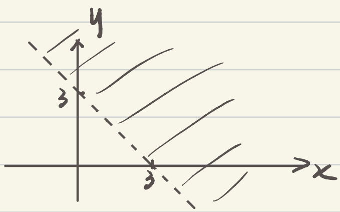
5.8. For the pair of random variables (X, Y) sketch the region of the plane corresponding to the following events. Identify which events are of product form.

(a) $\{X + Y > 3\}$.
(b) $\{e^X > Ye^3\}$.
(c) $\{\min(X, Y) > 0\} \cup \{\max\{X, Y\} < 0\}$.
(d) $\{|X - Y| \geq 1\}$.
(e) $\{|X/Y| > 2\}$.
(f) $\{X/Y < 2\}$.
(g) $\{X^3 > Y\}$.
(h) $\{XY < 0\}$.
(i) $\{\max(|X|, Y) < 3\}$.

HW07 – Book

Leon Chapter #5. 5.8

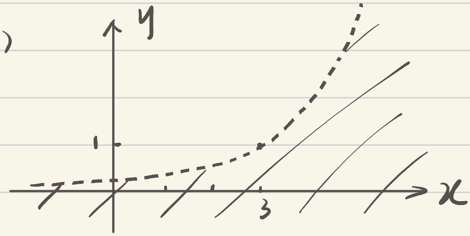
(a)



$\{X+Y>3\} \rightarrow \{Y>-X+3\}$

non-product event

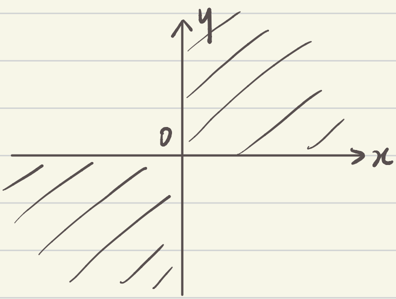
(b)



$\{e^X > Ye^3\} \rightarrow \{Y < e^{X-3}\}$

non-product event

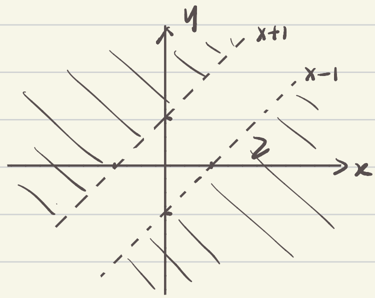
(c)



$\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}$

product event

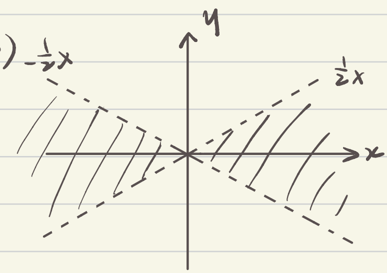
(d)



$\{|X-Y| \geq 1\} \rightarrow \{Y<X-1\} \cup \{Y>X+1\}$

non-product event

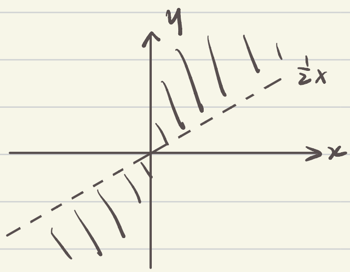
(e)



$\{|X/Y|>2\}$

non-product event

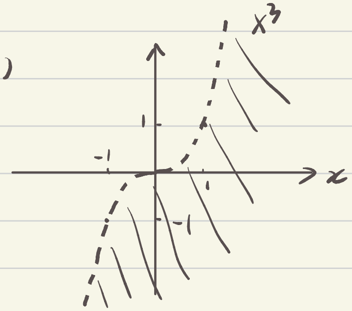
(f)



$\{X/Y<2\}$

non-product event

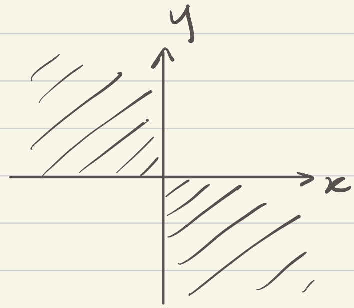
(g)



