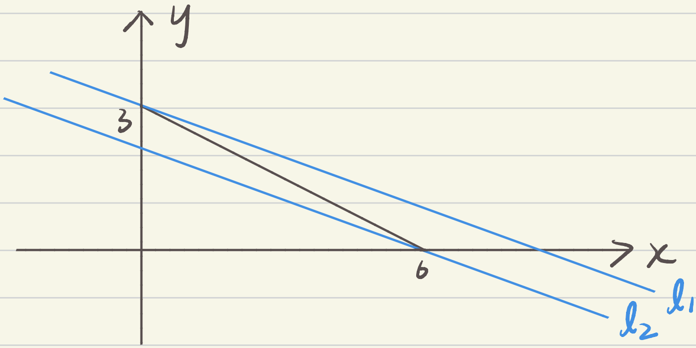
HW12 — Q1

Book Set 10.4 #1

1 Draw the region in the xy plane where $x+2y=6$ and $x\geq 0$ and $y\geq 0$. Which point in this "feasible set" minimizes the cost $c=x+3y$? Which point gives maximum cost? Those points are at corners.



$\because c = x+3y \Rightarrow k = -\frac{1}{3}$

$\therefore l_1$ gives maximum cost, point is $(0,3)$

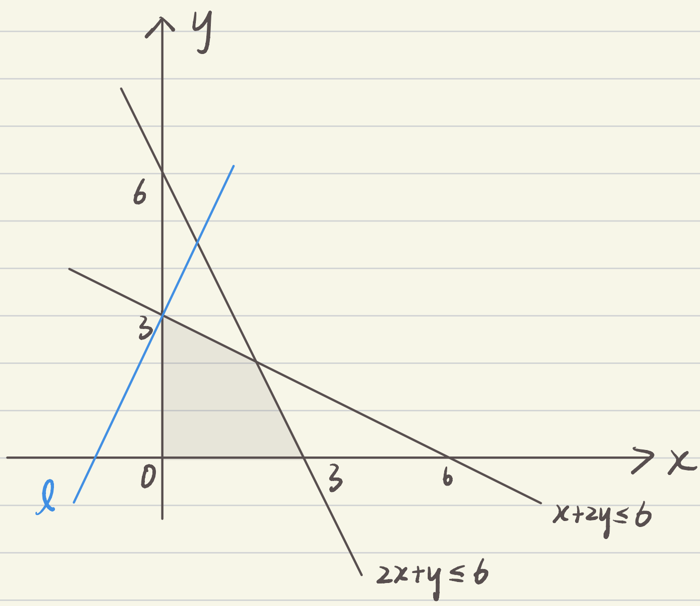
$c_{max} = 0+3\times 3 = 9$

$\therefore l_2$ gives minimum cost, point is $(6,0)$

$c_{min} = 6+3\times 0 = 6$

Book Set 10.4 #2

2 Draw the region in the xy plane where $x + 2y \leq 6,\ 2x + y \leq 6,\ x \geq 0,\ y \geq 0$. It has four corners. Which corner minimizes the cost $c = 2x - y$?



∴ l gives minimum cost, point is $(0,3)$

$C_{max} = 2x - y = 0 - 3 = -3$

HW12 — Q2

2. Derive the optimal strategies for the two person game with the payoff matrix A where

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = [a_{ij}]$$

and the matrix represents the payoff from player 2 to player 1. The term a_{ij} is the payoff when player 1 makes the i^{th} move and player 2 makes the j^{th} move.

Suppose mixed strategy for player 1 is $P^T = [p_1, p_2, p_3]$
for player 2 is $Q^T = [q_1, q_2, q_3]$

we what $\max_P (\min_q P^T A Q)$

∴ for player 1

$$\begin{cases} p_1 + p_2 + p_3 = 1 \\ p_1 \geq 0,\ p_2 \geq 0,\ p_3 \geq 0 \\ -p_2 + p_3 \geq V \\ p_1 - p_3 \geq V \\ -p_1 + p_2 \geq V \end{cases} \Rightarrow \begin{cases} p_1 = \frac{1}{3} \\ p_2 = \frac{1}{3} \\ p_3 = \frac{1}{3} \\ V = 0 \end{cases} \quad P = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

∴ for player 2

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ q_1 \geq 0,\ q_2 \geq 0,\ q_3 \geq 0 \\ q_2 - q_3 \leq V \\ -q_1 + q_3 \leq V \\ q_1 - q_2 \leq V \end{cases} \Rightarrow \begin{cases} q_1 = \frac{1}{3} \\ q_2 = \frac{1}{3} \\ q_3 = \frac{1}{3} \\ V = 0 \end{cases} \quad Q = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\therefore P^T A Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0$$

