HW10 — Handout — Q1

1. Suppose random variable X has a discrete probability density $P(X) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$ and random variable Y has density $P(Y) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ and both have joint density $P(X,Y)$ where the element in the ith row and jth column is $p(x_i, y_j)$:

$$
P(X,Y) = \begin{array}{c} \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{8} \end{array}
\begin{array}{c} \begin{array}{cccc} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array} \\
\begin{pmatrix} \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & \frac{1}{4} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{16} & 0 \end{pmatrix} \end{array} .
$$

Define the entropy of X in bits as $H(X) = -\sum_{i=1}^{n} p_i \log_2 p_i$, the conditional entropy $H(X|Y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, y_j) \log_2 p(x_i|y_j)$, and the joint entropy as $H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, y_j) \log_2 p(x_i, y_j)$. Define the mutual information $I(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, y_j) \log_2 \frac{p(x_i,y_j)}{p(x_i)p(y_j)}$. Then $I(X,Y) \geq 0$ holds. First compute $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, $H(Y|X)$, and $I(X,Y)$ for the above densities. Then prove that $H(X,Y) = H(X) + H(Y|X)$ and $I(X,Y) = H(X) - H(X|Y)$ and verify for the above densities. Thus $I(X,Y) = H(X) + H(Y) - H(X,Y)$. Verify this too.

$$H(X) = -\sum_{i=1}^{n} P_i \log_2 P_i = -\left(\tfrac{1}{2}\log_2^{\frac{1}{2}} + \tfrac{1}{4}\log_2^{\frac{1}{4}} + \tfrac{1}{8}\log_2^{\frac{1}{8}} + \tfrac{1}{8}\log_2^{\frac{1}{8}}\right) = 1.75$$

$$H(Y) = -\sum_{i=1}^{n} P_i \log_2 P_i = -\left(4 \cdot \tfrac{1}{4}\log_2^{\frac{1}{4}}\right) = -\log_2^{\frac{1}{4}} = 2$$

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} P(x_i, y_i) \log_2 P(x_i, y_i)$$

$$= -\left(\tfrac{2}{8}\log_2(\tfrac{1}{8}) + \tfrac{6}{16}\log_2(\tfrac{1}{16}) + \tfrac{4}{32}\log_2(\tfrac{1}{32}) + \tfrac{1}{4}\log_2(\tfrac{1}{4})\right)$$

$$= 3.375$$

$$P(x_i | y_i) = \frac{P(x_i, y_i)}{P(y_i)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & 0 \end{bmatrix}$$

$$P(y_i | x_i) = \frac{P(x_i, y_i)}{P(x_i)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 \end{bmatrix}$$

$$\therefore H(X|Y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} P(X_i, Y_i)\log_2 P(X_i|Y_i)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n} P(Y_i)P(X_i|Y_i)\log_2 P(X_i|Y_i)$$

$$= -\left(\frac{2}{4}\cdot\frac{1}{2}\cdot\log_2\left(\frac{1}{2}\right) + \frac{6}{4}\cdot\frac{1}{4}\cdot\log_2\left(\frac{1}{4}\right) + \frac{4}{4}\cdot\frac{1}{8}\log_2\left(\frac{1}{8}\right) + \frac{1}{4}\cdot 1\cdot\log_2(1)\right)$$

$$= 1.375$$

$$H(Y|X) = -\left(\frac{1}{2}\left[\frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{2}{8}\log_2\left(\frac{1}{8}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right] + \frac{1}{4}\left[\frac{2}{4}\log_2\left(\frac{1}{4}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right]\right.$$

$$\left. + \frac{1}{8}\left[\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{4}{4}\log_2\left(\frac{1}{4}\right) + \frac{1}{8}\log_2\left(\frac{1}{8}\right)\right]\right)$$

