HW08 — Q1

1 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

Book Set 6.3 #1

$\because \frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u = Au$

$\therefore Ax = \lambda x$ give the solutions $u = e^{\lambda t} x$

$\because P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)$

$\therefore$ Let $P_A(\lambda) = 0$, $\lambda_1 = 4$, $\lambda_2 = 1$

① For $\lambda_1 = 4$: $(A - \lambda I)x = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ $x_1 \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $x_1 = 1$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② For $\lambda_2 = 1$: $(A - \lambda I)x = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ $x_1 \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $x_1 = 1$, $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore u(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\because u(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

$\therefore \begin{cases} c_1 + c_2 = 5 \\ -c_2 = -2 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}$

$\therefore u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

Book Set 6.3 #4

$\because \frac{dv}{dt} = w - v \qquad \frac{dw}{dt} = v - w$

$\therefore \frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$

$\therefore \frac{du}{dt} = Au = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u$

$\because P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - 1 = \lambda^2 + 2\lambda$

$\therefore$ Let $P_A(\lambda) = 0$, $\lambda_1 = 0$, $\lambda_2 = -2$

① For $\lambda_1 = 0$, $(A - \lambda I)x = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$ $x_1 \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $x_1 = 1$, $e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② For $\lambda_2 = -2$, $(A - \lambda I)x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ $x_1 \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $x_1 = 1$, $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore u(t) = c_1 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\because u(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 20 \\ c_2 = 10 \end{cases}$$

$$\therefore u(t) = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

when $t=1$: $u(1) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + 10 e^{-2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 20+10e^{-2} \\ 20-10e^{-2} \end{bmatrix} \Rightarrow \begin{matrix} v = 20+10e^{-2} \\ w = 20-10e^{-2} \end{matrix}$

when $t=\infty$: $u(\infty) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \Rightarrow v=20 , w=20$

Questions 10–13 reduce second-order equations to first-order systems for (y, y').

10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

Book Set 6.3 #10

① $\because y'' - 5y' - 4y = 0$

$\therefore (\lambda^2 - 5\lambda - 4)e^{\lambda t} = 0$

$\therefore \frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 5y' + 4y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$

$\therefore P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = -\lambda(5-\lambda) - 4 = \lambda^2 - 5\lambda - 4$

$\therefore$ let $P_A(\lambda) = 0$, $\lambda_1 = \frac{5+\sqrt{41}}{2}$, $\lambda_2 = \frac{5-\sqrt{41}}{2}$

② suppose $y = e^{\lambda t}$

$\therefore y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$

$\therefore \lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}$

$\therefore \lambda^2 = 5\lambda + 4$

$\therefore \lambda^2 - 5\lambda - 4 = 0$

$\therefore \lambda_1 = \frac{5+\sqrt{41}}{2}$, $\lambda_2 = \frac{5-\sqrt{41}}{2}$

HW08 — Q2

4 If C is symmetric prove that $A^{\mathrm{T}}CA$ is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and $A^{\mathrm{T}}CA$?

Book Set 6.4 # 4

① $(A^TCA)^T = (A^T(CA))^T = (CA)^T(A^T)^T = A^TC^TA$

$\because C$ is symmetric

$\therefore C = C^T$

$\therefore (A^TCA)^T = A^TC^TA = A^TCA$

$\therefore A^TCA$ is symmetric

② if A is 6 by 3

$\therefore A^T$ is 3 by 6

$\because A^TCA$

$\therefore$ # of rows of C = # of columns of $A^T = 6$

# of columns of C = # of rows of $A = 6$

$\therefore C$ is 6 by 6

$\therefore A^T_{3\times6}\, C_{6\times6}\, A_{6\times3}$ is 3 by 3

5 Find the eigenvalues and the unit eigenvectors of

$$S = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Book Set 6.4 #5

$$P_B(\lambda) = \det(S-\lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ -\lambda & 0 \end{vmatrix} - \lambda \begin{vmatrix} 2-\lambda & 2 \\ 2 & -\lambda \end{vmatrix}$$

$$= 2(2\lambda) - \lambda((2-\lambda)(-\lambda)-4)$$

$$= \lambda(4-(\lambda^2-2\lambda-4))$$

$$= -\lambda(\lambda^2-2\lambda-8)$$

$$= -\lambda(\lambda-4)(\lambda+2)$$

$\therefore$ Let $P_B(\lambda)=0$, $\lambda_1=0$, $\lambda_2=4$, $\lambda_3=-2$

① For $\lambda_1=0$: $(A-\lambda I)x = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} x = 0$

