Problem Set 1

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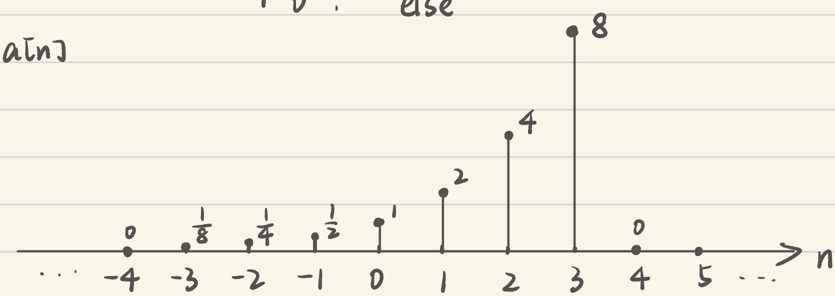
1. **Signal sketching. (15pts, 3pt each)** Sketch by hand the following signals. Here n is the discrete time variable, $u[n]$ is the unit step function and $\delta[n]$ is the delta function. The signals $a[n]$ and $c[n]$ defined in part (a) and (c), respectively, are also used in parts (d) and (e).

   (a) $a[n] = 2^n(u[n+3] - u[n-4])$
   (b) $b[n] = u[-3n+12]u[n+1]$
   (c) $c[n] = 2u[n+2] + 2u[n-2] - u[n-4]$
   (d) $d[n] = a[-n+2] + \delta[n-3]$
   (e) $e[n] = c[1-n^2]$

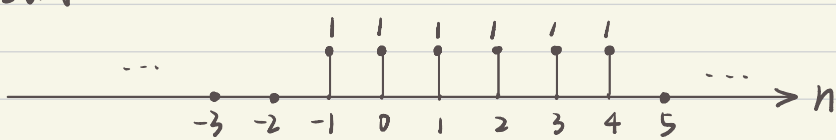
(a) $u[n+3]-u[n-4] = \begin{cases} 1, & -3 \le n < 4 \\ 0, & \text{else} \end{cases}$

$\therefore a[n]$



(b) $u[-3n+12] = \begin{cases} 1, & n \le 4 \\ 0, & \text{else} \end{cases}$ $\quad u[n+1] = \begin{cases} 1, & n \ge -1 \\ 0, & \text{else} \end{cases}$

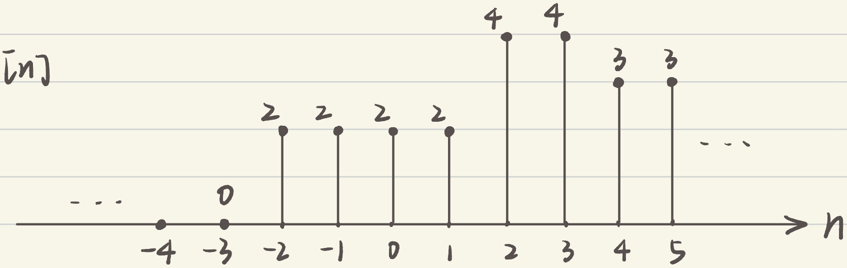
$\therefore b[n]$



(c) $2u[n+2] = \begin{cases} 2, & n \ge -2 \\ 0, & \text{else} \end{cases}$ $\quad 2u[n-2] = \begin{cases} 2, & n \ge 2 \\ 0, & \text{else} \end{cases}$