$$= 1.625$$

$$\because \frac{P(x_i, Y_i)}{P(X_i)P(Y_i)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\therefore I(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{n} P(x_i, y_j)\log_2\frac{P(X_i, Y_i)}{P(X_i)P(Y_j)}$$

$$= \frac{2}{16}\log_2\left(\frac{1}{2}\right) + \frac{1}{4}\log_2(2) + \frac{1}{8}\log_2(2) + \frac{2}{16}\log_2(2)$$

$$= 0.375$$

$$\because H(x, y) = -\sum_x\sum_y P(X, Y)\log_2 P(X, Y)$$

$$\therefore H(X) + H(Y|X) = -\sum_k P(X_k)\log_2 P(X_k) - \sum_x\sum_y P(X, Y)\log_2 P(Y|X)$$

$$= -\sum_k P(X_k)\log_2 P(X_k) - \sum_x\sum_y P(X, Y)\log_2\frac{P(X, Y)}{P(X)}$$

$$= -\sum_k P(X_k)\log_2 P(X_k) - \sum_x\sum_y P(X, Y)\log_2 P(X, Y)$$

$$+ \sum_x\sum_y P(X, Y)\log_2 P(X)$$

$$= -\sum_k P(X_k)\log_2 P(X_k) + \sum_x P(x)\log_2 P(x) - \sum_x\sum_y P(X, Y)\log_2 P(x, Y)$$

$$= H(x, Y)$$

$$= 1.75 + 1.625 = 3.375$$

$$I(X,Y) = \sum_x \sum_y P(X,Y) \log_2 \frac{P(X,Y)}{P(X)\,P(Y)}$$

$$= \sum_x \sum_y P(X,Y) \log_2 \frac{P(X|Y)}{P(X)}$$

$$= -\sum_x \sum_y P(X,Y) \log_2 P(X) - \left(-\sum_x \sum_y P(X,Y) \log_2 P(X|Y)\right)$$

$$= H(X) - H(X|Y)$$

$$= 1.75 - 1.375 = 0.375$$

HW10 — Handout — Q2

2. A subset $A \subset X$ is fuzzy iff it has a multivalues indicator function $I_A : X \to [0,1]$ or more simply $a : X \to [0,1]$. So a finite fuzzy set A defines a point in the n-dimensional unit hypercube: $A \subset [0,1]^n$. Then A defines a fit or "fuzzy unit" vector of length n: $A = (a(x_1), \ldots, a(x_n)) = (a_1, \ldots, a_n)$ where $a_i = Degree(x_i \in A)$. Pointwise operations define set operations: $A^C = (1 - a_1, \ldots, 1 - a_n)$, $A \cap B = (\min(a_1, b_1), \ldots, \min(a_n, b_n))$, and $A \cup B = (\max(a_1, b_1), \ldots, \max(a_n, b_n))$. Find A^C, $A \cap A^C$, $A \cup A^C$ for eachg of the following eight sets below.

$C = (1\ 0\ 1)$	$G = (0.3\ 0.3\ 0.3)$
$D = (0.4\ 0.5\ 0.6)$	$H = (0.5\ 0.1\ 0.4)$
$E = (0.3\ 0\ 0.1)$	$I = (0.5\ 0.9\ 0.6)$
$F = (0.6\ 0\ 0)$	$J = (0.5\ 0.5\ 0.5)$

1 $\because C = (1\ 0\ 1)$
$\therefore C^c = (0\ 1\ 0)$
$C \cap C^c = (0\ 0\ 0)$
$C \cup C^c = (1\ 1\ 1)$

2 $\because D = (0.4\ 0.5\ 0.6)$
$\therefore D^c = (0.6\ 0.5\ 0.4)$
$D \cap D^c = (0.4\ 0.5\ 0.4)$
$D \cup D^c = (0.6\ 0.5\ 0.6)$

3 $\because E = (0.3\ 0\ 0.1)$
$\therefore E^c = (0.7\ 1\ 0.9)$
$E \cap E^c = (0.3\ 0\ 0.1)$
$E \cup E^c = (0.7\ 1\ 0.9)$

