Problem Set 5

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1. **Sampling (65 pts)** A continuous time signal $x(t)$ has the CTFT $X(j\Omega)$ with a larger triangle of height 1 and two smaller triangles of height α as shown in Figure 1. The important frequencies are given in terms of Ω_0 in the plot.

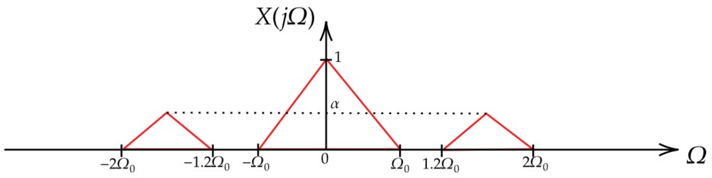


Figure 1: Plot of $X(j\Omega)$ for Problem 1.

(a) i. (**5pts**) What is the value of the maximum frequency $f_{\max_1}$ present in $X(j\Omega)$?, what is the Nyquist rate f_{Nyq_1}?. Express your answer as a function of Ω_0.

ii. (**5pts**) Let $x[n]$ be the sampled version of $x(t)$ at the Nyquist rate f_{Nyq_1}. Plot the DTFT of $x[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

iii. (**5pts**) Compute $x[n]$ from the above DTFT plot using the known DTFT pairs and the properties from the tables. (Hint: Homework 3, Problem 4)

(a) i. $\because \Omega_{m_1} = 2\Omega_0$

$$\therefore f_{max_1} = \frac{\Omega_{m_1}}{2\pi} = \frac{\Omega_0}{\pi}, \quad f_{Nyq_1} = \frac{2\Omega_{m_1}}{2\pi} = \frac{\Omega_m}{\pi} = \frac{2\Omega_0}{\pi}$$

ii. $\because x[n] = x(nT) \quad -\infty < n < \infty$

$$T_1 = \frac{1}{f_{Nyq_1}} = \frac{\pi}{2\Omega_0}$$

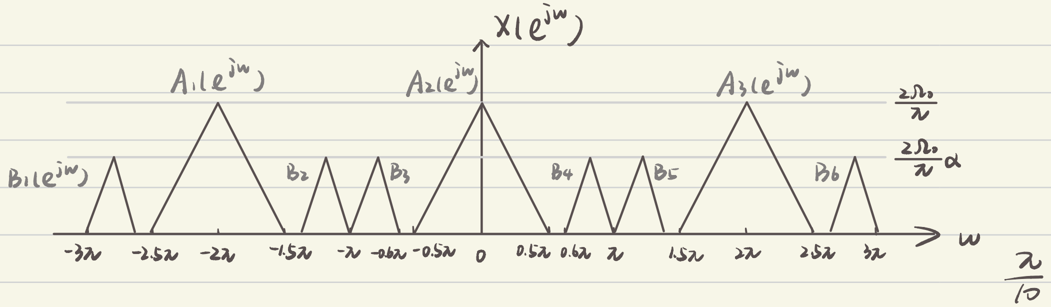
$$\therefore X(e^{j\omega}) = \frac{1}{T_1}\sum_{k=-\infty}^{+\infty} X(j\Omega)\Big|_{\Omega = \frac{\omega + 2\pi k}{T_1}}$$

$$= \frac{2\Omega_0}{\pi}\sum_{k=-\infty}^{+\infty} X(j\Omega)\Big|_{\Omega = \frac{\omega+2\pi k}{T_1}}$$

when $k=0$: $\Omega = \frac{\omega}{T_1} \Rightarrow \omega = \Omega T_1 = \frac{\Omega\pi}{2\Omega_0}$

$\Big(\therefore 2\Omega_0 \longrightarrow \omega = \pi, \quad 1.2\Omega_0 \longrightarrow \omega = 0.6\pi, \quad \Omega_0 \longrightarrow \omega = 0.5\pi$

