HW10-Q1

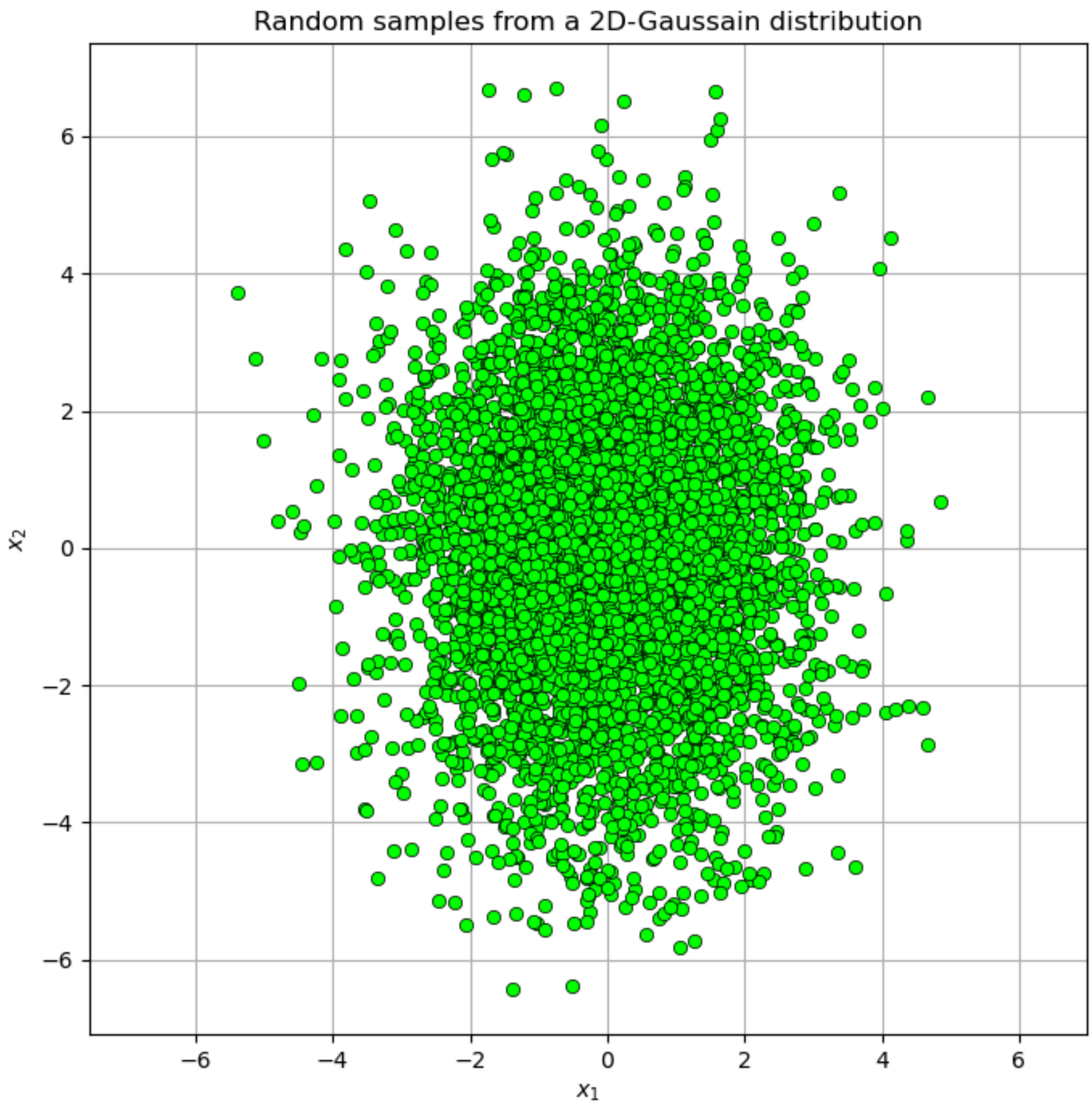
In [123…

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Ellipse
from scipy.stats import multivariate_normal

plt.rcParams['figure.figsize']=8,8

def generate_and_plot(kx,mu):
    distr=multivariate_normal(
        cov=kx,mean=mu,
        seed=1000
    )

    data=distr.rvs(size=5000)
    plt.grid()

    plt.plot(data[:,0],data[:,1],'o',color='lime',
                markeredgewidth=0.5,
                markeredgecolor='black')

    plt.title(r'Random samples from a 2D-Gaussain distribution')
    plt.xlabel(r'$x_1$')
    plt.ylabel(r'$x_2$')
    plt.axis('equal')

def ellipsoid(x,mu,kx):
    return ((x-mu) @ np.linalg.inv(kx) @ (x-mu).T)<=1

def plot_ellipsoid(mu,kx,ax):
    #repeat generate 5000 points
    distr=multivariate_normal(
        cov=kx,mean=mu,
        seed=1000
    )
    data=distr.rvs(size=5000)

