Problem Set 4

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1. **DTFT practice.(16 pts, 2, 2, 2, 5, 5)** Compute the DTFT and the inverse DTFT for the following sequences using DTFT properties. It should be obvious which one to compute for each case.

   (a) $x[n] = n^2(\frac{1}{3})^n u[n]$

   (b) $x[n] = \left(\frac{j}{4}\right)^n u[n+5]$

   (c) $x[n] = n(u[n+3] - u[n-4])$

   (d) $X(e^{j\omega}) = \frac{1+e^{j\omega}}{2-e^{-j\omega}}$ for $-\pi \leq \omega \leq \pi$

   (e) $X(e^{j\omega}) = \frac{e^{j3\omega}}{1-(2/3)e^{-j(\omega-\pi/5)}} - \frac{2}{1+(1/2)e^{-j(\omega+\pi/5)}}$ for $-\pi \leq \omega \leq \pi$

(a) $x[n] = n^2 (\frac{1}{3})^n u[n]$

$$(\tfrac{1}{3})^n u[n] \rightleftharpoons \frac{1}{1-\frac{1}{3}e^{-jw}} \quad \text{(DTFT of } a^n u[n])$$

$$n^2 f[n] \rightleftharpoons j^2 \frac{d^2 F(e^{jw})}{dw^2}$$

$$\because \frac{dF(e^{jw})}{dw} = \frac{-\frac{d}{dw}(1-\frac{1}{3}e^{-jw})}{(1-\frac{1}{3}e^{-jw})^2} = \frac{-\frac{1}{3}je^{-jw}}{(1-\frac{1}{3}e^{-jw})^2}$$

$$\frac{d^2F(e^{jw})}{dw^2} = \frac{(1-\frac{1}{3}e^{-jw})^{\cancel{2}} \cdot (-\frac{1}{3}e^{-jw}) - (-\frac{1}{3}je^{-jw}) \cdot 2(1-\frac{1}{3}e^{-jw}) \cdot \frac{1}{3}je^{-jw}}{(1-\frac{1}{3}e^{-jw})^{\cancel{4}\,3}}$$

$$= \frac{(1-\frac{1}{3}e^{-jw})(-\frac{1}{3}e^{-jw}) + (\frac{2}{3}e^{-jw})(-\frac{1}{3}e^{-jw})}{(1-\frac{1}{3}e^{-jw})^3}$$

$$= \frac{(1+\frac{1}{3}e^{-jw})(-\frac{1}{3}e^{-jw})}{(1-\frac{1}{3}e^{-jw})^3}$$

$$\therefore X(e^{jw}) = j^2 \frac{(1+\frac{1}{3}e^{-jw})(-\frac{1}{3}e^{-jw})}{(1-\frac{1}{3}e^{-jw})^3} = \boxed{\frac{\frac{1}{3}e^{-jw}(1+\frac{1}{3}e^{-jw})}{(1-\frac{1}{3}e^{-jw})^3}}$$

(b) $x[n] = (\frac{j}{4})^n u[n+5]$

$$(\tfrac{j}{4})^n u[n] \rightleftharpoons \frac{1}{1-\frac{j}{4}e^{-jw}} \quad \text{(DTFT of } a^n u[n])$$

$$\left(\frac{j}{4}\right)^{n+5} u[n+5] \rightleftharpoons e^{5jw} \frac{1}{1-\frac{j}{4}e^{-jw}} \quad \text{(Tim- shift property)}$$

$$\left(\frac{j}{4}\right)^{n+5} u[n+5] \rightleftharpoons \left(\frac{4}{j}\right)^{5} \frac{e^{5jw}}{1-\frac{j}{4}e^{-jw}} \quad \text{(Linearity)}$$

$$\therefore X(e^{jw}) = \frac{4^5 e^{5jw}}{j + \frac{1}{4}e^{-jw}}$$

(c) $x[n] = n(u[n+3] - u[n-4])$

$$\because u[n+3]-u[n-4] = \begin{cases} 1, & -3 \le n < 4 \\ 0, & \text{else} \end{cases}$$

$$\begin{aligned}
\therefore X(e^{jw}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \\
&= \sum_{n=-3}^{3} n e^{-jwn} \\
&= -3e^{3jw} - 2e^{2jw} - e^{jw} + 0 + e^{-jw} + 2e^{-2jw} + 3e^{-3jw} \\
&= -3(e^{3jw} - e^{-3jw}) - 2(e^{2jw} - e^{-2jw}) - (e^{jw} - e^{-jw}) \\
&= -6j\sin(3w) - 4j\sin(2w) - 2j\sin(w) \\
&= -2j(3\sin(3w) + 2\sin(2w) + \sin(w))
\end{aligned}$$