$X(j\Omega) = 1 \longrightarrow X(e^{j\omega}) = \frac{2\Omega_0}{\pi}, \quad X(j\Omega) = \alpha \longrightarrow X(e^{j\omega}) = \frac{2\Omega_0}{\pi}\alpha \Big)$



iii

$$A_2(e^{jw}) = \begin{cases} \frac{4\Omega_0}{\pi^2}\left(\frac{1}{2}\pi - |w|\right), & |w| \le \frac{1}{2}\pi \\ 0, & \text{else} \end{cases} \rightleftharpoons a_2[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}\right)^2, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$A_1(e^{jw}) = A_2(e^{j(w+2\pi)}) \rightleftharpoons a_1[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}\right)^2 e^{-j2\pi n}, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$A_3(e^{jw}) = A_3(e^{j(w-2\pi)}) \rightleftharpoons a_3[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}\right)^2 e^{j2\pi n}, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$B'(e^{jw}) = \begin{cases} \frac{10\Omega_0}{\pi^2}\alpha\left(\frac{1}{5}\pi - |w|\right), & |w| \le \frac{1}{5}\pi \\ 0, & \text{else} \end{cases} \rightleftharpoons b'[n] = \begin{cases} \frac{20\Omega_0}{\pi}\alpha\left(\frac{\sin\left(\frac{\pi n}{10}\right)}{\pi n}\right)^2, & n \neq 0 \\ \frac{\Omega_0}{5\pi}\alpha, & n = 0 \end{cases}$$

$$B_1(e^{jw}) = B'(e^{j(w+2.8\pi)}) \rightleftharpoons b_1[n] = \begin{cases} \frac{20\Omega_0}{\pi}\alpha\left(\frac{\sin\left(\frac{\pi n}{10}\right)}{\pi n}\right)^2 e^{-j2.8\pi n}, & n \neq 0 \\ \frac{\Omega_0}{5\pi}\alpha, & n = 0 \end{cases}$$

$$B_6(e^{jw}) = B'(e^{j(w-2.8\pi)}) \rightleftharpoons b_6[n] = \begin{cases} \frac{20\Omega_0}{\pi}\alpha\left(\frac{\sin\left(\frac{\pi n}{10}\right)}{\pi n}\right)^2 e^{j2.8\pi n}, & n \neq 0 \\ \frac{\Omega_0}{5\pi}\alpha, & n = 0 \end{cases}$$

$$B_2(e^{jw}) = B'(e^{j(w+1.2\pi)}) \qquad B_5(e^{jw}) = B'(e^{j(w-1.2\pi)})$$

(same as above)

$$B_3(e^{jw}) = B'(e^{j(w+0.8\pi)}) \qquad B_4(e^{jw}) = B'(e^{j(w-0.8\pi)})$$

$$\therefore x[n] = b_1[n] + a_1[n] + b_2[n] + b_3[n] + a_2[n] + b_4[n] + b_5[n] + a_3[n] + b_6[n]$$
$$= (a_1[n] + a_3[n]) + a_2[n] + (b_1[n] + b_6[n]) + (b_2[n] + b_5[n]) + (b_3[n] + b_4[n])$$

(b) Now you are told that due to some hardware limitations, you are not allowed to sample at the rate f_{Nyq_1}, and you are expected to sample at a lower rate. For this, you are given an ideal analog low-pass filter $g(t)$ with the frequency response $G(j\Omega)$,

$$G(j\Omega) = \begin{cases} 1 & |\Omega| \leq 1.1\Omega_0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $y(t)$ be the signal obtained by applying the filter $g(t)$ to $x(t)$.

