HW09 — Q1

17 Diagonalize this unitary matrix to reach $U = Q\Lambda Q^{\mathrm{H}}$. Again all λ's are ______:

$$U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

Book Set 9.2 #17

$$P_U(\lambda) = \det(U-\lambda I) = \begin{vmatrix} \frac{1}{\sqrt{3}}-\lambda & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}-\lambda \end{vmatrix} = \left(\frac{1}{\sqrt{3}}-\lambda\right)\left(-\frac{1}{\sqrt{3}}-\lambda\right) - \left(\frac{1-i}{\sqrt{3}}\right)\left(\frac{1+i}{\sqrt{3}}\right)$$

$$= \lambda^2 - \frac{1}{3} - \frac{2}{3} = \lambda^2 - 1$$

$\therefore$ Let $P_U(\lambda) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

For $\lambda_1 = 1$: $(U-\lambda I)x = \begin{bmatrix} \frac{1}{\sqrt{3}}-1 & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}-1 \end{bmatrix} x = \begin{bmatrix} \frac{1-\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1-\sqrt{3}}{\sqrt{3}} \end{bmatrix} x = 0$

$$\therefore x = \begin{bmatrix} 1+\sqrt{3} \\ 1+i \end{bmatrix} \Rightarrow e_1 = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} 1+\sqrt{3} \\ 1+i \end{bmatrix}$$

For $\lambda_2 = -1$: $(U-\lambda I)x = \begin{bmatrix} \frac{1}{\sqrt{3}}+1 & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}+1 \end{bmatrix} x = \begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix} x = 0$

$$\therefore x = \begin{bmatrix} 1-i \\ -\sqrt{3}-1 \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} 1-i \\ -\sqrt{3}-1 \end{bmatrix}$$

$$\therefore \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ 1+i & -\sqrt{3}-1 \end{bmatrix}$$

$U = Q\Lambda Q^H$

$\therefore$ for all λ's, $\lambda^2 = 1$ $(|\lambda| = 1)$

25 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is an orthogonal matrix.

Book Set 9.2 #25

$\because$ $A+iB$ is a unitary matrix

$\therefore (A+iB)^H = A^T - iB^T$

$$(A+iB)^H(A+iB) = (A^T - iB^T)(A+iB)$$
$$= A^TA + iA^TB - iB^TA + B^TB$$
$$= (A^TA + B^TB) + i(A^TB - B^TA)$$
$$= I$$

$$\therefore \begin{cases} A^TA + B^TB = I \\ A^TB - B^TA = 0 \end{cases}$$

$$\because Q^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}$$

$$\therefore Q^TQ = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A^TA + B^TB & -A^TB + AB^T \\ -B^TA + A^TB & B^TB + A^TA \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\therefore Q$ is an orthogonal matrix

26 If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is symmetric.

Book Set 9.2 #26

$\because$ $A+iB$ is Hermitian

$\therefore (A+iB)^H = A+iB$

$\therefore A^T - iB^T = A + iB$

$$\therefore \begin{cases} A = A^T \\ -iB^T = iB \Rightarrow -B^T = B \end{cases}$$

$\therefore$ Let $C = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$\therefore C^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = C$

$\therefore$ C is symmetric

HW09 — Q2

2. Find a Schur decomposition for:

$$A = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix}.$$

$$T_0 = A = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix} \qquad n=4 \text{ and } k=\{1,2,3\}$$

① $k=1 \Rightarrow n-k+1=4$

Step 1.1:

$$\underset{4\times 4}{A_1} = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix}$$

Step 1.2:

$$P_{A_1}(\lambda) = \det(A_1 - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 & -1 \\ 1 & 2-\lambda & 0 & 1 \\ 2 & 0 & 4-\lambda & 2 \\ -1 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 4-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2-\lambda & 0 \\ 2 & 0 & 4-\lambda \\ -1 & 0 & 0 \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda)(4-\lambda)(3-\lambda) + (-1)(2-\lambda)(4-\lambda)$$

$$= (4-\lambda)(2-\lambda)\left[(3-\lambda)^2 - 1\right]$$

$$= (4-\lambda)(2-\lambda)(\lambda^2 - 6\lambda + 8)$$

$$= (4-\lambda)(2-\lambda)(\lambda-2)(\lambda-4)$$

$$= (4-\lambda)^2(2-\lambda)^2$$

$\therefore$ Let $P_{A_1}(\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = 4,\ \lambda_3 = \lambda_4 = 2$

pick $\lambda = 4$

$$\therefore (A_1-\lambda I)x=\begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} x=0$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1+x_4=0 \\ x_2=0 \end{cases} \Rightarrow x=\begin{bmatrix} -x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

Let $x_3=1,\ x_4=0 \Rightarrow x=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\qquad A_1x=4x$

Step 1.3:

Let $E=\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{e_3}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{e_4} \right\}$