(d) $X(e^{jw}) = \frac{1+e^{jw}}{2-e^{-jw}}$ for $-\pi \le w \le \pi$

$$= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{-jw}} + \frac{e^{jw}}{1-\frac{1}{2}e^{-jw}}\right)$$

$$\because \frac{1}{1-\frac{1}{2}e^{-jw}} \xrightleftharpoons{\text{IDTFT}} \left(\frac{1}{2}\right)^{n} u[n] \quad \text{(DTFT of } a^n u[n])$$

$$\frac{e^{jw}}{1-\frac{1}{2}e^{-jw}} \xrightleftharpoons{\text{IDTFT}} \left(\frac{1}{2}\right)^{n+1} u[n+1] \quad \text{(Time - Shifting)}$$

$$\begin{aligned}
\therefore x[n] &= \frac{1}{2}\left(\left(\frac{1}{2}\right)^{n} u[n] + \left(\frac{1}{2}\right)^{n+1} u[n+1]\right) \\
&= \left(\frac{1}{2}\right)^{n+1} u[n] + \left(\frac{1}{2}\right)^{n+2} u[n+1]
\end{aligned}$$

(e) $X(e^{jw}) = \dfrac{e^{j3w}}{1-(\frac{2}{3})e^{-j(w-\frac{\pi}{5})}} - \dfrac{2}{1+(\frac{1}{2})e^{-j(w+\frac{\pi}{5})}}$ for $-\pi \le w \le \pi$

$$\because \frac{1}{1-(\frac{2}{3})e^{-jw}} \xrightleftharpoons{IDTFT} (\tfrac{2}{3})^n u[n] \quad (\text{DTFT of } a^n u[n])$$

$$\frac{e^{j3w}}{1-(\frac{2}{3})e^{-jw}} \xrightleftharpoons{IDTFT} (\tfrac{2}{3})^{n+3} u[n+3] \quad (\text{Time-Shifting})$$

$$\frac{e^{j3(w-\frac{\pi}{5})}}{1-(\frac{2}{3})e^{-j(w-\frac{\pi}{5})}} \xrightleftharpoons{IDTFT} (\tfrac{2}{3})^{n+3} e^{j\frac{\pi}{5}(n+3)} u[n+3] \quad (\text{Frequency-Shifting})$$

$$\therefore \frac{e^{j3w}}{1-(\frac{2}{3})e^{-j(w-\frac{\pi}{5})}} \xrightleftharpoons{IDTFT} (\tfrac{2}{3})^{n+3} e^{j\frac{\pi}{5}(n+6)} u[n+3] \quad (\text{Linearity})$$

$$\because \frac{1}{1-(-\frac{1}{2})e^{-jw}} \xrightleftharpoons{IDTFT} (-\tfrac{1}{2})^n u[n] \quad (\text{DTFT of } a^n u[n])$$

$$\frac{1}{1-(-\frac{1}{2})e^{-j(w+\frac{\pi}{5})}} \xrightleftharpoons{IDTFT} (-\tfrac{1}{2})^n e^{j\frac{\pi}{5}n} u[n] \quad (\text{Frequency-shifting})$$

$$\frac{2}{1+\frac{1}{2}e^{-j(w+\frac{\pi}{5})}} \xrightleftharpoons{IDTFT} 2(-\tfrac{1}{2})^n e^{j\frac{\pi}{5}n} u[n] \quad (\text{Linearity})$$

$$\therefore x[n] = (\tfrac{2}{3})^{n+3} e^{j\frac{\pi}{5}(n+6)} u[n+3] + 2(-\tfrac{1}{2})^n e^{j\frac{\pi}{5}n} u[n]$$

2. **Frequency response (57 pts).** Consider the following input-output relationships describing different LTI causal systems. Use MATLAB for plotting for this problem.

1) **(18 pts, 2pts for each)** For each system, find a closed form expression for its frequency response $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$.

(a) $y[n] = -x[n-3]$

Freq. Resp. $H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{q=0}^{M} P_q e^{-jwq}}{\sum_{k=0}^{N} d_k e^{-jwk}} = \frac{-e^{-j3w}}{1} = \boxed{-e^{-j3w}}$

(b) $y[n] - y[n-2] = x[n] + x[n-2]$

$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \boxed{\frac{1+e^{-j2w}}{1-e^{-j2w}}}$

(c) $y[n] = 4x[n] + 3x[n-1] + 2x[n-2] + x[n-4]$

$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \boxed{4 + 3e^{-jw} + 2e^{-j2w} + e^{-j4w}}$

(d) $y[n] = x[n] + 4x[n-1] + 3x[n-2] + 4x[n-3] + x[n-4]$

$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \boxed{1 + 4e^{-jw} + 3e^{-j2w} + 4e^{-j3w} + e^{-j4w}}$

(e) $5y[n-2] + 2y[n] = 5x[n] + 2x[n-2]$

$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \boxed{\frac{5+2e^{-j2w}}{5e^{-j2w}+2}}$