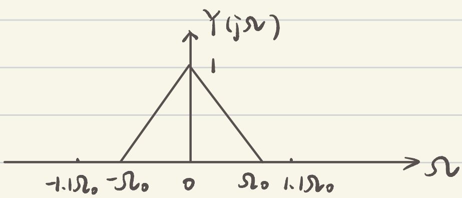
i. (**5pts**) Let the CTFT of $y(t)$ be $Y(j\Omega)$. Plot $Y(j\Omega)$ and label all the important frequencies and scaling factors.

ii. (**6 pts**) What is the value of the maximum frequency f_{max_2} present in $Y(j\Omega)$?, what is the Nyquist rate f_{Nyq_2} for $y(t)$?. By what factor the Nyquist rate has reduced compared to part (a)?

iii. (**5pts**) Let $y[n]$ be the sampled version of $y(t)$ at the Nyquist rate f_{Nyq_2}. Plot the DTFT of $y[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

iv. (**4pts**) Compute $y[n]$ from the above DTFT plot using the known DTFT pairs.

(b) i. $Y(j\Omega) = X(j\Omega)G(j\Omega)$



ii $\because \Omega_{m_2} = \Omega_0$

$\therefore f_{max_2} = \frac{\Omega_{m_2}}{2\pi} = \frac{\Omega_0}{2\pi}$, $f_{Nyq_2} = \frac{2\Omega_{m_2}}{2\pi} = \frac{\Omega_0}{\pi}$

Because the low-pass filter g(t) filt the high frequency signal

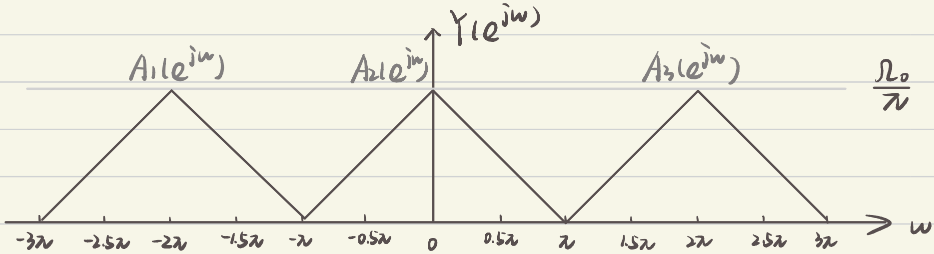
iii $T_2 = \frac{1}{f_{Nyq_2}} = \frac{\pi}{\Omega_0}$

$\therefore Y(e^{j\omega}) = \frac{1}{T_2} \sum_{k=-\infty}^{+\infty} Y(j\Omega)\Big|_{\Omega=\frac{\omega+2\pi k}{T_2}}$

$= \frac{\Omega_0}{\pi} \sum_{k=-\infty}^{+\infty} Y(j\Omega)\Big|_{\Omega=\frac{\omega+2\pi k}{T_2}}$

when $k=0$, $\Omega = \frac{w}{T_2} \Rightarrow w = \Omega T_2 = \frac{\Omega\pi}{\Omega_0}$

$$\left(\begin{array}{l} \therefore \Omega = \Omega_0 \rightarrow w = \pi \\ Y(j\Omega) = 1 \longrightarrow Y(e^{jw}) = \frac{\Omega_0}{\pi} \end{array} \right)$$



iv. $$A_2(e^{jw}) = \begin{cases} \frac{\Omega_0}{\pi^2}(\pi - |w|) & , |w| \le \pi \\ 0 & , \text{else} \end{cases} \rightleftharpoons a_2[n] = \begin{cases} \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 & , n \neq 0 \\ \frac{\Omega_0}{2\pi} & , n = 0 \end{cases}$$

$$A_1(e^{jw}) = A_2(e^{j(w+2\pi)}) \rightleftharpoons a_1[n] = \begin{cases} \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 e^{-j2\pi n} & , n \neq 0 \\ \frac{\Omega_0}{2\pi} & , n = 0 \end{cases}$$

$$A_3(e^{jw}) = A_2(e^{j(w-2\pi)}) \rightleftharpoons a_3[n] = \begin{cases} \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 e^{j2\pi n} & , n \neq 0 \\ \frac{\Omega_0}{2\pi} & , n = 0 \end{cases}$$