$\therefore L=\{e_1, e_2, x, e_4\}$

$\therefore S=U=\{x, e_1, e_2, e_4\}$

$$N_1=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 1.4: $U_1=N_1$

Step 1.5: $T_1 = U_1^H T_0 U_1 = U_1^T T_0 U_1$ —— $U_1 \in \mathbb{R}^{n\times n}$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 4 & 2 & 0 & 2 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 0 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

② $k=2 \Rightarrow n-k+1=3$

Step 2.1: $\underset{3\times 3}{A_2} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$

Step 2.2: $P_{A_2}(\lambda) = \det(A_2 - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2-\lambda \\ -1 & 0 \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda)(3-\lambda) + (2-\lambda)(-1)$$

$$= (2-\lambda)\left[(3-\lambda)^2 - 1\right]$$

$$= (2-\lambda)(\lambda-2)(\lambda-4)$$

$\therefore$ Let $P_{A_2}(\lambda)=0 \Rightarrow \lambda_1=\lambda_2=2,\ \lambda_3=4$

Pick $\lambda=2$

$\therefore (A-\lambda I)x = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} x = 0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1=0 \\ x_3=0 \end{cases}$$

$\therefore x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad A_2x=2x$

Step 2.3: Let $E = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$\therefore L = \{ e_1, x, e_3 \}$

$\therefore S = U = \{ x, e_1, e_3 \}$

$N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 2.4:

$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Step 2.5: $T_2 = U_2^H T_1 U_2 = U_2^T T_1 U_2$ —— $U_2 \in \mathbb{R}^{4\times 4}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 2 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

③ $k=3 \Rightarrow n-k+1=2$

Step 3.1: $\underset{2\times 2}{A_3} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

Step 3.2: $\lambda = 2$ is an eigenvalue of A_3

$$\therefore (A_3 - \lambda I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x = 0 \quad \Rightarrow \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 3.3: Let $E = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$

$\therefore L = \{ e_1, x \}$

$$\therefore \beta_1 = x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\beta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \alpha_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\alpha_2 = \frac{\beta_2}{\|\beta_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore N_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 3.4:

$$U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 3.5: $T_3 = U_3^H T_2 U_3 = U_3^T T_2 U_3$ —— $U_3 \in \mathbb{R}^{3\times 3}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & \frac{4}{\sqrt{2}} & 0 \\ 0 & 2 & \frac{2}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & \frac{4}{\sqrt{2}} & 0 \\ 0 & 2 & \frac{2}{\sqrt{2}} & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

④ $U = U_1 U_2 U_3$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore A = U^H T_3 U$

HW09 — Q3

3. Show that if U is unitary and $A = U^H BU$, then B is normal if and only if A is normal.

Proof: Claim 1: B is normal $\rightarrow$ A is normal

proof: $\because B$ is normal

$\therefore B^H B = BB^H$

$\therefore A^H A = (U^H BU)^H (U^H BU) = U^H B^H U \cdot U^H BU$

$\because U$ is unitary

$\therefore U^H U = UU^H = I$

$\therefore A^H A = U^H B^H IBU = U^H B^H BU = U^H BB^H U$

$= U^H BIB^H U = U^H BUU^H B^H U$

$= (U^H BU)(U^H BU)^H$

$= AA^H$

$\therefore A$ is normal QED — Claim 1

Claim 2: A is normal $\longrightarrow$ B is normal

proof: $\because A$ is normal

$\therefore A^H A = AA^H$

$\because A^H A = (U^H BU)^H (U^H BU) = U^H B^H UU^H BU$

$= U^H B^H IBU = U^H B^H BU$

$AA^H = (U^H BU)(U^H BU)^H = U^H BUU^H B^H U$

$= U^H BIB^H U = U^H BB^H U$

$\therefore U^H B^H BU = U^H BB^H U$

$\therefore U(U^H B^H B U)U^H = U(U^H B B^H U)U^H$

$\therefore B^H B = B B^H$

$\therefore B$ is normal QED - Claim 2

$\therefore$ from Claim 1, 2 : B is normal IFF A is normal

QED

HW09 — Q4

4. Show that if x is an eigenvector of a normal matrix A corresponding to eigenvalue λ, then x is an eigenvector of A^H corresponding to the conjugate λ^*.