(f) $y[n] = \frac{1}{10}\sum_{k=0}^{9} x[n-k]$

$H(e^{jw}) = \frac{1}{10}\sum_{k=0}^{9} e^{-jwk} = \boxed{\frac{1}{10}\,\frac{1-e^{-j10w}}{1-e^{-jw}}}$

(g) $y[n] = \frac{1}{10}\sum_{k=0}^{9} (-1)^k x[n-k]$

$H(e^{jw}) = \frac{1}{10}\sum_{k=0}^{9} (-1)^k e^{-jwk} = \frac{1}{10}\sum_{k=0}^{9} (-e^{-jw})^k = \boxed{\frac{1}{10}\,\frac{1-e^{-j10w}}{1+e^{-jw}}}$

(h) $y[n] = \sum_{k=0}^{41} \left(\frac{4}{7}\right)^k x[n-k]$

$H(e^{jw}) = \sum_{k=0}^{41} \left(\frac{4}{7}\right)^k e^{-jwk} = \sum_{k=0}^{41} \left(\frac{4}{7}e^{-jw}\right)^k = \boxed{\frac{1-\left(\frac{4}{7}e^{-jw}\right)^{42}}{1-\frac{4}{7}e^{-jw}}}$

(i) $y[n] = \sum_{k=0}^{41} (-\frac{4}{7})^k x[n-k]$

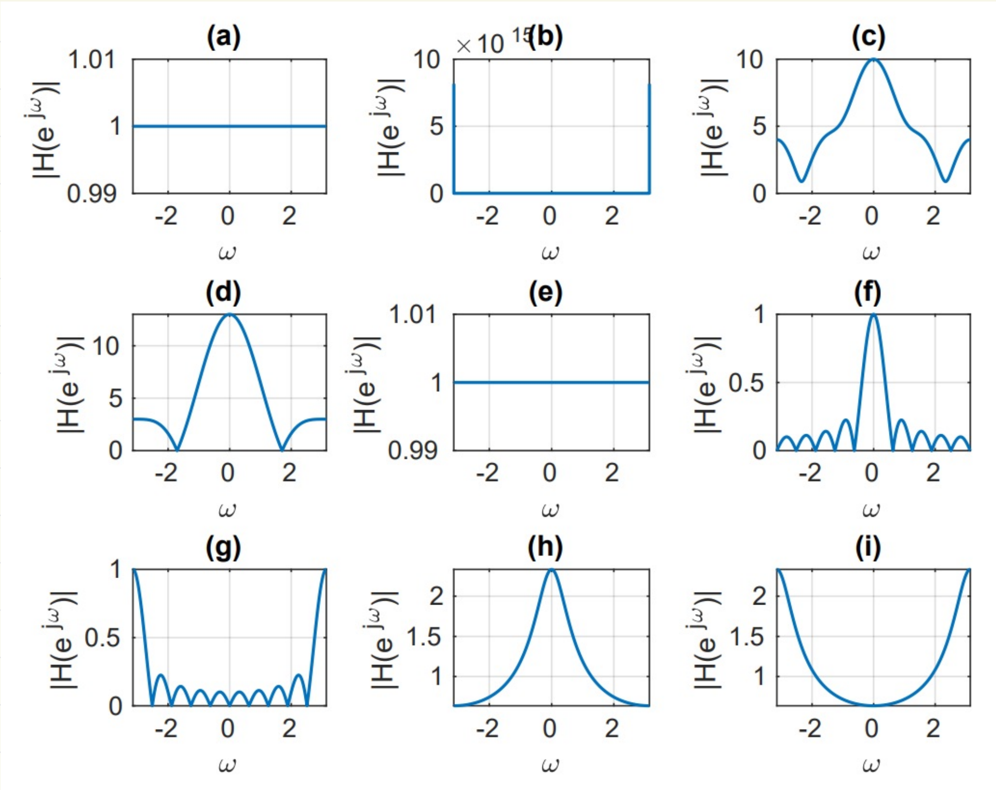
$$H(e^{jw}) = \sum_{k=0}^{41} (-\frac{4}{7})^k e^{-jwk} = \sum_{k=0}^{41} (-\frac{4}{7}e^{-jw})^k = \frac{1-(\frac{4}{7}e^{-jw})^{42}}{1+\frac{4}{7}e^{-jw}}$$

2) **(18 pts, 1 pt each plot, 1pt each filter type)** Plot the magnitude of the frequency response in the interval $[-\pi, \pi]$. Based on your plots, would you say the filters are low pass (they attenuate frequencies closer to π and $-\pi$, while preserving frequencies closer to 0), high pass (the opposite of low pass), all-pass or something else?.

