Homework #9

Sunday, October 27, 2024 12:34 PM

1. Sec 9.2:

#17:

$$U = Q \Lambda Q^H$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$T = \frac{1}{\sqrt{2}}(1+1-i) = 0 \quad ,$$

$$D = \frac{1}{2}(1)(-1) - \frac{1}{2}(1+i)(1-i)$$

$$= \frac{1}{2}\left[-1 - (1-i^2)\right] = \frac{1}{2}(-1-(1+1)) = \frac{-2}{2} = -1$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{0 \pm \sqrt{0 - 4(-1)}}{2} = \frac{\pm\sqrt{4}}{2} = \pm 1$$

$$\lambda_1 = 1, \quad U\alpha_1 = \lambda_1 \alpha_1 \quad : \quad (U - \lambda_1 I)\alpha_1 = 0$$

$$U\alpha_1 = \lambda_1 \alpha_1 \qquad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1-i \\ 1+i & -1-i \end{bmatrix} \alpha_1 = 0$$

$$\therefore \begin{bmatrix} -1 & 1-i \\ 1+i & -1-i \end{bmatrix} \alpha_1 = 0 \qquad \alpha_1 = \begin{bmatrix} 1+\sqrt{2} \\ 1+i \end{bmatrix}$$

$$\alpha_1^H \alpha_1 = \begin{bmatrix} 1+\sqrt{2} & 1-i \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} \\ 1+i \end{bmatrix} = 1+3+2\sqrt{2}+1-i^2 = 6+2\sqrt{2}.$$

$$\therefore \alpha_1 = \frac{1}{\sqrt{6+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} \\ 1+i \end{bmatrix}$$

$$\lambda_2 = -1 : \qquad U\alpha_2 = \lambda_2 \alpha_2 \quad , \quad (U - \lambda_2 I)\alpha_2 = 0$$

$$U\alpha_2 = \lambda_2 \alpha_2 \qquad \frac{1}{\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} & 1-i \\ 1+i & -1+\sqrt{2} \end{bmatrix} \alpha_2 = 0$$

$$\therefore \alpha_2 = \begin{bmatrix} -1+i \\ 1+\sqrt{2} \end{bmatrix}$$

$$\alpha_2^H \alpha_2 = \begin{bmatrix} -1+i & 1-i \\ 1+\sqrt{2} & -1+\sqrt{2} \end{bmatrix}^H \begin{bmatrix} -1+i \\ 1+\sqrt{2} \end{bmatrix} = 1-i^2 + 1+3+2\sqrt{2} = \text{interesting}+2\sqrt{2}$$

$$= 6+2\sqrt{2}$$

$$U = \frac{1}{\sqrt{6+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} & -1+i \\ 1+i & 1+\sqrt{2} \end{bmatrix}$$

#25:

$U = A + iB$ is unitary. (A and B are real.)

$$\therefore UU^H = (A+iB)(A^T - iB^T) = AA^T - iAB^T + iBA^T + BB^T$$

$$= (AA^T + BB^T) + i\underbrace{(BA^T - AB^T)}_{0}$$

$$= AA^T + BB^T = I \quad \text{—— ①}$$

Also,

$$U^H U = (A^T - iB^T)(A + iB) = A^T A + iA^T B - iB^T A + B^T B$$

$$= (A^T A + B^T A) + i\underbrace{(A^T B - B^T A)}_{0}$$

$$= (A^T A + B^T A) = I \quad \text{—— ②}$$

Let $Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

Then

$$QQ^T = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = \begin{bmatrix} AA^T + BB^T & AB^T - BA^T \\ BA^T - AB^T & BB^T + AA^T \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

#26: $A + iB$ (A and B are real.)

$$(A+iB)^H = ((A+iB)^T)^* = (A^T - iB^T)^* = (A - iB)^T$$

$$= A^T - iB^T$$

$$\therefore A + iB = A^T - iB^T \Rightarrow \underline{A = A^T} \text{ and } \underline{-B = B^T}$$

Let $C = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$$C^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = \begin{bmatrix} A & B^T \\ -B^T & A \end{bmatrix} \quad \text{— } A = A^T$$

$$= \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad \text{—— } B = -B^T$$

$$= C$$

$\therefore$ C is symmetric.

