Problem Set 7

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1. **(25 pts, 5 pts each) Region of Convergence** Consider the discrete-time signals $x_1[n] = (-5)^{-n}u[-n-1]$ and $x_2[n] = -(-5)^{-n}u[-n-6]$.

   (a) Compute the z-transforms of $x_1[n]$ and $x_2[n]$.

   (b) Compute and sketch the ROC, zeros and poles for $X_1(z)$ and $X_2(z)$.

   (c) Define $x[n] = x_1[n] + x_2[n]$. Compute the z-transform $X(z)$ and express it in rational form.

   (d) Compute and sketch the ROC, zeros and poles of $X(z)$.

   (e) You should see a big difference between your answers to (b) and (c). Explain what happened.

(a) $X_1(z) = Z(x_1[n]) = Z((-5)^{-n}u[-n-1])$

$$= \sum_{n=-\infty}^{+\infty} (-5)^{-n} u[-n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-5)^{-n} z^{-n}$$

$$= \sum_{n=1}^{+\infty} (-5z)^{n}$$

$$= (-5z)\sum_{n=1}^{+\infty} (-5z)^{n-1}$$

$$= (-5z)\sum_{n=0}^{+\infty} (-5z)^{n}$$

$$= (-5z)\frac{1}{1-(-5z)} \quad \text{if } |-5z|<1 \Rightarrow |z|<\frac{1}{5}$$

$$= \frac{-5z}{1+5z}$$

$X_2(z) = Z(x_2[n]) = Z(-(-5)^{-n}u[-n-6])$

$$= \sum_{n=-\infty}^{-6} -(-5)^{-n} z^{-n}$$

$$= -\sum_{n=6}^{+\infty} (-5z)^{n}$$

$$= -(-5z)^{6}\sum_{n=6}^{+\infty} (-5z)^{n-6}$$

$$= -(-5z)^{6}\sum_{n=0}^{+\infty} (-5z)^{n}$$

$$= -(-5z)^{6}\frac{1}{1-(-5z)} \quad \text{if } |-5z|<1 \Rightarrow |z|<\frac{1}{5}$$

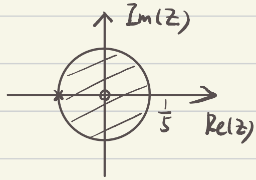
$$= \frac{-(5z)^{6}}{1+5z}$$

(b) for $X_1(z) = \frac{-5z}{1+5z}$

ROC: $|-5z| < 1 \Rightarrow |z| < \frac{1}{5}$

zeros: $z = 0$

poles: $1+5z = 0 \Rightarrow z = -\frac{1}{5}$

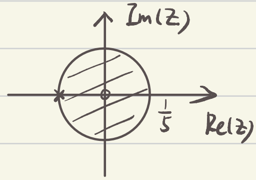


for $X_2(z) = \frac{-(5z)^6}{1+5z}$

ROC: $|-5z| < 1 \Rightarrow |z| < \frac{1}{5}$

zeros: $z = 0$ (multiplicity of 6)

poles: $1+5z = 0 \Rightarrow z = -\frac{1}{5}$



(c) $\because x[n] = x_1[n] + x_2[n] = (-5)^{-n} u[-n-1] - (-5)^{-n} u[-n-6]$

$$= \begin{cases} (-5)^{-n}, & -5 \le n \le -1 \\ 0, & \text{else} \end{cases}$$

$$\therefore X(z) = \sum_{n=-5}^{-1} (-5)^{-n} z^{-n} = \sum_{n=1}^{5} (-5z)^n$$

$$= \frac{(-5z)^1 (1 - (-5z)^5)}{1 - (-5z)}$$

$$= \frac{-5z - (-5z)^6}{1+5z} = X_1(z) + X_2(z)$$

$$= -5z + 25z^2 - 125z^3 + 625z^4 - 3125z^5$$

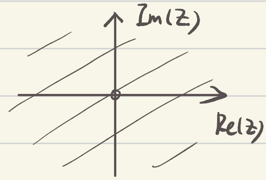
(d) for $X(z)$:

ROC: $\because x[n]$ is a finite-length signal

$\therefore$ ROC is the entire z-plane $|z|>0$

poles: not exist

zeros: $z=0$



(e) we find ROC extends from $|z|<\frac{1}{5}$ to $|z|>0$

$\because$ the combination of $X_1(z)$ and $X_2(z)$ turned two infinite sequences into a finite sequence

$\therefore$ this combination cancels one of the poles that was present

$\therefore$ ROC is enlarged into territory that the pole was previously block off

2. **(32 pts, 4,4,4,4,16) ROC and DTFT** Let $x[n] = e^{-\alpha n}u[n]$, where $\alpha \in \mathbb{R}$.