4 $\because F = (0.6\ 0\ 0)$
$\therefore F^c = (0.4\ 1\ 1)$
$F \cap F^c = (0.4\ 0\ 0)$
$F \cup F^c = (0.6\ 1\ 1)$

5 $\because G = (0.3\ 0.3\ 0.3)$
$\therefore G^c = (0.7\ 0.7\ 0.7)$
$G \cap G^c = (0.3\ 0.3\ 0.3)$
$G \cup G^c = (0.7\ 0.7\ 0.7)$

6 $\because H = (0.5\ 0.1\ 0.4)$
$\therefore H^c = (0.5\ 0.9\ 0.6)$
$H \cap H^c = (0.5\ 0.1\ 0.4)$
$H \cup H^c = (0.5\ 0.9\ 0.6)$

7 $\because I = (0.5\ 0.9\ 0.6)$
$\therefore I^c = (0.5\ 0.1\ 0.4)$
$I \cap I^c = (0.5\ 0.1\ 0.4)$
$I \cup I^c = (0.5\ 0.9\ 0.6)$

8 $\because J = (0.5\ 0.5\ 0.5)$
$\therefore J^c = (0.5\ 0.5\ 0.5)$
$J \cap J^c = (0.5\ 0.5\ 0.5)$
$J \cup J^c = (0.5\ 0.5\ 0.5)$

HW10 — Handout — Q3

3. (a) Define the count $c(A)$ of a fuzzy set A as $c(A) = \sum_i a_i$ if A is discrete and $c(A) = \int_c^d a(x)dx$ if A is continuous and a subset of the real interval $[c, d]$. Compute the counts of the eight sets in Problem 2.

(a) $c(C) = 1+0+1 = 2$

$c(D) = 04+0.5+0.6 = 1.5$

$c(E) = 0.3+0+0.1 = 0.4$

$c(F) = 0.6+0+0 = 0.6$

$c(G) = 0.3+0.3+0.3 = 0.9$

$c(H) = 0.5+0.1+0.4 = 1$

$c(I) = 0.5+0.9+0.6 = 2$

$c(J) = 0.5+0.5+0.5 = 1.5$

(b) Define the fuzzy equality or the degree to which two fuzzy sets equal each other as follows: $E(A, B) = \frac{c(A \cap B)}{c(A \cup B)}$. Compute the fuzzy equalities of all 8 sets with one another and list the 64 results in an 8-by-8 matrix.

(b) $E(C,C) = \frac{c(C \cap C)}{c(C \cup C)} = \frac{c((1\ 0\ 1))}{c((1\ 0\ 1))} = 1$

$E(C,D) = \frac{c(C \cap D)}{c(C \cup D)} = \frac{c((0.4\ 0\ 0.6))}{c((1\ 0.5\ 1))} = \frac{1}{2.5} = 0.4 = E(D,C)$

$E(C,E) = \frac{c(C \cap E)}{c(C \cup E)} = \frac{c((0.3\ 0\ 0.1))}{c((1\ 0\ 1))} = \frac{0.4}{2} = 0.2 = E(E,C)$

$E(C,F) = \frac{c(C \cap F)}{c(C \cup F)} = \frac{c((0.6\ 0\ 0))}{c((1\ 0\ 1))} = \frac{0.6}{2} = 0.3 = E(F,C)$

$E(C,G) = \frac{c(C \cap G)}{c(C \cup G)} = \frac{c((0.3\ 0\ 0.3))}{c((1\ 0.3\ 1))} = \frac{0.6}{2.3} \approx 0.26 = E(G,C)$

$E(C,H) = \frac{c(C \cap H)}{c(C \cup H)} = \frac{c((0.5\ 0\ 0.4))}{c((1\ 0.1\ 1))} = \frac{0.9}{2.1} = 0.43 = E(H,C)$