    #filter the points which are belong to the ellipsoid and plot
    filtered_data=np.array([point for point in data if ellipsoid(point,mu,kx)])

    plt.grid()
    plt.plot(filtered_data[:,0],filtered_data[:,1],'o',color='lime',
                markeredgewidth=0.5,
                markeredgecolor='black')
    plt.title(r'filtered data and ellipsoid')
    plt.xlabel(r'$x_1$')
    plt.ylabel(r'$x_2$')
    plt.axis('equal')

    eigenvalues,eigenvectors=np.linalg.eigh(kx)
    axis_lengths=2*np.sqrt(eigenvalues)
    angle=np.degrees(np.arctan2(*eigenvectors[:, 0][::-1]))
    #print(angle)
    ell=Ellipse(xy=mu,width=axis_lengths[0],height=axis_lengths[1],angle=angle,
                edgecolor='red', facecolor='none', lw=2)
    ax.add_patch(ell)
```

when mu = $\begin{bmatrix} 0 & 0 \end{bmatrix}$ Kx = $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

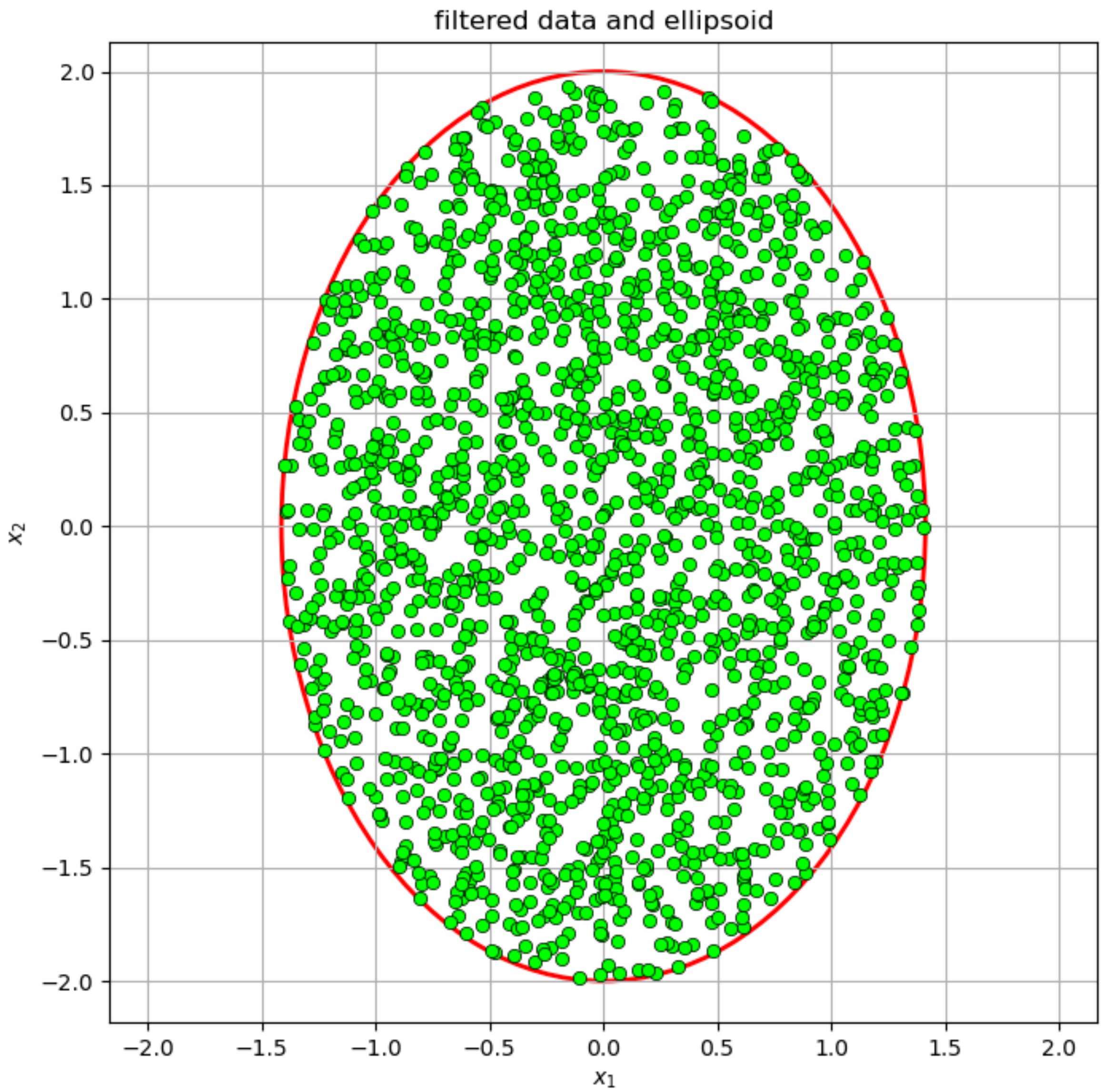
In [124…