code for 2) and 3)
(magnitude) (phase)

```
b_a = [0 0 0 -1];
a_a = 1;

b_b = [1 0 -1];
a_b = [1 0 -1];

b_c = [4 3 2 0 1];
a_c = 1;

b_d = [1 4 3 4 1];
a_d = 1;

b_e = [5 0 -2];
a_e = [2 0 5];

b_f = (1/10) * (1).^(0:9);
a_f = 1;

b_g = (1/10) * (-1).^(0:9);
a_g = 1;

b_h = (4/7).^(0:41);
a_h = 1;

b_i = (-4/7).^(0:41);
a_i = 1;


b_filters = {b_a, b_b, b_c, b_d, b_e, b_f, b_g, b_h, b_i};
a_filters = {a_a, a_b, a_c, a_d, a_e, a_f, a_g, a_h, a_i};
```

```
n = 512;
for i = 1:9
    figure;
    [H, w] = freqz(b_filters{i}, a_filters{i}, n, 'whole');
    w = w - pi;

    subplot(2, 1, 1);
    plot(w, abs(H));
    title(['Magnitude Response (', char(96+i), ')']);
    xlabel('\omega');
    ylabel('|H(e^{j\omega})|');
    grid on;
    if i == 1 || i == 5  % (a) and (e)
        ylim([0.99 1.01]);
        yticks([0.99 1.00 1.01]);
    end

    subplot(2, 1, 2);
    plot(w, angle(H));
    title(['Phase Response (', char(96+i), ')']);
    xlabel('\omega');
    ylabel('\angle H(e^{j\omega})');
    grid on;
end
```



(a) all-pass

(b) high pass, only when $w = \pm\pi$ pass

(c) band pass

(d) band pass

(e) all-pass

(f) low pass

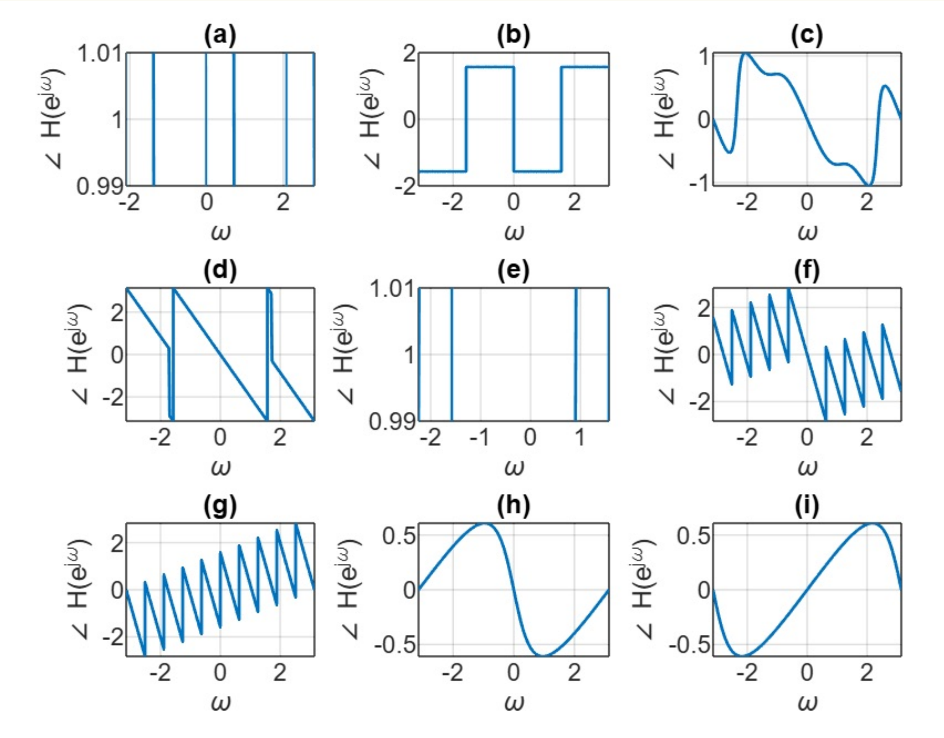
(g) high pass

(h) low pass

(i) high pass

3) **(18 pts, 1 pt each plot, 1pt each filter type)** Plot all the phase responses. From the phase plots of (a)-(i), do they appear to have generalized linear phase or non linear phase?

code is shown in 2)



(a) non-linear . sudden phase changes

(b) non-linear . sudden phase changes

(c) non-linear . smooth but nonlinear curve

(d) linear . periodic jumps but nearly linear

(e) non-linear . sudden phase changes

(f) linear . linear with periodic oscillations

(g) non-linear . oscillatory phase

(h) linear . smooth curve and nearly linear

(i) linear . smooth curve and nearly linear

4) (**3 points**) For the cases that appear to have linear phase, do they satisfy the sufficient conditions for GLP discussed in class?