$\therefore y[n] = a_1[n] + a_2[n] + a_3[n]$

$$= \begin{cases} \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 + \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 e^{-j2\pi n} + \frac{2\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 e^{j2\pi n} & , n \neq 0 \\ \frac{\Omega_0}{2\pi} + \frac{\Omega_0}{2\pi} + \frac{\Omega_0}{2\pi} & , n = 0 \end{cases}$$

$$= \begin{cases} \left(\frac{2\Omega_0}{\pi} + \frac{4\Omega_0}{\pi}\cos(2\pi n)\right)\left(\frac{\sin(\frac{\pi n}{2})}{\pi n}\right)^2 & , n \neq 0 \\ \frac{3\Omega_0}{2\pi} & , n = 0 \end{cases}$$

(c) Now consider sampling of $y(t)$ in (b) at the rate f_{Nyq_1} from (a). Let this sampled signal be $z[n]$.

i. (**5pts**) Plot the DTFT of $z[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

ii. (**5pts**) Compute $z[n]$ from the above DTFT plot using the known DTFT pairs.

iii. (**15pts**) Let the normalized error e, a measure of information loss due to the low-pass filtering, be defined as follows:

$$e = \frac{\sum_{n=-\infty}^{\infty} |x[n] - z[n]|^2}{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

Find a closed-form expression for e in terms of α and Ω_0. (Hint: Parseval's theorem)

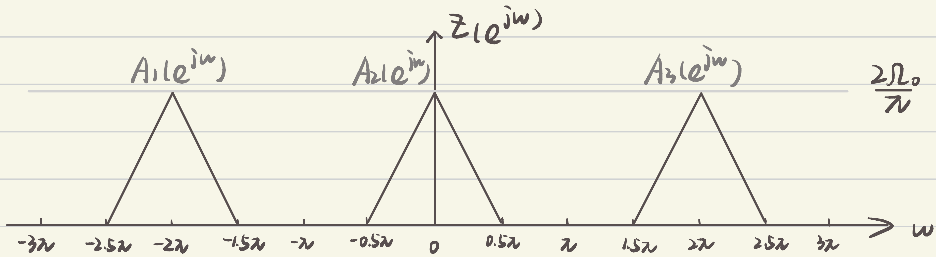
(c) i. $\because 2\Omega_{m_2} = 2\Omega_0$

$$f_{Nyq_1} = \frac{2\Omega_0}{\pi}, \quad T_1 = \frac{1}{f_{Nyq_1}} = \frac{\pi}{2\Omega_0}$$

$$\therefore Z(e^{jw}) = \frac{1}{T_1} \sum_{k=-\infty}^{+\infty} Y(j\Omega)\Big|_{\Omega = \frac{w+2\pi k}{T_1}}$$

when $k=0$: $\Omega = \frac{w}{T_2} \Rightarrow w = \Omega T_2 = \frac{\Omega\pi}{2\Omega_0}$

$$\left(\begin{array}{l} \therefore \Omega = \Omega_0 \rightarrow w = 0.5\pi \\ Y(j\Omega) = 1 \longrightarrow Z(j\Omega) = \frac{2\Omega_0}{\pi} \end{array} \right)$$



ii.
$$A_2(e^{jw}) = \begin{cases} \frac{4\Omega_0}{\pi^2}\left(\frac{1}{2}\pi - |w|\right), & |w| \le \frac{1}{2}\pi \\ 0, & \text{else} \end{cases} \rightleftharpoons a_2[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$A_1(e^{jw}) = A_2(e^{j(w+2\pi)}) \rightleftharpoons a_1[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2 e^{-j2\pi n}, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$A_3(e^{jw}) = A_2(e^{j(w-2\pi)}) \Longleftrightarrow a_3[n] = \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2 e^{j2\pi n}, & n \neq 0 \\ \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$\therefore z[n] = a_1[n] + a_2[n] + a_3[n]$$