```
Kx=np.array([[2.0,0.0],[0.0,4.0]])
mu=np.array([0,0])
random_seed=10

generate_and_plot(Kx,mu)
```



In [125…

```
ax=plt.gca()
plot_ellipsoid(mu,Kx,ax)
plt.show()
```



Compare

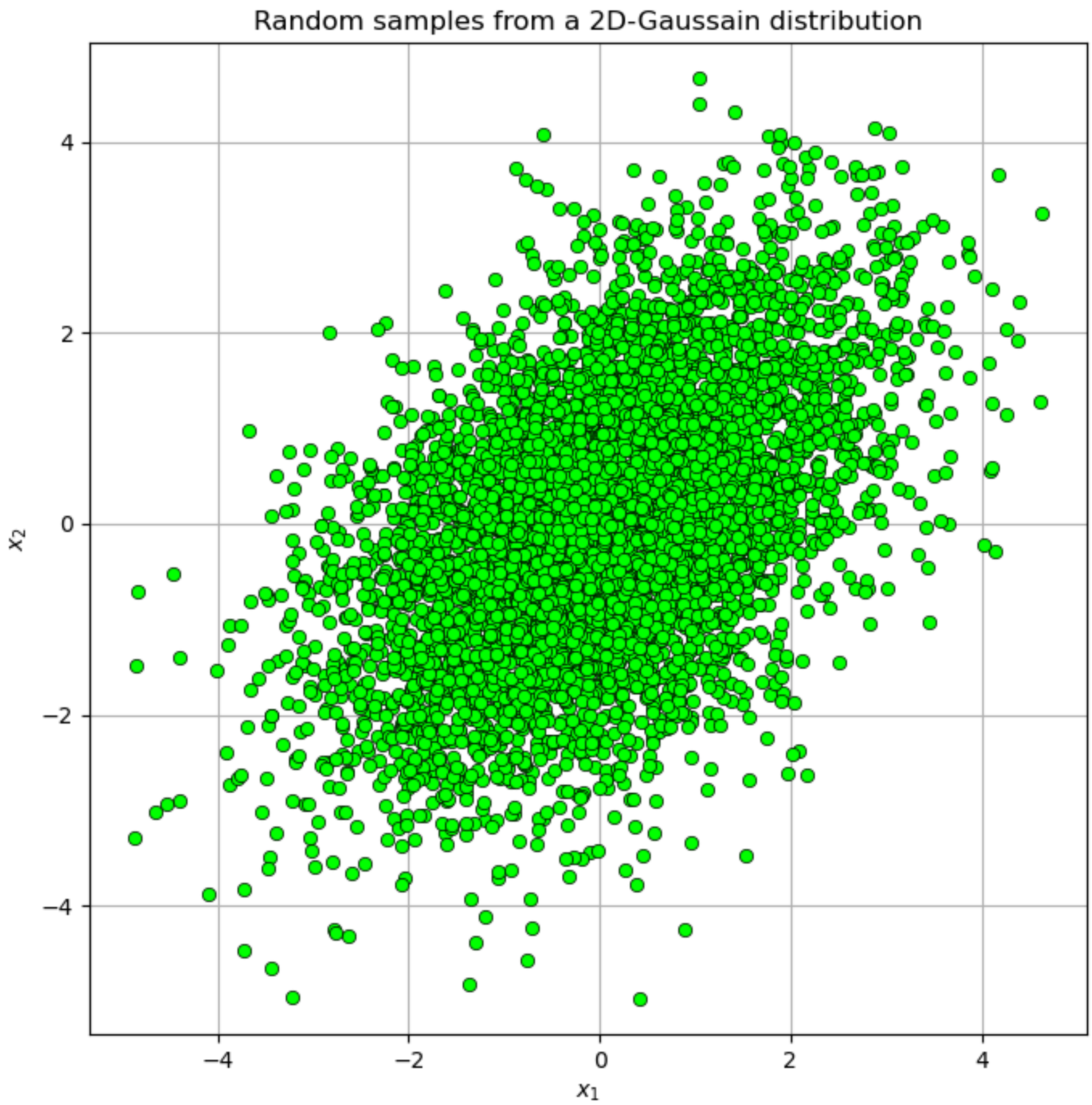
- I observe that when I use the definition of an ellipsoid to filter the random points, they lie precisely within the ellipsoid we draw based on the eigenvalues and eigenvectors.
