}{12} = 175/3$

$\therefore P[60 \le N_{20} \le 80] = P[|N_{20} - 70| < 10] \overset{M.I.}{\geq} 1 - \frac{V[N_{20}]}{10^2}$

$= 1 - \frac{175}{300}$

≈ 0.417

7.41. Let ζ be selected at random from the interval $S = [0, 1]$, and let the probability that ζ is in a subinterval of S be given by the length of the subinterval. Define the following sequences of random variables for $n \geq 1$:

$$X_n(\zeta) = \zeta^n, \; Y_n(\zeta) = \cos^2 2\pi\zeta, \; Z_n(\zeta) = \cos^n 2\pi\zeta.$$

Do the sequences converge, and if so, in what sense and to what limiting random variable?

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① $X_n(\zeta) = \zeta^n$

when $\zeta = 1$, $X_n(1) = 1$

when $0 \leq \zeta < 1$, $\lim\limits_{n\to\infty} X_n(\zeta) = \lim\limits_{n\to\infty} \zeta^n = 0$

$\therefore$ converge with prob 1 (almost everywhere) $X_n(\zeta) \xrightarrow{0} 0$

② $Y_n(\zeta) = \cos^2 2\pi\zeta$

$\therefore \forall \zeta \in S$, $Y_n(\zeta) \xrightarrow{u} \cos^2 2\pi\zeta$

③ $Z_n(\zeta) = \cos^n 2\pi\zeta$

when $\zeta = 1$, $Z_n(\zeta) = \cos^n 2\pi = 1$

when $\zeta = 0$, $Z_n(\zeta) = \cos^n 0 = 1$

when $\zeta = \frac{1}{2}$, $Z_n(\zeta) = \cos^n \pi = (-1)^n$

when $0 < \zeta < \frac{1}{2}$ or $\frac{1}{2} < \zeta < 1$, $\cos 2\pi\zeta \in (-1, 1)$, $\lim\limits_{n\to\infty} Z_n(\zeta) \to 0$

$\therefore$ converge with prob. 1 (almost everywhere)

$Z_n(\zeta) \xrightarrow{0} 0$

7.44. Let X_n be a sequence of iid random variables with mean m and variance $\sigma^2 < \infty$. Let M_n be the associated sequence of arithmetic averages,

$$M_n = \frac{1}{n}\sum_{i=0}^{n} X_i.$$

Show that M_n converges to m in the mean square sense.

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$\because M_n = \frac{1}{n}\sum_{i=0}^{n} X_i = \bar{X}$

$\therefore E[M_n] = m$

$V[M_n] = \sigma^2/n$

$\therefore E[M_n - m] = m - m = 0$

$V[M_n - m] = V[M_n] = \sigma^2/n$

$$\therefore \lim_{n\to\infty} E[(M_n - m)^2] = \lim_{n\to\infty}\left[V[M_n - m] - E^2[M_n - m]\right]$$

$$= \lim_{n\to\infty}\left[\frac{\sigma^2}{n} - 0\right]$$

$$= 0$$

$\therefore M_n \xrightarrow{m} 0$

7.45. Let X_n and Y_n be two (possibly dependent) sequences of random variables that converge in the mean square sense to X and Y, respectively. Does the sequence $X_n + Y_n$ converge in the mean square sense, and if so, to what limit?

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$\because X_n \xrightarrow{m} X,\ Y_n \xrightarrow{m} Y$

$\therefore \lim_{n\to\infty} E[(X_n-X)^2]=0,\ \lim_{n\to\infty} E[(Y_n-Y)^2]=0$

$$\begin{aligned}
\lim_{n\to\infty} E[((X_n+Y_n)-(X+Y))^2] &= \lim_{n\to\infty} E[((X_n-X)+(Y_n-Y))^2] \\
&= \lim_{n\to\infty} E[(X_n-X)^2] + \lim_{n\to\infty} E[(Y_n-Y)^2] + \lim_{n\to\infty} 2E[(X_n-X)(Y_n-Y)] \\
&= 0+0+\lim_{n\to\infty} 2E[(X_n-X)(Y_n-Y)] \\
&\leq 2\lim_{n\to\infty} \sqrt{E[(X_n-X)^2]}\sqrt{E[(Y_n-Y)^2]} \\
&= 2\cdot 0 \\
&= 0
\end{aligned}$$

$\therefore X_n+Y_n \xrightarrow{m} X+Y$, without additional limit

7.50. Let X_n be a sequence of Laplacian random variables with parameter $\alpha = n$. Does this sequence converge in distribution?