According to the question : $Ax = \lambda x \Rightarrow (A-\lambda I)x = 0$

$\therefore \langle (A-\lambda I)x, (A-\lambda I)x \rangle = \|(A-\lambda I)x\|^2 = 0$

$\because \langle Tx, y \rangle = \langle x, T^H y \rangle$

$$\therefore \langle (A-\lambda I)x, (A-\lambda I)x \rangle = \langle x, (A-\lambda I)^H (A-\lambda I)x \rangle$$

$$= \langle x, (A^H - \lambda^* I)(A-\lambda I)x \rangle$$

$$= \langle x, (A^H A - \lambda A^H - \lambda^* A + \lambda\lambda^* I)x \rangle$$

$\because$ A is normal. $A^H A = AA^H$

$$\therefore = \langle x, (AA^H - \lambda A^H - \lambda^* A + \lambda\lambda^* I)x \rangle$$

$$= \langle x, (A-\lambda I)(A^H - \lambda^* I)x \rangle$$

$$= \langle (A-\lambda I)^H x, (A^H - \lambda^* I)x \rangle$$

$$= \langle (A^H - \lambda^* I)x, (A^H - \lambda^* I)x \rangle$$

$$= \|(A^H - \lambda^* I)x\|^2$$

$\therefore \|(A^H - \lambda^* I)x\|^2 = 0$

$\therefore (A^H - \lambda^* I)x = 0 \Rightarrow A^H x = \lambda^* x$

$\therefore$ x is eigenvector of A^H with eigenvalue λ^*

QED

Q5 Image Compression with Discrete Fourier Transform (DFT)

In [14]:
```
import numpy as np
import matplotlib.pyplot as plt

compression_ratio=[0.001,0.003,0.01,0.03]

#load image
img=plt.imread("baboon.png")

#display the original image
plt.figure()
plt.imshow(img)
plt.title("Original Image")
_=plt.axis("off")
```

Original Image

In [17]:
```
#Normalization
if img.max()>1:
    img=img/255.0

for ratio in compression_ratio:
    img_compressed=np.zeros_like(img)

    #apply to each of the channels
    for channel in range(img.shape[2]):
        #compute 2D DFT
        dft=np.fft.fft2(img[:,:,channel])
        #shift the zero frequency component to the center
        dft_shifted=np.fft.fftshift(dft)
        #calculate the magnitude spectrum
        magnitude_spectrum=np.abs(dft_shifted)
```

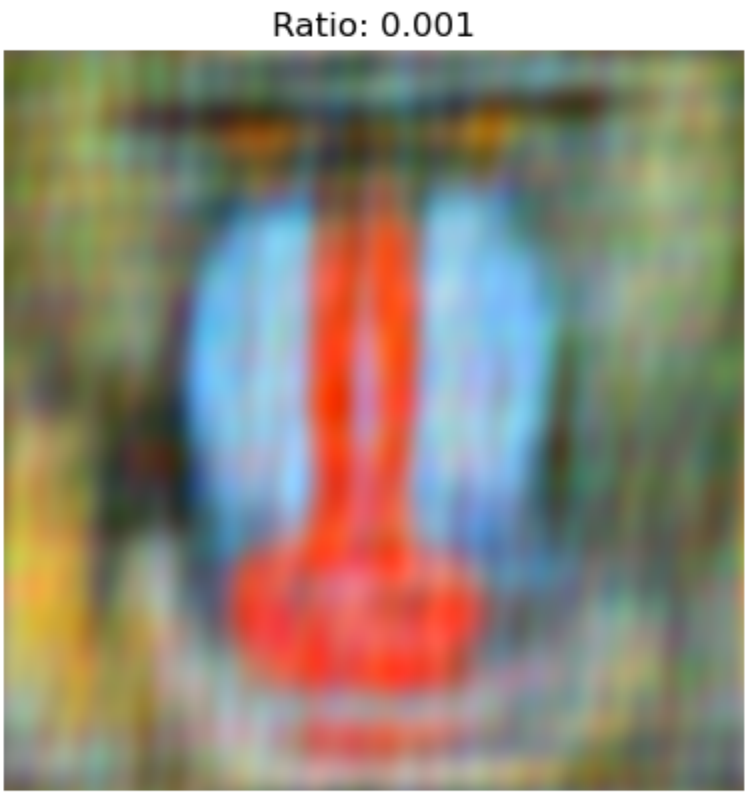
```
        #calculate the threshold
        threshold=np.percentile(magnitude_spectrum,100*(1-ratio))
        #select large coefficients
        dft_compressed=dft_shifted*(magnitude_spectrum>=threshold)
        #shift the zero frequency component back
        dft_inverse_shifted=np.fft.ifftshift(dft_compressed)
        #compute the inverse DFT
        channel_compressed=np.fft.ifft2(dft_inverse_shifted)
        #extract the real part
        img_compressed[:,:,channel]=np.real(channel_compressed)

    img_compressed=np.clip(img_compressed,0,1)

    #display the compressed image
    plt.figure()
    plt.imshow(img_compressed)
    plt.title(f"Ratio: {ratio}")
    plt.axis("off")

    #compute the mean squared error
    mse=np.mean((img-img_compressed)**2)
    print(f"Compression Ratio: {ratio}, MSE: {mse:.4f}")
```

```
Compression Ratio: 0.001, MSE: 0.0155
Compression Ratio: 0.003, MSE: 0.0131
Compression Ratio: 0.01, MSE: 0.0107
Compression Ratio: 0.03, MSE: 0.0082
```



Ratio: 0.003

Ratio: 0.01

Ratio: 0.03