(d) $h[n] = -h[-n]$, satisfy the sufficient conditions for GLP

(f) $h[n] = -h[-n]$, satisfy the sufficient conditions for GLP

(h) $h[n] = -h[-n]$, satisfy the sufficient conditions for GLP

(i) $h[n] = -h[-n]$, satisfy the sufficient conditions for GLP

3. **Filtering, 25pts** Use MATLAB for plotting for this problem. Consider the following signal.

$$x[n] = u[n] + 2u[n-10] - 2u[n-30] - u[n-40]$$

(a) (**2pts**)Plot the signal $x[n]$ using the `stem` command.

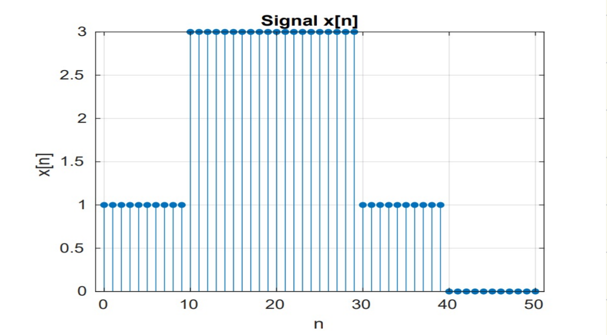
(b) (**2pts**)Plot the magnitude and phase of the DTFT of $x[n]$ in the interval $[-\pi, \pi]$.

(c) Using the `filter` command, apply each of the filters (a)-(i) from Problem 2 and
(i) (**9pts**) plot the resulting signals using the `stem` command.
(ii) (**9pts**)plot the magnitude and phase of the DTFT of the resulting signals in the interval $[-\pi, \pi]$.

(d) (**3pts**) Do you see any significant differences between linear and non-linear phase filters that would make you prefer one over the other?

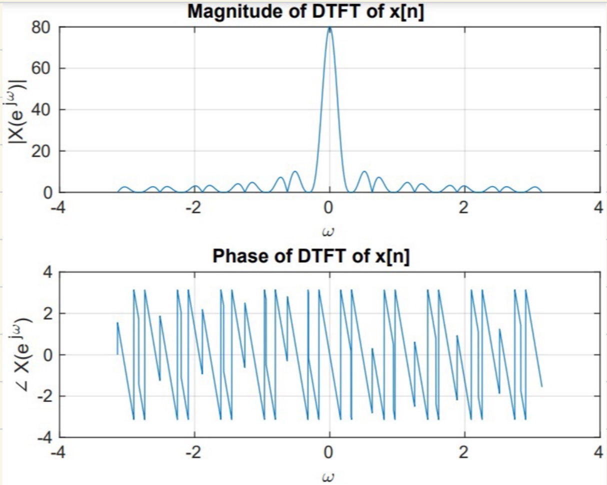
(a)

```
n = 0:50;
u = @(n) double(n >= 0);  % unit step function u[n]
x_n = u(n) + 2*u(n-10) - 2*u(n-30) - u(n-40);

figure;
stem(n, x_n, 'filled', 'MarkerSize', 4);
title('Signal x[n]');
xlabel('n');
ylabel('x[n]');
grid on;
```



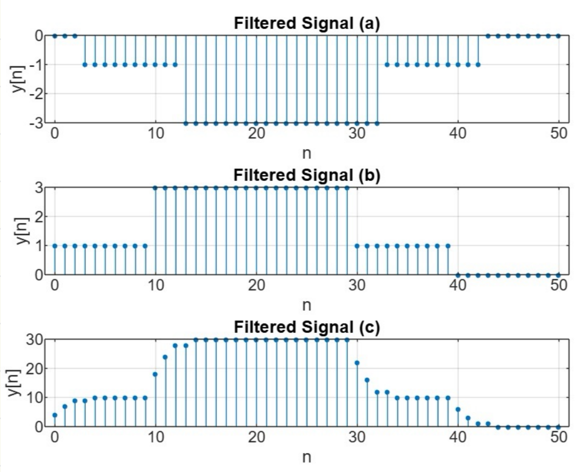
(b)

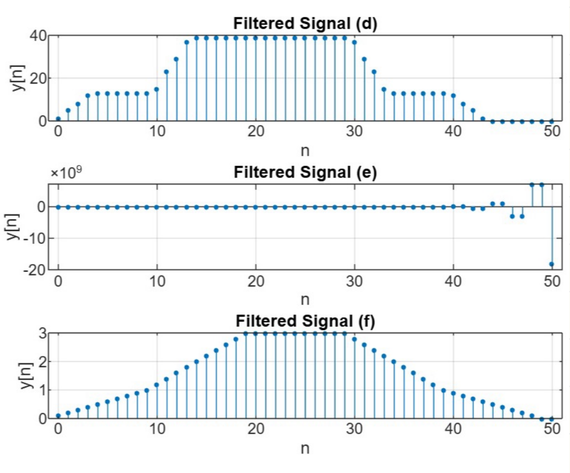
```
omega = linspace(-pi, pi, 10000);

% Compute DTFT using FFT
Xw = fftshift(fft(x_n, 10000)); % x_n from Q(a)

figure;
subplot(2,1,1);
plot(omega, abs(Xw)); % magnitude response
title('Magnitude of DTFT of x[n]');
xlabel('\omega');
ylabel('|X(e^{j\omega})|');
grid on;

subplot(2,1,2);
plot(omega, angle(Xw)); % phase response
title('Phase of DTFT of x[n]');
xlabel('\omega');
ylabel('\angle X(e^{j\omega})');
grid on;
```