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$\because X_n \sim \text{Laplacian}(n)$

$f(x) = \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}} = \frac{1}{2n} e^{-\frac{|x|}{n}}$

$\therefore E[X_n] = 0 \ , \quad V[X_n] = 2n^2$

$\because \lim_{n\to\infty} P[\,|X_n - 0| \geq \varepsilon\,] = \lim_{n\to\infty} P[\,|X_n - E[X_n]| \geq \varepsilon\,]$

$\overset{C.I}{\leq} \lim_{n\to\infty} \frac{V[X_n]}{\varepsilon^2}$

$= \lim_{n\to\infty} \frac{2n^2}{\varepsilon^2}$

$= \infty$

$\therefore X_n$ does not converge in probability

$\therefore X_n$ does not converge in distribution

Guber Chapter #14 14.1

1. Let $X_n \sim \text{Cauchy}(1/n)$. Show that X_n converges in probability to zero.

$X_n \sim \text{Cauchy}(1/n)$

$\therefore f_{X_n}(x) = \dfrac{n}{\pi(1+n^2x^2)}$

$$\therefore \lim_{n\to\infty} P(|X_n - 0| > \varepsilon) = \lim_{n\to\infty} P(|X_n| > \varepsilon)$$

$$= \lim_{n\to\infty} \int_{|x|>\varepsilon} f_{X_n}(x)\,dx$$

$$= \lim_{n\to\infty} 2\int_{\varepsilon}^{\infty} \frac{n}{\pi(1+n^2x^2)}\,dx$$

let $u = nx$, $du = n\,dx$

$$= \lim_{n\to\infty} \frac{2}{\pi}\int_{n\varepsilon}^{\infty} \frac{1}{1+u^2}\,du$$

$$= \lim_{n\to\infty} \frac{2}{\pi}\arctan\left(\frac{1}{n\varepsilon}\right)$$

$$= \frac{2}{\pi}\arctan 0$$

$$= 0$$

$$\therefore X_n \xrightarrow{P} 0$$

2. Let c_n be a converging sequence of real numbers with limit c. Define the constant random variables $Y_n \equiv c_n$ and $Y \equiv c$. Show by direct analysis of $\mathsf{P}(|Y_n - Y| \geq \varepsilon)$ that Y_n converges in probability to Y.

Remark. Here is an easier approach (that you are *not* to use for your solution of this problem). Since c_n and c are deterministic,

$$\mathsf{E}[|c_n - c|] = |c_n - c| \rightarrow 0.$$

Since convergence in mean implies convergence in probability, Y_n converges in probability to Y.

Guber Chapter #14 14.2

$\because \lim_{n\to\infty} c_n = c$

$\therefore \lim_{n\to\infty} P(|Y_n - Y| \geq \varepsilon) = \lim_{n\to\infty} P(|c_n - c| \geq \varepsilon)$

$= \lim_{n\to\infty} P(0 \geq \varepsilon)$

$= 0$

$\therefore Y_n \xrightarrow{P} Y$

3. Let $U \sim \text{uniform}[0,1]$, and put

$$X_n := nI_{[0,1/\sqrt{n}]}(U), \quad n = 1,2,\ldots.$$

Does X_n converge in probability to zero?

Guber Chapter #14 14.3

$\because U \sim \text{uniform}[0,1]$

$$X_n = nI_{[0,1/\sqrt{n}]}(U) = \begin{cases} n, & \text{if } U \in [0, \frac{1}{\sqrt{n}}] \\ 0, & \text{if } U \notin [0, \frac{1}{\sqrt{n}}] \end{cases}$$

$$\therefore \lim_{n\to\infty} P(|X_n - 0| > \varepsilon) = \lim_{n\to\infty} P(|X_n| > \varepsilon)$$

$$= \lim_{n\to\infty} P(X_n = n)$$

$$= \lim_{n\to\infty} P(U \in [0, \tfrac{1}{\sqrt{n}}])$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}$$

$$= 0$$

$$\therefore X_n \xrightarrow{P} 0$$