$-u[n-4] = \begin{cases} -1, & n \ge 4 \\ 0, & \text{else} \end{cases}$

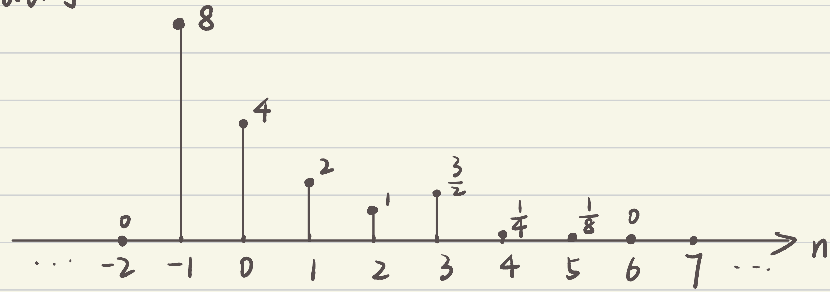
$\therefore c[n]$



(d) $a[-n+2] = a[-(n-2)]$

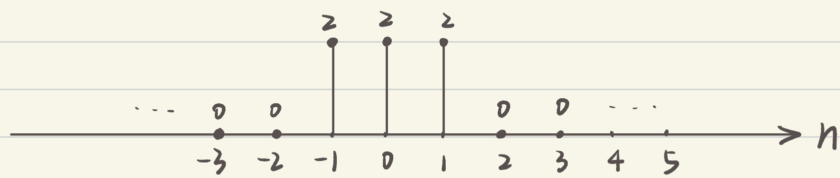
$\delta[n-3] = \begin{cases} 1, & n=3 \\ 0, & \text{else} \end{cases}$

$\therefore d[n]$



(e) $1-n^2 \geq -2 \Rightarrow -\sqrt{3} < n < \sqrt{3}$

$\therefore e[n]$



2. **Alternative signal representation.(10 pts, 2pt each)** Represent each signal from part 1 as an ordered list of numbers. For example, the unit impulse $\delta[n]$ is represented by

$$\{\cdots, 0, 0, \underset{\uparrow}{1}, 0, , 0 \cdots\} \tag{1}$$

(a) $a[n] = \{\cdots, 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \underset{\uparrow}{1}, 2, 4, 8, 0, \cdots\}$

(b) $b[n] = \{\cdots, 0, 1, \underset{\uparrow}{1}, 1, 1, 1, 1, 0, \cdots\}$

(c) $c[n] = \{\cdots, 0, 2, 2, \underset{\uparrow}{2}, 2, 4, 4, 3, 3, \cdots\}$

(d) $d[n] = \{\cdots 0, 8, \underset{\uparrow}{4}, 2, 1, \frac{3}{2}, \frac{1}{4}, \frac{1}{8}, 0, 0, \cdots\}$

(e) $e[n] = \{\cdots, 0, 0, 2, \underset{\uparrow}{2}, 2, 0, 0, \cdots\}$

3. **Sinusoidal sequence (20 pts, 4pt each)**. Consider the sequences

$$x[n] = \sin\left(\frac{\pi}{6}n\right), \quad y[n] = x[n](u[n] - u[n-N]) \tag{2}$$

for some fixed $N > 0$. $u[n]$ is the step function.

(a) Sketch by hand $x[n]$.

(b) $x[n]$ is T-periodic, what is is the smallest possible value of T?.
A discrete time signal $a[n]$ is T-periodic with period $T \in \mathbb{N}$ if for all $n \in \mathbb{Z}$

$$a[n] = a[n+T]. \tag{3}$$

(c) For $N = 4, 8, 12$ sketch by hand $y[n]$

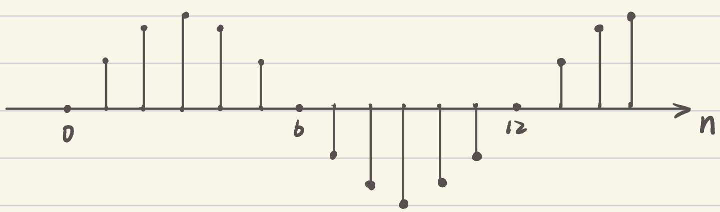
(d) For which of the above three values of N the identity

$$\sum_{k\in\mathbb{Z}} y[n-kN] = x[n] \tag{4}$$

is true?. Give a brief explanation.

(e) Consider the signal $z[n] = 3^{|x[n]|}$. Briefly explain whether z[n] is a periodic signal? If yes, find the period of this signal z[n].