$\{X^3 > Y\}$

non-product event

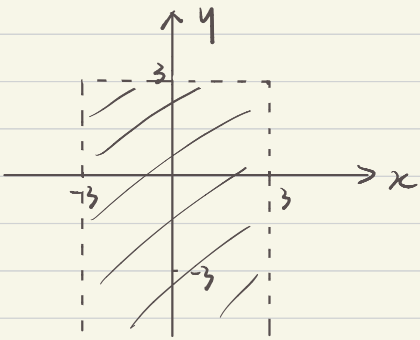
(h)



$\{XY < 0\}$

product event

(i)



$\{\max(|X|, Y) < 3\} \rightarrow \{-3 < X < 3\} \cap \{Y < 3\}$

product event

5.12. A modem transmits a two-dimensional signal (X, Y) given by:

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8)$$

where Θ is a discrete uniform random variable in the set $\{0, 1, 2, \ldots, 7\}$.

(a) Show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the joint pmf of X and Y.

(c) Find the marginal pmf of X and of Y.

(d) Find the probability of the following events: $A = \{X = 0\}$, $B = \{Y \le r/\sqrt{2}\}$, $C = \{X \ge r/\sqrt{2}, Y \ge r/\sqrt{2}\}$, $D = \{X < -r/\sqrt{2}\}$.

Leon Chapter #5. 5.12

(a) when $\theta=0$, $(X,Y)=(r\cos 0, r\sin 0)=(r,0)$

$\theta=1$, $(X,Y)=(r\cos\frac{\pi}{4}, r\sin\frac{\pi}{4})=(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)$

$\theta=2$, $(X,Y)=(r\cos\frac{\pi}{2}, r\sin\frac{\pi}{2})=(0,r)$

$\theta=3$, $(X,Y)=(r\cos\frac{3\pi}{4}, r\sin\frac{3\pi}{4})=(-\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)$

$\theta=4$, $(X,Y)=(r\cos\pi, r\sin\pi)=(-r,0)$

$\theta=5$, $(X,Y)=(r\cos\frac{5\pi}{4}, r\sin\frac{5\pi}{4})=(-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)$

$\theta=6$, $(X,Y)=(r\cos\frac{3\pi}{2}, r\sin\frac{3\pi}{2})=(0,-r)$

$\theta=7$, $(X,Y)=(r\cos\frac{7\pi}{4}, r\sin\frac{7\pi}{4})=(\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)$

$\therefore S_{XY}=\{(0,0), (\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r), (0,r), (-\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r), (-r,0), (-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r), (0,-r), (\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)\}$

(b) $\because$ θ is a discrete uniform random variable

$\therefore p_{XY}(x,y)=p_{XY}(r\cos(2\pi\theta/8), r\sin(2\pi\theta/8))=\frac{1}{8}$ $\qquad \theta=0,1,2,\cdots,7$

(c) $p_X(x)=\sum_y p_{XY}(x,y)$

$\therefore$ for $x=r$, $p_X(r)=p_{XY}(r,0)=1/8$

$x=\frac{\sqrt{2}}{2}r$, $p_X(\frac{\sqrt{2}}{2}r)=p_{XY}(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)+p_{XY}(\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)=1/8+1/8=1/4$

$x=0$, $p_X(0)=p_{XY}(0,r)+p_{XY}(0,-r)=1/8+1/8=1/4$

$x=-\frac{\sqrt{2}}{2}r$, $p_X(-\frac{\sqrt{2}}{2}r)=p_{XY}(-\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)+p_{XY}(-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)=1/8+1/8=1/4$

$x=-r$, $p_X(-r)=p_{XY}(-r,0)=1/8$

$p_Y(y)=\sum_x p_{XY}(x,y)$

$\therefore$ for $y=r$, $p_Y(r)=p_{XY}(0,r)=1/8$

$y=\frac{\sqrt{2}}{2}r$, $p_Y(\frac{\sqrt{2}}{2}r)=p_{XY}(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)+p_{XY}(-\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r)=1/8+1/8=1/4$