- Additionally, the lengths of the principal axes are shorter than the range of the random points, approximately by a factor of one-third.

(The comparison results of the last two experiments are the same as this one.)

when mu = $\begin{bmatrix} 0 & 0 \end{bmatrix}$ Kx = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

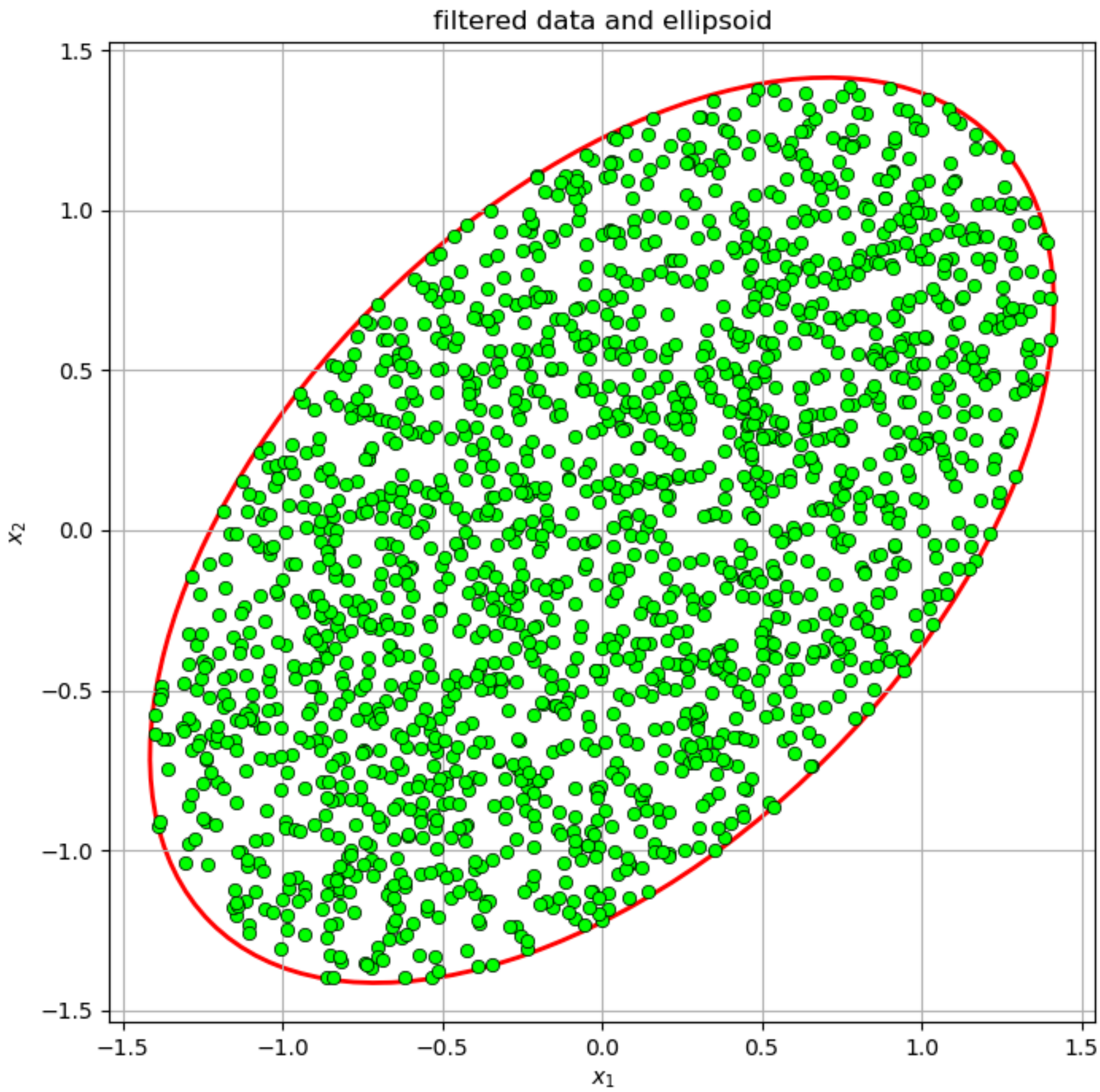
In [126…