($\because A = -A^T$ symmetric game
$\therefore$ the game is fair
best strategy for player 1 = best strategy for player 2)

HW12 – Q3

3. Matrix A is the payoff matrix for a two-player *zero-sum* game:

$$P_1 \quad A = \begin{array}{c} P_2 \\ \begin{bmatrix} -5 & 2 & -7 & 8 & 5 \\ 3 & 6 & 2 & 2 & 10 \\ -1 & 0 & -9 & 0 & 3 \\ 10 & 3 & 8 & 9 & 4 \\ 3 & 2 & 4 & 5 & -3 \end{bmatrix} \end{array}.$$

Entry a_{ij} is the *payoff* for *Player 1* when Player 1 makes move i and Player 2 makes move j. Find the value $v(A)$ of the game for *stochastic vectors* $\mathbf{x}$ and $\mathbf{y}$:

$$v(A) = \max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T A \mathbf{y} \right).$$

$A \equiv$ payoff from P_2 to P_1

$v(A) = \max_x \left(\min_y x^T A y \right)$

$$A = \begin{array}{c|ccccc} & \#1 & \#2 & \#3 & \#4 & \#5 \\ \hline \#1 & -5 & 2 & -7 & 8 & 5 \\ \#2 & 3 & 6 & 2 & 2 & 10 \\ \#3 & -1 & 0 & -9 & 0 & 3 \\ \#4 & 10 & 3 & 8 & 9 & 4 \\ \#5 & 3 & 2 & 4 & 5 & -3 \\ & \times & & & \times & \end{array}$$

$$\begin{cases} y_1^* = 0 \\ y_4^* = 0 \end{cases}$$

$$A' = \begin{array}{c|ccc|c} & \#2 & \#3 & \#5 & \\ \hline \#1 & 2 & -7 & 5 & \times \\ \#2 & 6 & 2 & 10 & \\ \#3 & 0 & -9 & 3 & \times \\ \#4 & 3 & 8 & 4 & \\ \#5 & 2 & 4 & -3 & \times \end{array}$$

$$\begin{cases} x_1^* = 0 \\ x_3^* = 0 \\ x_5^* = 0 \end{cases}$$

$$A'' = \begin{array}{c|ccc} & \#2 & \#3 & \#5 \\ \hline \#2 & 6 & 2 & 10 \\ \#4 & 3 & 8 & 4 \\ & & & \times \end{array}$$

$$\{ y_5^* = 0$$

$$A''' = \begin{array}{c|cc|c} & \#2 & \#3 & \\ \#2 & 6 & 2 & p \\ \#4 & 3 & 8 & 1-p \\ & q & 1-q & \end{array}$$

$\therefore$ for P_1 :

$$6p + 3(1-p) = 2p + 8(1-p)$$
$$3p + 3 = -6p + 8$$
$$9p = 5$$
$$\therefore\ p = 5/9\ ,\quad 1-p = 4/9 \quad \Rightarrow \quad x^* = \begin{bmatrix} 0 \\ 5/9 \\ 0 \\ 4/9 \\ 0 \end{bmatrix}$$

for P_2 :

$$6q + 2(1-q) = 3q + 8(1-q)$$
$$4q + 2 = -5q + 8$$
$$9q = 6$$
$$\therefore\ q = 2/3\ ,\quad 1-q = 1/3 \quad \Rightarrow \quad y^* = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore V(A) = \max_x \left(\min_y x^T A y \right) = x^{*T} A y^*$$

$$= \begin{bmatrix} 0 & \frac{5}{9} & 0 & \frac{4}{9} & 0 \end{bmatrix} \begin{bmatrix} -5 & 2 & -7 & 8 & 5 \\ 3 & 6 & 2 & 2 & 10 \\ -1 & 0 & -9 & 0 & 3 \\ 10 & 3 & 8 & 9 & 4 \\ 3 & 2 & 4 & 5 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{5}{9} & 0 & \frac{4}{9} & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{3} - \frac{7}{3} \\ \frac{12}{3} + \frac{2}{3} \\ 0 - \frac{9}{3} \\ 2 + \frac{8}{3} \\ \frac{4}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{5}{9} & 0 & \frac{4}{9} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{14}{3} \\ -3 \\ \frac{14}{3} \\ \frac{8}{3} \end{bmatrix}$$

$$= \frac{5}{9}\,\frac{14}{3} + \frac{4}{9}\,\frac{14}{3}$$

$$= \frac{14}{3}$$