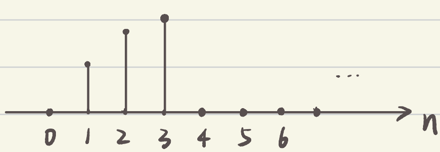
(a) $x[n] = \sin(\frac{\pi}{6}n)$



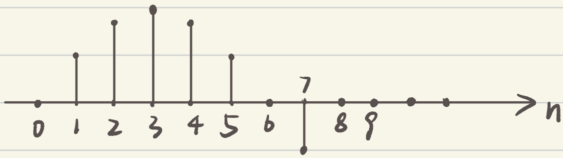
(b) $\because x[n+12] = \sin(\frac{\pi}{6}(n+12)) = \sin(\frac{\pi}{6}n + 2\pi) = \sin(\frac{\pi}{6}n) = x[n]$

$\therefore T = 12$

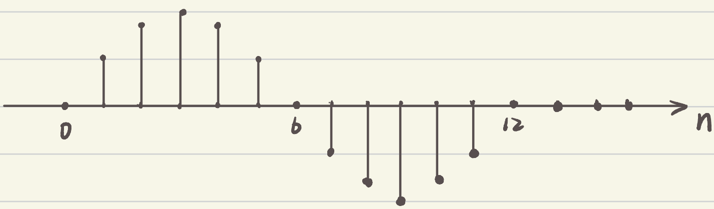
(c) when $N=4$: $y[n] = x[n] \cdot (u[n] - u[n-4])$



when $N=8$: $y[n]=x[n]\cdot(u[n]-u[n-8])$



when $N=12$: $y[n]=x[n]\cdot(u[n]-u[n-12])$



(d) when $N=4$, let $n=7$

$$\sum_{k\in\mathbb{Z}} y[7-4k] = \cdots + \cancelto{0}{y[-1]} + y[3] + \cancelto{0}{y[7]} + \cdots = y[3] \neq x[7]$$

$\therefore$ false

when $N=8$, let $n=9$

$$\sum_{k\in\mathbb{Z}} y[9-8k] = \cdots + \cancelto{0}{y[-7]} + y[1] + \cancelto{0}{y[17]} + \cdots = y[1] \neq x[9]$$

$\therefore$ false

when $N=12$, $\sum_{k\in\mathbb{Z}} y[n-12k] = \cancelto{0}{\cdots} + y[n \bmod 12] + \cancelto{0}{\cdots}$

$$= y[n \bmod 12]$$

$$= x[n \bmod 12]$$

$$= x[n] \qquad (\because T=12)$$

$\therefore$ true

$\therefore$ only when $N = 12m \quad m \in \mathbb{Z}$, $\sum_{k\in\mathbb{Z}} y[n-kN] = x[n]$ is true

(e) $\because x[n]$ is 12-periodic and $x[n] = \sin(\frac{\pi}{6}n)$

$\therefore |x[n]|$ is 6-periodic

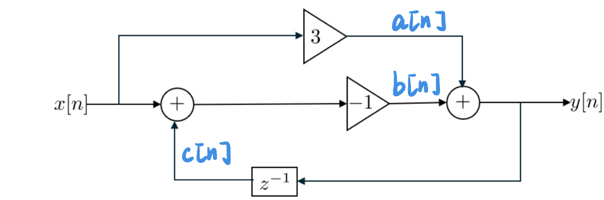
$\therefore |x[n]| = |x[n+6]|$

$\because z[n+6] = 3^{|x[n+6]|} = 3^{|x[n]|} = z[n]$

$\therefore z[n]$ is 6-periodic

4. **System diagram (20 pts, 5pt each)** For each of the systems below, write a linear difference equation that describes $y[n]$ as a function of its past values, and the present and past values of $x[n]$

(a)



$a[n] = 3x[n]$

$b[n] = -(x[n]+c[n]) = -x[n] - c[n]$

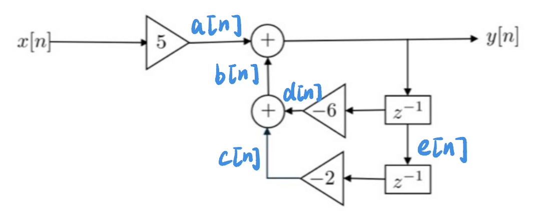
$c[n] = y[n-1]$

$\therefore y[n] = a[n] + b[n]$

$= 3x[n] - x[n] - c[n]$

$= 2x[n] - y[n-1]$

(b)