$y=0$, $p_Y(0)=p_{XY}(r,0)+p_{XY}(-r,0)=1/8+1/8=1/4$

$y=-\frac{\sqrt{2}}{2}r$, $p_Y(-\frac{\sqrt{2}}{2}r)=p_{XY}(\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)+p_{XY}(-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r)=1/8+1/8=1/4$

$y=-r$, $p_Y(-r)=p_{XY}(0,-r)=1/8$

(d) $P(A) = P(X=0) = p_X(0) = 1/4$

$P(B) = P(Y \leq \frac{r}{\sqrt{2}}) = 1 - P(Y > \frac{r}{\sqrt{2}}) = 1 - p_Y(r) = 1 - 1/8 = 7/8$

$P(C) = P(X \geq \frac{r}{\sqrt{2}}, Y \geq \frac{r}{\sqrt{2}}) = p_{XY}(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r) = 1/8$

$P(D) = P(X < -\frac{r}{\sqrt{2}}) = p_X(-r) = 1/8$

5.13. Let N_1 be the number of Web page requests arriving at a server in a 100-ms period and let N_2 be the number of Web page requests arriving at a server in the next 100-ms period. Assume that in a 1-ms interval either zero or one page request takes place with respective probabilities $1 - p = 0.95$ and $p = 0.05$, and that the requests in different 1-ms intervals are independent of each other.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the joint pmf of X and Y.

(c) Find the marginal pmf for X and for Y.

(d) Find the probability of the events $A = \{X \geq Y\}$, $B = \{X = Y = 0\}$, $C = \{X > 5, Y > 3\}$.

(e) Find the probability of the event $D = \{X + Y = 10\}$.

Leon Chapter #5. 5.13

(a) $N_1, N_2 \sim \text{Binomial}(100, 0.05)$

$S_{XY} = \{(X,Y) \mid X \in [0,100], Y \in [0,100]\}$

(b) $\because N_1, N_2$ are in the different period

$\therefore N_1$ and N_2 are independent

$$\therefore P_{XY}(X,Y) = P_X(X)\cdot P_Y(y) = \binom{100}{x} p^x (1-p)^{100-x} \cdot \binom{100}{y} p\,(1-p)^{100-y}$$

$$= \binom{100}{x}\binom{100}{y} p^{x+y} (1-p)^{200-(x+y)}$$

$$= \binom{100}{x}\binom{100}{y} 0.05^{x+y}\, 0.95^{200-(x+y)}$$

(c) $p_X(x) = \binom{100}{x} 0.05^x\, 0.95^{100-x}$

$p_Y(y) = \binom{100}{y} 0.05^y\, 0.95^{100-y}$

(d) $P(A) = P(X \geq Y) = \sum_{x=0}^{100}\sum_{y=0}^{x} p_{XY}(x,y) = \sum_{x=0}^{100}\sum_{y=0}^{x} \binom{100}{x}\binom{100}{y} 0.05^{x+y}\, 0.95^{200-(x+y)}$

$P(B) = P(X=Y=0) = \binom{100}{0}\binom{100}{0} 0.05^0\, 0.95^{200} = 0.95^{200}$

$P(C) = P(X>5, Y>3) = \sum_{x=6}^{100}\sum_{y=4}^{100} \binom{100}{x}\binom{100}{y} 0.05^{x+y}\, 0.95^{200-(x+y)}$

(e) $P(D) = P(X+Y=10) = P(X=0,Y=10) + P(X=1,Y=9) + \cdots + P(X=9,Y=1) + P(X=10,Y=0)$

$$= \binom{100}{0}\binom{100}{10} 0.05^{10}\, 0.95^{200-10} + \binom{100}{1}\binom{100}{9} 0.05^{10}\, 0.95^{200-10} + \cdots$$

$$+ \binom{100}{9}\binom{100}{1} 0.05^{10}\, 0.95^{200-10} + \binom{100}{10}\binom{100}{0} 0.05^{10}\, 0.95^{200-10}$$

$$= 0.05^{10}\, 0.95^{190} \sum_{k=0}^{10} \binom{100}{k}\binom{100}{10-k}$$

5.14. Let N_1 be the number of Web page requests arriving at a server in the period (0, 100) ms and let N_2 be the *total* combined number of Web page requests arriving at a server in the period (0, 200) ms. Assume arrivals occur as in Problem 5.13.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the joint pmf of N_1 and N_2.