$E(C,I) = \frac{c(C \cap I)}{c(C \cup I)} = \frac{c((0.5\ 0\ 0.6))}{c((1\ 0.9\ 1))} = \frac{1.1}{2.9} = 0.38 = E(I,C)$

$E(C,J) = \frac{c(C \cap J)}{c(C \cup J)} = \frac{c((0.5\ 0\ 0.5))}{c((1\ 0.5\ 1))} = \frac{1}{2.5} = 0.4 = E(J,C)$

Similarly: $E(D,D)=1$

$E(D,E)=\frac{0.4}{1.5}\approx 0.67=E(E,D)$

$E(D,F)=\frac{0.4}{1.7}\approx 0.24=E(F,D)$

$E(D,G)=\frac{0.9}{1.5}=0.6=E(G,D)$

$E(D,H)=\frac{0.9}{1.6}\approx 0.56=E(H,D)$

$E(D,I)=\frac{1.5}{2.0}=0.75=E(I,D)$

$E(D,J)=\frac{1.4}{1.6}=0.875=E(J,D)$

$E(E,E)=1$

$E(E,F)=E(F,E)\approx 0.43$

$E(E,G)=E(G,E)\approx 0.44$

$E(E,H)=E(H,E)\approx 0.4$

$E(E,I)=E(I,E)=0.2$

$E(E,J)=E(J,E)=0.27$

$E(G,G)=1$

$E(G,H)=E(H,G)\approx 0.58$

$E(G,I)=E(I,G)=0.45$

$E(G,J)=E(J,G)=0.6$

$E(F,F)=1$

$E(F,G)=E(G,F)=0.25$

$E(F,H)=E(H,F)\approx 0.45$

$E(F,I)=E(I,F)\approx 0.24$

$E(F,J)=E(J,F)\approx 0.31$

$E(H,H)=1$

$E(H,I)=E(I,H)=0.5$

$E(H,J)=E(J,H)\approx 0.67$

$E(I,I)=1$

$E(I,J)=E(J,I)=0.75$

$E(J,J)=1$

$$\therefore \begin{bmatrix} 1 & 0.4 & 0.2 & 0.3 & 0.26 & 0.43 & 0.38 & 0.4 \\ 0.4 & 1.0 & 0.27 & 0.24 & 0.6 & 0.56 & 0.75 & 0.875 \\ 0.2 & 0.27 & 1.0 & 0.43 & 0.44 & 0.4 & 0.2 & 0.27 \\ 0.3 & 0.24 & 0.43 & 1.0 & 0.25 & 0.45 & 0.24 & 0.31 \\ 0.26 & 0.6 & 0.44 & 0.25 & 1.0 & 0.58 & 0.45 & 0.6 \\ 0.43 & 0.56 & 0.4 & 0.45 & 0.58 & 1.0 & 0.5 & 0.67 \\ 0.38 & 0.75 & 0.2 & 0.24 & 0.45 & 0.5 & 1.0 & 0.75 \\ 0.4 & 0.875 & 0.27 & 0.31 & 0.6 & 0.67 & 0.75 & 1.0 \end{bmatrix}$$

(c) Define the fuzzy subsethood $S(A, B) = Degree(A \subset B)$ as the degree to which A equals the set intersection $A \cap B : S(A, B) = E(A, A \cap B)$. Hence $S(A, B) = \frac{c(A \cap B)}{c(A)}$. Prove that we can reverse things and write fuzzy equality in terms of fuzzy subsethood. Then compute the subsethoods of all 8 sets above with one another and list the 64 results in an 8-by-8 matrix. Note that if A is binary then $S(X, A) = \frac{n_A}{n}$ is the relative frequency of n_A successes in n trials.