```
Kx=np.array([[2.0,1.0],[1.0,2.0]])
mu=np.array([0,0])
random_seed=10

generate_and_plot(Kx,mu)
```



In [127…

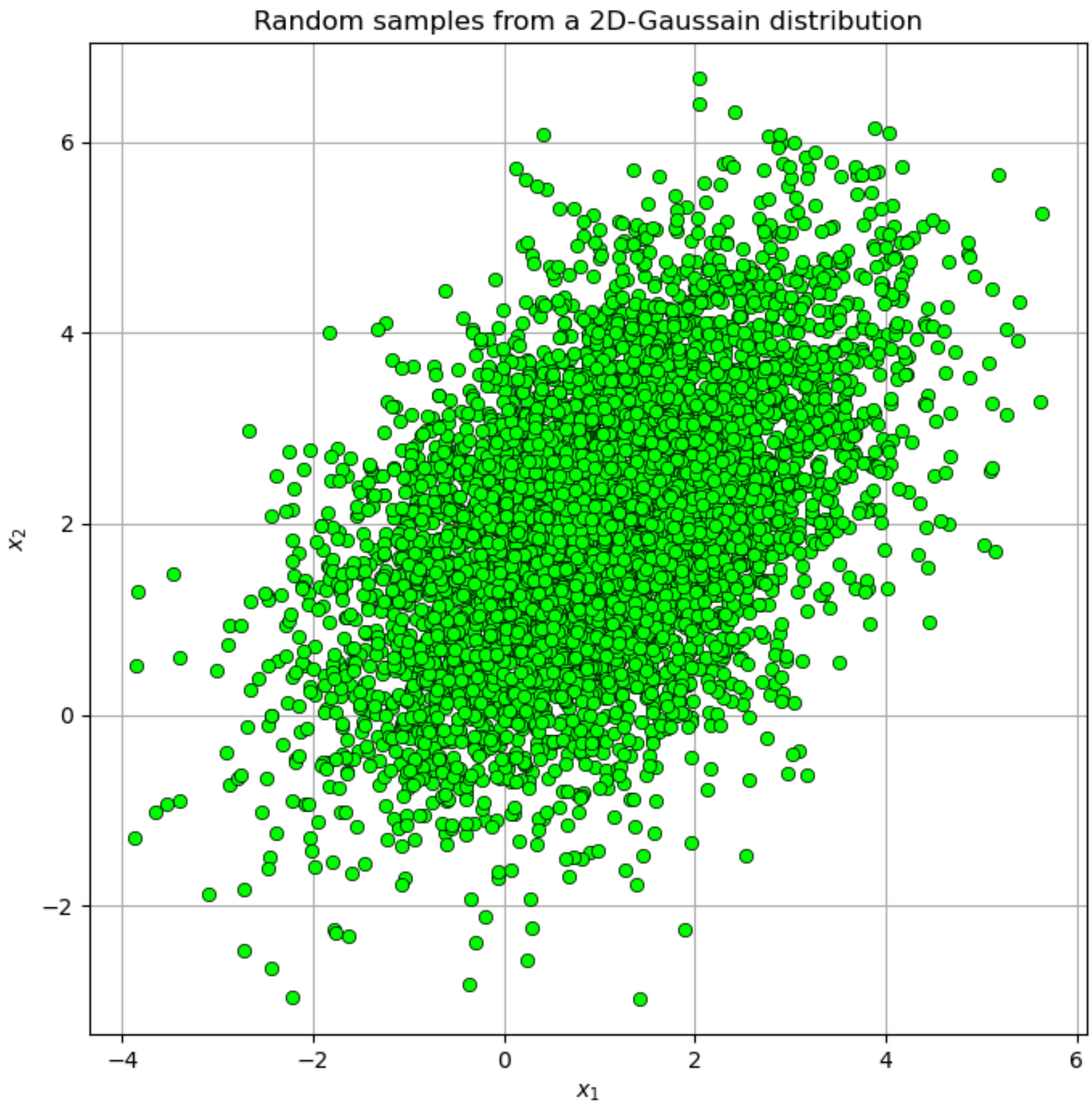
```
ax=plt.gca()
plot_ellipsoid(mu,Kx,ax)
plt.show()
```



when mu = $\begin{bmatrix} 2 & 1 \end{bmatrix}$ Kx = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

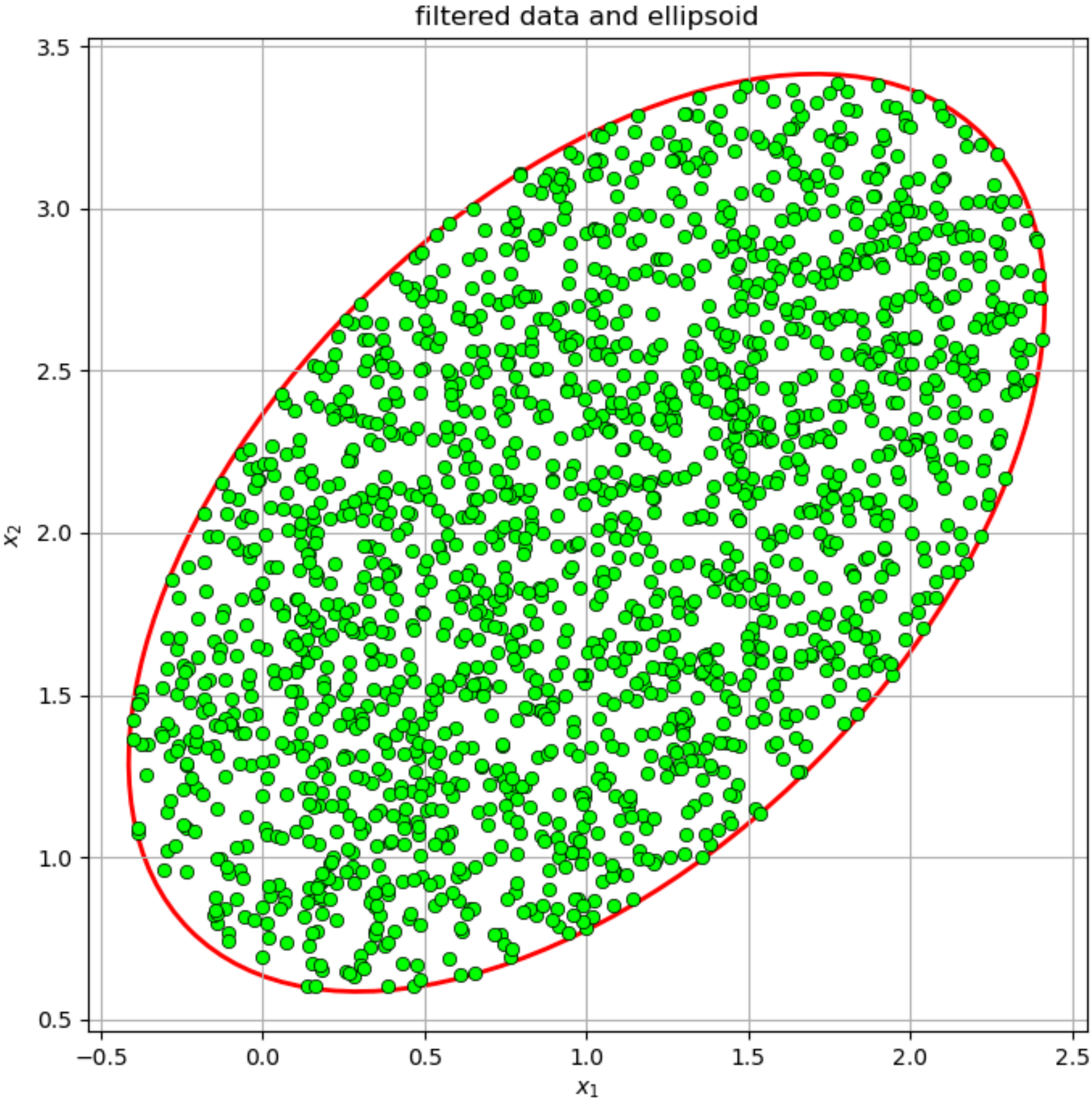
In [128…