$$= \begin{cases} \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2 + \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2 e^{-j2\pi n} + \frac{8\Omega_0}{\pi}\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2 e^{j2\pi n}, & n \neq 0 \\ \frac{\Omega_0}{2\pi} + \frac{\Omega_0}{2\pi} + \frac{\Omega_0}{2\pi}, & n = 0 \end{cases}$$

$$= \begin{cases} \left(\frac{8\Omega_0}{\pi} + \frac{16\Omega_0}{\pi}\cos(2\pi n)\right)\left(\frac{\sin(\frac{\pi n}{4})}{\pi n}\right)^2, & n \neq 0 \\ \frac{3\Omega_0}{2\pi}, & n = 0 \end{cases}$$

iii $$e = \frac{\sum_{n=-\infty}^{\infty} |x[n] - z[n]|^2}{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

$$= \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw}) - Z(e^{jw})|^2 dw}{\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw}$$

$\because$ $z[n]$ only reserve the low frequency part of $x[n]$

$\therefore$ $x[n] - z[n]$ is the high frequency part (which is filted)

$\therefore$ $Z(e^{jw}) = H(e^{jw}) X(e^{jw})$.

$H(e^{jw})$ is an ideal analog low pass filter

$$H(e^{jw}) = \begin{cases} 1, & |w| \leq 0.5\pi \\ 0, & |w| > 0.5\pi \end{cases}$$

$$\therefore X(e^{jw}) - Z(e^{jw}) = (1 - H(e^{jw}))X(e^{jw})$$

$$\therefore e = \frac{\int_{|w|>0.5\pi} |X(e^{jw})|^2 dw}{\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw}$$

$$= \frac{\int_{\Omega_0}^{2\Omega_0} |X(j\Omega)|^2 d\Omega}{\int_{0}^{2\Omega_0} |X(j\Omega)|^2 d\Omega}$$

(d) (**5pts**) Consider the normalized error $\hat{e}$ defined as follows:

$$\hat{e} = \frac{\sum_{n=-\infty}^{\infty} |x[n] - y[n]|^2}{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

Do you think $\hat{e}$ is a good measure of information loss due to low-pass filtering over e in part (c)? Briefly explain. You don't have to compute $\hat{e}$. A qualitative explanation is sufficient.

e is a better measure of information loss due to low-pass filtering because it uses the same highest frequency for CTFT, directly capturing the energy lost from the removed high-frequency components.

In contrast, $\hat{e}$ is affected by the change in highest frequency, making it less reliable for isolating the impact of filtering alone.

2. **Sub-Nyquist Sampling** (36 pts) The goal of this problem is to obtain an alternative sampling strategy for a bandlimited signal $x(t)$ when we have additional knowledge about its frequency content. Consider a continuous time signal $x(t)$ with CTFT $X(j\Omega)$ (see Figure 2). Note that $X(j\Omega)$ is zero outside the interval $[\Omega_0 - B/2, \Omega_0 + B/2]$, where $\Omega_0 = 4000\pi$ and $0 < B \leq 8000\pi$ is a positive real value.

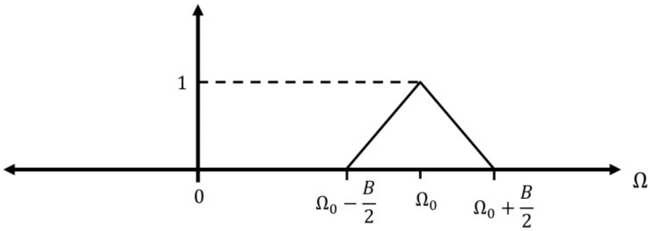


Figure 2: Plot of $X(j\Omega)$ for Problem 2.