(c) $S(C,C) = \dfrac{c(C \cap C)}{c(C)} = \dfrac{C((1,0,1))}{C((1,0,1))} = \dfrac{2}{2} = 1$

$S(C,D) = \dfrac{c(C \cap D)}{c(C)} = \dfrac{C((0.4 \;\; 0 \;\; 0.6))}{2} = \dfrac{1}{2} = 0.5$

$S(C,E) = \dfrac{c(C \cap E)}{c(C)} = \dfrac{C((0.3 \;\; 0 \;\; 0.1))}{2} = \dfrac{0.4}{2} = 0.2$

$S(C,F) = \dfrac{C(C \cap F)}{C(C)} = \dfrac{C((0.6 \;\; 0 \;\; 0))}{2} = \dfrac{0.6}{2} = 0.3$

$S(C,G) = \dfrac{C(C \cap G)}{C(C)} = \dfrac{C((0.3 \;\; 0 \;\; 0.3))}{2} = \dfrac{0.6}{2} = 0.3$

$S(C,H) = \dfrac{C(C \cap H)}{C(C)} = \dfrac{C((0.5 \;\; 0 \;\; 0.4))}{2} = \dfrac{0.9}{2} = 0.45$

$S(C,I) = \dfrac{C(C \cap I)}{C(C)} = \dfrac{C((0.5 \;\; 0 \;\; 0.6))}{2} = \dfrac{1.1}{2} = 0.55$

$S(C,J) = \dfrac{C(C \cap J)}{C(C)} = \dfrac{C((0.5 \;\; 0 \;\; 0.5))}{2} = \dfrac{1}{2} = 0.5$

∴ Similarly: $S(D,C) \approx 0.67$ $S(D,D) = 1$ $S(D,E) \approx 0.27$ $S(D,F) \approx 0.27$

$S(D,G) = 0.6$ $S(D,H) = 0.6$ $S(D,I) = 1$ $S(D,J) \approx 0.93$

$S(E,C) = 1$ $S(E,D) = 1$ $S(E,E) = 1$ $S(E,F) = 0.75$

$S(E,G) = 1$ $S(E,H) = 1$ $S(E,I) = 1$ $S(E,J) = 1$

$S(F,C) = 1$ $S(F,D) \approx 0.67$ $S(F,E) = 0.5$ $S(F,F) = 1$

$S(F,G) = 0.5$ $S(F,H) \approx 0.83$ $S(F,I) \approx 0.83$ $S(F,J) \approx 0.83$

$S(G,C) \approx 0.67$ $S(G,D) = 1$ $S(G,E) \approx 0.44$ $S(G,F) \approx 0.33$

$S(G,G) = 1$ $S(G,H) \approx 0.78$ $S(G,I) = 1$ $S(G,J) = 1$

$S(H,C) = 0.9$ $S(H,D) = 0.9$ $S(H,E) = 0.4$ $S(H,F) = 0.5$

$S(H,G) = 0.7$ $S(H,H) = 1$ $S(H,I) = 1$ $S(H,J) = 1$

$S(I,C) = 0.55$ $S(I,D) = 0.75$ $S(I,E) = 0.2$ $S(I,F) = 0.25$

$S(I,G) = 0.45$ $S(I,H) = 0.5$ $S(I,I) = 1$ $S(I,J) = 0.75$

$S(J,C) \approx 0.67$ $S(J,D) \approx 0.93$ $S(J,E) \approx 0.27$ $S(J,F) \approx 0.33$

$S(J,G) = 0.6$ $S(J,H) \approx 0.67$ $S(J,I) = 1$ $S(J,J) = 1$

$$\therefore \begin{bmatrix} 1 & 0.5 & 0.2 & 0.3 & 0.3 & 0.45 & 0.55 & 0.5 \\ 0.67 & 1 & 0.27 & 0.27 & 0.6 & 0.6 & 1 & 0.93 \\ 1 & 1 & 1 & 0.75 & 1 & 1 & 1 & 1 \\ 1 & 0.67 & 0.5 & 1 & 0.5 & 0.83 & 0.83 & 0.83 \\ 0.67 & 1 & 0.44 & 0.33 & 1 & 0.78 & 1 & 1 \\ 0.9 & 0.9 & 0.4 & 0.5 & 0.7 & 1 & 1 & 1 \\ 0.55 & 0.75 & 0.2 & 0.25 & 0.45 & 0.5 & 1 & 0.75 \\ 0.67 & 0.93 & 0.27 & 0.33 & 0.6 & 0.67 & 1 & 1 \end{bmatrix}$$