$\therefore x = \begin{bmatrix} 0 \\ z \\ -z \end{bmatrix}$ $z \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $z=1$, $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

② For $\lambda_2=4$: $(A-\lambda I)x = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} x = 0$

$\because \begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 0 \\ 0 & 4 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\therefore x = \begin{bmatrix} 2z \\ z \\ z \end{bmatrix} \quad z \in \mathbb{C} \ \& \neq 0$

$\therefore$ let $z = 1$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

③ For $\lambda_3 = -2$: $(A - \lambda I)x = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} x = 0$

$\because \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\therefore x = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} \quad z \in \mathbb{C} \ \& \neq 0$

$\therefore$ let $z = -1$, $v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow e_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

7 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

Book Set 6.4 #7

$$P_S(\lambda) = \det(S-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ -2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & -1-\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (1-\lambda)(\lambda(1+\lambda)-4) + 2\cdot 2(1+\lambda)$$

$$= (1-\lambda)(\lambda^2+\lambda-4) + 4(1+\lambda)$$

$$= \cancel{\lambda^2}+\lambda\cancel{-4}-\lambda^3\cancel{-\lambda^2}+4\lambda\cancel{+4}+4\lambda$$

$$= -\lambda^3+9\lambda$$

$$= \lambda(9-\lambda^2)$$

$\therefore$ Let $P_S(\lambda)=0$, $\lambda_1=0$, $\lambda_2=3$, $\lambda_3=-3$

① For $\lambda_1=0$: $(S-\lambda I)x = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} x = 0$

$$\because \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x = \begin{bmatrix} -2z \\ -2z \\ z \end{bmatrix}$ $z \in \mathbb{C}$ & $\neq 0$

$\therefore$ Let $z=-1$, $v_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \Rightarrow e_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

② For $\lambda_2=3$: $(S-\lambda I)x = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} x = 0$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} z \\ -\frac{1}{2}z \\ z \end{bmatrix} \quad z \in \mathbb{C} \ \& \ z \neq 0$$

$$\therefore \text{let } z = 2, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \Rightarrow e_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

③ For $\lambda_3 = -3$, $(S - \lambda I)x = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} x = 0$

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} -\frac{1}{2}z \\ z \\ z \end{bmatrix} \quad z \in \mathbb{C} \ \& \neq 0$$

$$\therefore \text{let } z = -2, \quad v_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \Rightarrow e_3 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$\therefore Q = [e_1 \ e_2 \ e_3] = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow S = Q \Lambda Q^T$$

13 Write S and B in the form $\lambda_1 \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1).$$

Book Set 6.4 #13

① $P_S(\lambda) = \det \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$

$\therefore$ Let $P_S(\lambda) = 0$, $\lambda_1 = 2$, $\lambda_2 = 4$

For $\lambda_1 = 2$: $(S - \lambda I)x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 4$: $(S - \lambda I)x = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = 0 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$

$= 2 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} [1, -1] + 4 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} [1, 1]$

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$= S$

② $P_B(\lambda) = \det \begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix} = (9-\lambda)(16-\lambda) - 12^2 = \lambda(\lambda - 25)$

$\therefore$ Let $P_B(\lambda) = 0$, $\lambda_1 = 0$, $\lambda_2 = 25$

For $\lambda_1=0$: $(B-\lambda I)x=\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix}x=0 \Rightarrow v_1=\begin{bmatrix}4\\-3\end{bmatrix} \Rightarrow x_1=\frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}$

For $\lambda_2=25$: $(B-\lambda I)x=\begin{bmatrix}-16 & 12\\12 & -9\end{bmatrix}x=0 \Rightarrow v_2=\begin{bmatrix}3\\4\end{bmatrix} \Rightarrow x_2=\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$

$\therefore \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$

$= 0\cdot\frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}\cdot\frac{1}{5}[4,-3] + 25\cdot\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\cdot\frac{1}{5}[3,4]$

$= \begin{bmatrix}9 & 12\\12 & 16\end{bmatrix}$

$= B$

HW08 — Q3

8 Which classes of matrices does P belong to: invertible, Hermitian, unitary?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

Book Set 9.2 #8

① $\because P^H = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$

$$P^H P = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore P^H = P^{-1}$

$\therefore P$ is unitary Matrices (invertible)

② $P^2 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$$P^3 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} = (-i)I$$

$$P^4 = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (-i)P$$

$$P^5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix} = (-i)P^2$$

$$P^6 = \begin{bmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-i)^2 I$$

$$P^7 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{bmatrix} = (-i)^2 P$$

$\therefore 100 = 3 \times 33 + 1$

$$\therefore P^{100} = (-i)^{33} P = -iP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