(c) Find the marginal pmf for N_1 and N_2.

(d) Find the probability of the events $A = \{N_1 < N_2\}$, $B = \{N_2 = 0\}$, $C = \{N_1 > 5, N_2 > 3\}$, $D = \{|N_2 - 2N_1| < 2\}$.

Leon Chapter #5. 5.14

(a) $N_1 \sim \text{Binomial}(100, 0.05)$

$N_2 - N_1 \sim \text{Binomial}(100, 0.05)$

$S_{XY} = \{(X,Y) \mid X \in [0,100],\ Y = X + X',\ X' \in [0,100]\}$

(b) $p_{N_1,N_2}(N_1,N_2) = p_X(N_1)\cdot p_Y(N_2-N_1) = \binom{100}{N_1} 0.05^{N_1} 0.95^{100-N_1} \cdot \binom{100}{N_2-N_1} 0.05^{N_2-N_1} 0.95^{100-(N_2-N_1)}$

$= \binom{100}{N_1}\binom{100}{N_2-N_1} 0.05^{N_2} 0.95^{200-N_2} \quad (N_2 \geq N_1)$

(c) $p_{N_1}(N_1) = \binom{100}{N_1} 0.05^{N_1} 0.95^{100-N_1}$

$p_{N_2}(N_2) = \sum_{n=0}^{N_2} \binom{100}{N_2-n} 0.05^{N_2-n} 0.95^{100-(N_2-n)} = \binom{200}{N_2} 0.05^{N_2} 0.95^{200-N_2}$

(d) $P(A) = P(N_1 < N_2) = 1 - P(N_1 = N_2) = 1 - \sum_{n=0}^{100} \binom{100}{n}\binom{100}{0} 0.05^{n} 0.95^{200-n}$

$P(B) = P(N_2 = 0) = \binom{100}{0}\binom{100}{0} 0.05^0 0.95^{200} = 0.95^{200}$

$P(C) = P(N_1 > 5, N_2 > 3) = P(N_1 > 5, N_2 > 5) = \sum_{n_1=6}^{100}\sum_{n_2=n_1}^{100} \binom{100}{n_1}\binom{100}{n_2-n_1} 0.05^{n_2} 0.95^{200-n_2}$

$P(D) = P(|N_2 - 2N_1| < 2) = P(N_2 < 2 + 2N_1) + P(N_2 > 2N_1 - 2)$

$= P(N_2 - N_1 < 2 + N_1) + P(N_2 - N_1 > N_1 - 2)$

$= \sum_{n_1=0}^{98}\sum_{n_2=n_1}^{2+2n_1} 0.05^{n_2} 0.95^{200-n_2} + \sum_{n_1=2}^{100}\sum_{n_2=2n_1-2}^{200} 0.05^{n_2} 0.95^{200-n_2}$

5.25. The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x,y) = e^{-x/2} y e^{-y^2} \qquad \text{for } x > 0, y > 0.$$

(a) Find the joint cdf.

(b) Find $P[X^{1/2} > Y]$.

(c) Find the marginal pdfs.

