10 A linear transformation from $\mathbf{V}$ to $\mathbf{W}$ has an *inverse* from $\mathbf{W}$ to $\mathbf{V}$ when the range is all of $\mathbf{W}$ and the kernel contains only $\boldsymbol{v} = \mathbf{0}$. Then $T(\boldsymbol{v}) = \boldsymbol{w}$ has one solution $\boldsymbol{v}$ for each $\boldsymbol{w}$ in $\mathbf{W}$. Why are these T's not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\quad \mathbf{W} = \mathbf{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\quad \mathbf{W} = \mathbf{R}^3$

(c) $T(v_1, v_2) = v_1$ $\quad \mathbf{W} = \mathbf{R}^1$

HW05 — Q1

(1) Book Set 8.1 #10

(a) $T(V_1, V_2) = (V_2, V_2)$, $W = R^2$

$\because T(0,0) = (0,0)$

$T(1,0) = (0,0)$

$\therefore$ the kernel does not contain only $v=0$

$\therefore$ T is not invertible

(b) $T(V_1, V_2) = (V_1, V_2, V_1 + V_2)$, $W = R^3$

Suppose $w = (1,2,0) \in R^3$

we cannot find v s.t. $T(v) = (1,2,0)$

$\therefore$ T is not invertible

(c) $T(V_1, V_2) = V_1$, $W = R^1$

$\because T(0,0) = 0$

$T(0,1) = 0$

$\therefore$ the kernel does not contain only $v=0$

$\therefore$ T is not invertible

12 Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(\boldsymbol{v})$:

(a) $\boldsymbol{v} = (2,2)$ (b) $\boldsymbol{v} = (3,1)$ (c) $\boldsymbol{v} = (-1,1)$ (d) $\boldsymbol{v} = (a,b)$.

(2) Book Set 8.1 #12

Suppose $T(v) = Av$. $A \equiv [a_{ij}]$

$\therefore T(1,1) = A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \Rightarrow \begin{cases} a_{11}+a_{12}=2 \\ a_{21}+a_{22}=2 \end{cases}$

$T(2,0) = A\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{cases} 2a_{11}=0 \\ 2a_{21}=0 \end{cases}$

$\therefore a_{11}=0,\ a_{12}=2,\ a_{21}=0,\ a_{22}=2$

$\therefore A = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$

(a) $v=(2,2) \Rightarrow Av = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\therefore T(2,2) = (4,4)$

(b) $v=(3,1) \Rightarrow Av = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\therefore T(3,1) = (2,2)$

(c) $v=(-1,1) \Rightarrow Av = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\therefore T(-1,1) = (2,2)$

(d) $v=(a,b) \Rightarrow Av = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ 2b \end{bmatrix}$ $\therefore T(a,b) = (2b,2b)$

14 Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Show that the range of T is the whole matrix space $\mathbf{V}$ and the kernel is the zero matrix:

(1) If $AM = 0$ prove that M must be the zero matrix.

(2) Find a solution to $AM = B$ for any 2 by 2 matrix B.

(3) Book Set 8.1 #14

① Suppose $M_{2\times2} = [m_{ij}]$

$\because AM = 0$

$$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11}+2m_{21} & m_{12}+2m_{22} \\ 3m_{11}+5m_{21} & 3m_{12}+5m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{matrix} R_1 = R_1 \\ R_2 = R_2 - 3R_1 \end{matrix} \quad \begin{bmatrix} m_{11}+2m_{21} & m_{12}+2m_{22} \\ -m_{21} & -m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{matrix} R_1 = R_1 + 2R_2 \\ R_2 = -R_2 \end{matrix} \quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = M$$

$\therefore$ M must be zero matrix

② Suppose $B_{2\times2} = [b_{ij}]$

$\because AM = B$

$$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11}+2m_{21} & m_{12}+2m_{22} \\ 3m_{11}+5m_{21} & 3m_{12}+5m_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\therefore \begin{matrix} R_1 = R_1 \\ R_2 = R_2 - 3R_1 \end{matrix} \quad \begin{bmatrix} m_{11}+2m_{21} & m_{12}+2m_{22} \\ -m_{21} & -m_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21}-3b_{11} & b_{22}-3b_{12} \end{bmatrix}$$

$$\therefore \begin{matrix} R_1 = R_1 + 2R_2 \\ R_2 = -R_2 \end{matrix} \quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 2b_{21}-5b_{11} & 2b_{22}-5b_{12} \\ 3b_{11}-b_{21} & 3b_{12}-b_{22} \end{bmatrix} = M$$

2. Let $F : V \to U$ and $G : U \to V$ be linear. Prove or disprove that if F and G are nonsingular, then $G \circ F$ is nonsingular.