(d) Define the degree of fuzziness or vagueness of a set as the degree to which the set equals its own set complement: $F(A) = E(A, A^C)$. Hence $F(A) = \frac{c(A \cap A^C)}{c(A \cup A^C)}$. Hence $F(A) = 0$ iff A is binary iff A is a cube vertex. Compute the degrees of fuzziness for each of the above 8 sets. Then prove that the three distinct concepts of equality, subsethood, and fuzziness coincide in the following case:

$$E(A, A^C) = F(A) = S(A \cup A^C, A \cap A^C).$$

(d) $F(C) = \frac{c(C \cap C^c)}{c(C \cup C^c)} = \frac{c((0,0,0))}{c((1,1,1))} = \frac{0}{3} = 0$

$F(D) = \frac{c(D \cap D^c)}{c(D \cup D^c)} = \frac{c((0.4\ 0.5\ 0.4))}{c((0.6\ 0.5\ 0.6))} = \frac{1.3}{1.7} \approx 0.76$

$F(E) = \frac{c(E \cap E^c)}{c(E \cup E^c)} = \frac{c((0.3\ 0\ 0.1))}{c((0.7\ 1\ 0.9))} = \frac{0.4}{2.6} \approx 0.15$

$F(F) = \frac{c(F \cap F^c)}{c(F \cup F^c)} = \frac{c((0.4\ 0\ 0))}{c((0.6\ 1\ 1))} = \frac{0.4}{2.6} \approx 0.15$

$F(G) = \frac{c(G \cap G^c)}{c(G \cup G^c)} = \frac{c((0.3\ 0.3\ 0.3))}{c((0.7\ 0.7\ 0.7))} = \frac{0.9}{2.1} \approx 0.43$

$F(H) = \frac{c(H \cap H^c)}{c(H \cup H^c)} = \frac{c((0.5\ 0.1\ 0.4))}{c((0.5\ 0.9\ 0.6))} = \frac{1}{2} = 0.5$

$F(I) = \frac{c(I \cap I^c)}{c(I \cup I^c)} = \frac{c((0.5\ 0.1\ 0.4))}{c((0.5\ 0.9\ 0.6))} = \frac{1}{2} = 0.5$

$F(J) = \frac{c(J \cap J^c)}{c(J \cup J^c)} = \frac{c((0.5\ 0.5\ 0.5))}{c((0.5\ 0.5\ 0.5))} = \frac{1.5}{1.5} = 1$

HW10 — Handout — Q4

> 4. A finite discrete probability density $P = (p_1, \ldots, p_n)$ is just a fuzzy set with a unit count: $c(P) = 1$. So we can compute the fuzziness of P as $F(P)$. Suppose first that every probability element is such that $p_i \le \frac{1}{2}$. Then find $F(P)$. Suppose next that there is one probability element p_j such that $p_j > \frac{1}{2}$. Then find $F(P)$. Use these results to prove that the fuzziness of a probability distribution linearly goes to zero with dimension $F(P) \le \frac{1}{n-1}$.