```
Kx=np.array([[2.0,1.0],[1.0,2.0]])
mu=np.array([1.0,2.0])
random_seed=10

generate_and_plot(Kx,mu)
```



In [129…

```
ax=plt.gca()
plot_ellipsoid(mu,Kx,ax)
plt.show()
```


filtered data and ellipsoid
3.5
3.0
2.5
2.0
1.5
1.0
0.5
x_2
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
x_1

HW10 — Q2

(a) Compute A^k for any k integer, and show that

$$\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

(a) $\because A = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$

$\therefore P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4}-\lambda \end{vmatrix} = (\frac{1}{2}-\lambda)(\frac{3}{4}-\lambda) - \frac{1}{8} = \lambda^2 - \frac{5}{4}\lambda + \frac{1}{4}$

$\therefore$ Let $P_A(\lambda) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = \frac{1}{4}$

$\therefore$ For $\lambda_1 = 1$: $(A-\lambda I)x = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda_2 = \frac{1}{4}$: $(A-\lambda I)x = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore Q = [e_1\ e_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$

$\therefore Q^{-1} = \frac{1}{-1-2} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$\therefore A = Q \Lambda Q^{-1}$

$$\begin{aligned}
\therefore A^k = (Q\Lambda Q^{-1})^k &= (Q\Lambda Q^{-1})(Q\Lambda Q^{-1}) \cdots (Q\Lambda Q^{-1}) \\
&= Q \Lambda I \Lambda I \cdots \Lambda Q^{-1} \\
&= Q \Lambda^k Q^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (\frac{1}{4})^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & (\frac{1}{4})^k \\ 2 & -(\frac{1}{4})^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1+2(\frac{1}{4})^k & 1-(\frac{1}{4})^k \\ 2-2(\frac{1}{4})^k & 2+(\frac{1}{4})^k \end{bmatrix}
\end{aligned}$$

$\therefore \lim_{k\to\infty} A^k = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

(b) Assuming initially a distribution $[a_0, b_0]^T = [\frac{1}{2}, \frac{1}{2}]^T$ If this migration pattern continues, compute the long-run population distribution. Will city A be desserted?

(b) $\because \begin{bmatrix} a_{k+1} \\ b_{k+1} \end{bmatrix} = A \begin{bmatrix} a_k \\ b_k \end{bmatrix} = A\left(A \begin{bmatrix} a_{k-1} \\ b_{k-1} \end{bmatrix}\right) = A^2\left(A \begin{bmatrix} a_{k-2} \\ b_{k-2} \end{bmatrix}\right) = \cdots = A^{k+1} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$

$$\therefore \lim_{k\to\infty} \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \lim_{k\to\infty} A^k \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$\therefore$ the long-run population is $[\frac{1}{3}, \frac{2}{3}]^T$