Leon Chapter #5. 5.25

(a) $F_{X,Y}(x,y) = \int_0^x \int_0^y e^{-\frac{x'}{2}} y e^{-y'^2} dy' dx' = \int_0^x e^{-\frac{x'}{2}} \int_0^y y e^{-y'^2} dy' dx'$

$= \int_0^x e^{-\frac{x'}{2}} \left[\left(-\frac{1}{2} e^{-y'^2}\right)\Big|_0^y \right] dx' = \int_0^x e^{-\frac{x'}{2}} \left(-\frac{1}{2} e^{-y^2} + \frac{1}{2}\right) dx'$

$= \frac{1}{2}(1-e^{-y^2}) \int_0^x e^{-\frac{x'}{2}} dx' = \frac{1}{2}(1-e^{-y^2})(-2e^{-\frac{x'}{2}})\Big|_0^x$

$= \frac{1}{2}(1-e^{-y^2}) \cdot 2(1-e^{-\frac{x}{2}})$

$= 1 - e^{-\frac{x}{2}} - e^{-y^2} + e^{-\frac{x}{2}-y^2}$

(b) $P[X^{\frac{1}{2}} > Y] = P[X > Y^2] = \int_0^\infty \int_{y^2}^\infty e^{-\frac{x}{2}} y e^{-y^2} dx dy$

$= \int_0^\infty \left(-2e^{-\frac{x}{2}} y e^{-y^2}\right)\Big|_{y^2}^\infty dy = \int_0^\infty 2e^{-\frac{y^2}{2}} y e^{-y^2} dy$

$= \int_0^\infty 2y e^{-\frac{3}{2}y^2} dy = -\frac{2}{3}\left(e^{-\frac{3}{2}y^2}\right)\Big|_0^\infty = \frac{2}{3}$

(c) $f_X(x) = \int_0^\infty e^{-\frac{x}{2}} y e^{-y^2} dy = e^{-\frac{x}{2}} \int_0^\infty y e^{-y^2} = -\frac{1}{2} e^{-\frac{x}{2}} (e^{-y^2})\Big|_0^\infty = \frac{1}{2} e^{-\frac{x}{2}}$

$f_Y(y) = \int_0^\infty e^{-\frac{x}{2}} y e^{-y^2} dx = y e^{-y^2} \int_0^\infty e^{-\frac{x}{2}} = -2y e^{-y^2} (e^{-\frac{x}{2}})\Big|_0^\infty = 2y e^{-y^2}$

5.26. Let X and Y have joint pdf:

$$f_{X,Y}(x, y) = k(x + y) \qquad \text{for } 0 \le x \le 1, 0 \le y \le 1.$$

(a) Find k.
(b) Find the joint cdf of (X, Y).
(c) Find the marginal pdf of X and of Y.
(d) Find $P[X < Y]$, $P[Y < X^2]$, $P[X + Y > 0.5]$.

Leon Chapter #5. 5.26

(a) $\int_0^1\int_0^1 f_{XY}(x,y) = \int_0^1\int_0^1 k(x+y)\,dy\,dx = k\int_0^1 (xy+\frac{1}{2}y^2)\Big|_0^1 dx$

$= k\int_0^1 x+\frac{1}{2}\,dx = k(\frac{1}{2}x^2+\frac{1}{2}x)\Big|_0^1 = k = 1$

$\therefore k=1$

(b) $\because f_{XY}(x,y) = x+y$

$\therefore F_{XY}(x,y) = \int_0^x\int_0^y f_{XY}(u,v)\,dv\,du = \int_0^x\int_0^y u+v\,dv\,du = \int_0^x (uv+\frac{1}{2}v^2)\Big|_0^y du$

$= \int_0^x uy+\frac{1}{2}y^2\,du = (\frac{1}{2}yu^2+\frac{1}{2}y^2u)\Big|_0^x = \frac{1}{2}x^2y+\frac{1}{2}y^2x$

(c) $f_X(x) = \int_0^1 x+y\,dy = (xy+\frac{1}{2}y^2)\Big|_0^1 = x+\frac{1}{2}$

$f_Y(y) = \int_0^1 x+y\,dx = (\frac{1}{2}x^2+xy)\Big|_0^1 = y+\frac{1}{2}$

(d) $P[X<Y] = \int_0^1\int_0^y x+y\,dx\,dy = \int_0^1(\frac{1}{2}x^2+xy)\Big|_0^y dy = \int_0^1 \frac{3}{2}y^2\,dy = (\frac{1}{2}y^3)\Big|_0^1 = \frac{1}{2}$