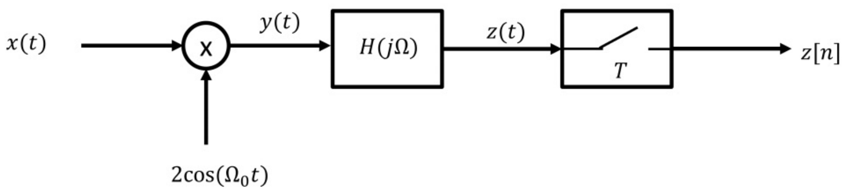


Figure 3: Modulated sub-Nyquist sampling

(a) (6pt) What is the value of the maximum frequency $f_{\max}$ present in $X(j\Omega)$?, what is the Nyquist rate?. Express your answer as a function of B.

(a) $\Omega_m = \Omega_0 + \frac{B}{2}$

$$\therefore f_{max} = \frac{\Omega_m}{2\pi} = \frac{\Omega_0 + \frac{B}{2}}{2\pi}, \quad f_{Nyq} = \frac{2\Omega_m}{2\pi} = \frac{\Omega_0 + \frac{B}{2}}{\pi}$$

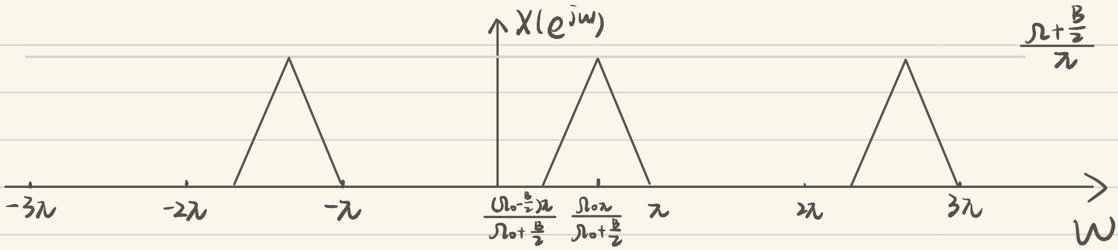
(b) (6pt) Let $x[n]$ be the sampled version of $x(t)$ at the Nyquist rate. Plot the DTFT of $x[n]$ for frequencies in the interval $[-3\pi, 3\pi]$ and label all the important frequencies and scaling factors.

(b) $T = \frac{1}{f_{Nyq}} = \frac{\pi}{\Omega_0 + \frac{B}{2}}$

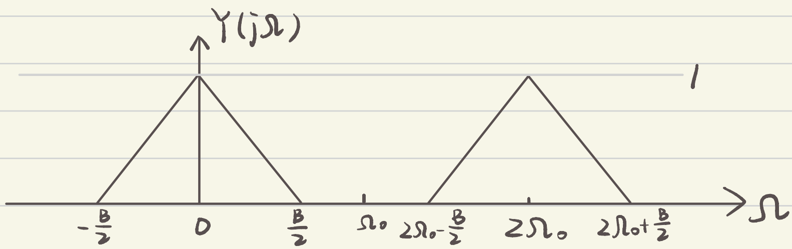
$$X(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j\Omega) \Big|_{\Omega = \frac{w + 2\pi k}{T}}$$

when $k=0$, $\Omega = \frac{w}{T} \Rightarrow w = \Omega T = \frac{\Omega\pi}{\Omega_0 + \frac{B}{2}}$

$$\left(\begin{array}{l} \therefore \Omega = \Omega_0 \rightarrow w = \Omega_0\pi / (\Omega_0 + \frac{B}{2}) ,\ \Omega = \Omega_0 + \frac{B}{2} \longrightarrow w = \pi \\ \Omega = \Omega_0 - \frac{B}{2} \rightarrow w = (\Omega_0 - \frac{B}{2})\pi / (\Omega_0 + \frac{B}{2}) \\ X(j\Omega) = 1 \longrightarrow X(e^{jw}) = \frac{\Omega + \frac{B}{2}}{\pi} \end{array} \right)$$