③ $$P_P(\lambda) = \det(P - \lambda I) = \begin{vmatrix} -\lambda & i & 0 \\ 0 & -\lambda & i \\ i & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & i \\ 0 & -\lambda \end{vmatrix} - i \begin{vmatrix} 0 & i \\ i & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 0) - i(0 - i^2)$$

$$= -\lambda^3 - i$$

Let $P_P(\lambda) = 0$, $\lambda^3 = -i = e^{-\frac{\pi}{2}i}$

$\therefore \lambda = \sqrt[3]{e^{-\frac{\pi}{2}i}} = e^{-\frac{\pi}{6}i + \frac{2k\pi}{3}i}$ $\quad k = 0, 1, 2$

$\therefore k = 0: \ \lambda_1 = e^{-\frac{\pi}{6}i}$

$k = 1: \ \lambda_2 = e^{-\frac{\pi}{6}i + \frac{2\pi}{3}i} = e^{\frac{\pi}{2}i} = i$

$k = 2: \ \lambda_3 = e^{-\frac{\pi}{6}i + \frac{4\pi}{3}i} = e^{\frac{7\pi}{6}i}$

9 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix Q. What property of P makes these eigenvectors orthogonal?

Book Set 9.2 #9

$\because \lambda_1 = e^{-\frac{\pi}{6}i}$, $\lambda_2 = i$, $\lambda_3 = e^{-\frac{5\pi}{6}i}$

$\therefore$ For $\lambda_1 = e^{-\frac{\pi}{6}i}$: $(P-\lambda I)x = \begin{bmatrix} -e^{-\frac{\pi}{6}i} & i & 0 \\ 0 & -e^{-\frac{\pi}{6}i} & i \\ i & 0 & -e^{-\frac{\pi}{6}i} \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ e^{\frac{4\pi}{3}i} \\ e^{\frac{8\pi}{3}i} \end{bmatrix}$

For $\lambda_2 = i$: $(P-\lambda I)x = \begin{bmatrix} -i & i & 0 \\ 0 & -i & i \\ i & 0 & -i \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_3 = e^{-\frac{5\pi}{6}i}$: $(P-\lambda I)x = \begin{bmatrix} -e^{-\frac{5\pi}{6}i} & i & 0 \\ 0 & -e^{-\frac{5\pi}{6}i} & i \\ i & 0 & -e^{-\frac{5\pi}{6}i} \end{bmatrix} x = 0 \Rightarrow x_3 = \begin{bmatrix} 1 \\ e^{\frac{2\pi}{3}i} \\ e^{\frac{4\pi}{3}i} \end{bmatrix}$

$\therefore Q = [x_1, x_2, x_3] = \begin{bmatrix} 1 & 1 & 1 \\ w^2 & 1 & w \\ w^4 & 1 & w^2 \end{bmatrix}$, $w = e^{\frac{2\pi}{3}i}$

is a Fourier matrix F_3

$\therefore$ The eigenvectors of any unitary matrix (P) are orthogonal

11 If Q and U are unitary matrices, show that Q^{-1} is unitary and also QU is unitary. Start from $Q^{\mathrm{H}}Q = I$ and $U^{\mathrm{H}}U = I$.

Book Set 9.2 #11

① $\because$ Q is unitary matrix

$\therefore$ $Q^H Q = I$

$\therefore$ $(Q^{-1})^H Q^{-1} = (Q^H)^{-1} Q^{-1} = (Q Q^H)^{-1} = (Q^H Q)^{-1} = I$

$\therefore$ Q^{-1} is unitary matrix

② $\because$ U is unitary matrix

$\therefore$ $U^H U = I$

$\therefore$ $(QU)^H QU = U^H Q^H Q U = U^H I U = U^H U = I$

$\therefore$ QU is unitry matrix

16 Diagonalize this orthogonal matrix to reach $U = Q\Lambda Q^{H}$. Now all λ's are _____:

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Book Set 9.2 #16

$\because P_U(\lambda) = \det(U-\lambda I) = \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = (\cos\theta-\lambda)^2 + \sin^2\theta$

$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1$

$\therefore$ Let $P_U(\lambda)=0$, $\lambda_1 = \cos\theta + i\sin\theta$, $\lambda_2 = \cos\theta - i\sin\theta$

For $\lambda_1 = \cos\theta + i\sin\theta$: $(U-\lambda I)x = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

For $\lambda_2 = \cos\theta - i\sin\theta$: $(U-\lambda I)x = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\therefore Q = [e_1, e_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ $\quad Q^H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}$

$\therefore U = Q\Lambda Q^H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

$\because \lambda_1 = \cos\theta + i\sin\theta = e^{i\theta}$, $\lambda_2 = \cos\theta - i\sin\theta = e^{-i\theta}$

$\therefore |\lambda_1| = |\lambda_2| = 1$

4. Let A be a unitary matrix such that $A \in \mathbb{C}^{n\times n}$.

a) Show that the absolute value of the determinant $|Det(A)| = 1$.

b) Show that if the eigenvalues of A are distinct then their eigenvectors are orthogonal.