$\therefore$ city A will not be deserted

3. Find the principal axes, their lengths, and sketch the following ellipsoids:

a) $2x_1^2 + 4x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3 = 1$

b) $5x_1^2 + 2x_2^2 + 4x_3^2 - 2\sqrt{2}x_1x_3 = 1$.

HW10 − Q3

a) $\because 2x_1^2 + 4x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3 = 1$

$$\therefore [x_1\ x_2\ x_3] \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T A x$$

$$\therefore P_A(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 4-\lambda & 1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 4-\lambda & 1 \\ 0 & 5-\lambda & 5-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (5-\lambda)\left[(2-\lambda)(3-\lambda)-2\right]$$

$$= (5-\lambda)(\lambda^2-5\lambda+4) = (5-\lambda)(\lambda-4)(\lambda-1)$$

$\therefore$ Let $P_A(\lambda)=0 \Rightarrow \lambda_1=5,\ \lambda_2=4,\ \lambda_3=1$

- For $\lambda_1=5$:

$$(A-\lambda I)x = \begin{bmatrix} -3 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -3 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- For $\lambda_2=4$:

$$(A-\lambda I)x = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

- For $\lambda_3 = 1$:

$$(A - \lambda I)x = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

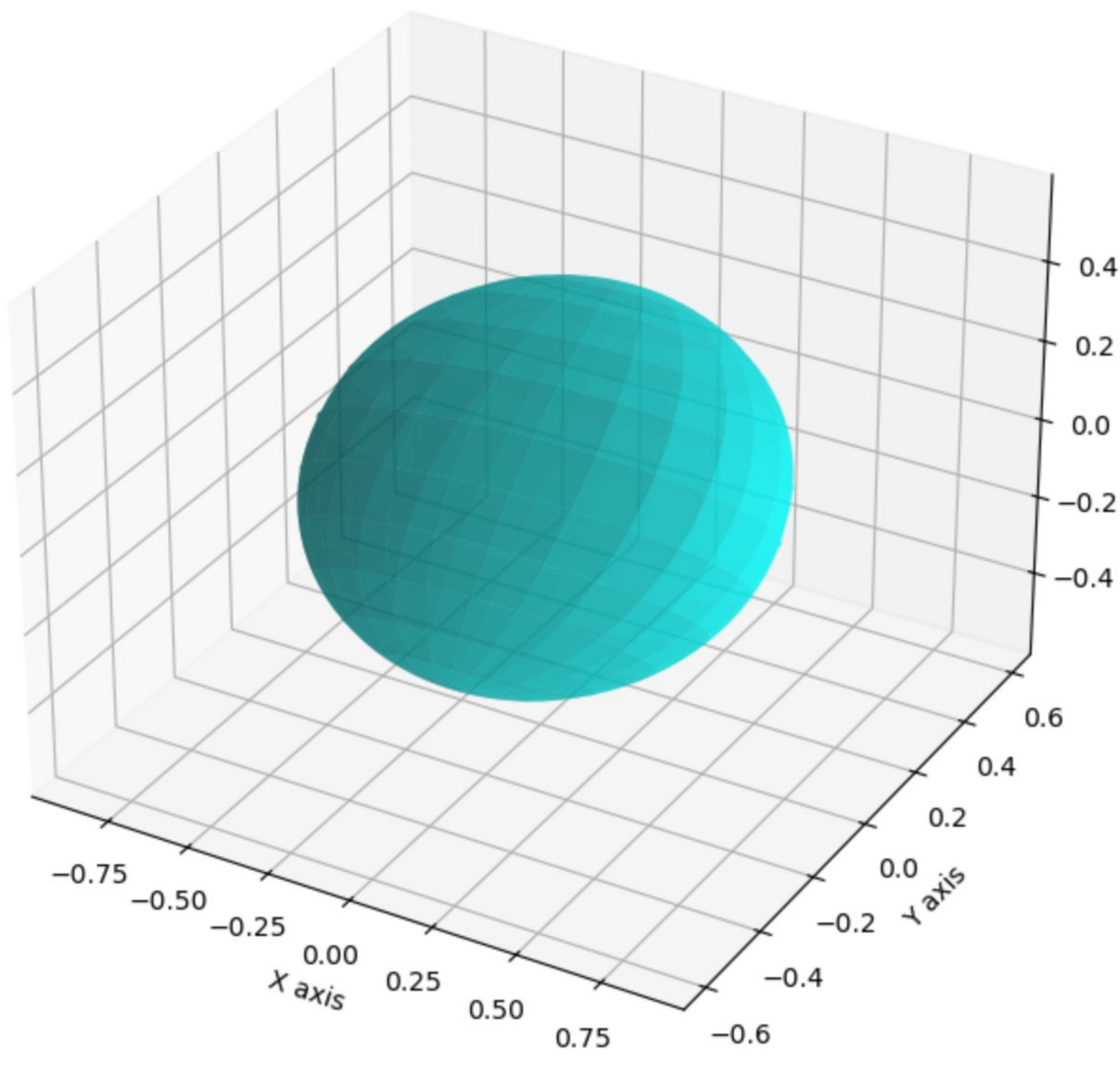
$$\therefore x = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow e_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore A = E \Lambda E^T \quad \text{when } E = [e_1\ e_2\ e_3] = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ \sqrt{3} & -\sqrt{2} & 1 \end{bmatrix}$$

$$\therefore x^T A x = x^T E \Lambda E^T x = y^T \Lambda y$$

$$\therefore y = E^T x = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3}x_2 + \sqrt{3}x_3 \\ \sqrt{2}x_1 + \sqrt{2}x_2 - \sqrt{2}x_3 \\ 2x_1 - x_2 + x_3 \end{bmatrix}$$

$$\begin{cases} y_1 = \frac{1}{\sqrt{2}}(x_2 + x_3) \\ y_2 = \frac{1}{\sqrt{3}}(x_1 + x_2 - x_3) \\ y_3 = \frac{1}{\sqrt{6}}(2x_1 - x_2 + x_3) \end{cases} \Rightarrow \begin{cases} \|P_1\| = \frac{1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{5}} \\ \|P_2\| = \frac{1}{\sqrt{\lambda_2}} = \frac{1}{2} \\ \|P_3\| = \frac{1}{\sqrt{\lambda_3}} = 1 \end{cases}$$



0.4
0.2
0.0
−0.2
−0.4
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
Y axis
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
X axis