HW05 - Q2

Proof: Suppose $G \circ F$ is singular,

$\therefore$ exist $\alpha \neq 0$, $(G \circ F)(\alpha) = 0$

$\therefore (G \circ F)(\alpha) = G(F(\alpha)) = 0$

$\because$ G is nonsingular

$\therefore F(\alpha) = 0$

$\because$ F is nonsingular

$\therefore \alpha = 0$

$\therefore$ contradiction

QED

3. Suppose $F : V \to U$ is linear. Show that the image of any subspace of V is a subspace of U.

HW05 - Q3

Proof: Suppose W is a subspace of V

① pick $w \in W \subset V$

$\because F$ is linear

$\therefore F(w) \in U$

② $\because W$ is a subspace

$\therefore 0 \in W$

$\therefore F(0) = 0 \in F(W)$

③ pick $\alpha \in F(W)$, $\beta \in F(W)$,

and suppose $F(x) = \alpha$, $F(y) = \beta$, $x \in W$, $y \in W$

$\therefore \alpha + \beta = F(x) + F(y)$

$= F(x+y)$ — F is linear

$\because x + y \in W$ — W is a subspace

$\therefore F(W)$ is closed under addition

④ pick $\alpha \in F(W)$, $k \in \mathbb{K}$

and suppose $F(x) = \alpha$, $x \in W$

$\therefore k\alpha = kF(x) = F(kx)$ — F is linear

$\because kx \in W$ — W is a subspace

$\therefore F(W)$ is closed under scalar multiplication

$\therefore$ the image of any subspace of V is a subspace of U

QED

4. Let $V = \mathbf{P}_{10}(t)$ be the vector space of polynomials of degree ≤ 10. Consider the linear map $\mathbf{D}^4 : V \to V$, where $\mathbf{D}^4$ is the fourth-order derivative. Find a basis for the image of $\mathbf{D}^4$ and determine whether $\mathbf{D}^4$ is nonsingular or not.

HW05 – Q4

① $\because V = P_{10}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_9 t^9 + a_{10} t^{10}$

$\therefore D^4(p(t)) = 0+0+0+0+4!a_4 + 5!a_5 t + \cdots + 9!a_9 t^5 + 10!a_{10} t^6$

$= 4!a_4 + 5!a_5 t + \cdots + 9!a_9 t^5 + 10!a_{10} t^6$

$\therefore$ a basis for the image of D^4 is $\{1, t, t^2, t^3, t^4, t^5, t^6\}$

② $\because$ when the highest degree term t^n, $n<4$

e.g. $D^4(a_1 t + a_2 t^2) = 0$

$D^4(a_0 + a_3 t^3) = 0$

$\therefore$ kernel of D^4 does not contain only 0

$\therefore D^4$ is singular

HW05 - Q5

5. Suppose that U, V, and W are vector spaces and $T: U \to V$ and $S: V \to W$ are linear transformations. Prove or disprove that $S \circ T$ is a linear transformation.

Proof: ① $(S\circ T)(\alpha+\beta) = S(T(\alpha+\beta))$, $\alpha \in U, \beta \in U$

$= S(T(\alpha)+T(\beta))$ — T is a lin. tran. , $T(\alpha) \in V$

$= S(T(\alpha)) + S(T(\beta))$ — S is a lin. tran.

$= (S\circ T)(\alpha) + (S\circ T)(\beta)$

② $(S\circ T)(k\alpha) = S(T(k\alpha))$, $\alpha \in U, k \in K$

$= S(kT(\alpha))$ — T is a lin. tran. , $T(\alpha) \in V$

$= kS(T(\alpha))$ — S is a lin. tran.

$= k(S\circ T)(\alpha)$

$\therefore S\circ T$ is a linear transformation

QED

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (y, x + z)$ and $G(x, y, z) = (2z, x - y)$:

a) Find a basis for $3F + 2G$.

b) Find the kernel of $3F + 2G$.

HW05 – Q6

(a) $3F+2G = 3(y, x+z) + 2(2z, x-y)$

$= (3y+4z, 5x-2y+3z)$

$= \begin{pmatrix} 3y+4z \\ 5x-2y+3z \end{pmatrix} = \begin{pmatrix} 0 & 3 & 4 \\ 5 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Let $A = \begin{bmatrix} 0 & 3 & 4 \\ 5 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

$\therefore \text{rank}(A) = 2$

$\therefore \{(5,-2,3), (0,3,4)\}$ can be a basis of $3F+2G$

(b) Let $3F+2G=0$

$\therefore \begin{bmatrix} 0 & 3 & 4 \\ 5 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\therefore [A \mid 0] \longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & \frac{17}{3} & 0 \\ 0 & 3 & 4 & 0 \end{array}\right]$

$\therefore z$ is free variable. and $\begin{cases} 5x + \frac{17}{3}z = 0 \\ 3y + 4z = 0 \end{cases}$

$\therefore x = -\frac{17}{15}z, \ y = -\frac{4}{3}z$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -\frac{17}{15} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$

$\therefore$ kernel of $3F+2F$ is $\text{span}\{(-\frac{17}{15}, -\frac{4}{3}, 1)\}$