$P[Y<X^2] = \int_0^1\int_0^{x^2} x+y\,dy\,dx = \int_0^1(xy+\frac{1}{2}y^2)\Big|_0^{x^2} dx = \int_0^1 x^3+\frac{1}{2}x^4\,dx = (\frac{1}{4}x^4+\frac{1}{10}x^5)\Big|_0^1 = \frac{7}{20}$

$P[X+Y>0.5] = \int_0^{0.5}\int_{0.5-x}^1 x+y\,dy\,dx + \int_{0.5}^1\int_0^1 x+y\,dy\,dx$

$= \int_0^{0.5}(xy+\frac{1}{2}y^2)\Big|_{0.5-x}^1 dx + \int_{0.5}^1 (xy+\frac{1}{2}y^2)\Big|_0^1 dx$

$= \int_0^{0.5} \frac{1}{2}x^2+x+\frac{3}{8}\,dx + \int_{0.5}^1 x+\frac{1}{2}\,dx$

$= (\frac{1}{6}x^3+\frac{1}{2}x^2+\frac{3}{8}x)\Big|_0^{0.5} + (\frac{1}{2}x^2+\frac{1}{2}x)\Big|_{0.5}^1$

$= \frac{1}{3}+\frac{5}{8} = \frac{23}{24}$

5.35. The input X to a communication channel is +1 or –1 with probability p and $1-p$, respectively. The received signal Y is the sum of X and noise N which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.25$.

(a) Find the joint probability $P[X = j, Y \le y]$.

(b) Find the marginal pmf of X and the marginal pdf of Y.

(c) Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

Leon Chapter #5. 5.35

(a) $N \sim N(0, 0.25)$ $\quad \sigma^2 = 0.25 \quad \sigma = 0.5$

$\therefore$ when $X=1$, $Y = X+N \sim N(1, 0.25)$, p

when $X=-1$, $Y = X+N \sim N(-1, 0.25)$, $1-p$

$\therefore P[X=1, Y\le y] = P[X=1]\,P[Y\le y \mid X=1] = p\,\Phi\left(\frac{y-1}{0.5}\right)$

$P[X=-1, Y\le y] = P[X=-1]\,P[Y\le y \mid X=-1] = (1-p)\,\Phi\left(\frac{y+1}{0.5}\right)$

(b) ① $P[X=1] = p \quad P[X=-1] = 1-p$

② $f_Y(y) = P[X=1, Y\le y] + P[X=-1, Y\le y]$

$= \sqrt{\frac{2}{\pi}}\left[p e^{-2(y-1)^2} + (1-p) e^{-2(y+1)^2}\right]$

(c) $P[Y>0 \mid X=1] = P\left[Z > \frac{0-1}{0.5}\right] = P[Z>-2] = 1-P[Z\le -2] = 1-0.0228 = 0.9772$

$P[Y>0 \mid X=-1] = P\left[Z > \frac{0-(-1)}{0.5}\right] = P[Z>2] = 1-P[Z\le 2] = 1-0.9772 = 0.0228$

$\therefore P[Y>0 \mid X=1] > P[Y>0 \mid X=-1]$

$\therefore X=1$ is more likely

5.74. Use the fact that $E[(tX + Y)^2] \geq 0$ for all t to prove the Cauchy-Schwarz inequality:

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Hint: Consider the discriminant of the quadratic equation in t that results from the above inequality.

Leon Chapter #5. 5.74

$\because 0 \leq E[(tX+Y)^2]$

$= E[t^2X^2 + 2tXY + Y^2]$

$= t^2E[X^2] + 2tE[XY] + E[Y^2]$

Let $t = -\dfrac{E[XY]}{E[X^2]}$

$= \dfrac{(E[XY])^2}{E[X^2]} - \dfrac{2(E[XY])^2}{E[X^2]} + E[Y^2]$

$= E[Y^2] - \dfrac{E[XY]^2}{E[X^2]}$

$\therefore \dfrac{E[XY]^2}{E[X^2]} \leq E[Y^2]$

$\therefore E[XY]^2 \leq E[X^2]E[Y^2]$