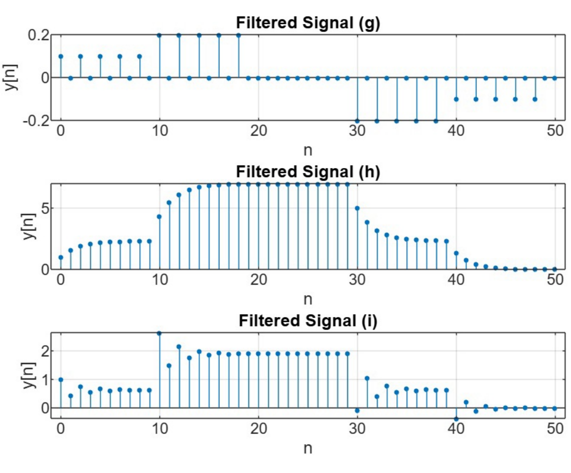


(c) (i)

```
b_a = [0 0 0 -1];
b_b = [1 0 -1];
b_c = [4 3 2 0 1];
b_d = [1 4 3 4 1];
b_e = [5 0 -2];
b_f = (1/10) * (1).^(0:9);
b_g = (1/10) * (-1).^(0:9);
b_h = (4/7).^(0:41);
b_i = (-4/7).^(0:41);

filters = {b_a, b_b, b_c, b_d, b_e, b_f, b_g, b_h, b_i};
filtered_signals = cell(1,9);

for i = 1:9
    if i==2
        filtered_signals{i} = filter(filters{i}, [1 0 -1], x_n);
    elseif i==5
        filtered_signals{i} = filter(filters{i}, [2 0 5], x_n);
    else
        filtered_signals{i} = filter(filters{i}, 1, x_n);
    end
end

figure;
for i = 1:9
    subplot(9,1,i);
    stem(n, filtered_signals{i}, 'filled', 'MarkerSize', 2);
    title(['Filtered Signal (', char(96+i), ')']);
    xlabel('n');
    ylabel('y[n]');
    grid on;
end
```