(a) Compute the z-transforms of $x[n]$.

(b) Compute and sketch the ROC, zeros and poles for $X(z)$.

(c) Find the values of α for which DTFT of $x[n]$ exist.

(d) Assume α satisfies the condition in (c), give the DTFT of $x[n]$.

(e) Let $y[n] = e^{(3-2\alpha)n}u[-n-1]$. Repeat part (a)-(d) for $y[n]$.

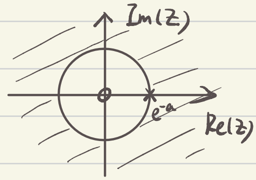
(a) $\because x[n] = e^{-an}u[n] = (e^{-a})^n u[n] \quad a \in \mathbb{R}$

$\therefore$ use table $X(z) = \dfrac{1}{1-e^{-a}z^{-1}} = \dfrac{z}{z-e^{-a}}$ if $|e^{-a}z^{-1}| > 1 \Rightarrow |z| > e^{-a}$

(b) ROC: $|z| > e^{-a}$

zeros: $z = 0$

poles: $z - e^{-a} = 0 \Rightarrow z = e^{-a}$



(c) $\because x[n] = e^{-an}u[n]$

$\therefore$ only when $|e^{-a}| < 1 \Rightarrow a > 0$, DTFT of $x[n]$ exist

(d) when $a > 0$, $e^{-a} < 1$, DTFT exist

use table: $X(e^{jw}) = \dfrac{1}{1-e^{-a}e^{-jw}} = \dfrac{1}{1-e^{-(a+jw)}}$

(e-a) $\because y[n] = e^{(3-2a)n}u[-n-1]$

$$\therefore Y(z) = \sum_{n=-\infty}^{+\infty} e^{(3-2a)n} u[-n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} e^{(3-2a)n} z^{-n}$$

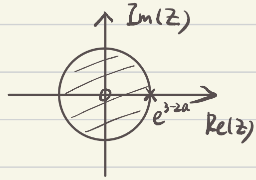
$$= \sum_{n=1}^{+\infty} (e^{(2a-3)} z)^n$$

$$= e^{(2a-3)} z \sum_{n=0}^{+\infty} (e^{(2a-3)} z)^n$$

$$= e^{(2a-3)} z \frac{1}{1-e^{(2a-3)} z}$$ if $|e^{(2a-3)} z| < 1 \Rightarrow |z| < e^{3-2a}$

$$\therefore Y(z) = \frac{e^{(2a-3)} z}{1 - e^{(2a-3)} z} = \frac{z}{e^{3-2a} - z}$$

(e-b) ROC : $|z| < e^{3-2a}$



zeros : $z = 0$

poles : $e^{3-2a} - z = 0 \Rightarrow z = e^{3-2a}$

(e-c) DTFT exist $\Longleftrightarrow$ unit circle $|z| = 1$ is in ROC

$\Longleftrightarrow e^{3-2a} > 1 \Rightarrow a$

$\Longleftrightarrow a < \frac{3}{2}$

(e-d) let $z = e^{jw}$

$\therefore$ when $a < \frac{3}{2}$, DTFT of $y[n]$ exist

$$Y(e^{jw}) = \frac{e^{jw}}{e^{3-2a} - e^{jw}} = \frac{1}{e^{3-2a-jw} - 1}$$

3. **(40 pts, 10 pts each)** Let $x_1[n]$ and $x_2[n]$ be defined as follows:

$$x_1[n] = \alpha^n u[n]$$

$$x_2[n] = \beta^n u[-n-1]$$

Let $x[n] = x_1[n] + x_2[n]$. For each value of α and β below, (i) find the z-transforms of $x[n]$ and express it in rational form, (ii) Compute and sketch the ROC, zeros and poles, (iii) give the DTFT if it exist.