HW08 — Q4

a) $\because$ A is unitary matrix

$\therefore A^H A = I$

$\therefore \det(A^H A) = \det(I) = 1$

$\because \det(A^H A) = \det(A^H)\det(A)$ — $\det(AB) = \det(A)\det(B)$

$= \det(A)^* \det(A)$ — $\det(A^H) = \det(A)^*$

$= \det^2(A)$ — unitary matrix A $\det(A)^* = \det(A)$

$\therefore \det^2(A) = 1$

$\therefore |\det(A)| = 1$

b) Suppose λ_1, λ_2 are two distinct eigenvalues of

$Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$

$\because (Av_1)^H v_2 = v_1^H A^H v_2$

$= v_1^H A v_2$ — A is unitary matrix

$= v_1^H (Av_2)$

$\begin{cases} (Av_1)^H v_2 = (\lambda_1 v_1)^H v_2 = \lambda_1^* v_1^H v_2 \\ v_1^H (Av_2) = v_1^H \lambda_2 v_2 = \lambda_2 v_1^H v_2 \end{cases}$

$\therefore \lambda_1^* v_1^H v_2 = \lambda_2 v_1^H v_2$

$\because \lambda_1, \lambda_2$ are distinct eigenvalues and A is unitary matrix

$\therefore \lambda_1^* \neq \lambda_2$

$\therefore v_1^H v_2 = 0$

$\therefore v_1, v_2$ are orthogonal

QED

5. Find the values of a, b, and c, all nonzero, such that the matrix A below is diagonalizable over $\mathbb{R}$:

$$A = \begin{bmatrix} 2 & a & b \\ 0 & -5 & c \\ 0 & 0 & 2 \end{bmatrix}.$$

HW08 — Q5

$\because P_A(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & a & b \\ 0 & -5-\lambda & c \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda)(2-\lambda)$

$\therefore$ Let $P_A(\lambda)=0$. $\lambda_1=\lambda_2=2$, $\lambda_3=-5$

① For $\lambda_1=\lambda_2=2$: $(A-\lambda I)x = \begin{bmatrix} 0 & a & b \\ 0 & -7 & c \\ 0 & 0 & 0 \end{bmatrix} x = 0$

$$\begin{bmatrix} 0 & a & b \\ 0 & -7 & c \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a & b \\ 0 & 0 & c+\frac{7b}{a} \\ 0 & 0 & 0 \end{bmatrix}$$

if A is diagonalizable, all eigenvactors of A are independent

$\therefore$ when $\lambda_1=\lambda_2=2$, A must have 2 independent eigenvactors

$\therefore \begin{cases} a \neq 0 \\ b \neq 0 \\ c+\frac{7b}{a}=0 \end{cases} \Rightarrow c = -\frac{7b}{a} \Rightarrow ac+7b=0$

② For $\lambda_3=-5$: $(A-\lambda I)x = \begin{bmatrix} 7 & a & b \\ 0 & 0 & c \\ 0 & 0 & 7 \end{bmatrix} x = 0$

$$\begin{bmatrix} 7 & a & b \\ 0 & 0 & c \\ 0 & 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 7 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} 7x_1 + ax_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow x_1 = -\frac{a}{7}x_2$$

$\therefore x_2 = \begin{bmatrix} -\frac{a}{7} \\ 1 \\ 0 \end{bmatrix}$

$\therefore ac+7b=0$

(e.g. $a=1$, $b=-1$, $c=7$)

6. If $A \in \mathbb{C}^{m \times n}$, prove or disprove that the eigenvalues of $A^H A$ are nonnegative.

HW08 — Q6

Proof: Suppose eigenvalue of $A^H A$ is λ, eigenvactor is v

$\therefore A^H A v = \lambda v$

$\because v^H (A^H A) v = v^H A^H A v = (Av)^H Av = \|Av\|^2 \geqslant 0$

$\therefore 0 \leq v^H (A^H A) v = v^H (A^H A v)$

$= v^H \lambda v$

$= \lambda v^H v$

$= \lambda \|v\|^2$

$\because \|v\|^2 \geqslant 0$

$\therefore \lambda \geqslant 0$ — nonnegative

QED