① $\because p_i \le \frac{1}{2}$

$\therefore p_i^c = 1 - p_i \ge \frac{1}{2} \ge p_i$

$\therefore P \cap P^c = P, \quad P \cup P^c = P^c$

$$\therefore F(P) = \frac{c(P \cap P^c)}{c(P \cup P^c)} = \frac{c(P)}{c(P^c)}$$

$$= \frac{\sum_{i=1}^n p_i}{\sum_{i=1}^n (1 - p_i)}$$

$$= \frac{\sum_{i=1}^n p_i}{n - \sum_{i=1}^n p_i}$$

$$= \frac{c(P)}{n - c(P)}$$

$$= \frac{1}{n-1}$$

② $\because \begin{cases} p_j > \frac{1}{2} \\ p_i < \frac{1}{2} \quad i \ne j \end{cases}$

$$\therefore c(P \cap P^c) = 1 - p_j + \sum_{i \ne j} p_i = 1 - 2p_j + \sum p_i = 1 - 2p_j + \underbrace{c(P)}_{=1}$$

$$= 2 - 2p_j = 2(1 - p_j) < 1$$

$\because c(P \cap P^c) + c(P \cup P^c) = n$

$\therefore c(P \cup P^c) = n - c(P \cap P^c) = n - 2(1 - p_j)$

$$\therefore F(P) = \frac{c(P \cap P^c)}{c(P \cup P^c)} = \frac{2(1-p_j)}{n-2(1-p_j)}$$

③ when ①: $\lim_{n\to\infty} F(P) = \lim_{n\to\infty} \frac{1}{n-1} = 0$

when ②: $\lim_{n\to\infty} F(P) = \lim_{n\to\infty} \frac{2(1-p_j)}{n-2(1-p_j)} = 0$ $\quad (\because 2(1-p_j) < 1)$

$\therefore \lim_{n\to\infty} F(P) = 0$

HW10 — Handout — Q5

5. Consider the continuous fuzzy set A_1 on the interval $[0, 1]$ with exponential set function $a_1(x) = e^{-2x}$. Find its fuzziness $F(A_1)$. Consider next the constant midpoint set function A_2 on the same interval with $a_2 = \frac{1}{2}$. What is its fuzziness? Find the two subsethood values $S(A_1, A_2)$ and $S(A_2, A_1)$.

$$F(A_1) = \int_0^1 a_1(x)dx = \int_0^1 e^{-2x} dx = -\frac{1}{2}e^{-2x}\Big|_0^1 = \frac{1}{2}(1-e^{-2})$$

$$F(A_2) = \int_0^1 \frac{1}{2} dx = \frac{1}{2}$$

$$\therefore S(A_1, A_2) = \frac{\int_0^1 \min(a_1(x), a_2(x))dx}{\int_0^1 a_1(x)dx}$$

$$S(A_2, A_1) = \frac{\int_0^1 \min(a_1(x), a_2(x))dx}{\int_0^1 a_2(x)dx}$$

$$\because e^{-2x} = \frac{1}{2}$$

$$\therefore -2x = \ln(\tfrac{1}{2})$$

$$x = \frac{1}{2}\ln 2$$

$$\because \text{when } [0, \tfrac{1}{2}\ln 2],\ \min(a_1(x), a_2(x)) = \frac{1}{2} \quad \int_0^{\frac{1}{2}\ln 2} \frac{1}{2}dx = \frac{1}{2}\cdot\frac{1}{2}\ln 2 = \frac{1}{4}\ln 2$$

$$\text{when } [\tfrac{1}{2}\ln 2, 1],\ \min(a_1(x), a_2(x)) = e^{-2x} \quad \int_{\frac{1}{2}\ln 2}^1 e^{-2x} dx = (-\tfrac{1}{2}e^{-2x})\Big|_{\frac{1}{2}\ln 2}^1$$

$$= -\frac{1}{2}e^{-2x} + \frac{1}{2}e^{-\ln 2}$$

$$= -\frac{1}{2}e^{-2} + 1$$

$$\therefore S(A_1, A_2) = \frac{\frac{1}{4}\ln 2 + 1 - \frac{1}{2}e^{-2}}{\frac{1}{2}(1-e^{-2})} \approx 0.9$$

$$S(A_2, A_1) = \frac{1}{2}\ln 2 + 2 - e^{-2} \approx 0.7$$