(a) $\alpha = -\frac{1}{3}, \quad \beta = \frac{1}{2}$
(b) $\alpha = \frac{1}{2}, \quad \beta = -2$
(c) $\alpha = -\frac{1}{2}, \quad \beta = \frac{1}{3}$
(d) $\alpha = -\frac{4}{3}, \quad \beta = \frac{5}{2}$

(a) (i) $x_1[n] = (-\frac{1}{3})^n u[n]$, $\quad x_2[n] = (\frac{1}{2})^n u[-n-1]$

$$\therefore X_1(z) = \sum_{n=0}^{+\infty} (-\tfrac{1}{3})^n z^{-n}$$
$$= \sum_{n=0}^{+\infty} (-\tfrac{1}{3} z^{-1})^n$$
$$= \frac{1}{1 + \frac{1}{3} z^{-1}} \quad \text{if } |-\tfrac{1}{3} z^{-1}| < 1 \Rightarrow |z| > \tfrac{1}{3}$$
$$= \frac{z}{z + \frac{1}{3}}$$

$$X_2(z) = \sum_{n=-\infty}^{-1} (\tfrac{1}{2})^n z^{-n}$$
$$= \sum_{n=1}^{+\infty} (2z)^n$$
$$= 2z \sum_{n=0}^{+\infty} (2z)^n$$
$$= \frac{2z}{1-2z} \quad \text{if } |2z| < 1 \Rightarrow |z| < \tfrac{1}{2}$$

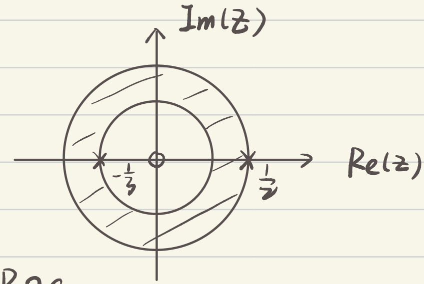
$$\therefore X(z) = X_1(z) + X_2(z) = \frac{z}{z + \frac{1}{3}} + \frac{2z}{1-2z} \qquad \tfrac{1}{3} < |z| < \tfrac{1}{2}$$

(ii) ROC $= \{|z| > \frac{1}{3}\} \cap \{|z| < \frac{1}{2}\}$

poles: $z + \frac{1}{3} = 0 \Rightarrow z = -\frac{1}{3}$

$1 - 2z = 0 \Rightarrow z = \frac{1}{2}$

zeros: $z = 0$ (multiplicity of 2)



(iii) $\because$ unit circle $|z| = 1$ is not in the ROC

$\therefore$ DTFT does not exist

(b) (i) $x_1[n] = (\frac{1}{2})^n u[n]$, $x_2[n] = (-2)^n u[-n-1]$

$$\therefore X_1(z) = \sum_{n=0}^{+\infty} (\tfrac{1}{2})^n z^{-n}$$
$$= \sum_{n=0}^{+\infty} (\tfrac{1}{2} z^{-1})^n$$
$$= \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{if } |\tfrac{1}{2} z^{-1}| < 1 \Rightarrow |z| > \tfrac{1}{2}$$
$$= \frac{z}{z-\frac{1}{2}}$$

$$X_2(z) = \sum_{n=-\infty}^{-1} (-2)^n z^{-n}$$
$$= \sum_{n=1}^{+\infty} (-\tfrac{1}{2} z)^n$$
$$= -\tfrac{1}{2} z \sum_{n=0}^{+\infty} (-\tfrac{1}{2} z)^n$$
$$= \frac{-\frac{1}{2}z}{1+\frac{1}{2}z} \quad \text{if } |-\tfrac{1}{2} z| < 1 \Rightarrow |z| < 2$$

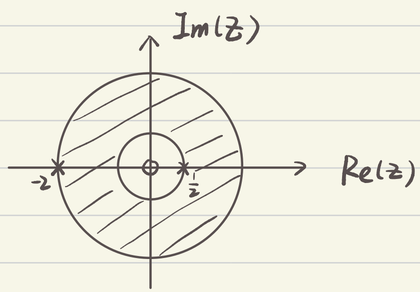
$$\therefore X(z) = X_1(z) + X_2(z) = \frac{z}{z-\frac{1}{2}} - \frac{\frac{1}{2}z}{1+\frac{1}{2}z} \qquad \tfrac{1}{2} < |z| < 2$$

(ii) ROC $= \{|z| > \frac{1}{2}\} \cap \{|z| < 2\}$

poles: $z - \frac{1}{2} = 0 \Rightarrow z = \frac{1}{2}$

$1 + \frac{1}{2}z = 0 \Rightarrow z = -2$

zeros: $z = 0$ (multiplicity of 2)



(iii) $\because$ unit circle $|z| = 1$ is in the ROC

$\therefore$ DTFT exist

$\therefore$ let $z = e^{jw}$

$$X(e^{jw}) = \frac{e^{jw}}{e^{jw} - \frac{1}{2}} - \frac{\frac{1}{2}e^{jw}}{1+\frac{1}{2}e^{jw}}$$