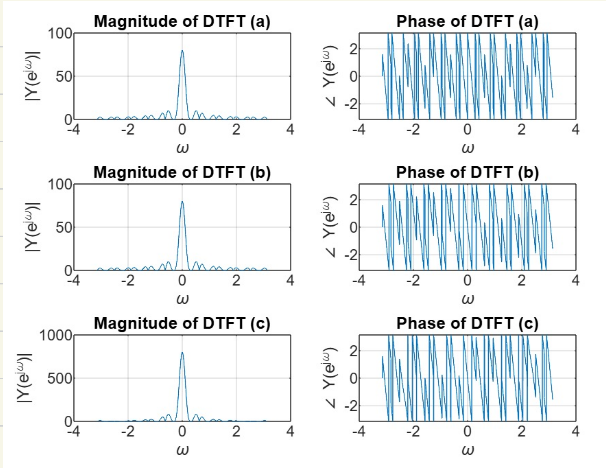


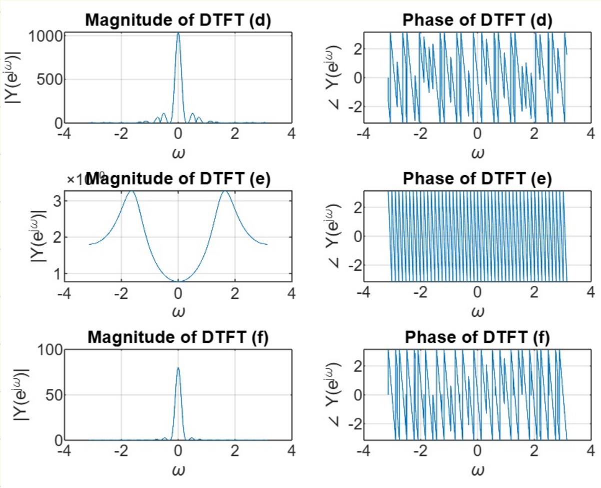


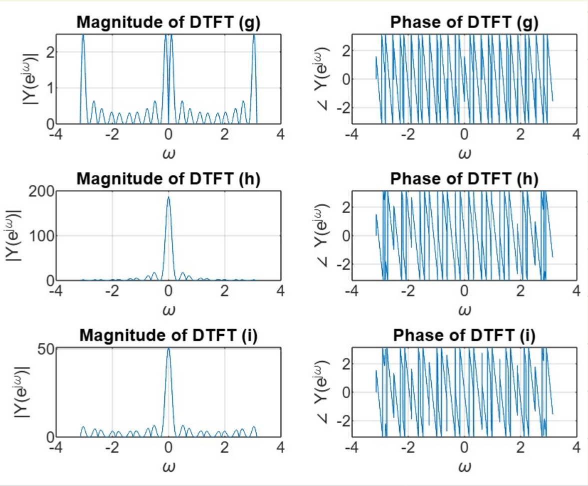


(C)(ii)

```
for figNum = 1:3
    figure('Position', [100, 100, 900, 700]);

    for localIdx = 1:3
        i = (figNum-1)*3 + localIdx;

        Yw = fftshift(fft(filtered_signals{i}, 10000));

        % magenitude
        subplot(3, 2, 2*localIdx-1);
        plot(omega, abs(Yw));
        title(['Magnitude of DTFT (', char(96+i), ')']);
        xlabel('\omega');
        ylabel('|Y(e^{j\omega})|');
        grid on;

        % phase
        subplot(3, 2, 2*localIdx);
        plot(omega, angle(Yw));
        title(['Phase of DTFT (', char(96+i), ')']);
        xlabel('\omega');
        ylabel('\angle Y(e^{j\omega})');
        grid on;
    end
end
```







(d) the magnitude of DTFT (d) (f) (h) (i)
has one main peak near 0
so I tend to prefer linear

4. **Non integer delay (9 pts, 3 pts each)** For an integer M, a system that produces a delay by M samples ($\delta[n-M]$) has DTFT $H(e^{j\omega}) = e^{-j\omega M}$. Now consider a system $h[n]$ with frequency response $H(e^{j\omega}) = e^{-j\omega\sqrt{5}}$ for $-\pi \leq \omega \leq \pi$. Whether $y[n]$ can be called a delay or not is subjective so you should answer based on an interpretation of your results.

(a) Compute the output for this system $y[n]$ with the input $x[n] = \alpha\cos(\omega_0 n) + \beta\sin((\omega_0/3)n)$. Compare $y[n]$ to $x[n]$, do you think this system should be called a non integer delay?.

(b) For $\omega_0 = \pi/10$, what is the period of $x[n]$?. Plot (using stem in MATLAB) $x[n]$, $x[n-1]$, $x[n-2]$ and $y[n]$ assuming that $\omega_0 = \pi/10$, $\alpha = -1/2$ and $\beta = 1$. Make sure your plots includes at least 2 full periods. By comparing the values of $y[n]$ to the other delayed inputs, can you say that $y[n]$ is also a delayed version of $x[n]$?

(c) Compute the impulse response $h[n]$ of this system.

(a) $\because g[n-n_0] \xrightleftharpoons{\text{DTFT}} e^{-j\omega n_0} G(e^{j\omega})$ (Time-shifting)

$\therefore x[n-\sqrt{5}] \xrightleftharpoons{\text{DTFT}} e^{-j\omega\sqrt{5}} X(e^{j\omega})$

$\because y[n] = x[n] * h[n]$

$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) = e^{-j\omega\sqrt{5}} X(e^{j\omega})$

$\therefore y[n] = x[n-\sqrt{5}] = \alpha\cos(w_0(n-\sqrt{5})) + \beta\sin((w_0/3)(n-\sqrt{5}))$