b) $5x_1^2 + 2x_2^2 + 4x_3^2 - 2\sqrt{2}\,x_1x_3 = 1$

$$\therefore [x_1\ x_2\ x_3]\begin{bmatrix} 5 & 0 & -\sqrt{2} \\ 0 & 2 & 0 \\ -\sqrt{2} & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T A x$$

$$\therefore P_A(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 0 & -\sqrt{2} \\ 0 & 2-\lambda & 0 \\ -\sqrt{2} & 0 & 4-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)(4-\lambda) - \sqrt{2}\cdot\sqrt{2}(2-\lambda)$$

$$= (2-\lambda)[(5-\lambda)(4-\lambda)-2] = (2-\lambda)(\lambda-3)(\lambda-6)$$

$\therefore$ Let $P_A(\lambda)=0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 6$

- For $\lambda_1 = 2$:

$$(A-\lambda I)x = \begin{bmatrix} 3 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 3 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 0 & -\sqrt{2} \\ 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- For $\lambda_2 = 3$:

$$(A-\lambda I)x = \begin{bmatrix} 2 & 0 & -\sqrt{2} \\ 0 & -1 & 0 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 2 & 0 & -\sqrt{2} \\ 0 & -1 & 0 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & -\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

• For $\lambda_3 = 6$:

$$(A-\lambda I)x = \begin{bmatrix} -1 & 0 & -\sqrt{2} \\ 0 & -4 & 0 \\ -\sqrt{2} & 0 & -2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -1 & 0 & -\sqrt{2} \\ 0 & -4 & 0 \\ -\sqrt{2} & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

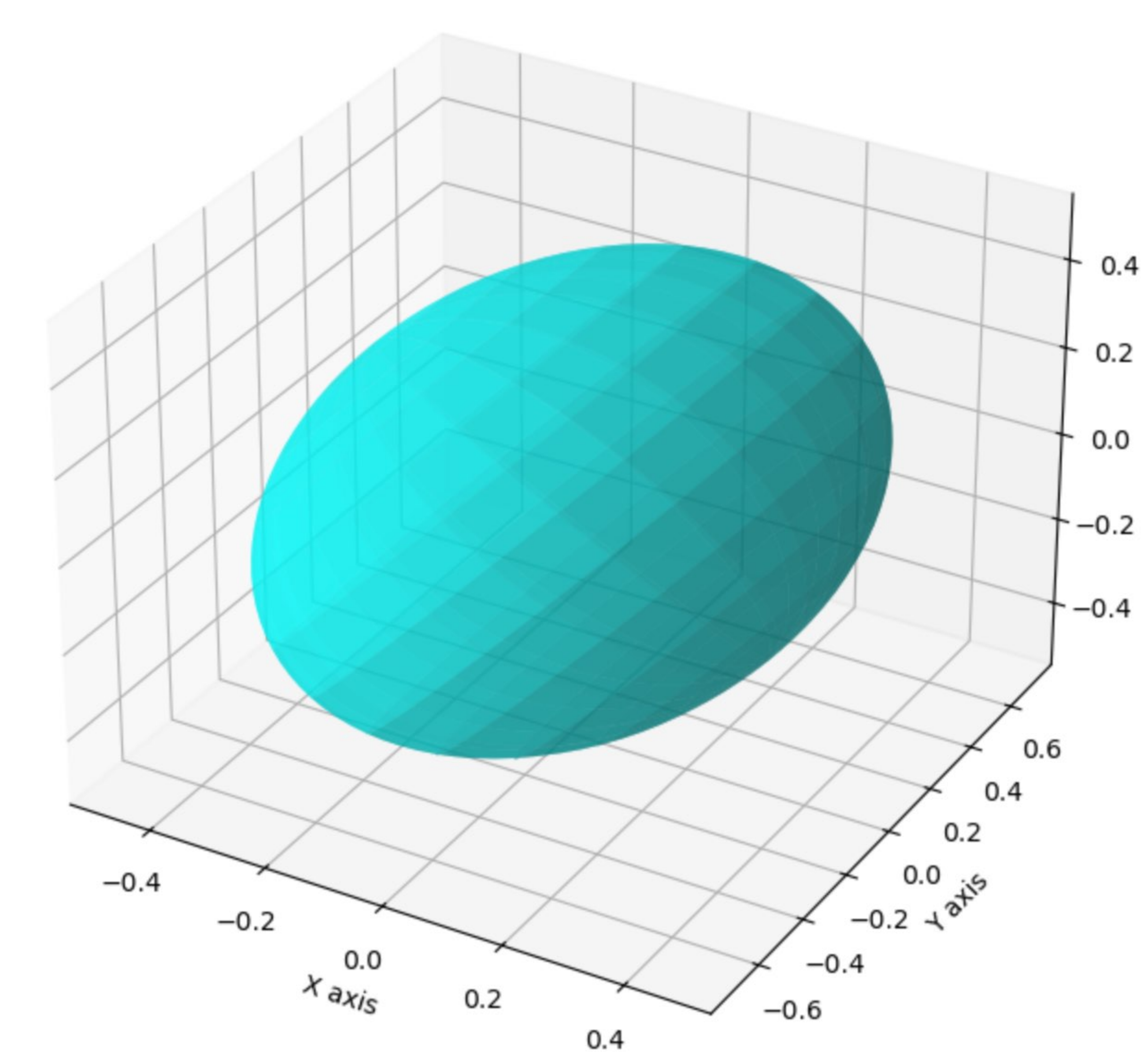
$$\therefore x = \begin{bmatrix} \sqrt{2} \\ 0 \\ -1 \end{bmatrix} \Rightarrow e_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore A = E \Lambda E^T \quad \text{when } E = [e_1\ e_2\ e_3] = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & \sqrt{2} \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\therefore x^T A x = x^T E \Lambda E^T x = y^T \Lambda y$$

$$\therefore y = E^T x = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} x_2 \\ x_1 + \sqrt{2} x_3 \\ \sqrt{2} x_1 + x_3 \end{bmatrix}$$

$$\begin{cases} y_1 = x_2 \\ y_2 = \frac{1}{\sqrt{3}}(x_1 + \sqrt{2} x_3) \\ y_3 = \frac{1}{\sqrt{3}}(\sqrt{2} x_1 + x_3) \end{cases} \Rightarrow \begin{cases} \|p_1\| = \frac{1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} \\ \|p_2\| = \frac{1}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{3}} \\ \|p_3\| = \frac{1}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{6}} \end{cases}$$



0.4
0.2
0.0
−0.2
−0.4
−0.4
−0.2
0.0
X axis
0.2
0.4
0.6
0.4
0.2
0.0
Y axis
−0.2
−0.4
−0.6