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2. $$A = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix}$$

$k = 1$:

$A_1 = T_1 = A$

$\lambda = 4$ is an eigenvalue of A_1

$A_1 x = \lambda x = 4x$, $\quad x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a solution.

$$\therefore N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U_1 = N_1$$

$$T_1 = U_1^H T_1 U_1 = \begin{bmatrix} 4 & 2 & 0 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

$k = 2$:

$$A_2 = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$\lambda = 2$ is an eigenvalue of A_2 and $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an the corresponding eigenvector.

$$A_2 x = \lambda x$$

$$\therefore N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ so that } T_2 = U_2^H T_1 U_2$$

$$T_2 = \begin{bmatrix} 4 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$k = 3$

$$\therefore A_3 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$\lambda = 2$ is an eigenvalue of A_3 and $x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the corresponding eigenvector.

$$N_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$T_3 = U_3^H T_2 U_3 = \begin{bmatrix} 4 & 2 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 2 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\therefore U = U_1 U_2 U_3$ and $U^H A U = T_3$

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3. $U^H U = I = UU^H$

$A = U^H B U$

B is normal if and only if A is normal.

Claim 1: If B is normal then A is normal.

Proof:

$$B^H B = BB^H$$

$$A = U^H B U, \qquad A^H = (U^H B U)^H = U^H B^H U$$

$$\therefore A^H A = U^H B^H \underbrace{UU^H}_{I} B U$$

$$= U^H B^H B U$$

$$\therefore AA^H = U^H B \underbrace{UU^H}_{I} B^H U$$

$$= U^H B B^H U$$

$$\therefore A^H A = U^H B^H B U = U^H B B^H U = AA^H \text{ —}$$

Q.E.D.

Claim 2: If A is normal then B is normal.

Proof:

$$A^H A = AA^H$$

$$A = U^H B U$$

$$UAU^H = UU^H B UU^H = B$$

$$\therefore B^H = (UAU^H)^H = (U^H)^H A^H U^H = UA^H U^H$$

$$\cdot \quad B^H B = UA^H U^H UAU^H$$

$$= UA^H \underbrace{U^H U}_{I} AU^H = UA^H A U^H$$

$$BB^H = UA \underbrace{U^H U}_{I} AU^H$$

$$= UAA^H U^H$$

$$\therefore B^H B = BB^H \quad \text{—— Q.E.D.}$$

4. A is such that $A^H A = AA^H$.

$$Ax = \lambda x$$

Recall: $(x, cy) = x^H cy \qquad \forall\ x, y \in \mathbb{C}^n,\ c \in \mathbb{C}$

$(cx, y) = c^* x^H y$

$(x, cy) = c^* x \cdot y$

$$Ax \cdot Ax = A^H Ax \cdot x$$

$$= AA^H x \cdot x$$

$$= A^H x \cdot A^H x$$

$$\therefore 0 = 0 \cdot 0$$

$$= (Ax - \lambda x) \cdot (Ax - \lambda x)$$

$$= Ax \cdot Ax - Ax \cdot \lambda x - \lambda x \cdot Ax + \lambda x \cdot \lambda x$$

$$= A^H x \cdot A^H x - \lambda^* (Ax \cdot x) - \lambda (x \cdot Ax) + (\lambda x \cdot \lambda x)$$

$$= A^H x \cdot A^H x - \lambda^* (x \cdot A^H x) - \lambda (A^H x \cdot x) + \lambda^2 (x \cdot x)$$

$$= (A^H x - \lambda^* x) \cdot (A^H x - \lambda^* x)$$

$$\therefore A^H x - \lambda^* x = 0$$

$$\Rightarrow A^H x = \lambda^* x$$