$\because \sqrt{5}$ is not integer

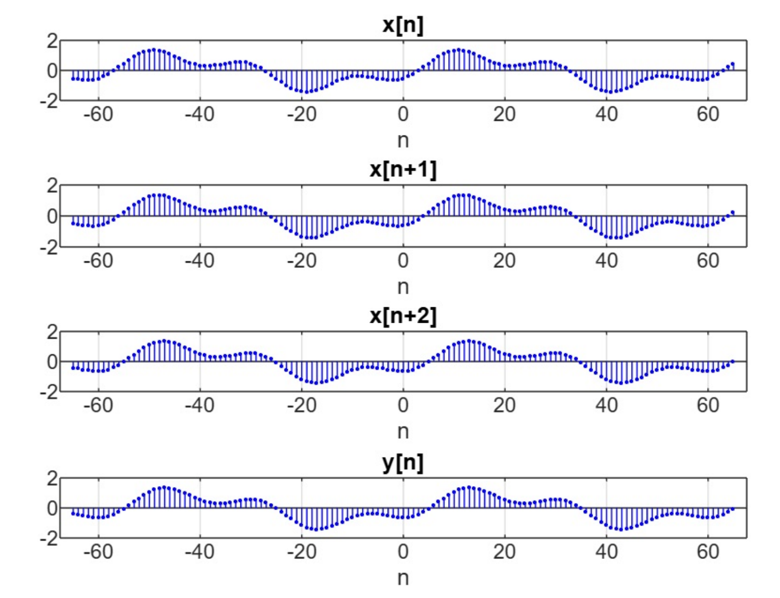
$\therefore$ is an non-integer delay

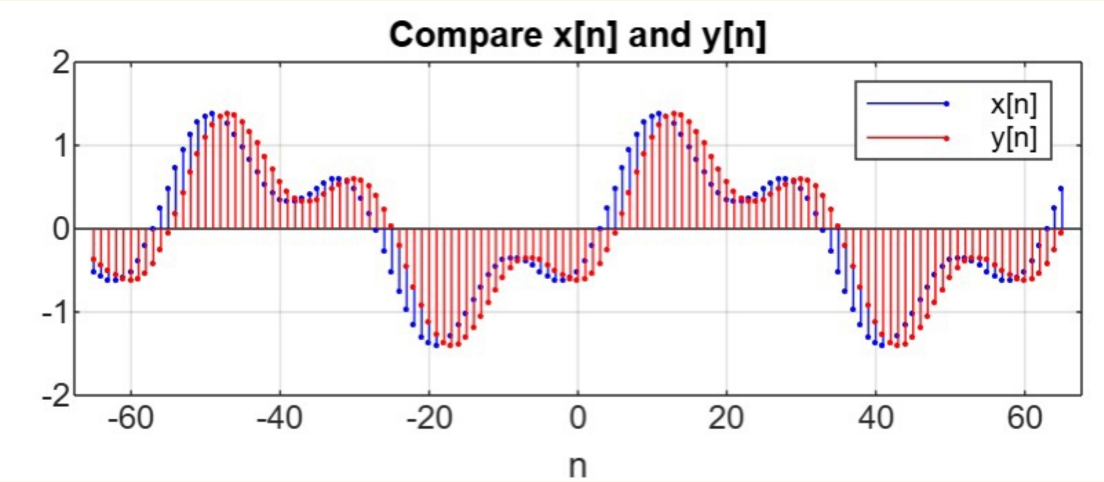
(b) for $\alpha\cos(w_0 n)$, $T_1 = \frac{2\pi}{w_0} = 2\pi / \frac{\pi}{10} = 20$

for $\beta\cos((w_0/3)n)$, $T_2 = \frac{2\pi}{\frac{w_0}{3}} = 6\pi / \frac{\pi}{10} = 60$

$\therefore T = \mathrm{lcm}(T_1, T_2) = 60$

```
n = -65:65; % T=60
omega = pi / 10;
alpha = -1/2;
beta = 1;

x = alpha * cos(omega * n) + beta * sin((omega / 3) * n);
x_n1 = alpha * cos(omega * (n - 1)) + beta * sin((omega / 3) * (n - 1));
x_n2 = alpha * cos(omega * (n - 2)) + beta * sin((omega / 3) * (n - 2));
y = alpha * cos(omega * (n - sqrt(5))) + beta * sin((omega / 3) * (n - sqrt(5)));

figure;
subplot(4,1,1);
stem(n, x, 'b', 'filled', 'MarkerSize', 1);
title('x[n]');  xlabel('n');
grid on;
subplot(4,1,2);
stem(n, x_n1, 'b', 'filled', 'MarkerSize', 1);
title('x[n+1]');  xlabel('n');
grid on;

subplot(4,1,3);
stem(n, x_n2, 'b', 'filled', 'MarkerSize', 1);
title('x[n+2]');  xlabel('n');
grid on;

subplot(4,1,4);
stem(n, y, 'b', 'filled', 'MarkerSize', 1);
title('y[n]');  xlabel('n');
grid on;
```





- $x[n-1]$ and $x[n-2]$ are exact shifted versions of $x[n]$
- $y[n]$ has a similar overall shape to $x[n]$, but it cannot be obtained from $x[n]$ through an integer shift. its sample values do not align perfectly with $x[n]$

$\therefore$ $y[n]$ is not a strictly delayed version of $x[n]$

(c) $\because H(e^{jw}) = e^{-jw\sqrt{5}}$

$$\therefore h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jw\sqrt{5}} \cdot e^{jwn} dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-\sqrt{5})} dw$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\pi(n-\sqrt{5})} - e^{-j\pi(n-\sqrt{5})}}{j(n-\sqrt{5})}\right)$$

$$= \frac{1}{2\pi} \frac{2j\sin(\pi(n-\sqrt{5}))}{j(n-\sqrt{5})}$$

$$= \boxed{\frac{\sin(\pi(n-\sqrt{5}))}{\pi(n-\sqrt{5})}}$$