Problem Set 5

Due: March 5th, 6pm PST on Gradescope EE483 Spring 2025

101 points. Please highlight or box your final answers.

1. **Sampling (65 pts)** A continuous time signal $x(t)$ has the CTFT $X(j\Omega)$ with a larger triangle of height 1 and two smaller triangles of height α as shown in Figure 1. The important frequencies are given in terms of Ω_0 in the plot.

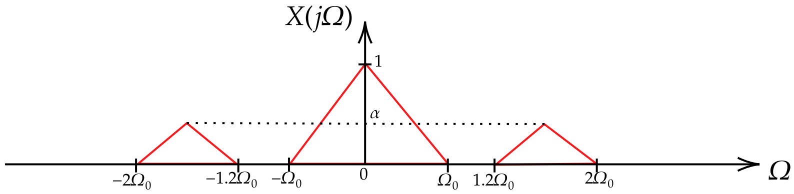

Figure 1: Plot of $X(j\Omega)$ for Problem 1.

(a) i. (**5pts**) What is the value of the maximum frequency $f_{\max_1}$ present in $X(j\Omega)$?, what is the Nyquist rate f_{Nyq_1}?. Express your answer as a function of Ω_0.

ii. (**5pts**) Let $x[n]$ be the sampled version of $x(t)$ at the Nyquist rate f_{Nyq_1}. Plot the DTFT of $x[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

iii. (**5pts**) Compute $x[n]$ from the above DTFT plot using the known DTFT pairs and the properties from the tables. (Hint: Homework 3, Problem 4)

(b) Now you are told that due to some hardware limitations, you are not allowed to sample at the rate f_{Nyq_1}, and you are expected to sample at a lower rate. For this, you are given an ideal analog low-pass filter $g(t)$ with the frequency response $G(j\Omega)$,

$$G(j\Omega) = \begin{cases} 1 & |\Omega| \leq 1.1\Omega_0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $y(t)$ be the signal obtained by applying the filter $g(t)$ to $x(t)$.

i. (**5pts**) Let the CTFT of $y(t)$ be $Y(j\Omega)$. Plot $Y(j\Omega)$ and label all the important frequencies and scaling factors.

ii. (**6 pts**) What is the value of the maximum frequency $f_{\max_2}$ present in $Y(j\Omega)$?, what is the Nyquist rate f_{Nyq_2} for $y(t)$?. By what factor the Nyquist rate has reduced compared to part (a)?

iii. (**5pts**) Let $y[n]$ be the sampled version of $y(t)$ at the Nyquist rate f_{Nyq_2}. Plot the DTFT of $y[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

iv. (**4pts**) Compute $y[n]$ from the above DTFT plot using the known DTFT pairs.

(c) Now consider sampling of $y(t)$ in (b) at the rate f_{Nyq_1} from (a). Let this sampled signal be $z[n]$.

i. (**5pts**) Plot the DTFT of $z[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

ii. (**5pts**) Compute $z[n]$ from the above DTFT plot using the known DTFT pairs.

iii. (**15pts**) Let the normalized error e, a measure of information loss due to the low-pass filtering, be defined as follows:

$$e = \frac{\sum_{n=-\infty}^{\infty} |x[n] - z[n]|^2}{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

Find a closed-form expression for e in terms of α and Ω_0. (Hint: Parseval's theorem)

(d) (**5pts**) Consider the normalized error $\hat{e}$ defined as follows:

$$\hat{e} = \frac{\sum_{n=-\infty}^{\infty} |x[n] - y[n]|^2}{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

Do you think $\hat{e}$ is a good measure of information loss due to low-pass filtering over e in part (c)? Briefly explain. You don't have to compute $\hat{e}$. A qualitative explanation is sufficient.

2. **Sub-Nyquist Sampling** (36 pts) The goal of this problem is to obtain an alternative sampling strategy for a bandlimited signal $x(t)$ when we have additional knowledge about its frequency content. Consider a continuous time signal $x(t)$ with CTFT $X(j\Omega)$ (see Figure 2). Note that $X(j\Omega)$ is zero outside the interval $[\Omega_0 - B/2, \Omega_0 + B/2]$, where $\Omega_0 = 4000\pi$ and $0 < B \leq 8000\pi$ is a positive real value.

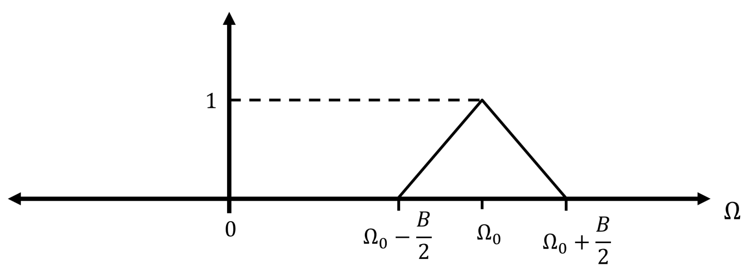

Figure 2: Plot of $X(j\Omega)$ for Problem 2.

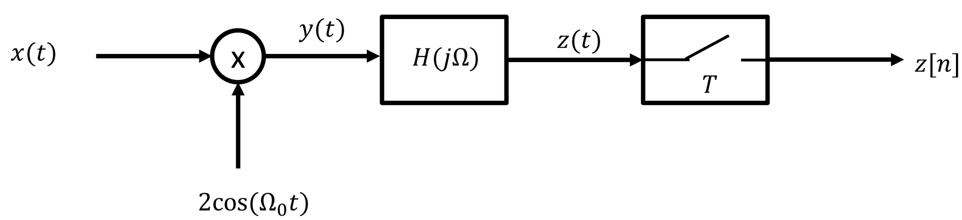

Figure 3: Modulated sub-Nyquist sampling

(a) (6pt) What is the value of the maximum frequency $f_{\max}$ present in $X(j\Omega)$?, what is the Nyquist rate?. Express your answer as a function of B.

(b) (6pt) Let $x[n]$ be the sampled version of $x(t)$ at the Nyquist rate. Plot the DTFT of $x[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

(c) (8pt) Consider the system from Figure 3. The signal $y(t) = 2\cos(\Omega_0 t)x(t)$ has CTFT $Y(j\Omega) = X(j(\Omega - \Omega_0)) + X(j(\Omega + \Omega_0))$. Plot $Y(j\Omega)$ labeling all important frequencies and scaling factors. Derive a precise condition for B so that $Y(j\Omega)$ preserves the shape of $X(j(\Omega)$. Hint: think about what would happen if the triangle is thin (B is small), or if the triangle is wide (B is big).

(d) (8pt)When the shape of $X(j\Omega)$ is preserved by $Y(j\Omega)$ (assuming the conditions from the previous section are true), design an ideal analog low pass filter $H(j\Omega)$ (indicate its gain and cut-off frequency), so that $z(t)$ is bandlimited and $Z(j\Omega)$ preserves the shape of $X(j\Omega)$.

(e) (8pt) What is the value of the maximum frequency of $z(t)$ and what is the Nyquist rate?, how does this sampling rate relate to the Nyquist rate of part (a)?. Let $z[n]$ be the sampled version of $z(t)$ at its Nyquist rate. Plot the DTFT of $z[n]$, make sure you label all important frequencies and include at least 2 periods in your plot.