(c) (i) $x_1[n] = (-\frac{1}{2})^n u[n]$, $x_2[n] = (\frac{1}{3})^n u[-n-1]$

$\therefore X_1(z) = \sum_{n=0}^{+\infty} (-\frac{1}{2})^n z^{-n}$

$= \sum_{n=0}^{+\infty} (-\frac{1}{2} z^{-1})^n$

$= \dfrac{1}{1+\frac{1}{2}z^{-1}}$ if $|-\frac{1}{2}z^{-1}| < 1 \Rightarrow |z| > \frac{1}{2}$

$= \dfrac{z}{z+\frac{1}{2}}$

$X_2(z) = \sum_{n=-\infty}^{-1} (\frac{1}{3})^n z^{-n}$

$= \sum_{n=1}^{+\infty} (3z)^n$

$= 3z \sum_{n=0}^{+\infty} (3z)^n$

$= \dfrac{3z}{1+3z}$ if $|3z| < 1 \Rightarrow |z| < \frac{1}{3}$

$\because \{|z| > \frac{1}{2}\} \cap \{|z| < \frac{1}{3}\} = \phi$

$\therefore$ no valid ROC

$\therefore$ $X(z)$ does not exist

(ii)(iii) $\therefore$ no valid ROC

$\therefore$ z-transform, zeros, poles, DTFT don't exist

(d) (i) $x_1[n] = (-\frac{4}{3})^n u[n]$, $x_2[n] = (\frac{5}{2})^n u[-n-1]$

$$\therefore X_1(z) = \sum_{n=0}^{+\infty} (-\tfrac{4}{3})^n z^{-n}$$
$$= \sum_{n=0}^{+\infty} (-\tfrac{4}{3} z^{-1})^n$$
$$= \frac{1}{1 + \frac{4}{3} z^{-1}} \quad \text{if } |-\tfrac{4}{3} z^{-1}| < 1 \Rightarrow |z| > \tfrac{4}{3}$$
$$= \frac{z}{z + \frac{4}{3}}$$

$$X_2(z) = \sum_{n=-\infty}^{-1} (\tfrac{5}{2})^n z^{-n}$$
$$= \sum_{n=1}^{+\infty} (\tfrac{2}{5} z)^n$$
$$= \tfrac{2}{5} z \sum_{n=0}^{+\infty} (\tfrac{2}{5} z)^n$$
$$= \frac{\frac{2}{5} z}{1 - \frac{2}{5} z} \quad \text{if } |\tfrac{2}{5} z| < 1 \Rightarrow |z| < \tfrac{5}{2}$$

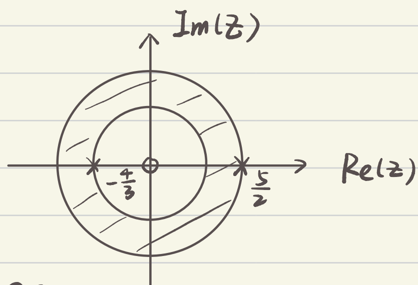
$$\therefore X(z) = X_1(z) + X_2(z) = \frac{z}{z + \frac{4}{3}} + \frac{\frac{2}{5} z}{1 - \frac{2}{5} z} \qquad \tfrac{4}{3} < |z| < \tfrac{5}{2}$$

(ii) $\text{ROC} = \{|z| > \frac{4}{3}\} \cap \{|z| < \frac{5}{2}\}$

poles: $z + \frac{4}{3} = 0 \Rightarrow z = -\frac{4}{3}$

$1 - \frac{2}{5} z = 0 \Rightarrow z = \frac{5}{2}$

zeros: $z = 0$ (multiplicity of 2)



(iii) $\because$ unit circle $|z| = 1$ is not in the ROC

$\therefore$ DTFT doesn't exist

4. **(10 pts) LTI systems and Z-transform** Let $h[n]$ be the impulse response of an LTI system and $H(z)$ be its $z-$transform. Prove that if the ROC of $H(z)$ contains the unit circle, then the LTI system is BIBO stable.

proof: Suppose z-transform of $h[n]$ exist, $H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$
and ROC of $H(z)$ contains the unit circle

$\therefore$ DTFT of $h[n]$ exist

$\therefore$ let $z = e^{jw}$

$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-jwn}$ is absolutely summable

$\therefore \sum_{n=-\infty}^{+\infty} |h[n] e^{-jwn}| < \infty$

$\therefore \sum_{n=-\infty}^{+\infty} |h[n]||e^{-jwn}| < \infty$

$\therefore \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

$\therefore$ this LTI system is BIBO stable