$a[n] = 5x[n]$

$b[n] = c[n] + d[n]$

$c[n] = -2e[n-1]$

$d[n] = -6y[n-1]$

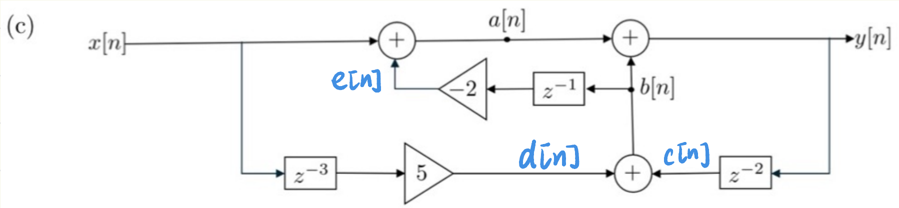
$e[n] = y[n-1]$

$\therefore y[n] = a[n] + b[n]$

$= 5x[n] - 2e[n-1] - 6y[n-1]$

$= 5x[n] - 2y[n-2] - 6y[n-1]$

(c)



$a[n] = x[n] + e[n]$

$b[n] = c[n] + d[n]$

$c[n] = y[n-2]$

$d[n] = 5x[n-3]$

$e[n] = -2b[n-1]$

$\therefore y[n] = a[n] + b[n]$

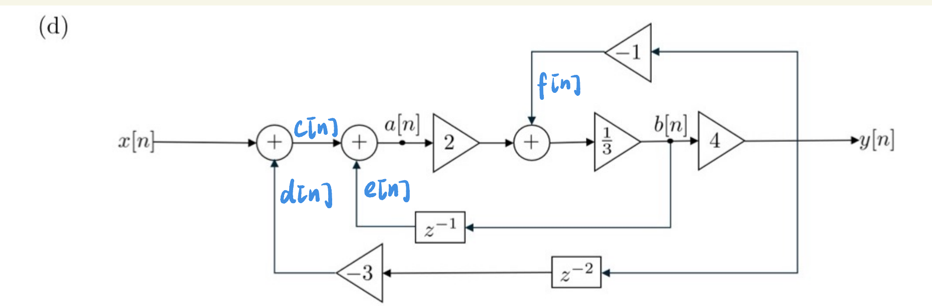
$= x[n] + e[n] + c[n] + d[n]$

$= x[n] - 2b[n-1] + y[n-2] + 5x[n-3]$

$= x[n] - 2c[n-1] - 2d[n-1] + y[n-2] + 5x[n-3]$

$= x[n] - 2y[n-3] - 10x[n-4] + y[n-2] + 5x[n-3]$

(d)



$a[n] = c[n] + e[n]$

$b[n] = \frac{1}{3}(2a[n] + f[n]) = \frac{2}{3}a[n] + \frac{1}{3}f[n]$

$c[n] = x[n] + d[n]$

$d[n] = -3y[n-2]$

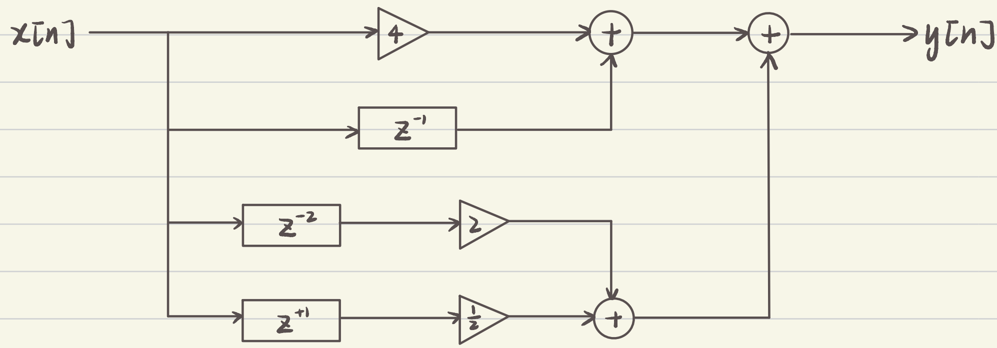
$e[n] = b[n-1]$

$f[n] = y[n-1]$

$$
\begin{aligned}
\therefore y[n] &= 4b[n] = \frac{8}{3}a[n] + \frac{4}{3}f[n] \\
&= \frac{8}{3}\left(c[n] + e[n] + 2y[n-1]\right) \\
&= \frac{8}{3}\left(x[n] + d[n] + b[n-1] + 2y[n-1]\right) \\
&= \frac{8}{3}\left(x[n] - 3y[n-2] + \frac{1}{4}y[n-1] + 2y[n-1]\right) \\
&= \frac{8}{3}x[n] - 8y[n-2] + 6y[n-1]
\end{aligned}
$$

5. **System diagram (5 pts)** Draw the system diagram for the following linear difference equation.

$$y[n] = 2x[n-2] - x[n-1] + 4x[n] + \frac{1}{2}x[n+1]$$



6. **Linear systems (30 pts, 5pt each)** For the following systems, determine whether they are linear or non-linear. Justify your answer with a proof or a counter example.