(c) (8pt) Consider the system from Figure 3. The signal $y(t) = 2\cos(\Omega_0 t)x(t)$ has CTFT $Y(j\Omega) = X(j(\Omega - \Omega_0)) + X(j(\Omega + \Omega_0))$. Plot $Y(j\Omega)$ labeling all important frequencies and scaling factors. Derive a precise condition for B so that $Y(j\Omega)$ preserves the shape of $X(j(\Omega)$. Hint: think about what would happen if the triangle is thin (B is small), or if the triangle is wide (B is big).



if $Y(j\Omega)$ preserves the shape of $X(j\Omega)$

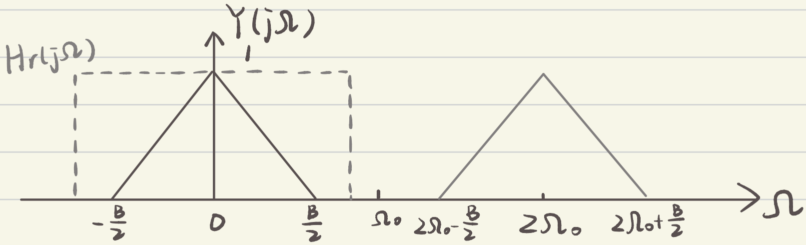
$X(j(\Omega - \Omega_0))$ and $X(j(\Omega + \Omega_0))$ must not have overlap

$\therefore 2\Omega_0 - \frac{B}{2} \geq \frac{B}{2}$ and $\Omega_0 = 4000\pi$

$\therefore 0 < B \leq 8000\pi$

(d) (8pt)When the shape of $X(j\Omega)$ is preserved by $Y(j\Omega)$ (assuming the conditions from the previous section are true), design an ideal analog low pass filter $H(j\Omega)$ (indicate its gain and cut-off frequency), so that $z(t)$ is bandlimited and $Z(j\Omega)$ preserves the shape of $X(j\Omega)$.

After low-pass filter $H(j\Omega)$, we hope to get $z(t)$:

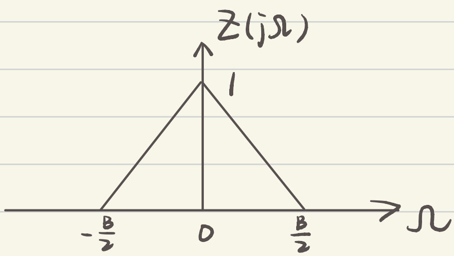


filt the high frequency signal

$\therefore \frac{B}{2} \leq \Omega_c < 2\Omega_0 - \frac{B}{2}$

$\therefore$ one idea filter : $H(j\Omega) = \begin{cases} 1, & |\Omega| \leq \frac{B}{2} \\ 0, & \text{else} \end{cases}$

$\therefore$



(e) (8pt) What is the value of the maximum frequency of $z(t)$ and what is the Nyquist rate?, how does this sampling rate relate to the Nyquist rate of part (a)?. Let $z[n]$ be the sampled version of $z(t)$ at its Nyquist rate. Plot the DTFT of $z[n]$, make sure you label all important frequencies and include at least 2 periods in your plot.

$\Omega_m' = \frac{B}{2}$

$\therefore f_{max}' = \frac{\Omega_m'}{2\pi} = \frac{B}{4\pi}$, $f_{Nyq}' = \frac{2\Omega_m'}{2\pi} = \frac{B}{2\pi} << 4000 + \frac{B}{2\pi} = f_{Nyq}$

this Nyquist rate is much smaller than the rate of part (a)

$\therefore T' = \frac{1}{f_{Nyq}'} = \frac{2\pi}{B}$

$\therefore Z(e^{jw}) = \frac{1}{T'} \sum_{k=-\infty}^{+\infty} Z(j\Omega) \Big|_{\Omega = \frac{w+2\pi k}{T'}}$

when $k=0$, $w = \Omega T' = 2\pi\Omega/B$

$\left(\begin{array}{l} \therefore \Omega = \frac{B}{2} \longrightarrow w = \pi \\ Z(j\Omega) = 1 \longrightarrow Z(e^{jw}) = \frac{B}{2\pi} \end{array} \right)$