(a) $y[n] = x[n^2]$
(b) $y[n] = x[n](u[n] - u[n-6])$
(c) $y[n] = \log_{10}(1 + |x[n]|)$
(d) $y[n] = (x[n])^2$
(e) $y[n] = nx[-3n+2]$
(f) $y[n] = -x[n] + 2x[n+1] + 3$

(a) $y[n] = x[n^2]$

Suppose $y_1[n] = x_1[n^2]$, $y_2[n] = x_2[n^2]$

$x[n] = a x_1[n] + b x_2[n]$, $a, b \in ¢$

$\therefore y[n] = x[n^2]$

$= a x_1[n^2] + b x_2[n^2]$

$= a y_1[n] + b y_2[n]$

$\therefore$ is linear

(b) $y[n] = x[n](u[n] - u[n-6])$

Suppose $y_1[n] = x_1[n](u[n] - u[n-6])$

$y_2[n] = x_2[n](u[n] - u[n-6])$

$x[n] = a x_1[n] + b x_2[n]$, $a, b \in ¢$

$\therefore y[n] = x[n](u[n] - u[n-6])$

$= (a x_1[n] + b x_2[n])(u[n] - u[n-6])$

$= a x_1[n](u[n] - u[n-6]) + b x_2[n](u[n] - u[n-6])$

$= a y_1[n] + b y_2[n]$

$\therefore$ is linear

(c) $y[n] = \log_{10}(1+|x[n]|)$

Suppose $y_1[n] = \log_{10}(1+|x_1[n]|)$, $y_2[n] = \log_{10}(1+|x_2[n]|)$

$x[n] = ax_1[n] + bx_2[n]$, $a, b \in ¢$

$\therefore y[n] = \log_{10}(1+|ax_1[n]+bx_2[n]|)$

$\neq \log_{10}(1+|x_1[n]|) + \log_{10}(1+|x_2[n]|)$

$\therefore$ non-linear

(d) $y[n] = (x[n])^2$

Suppose $y_1[n] = (x_1[n])^2$, $y_2[n] = (x_2[n])^2$

$x[n] = ax_1[n] + bx_2[n]$. $a, b \in ¢$

$\therefore y[n] = (x[n])^2$

$= (ax_1[n] + bx_2[n])^2$

$= a^2(x_1[n])^2 + 2ab\,x_1[n]x_2[n] + b^2(x_2[n])^2$

$\neq a(x_1[n])^2 + b(x_2[n])^2$

$\therefore$ non-linear

(e) $y[n] = nx[-3n+2]$

Suppose $y_1[n] = nx_1[-3n+2]$, $y_2[n] = nx_2[-3n+2]$

$x[n] = a\,x_1[n] + bx_2[n]$, $a, b \in ¢$

$\therefore y[n] = n(ax_1[-3n+2] + bx_2[-3n+2])$

$= anx_1[-3n+2] + bnx_2[-3n+2]$

$= ay_1[n] + by_2[n]$

$\therefore$ is linear

(f) $y[n] = -x[n] + 2x[n+1] + 3$

Suppose $y_1[n] = -x_1[n] + 2x_1[n+1] + 3$

$y_2[n] = -x_2[n] + 2x_2[n+1] + 3$

$x[n] = ax_1[n] + bx_2[n]$, $a, b \in \mathbb{C}$

$\therefore y[n] = -x[n] + 2x[n+1] + 3$

$= -(ax_1[n] + bx_2[n]) + 2(ax_1[n+1] + bx_2[n+1]) + 3$

$= a(-x_1[n] + 2x_1[n+1]) + b(-x_2[n] + 2x_2[n+1]) + 3$

$\neq ay_1[n] + by_2[n]$

$\therefore$ non-linear