Assigned: 17 September

Homework #3

EE 503: Fall 2024

Instructions: Write your solutions to these homework problems. Submit your work to Brightspace by the due date. Show all work and box answers where appropriate. Do not guess.

Due: Tuesday, 24 September at 12:00.

1. A function $f : X \to Y$ is *onto* (or *surjective*) iff for each "image" element $y \in Y$ there is a "pre-image" element $x \in X$ such that $y = f(x)$. A function f is *one-to-one* (or *injective*) iff distinct pre-images have distinct images: $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ for all $x_1 \in X$ and all $x_2 \in X$. Note that the contrapositive of the last statement states that $f(x_1) = f(x_2)$ only if $x_1 = x_2$ for all $x_1 \in X$ and all $x_2 \in X$. A function is *bijective* iff it is both injective and surjective (precisely when the inverse *point function* f^{-1} exists). Suppose $A \subset X$ and $B \subset Y$ for $f : X \to Y$. Then prove or disprove:
   (a) $f\left(f^{-1}\{B\}\right) \subset B$.
   (b) $f\left(f^{-1}\{B\}\right) = B$ if f is surjective.
   (c) $A \subset f^{-1}\{f(A)\}$.
   (d) $A = f^{-1}\{f(A)\}$ if f is injective.
   (e) $f : X \to Y$ is bijective implies $f : 2^X \to 2^Y$ is bijective.

2. Use the ratio test to determine whether the following infinite series diverge or converge:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$. (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$. (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$. (d) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$.

3. Find the interval of convergence for these power series (check both endpoints):
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{8^n}(x-3)^n$. (b) $\displaystyle\sum_{n=1}^{\infty} n!\left(\frac{x}{2}\right)^n$. (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$.

4. Use the ϵ-definition (*i.e., garden hose*) to evaluate the limit of these sequences. Given $\epsilon = 10^{-6}$ what is the smallest index n_0 such that $|a_n - L| < \epsilon$ for all $n \geq n_0$?
   (a) $\sqrt{n+1} - \sqrt{n}$. (b) $2^{-n}\cos(n\pi)$. (c) $\left(1 + \frac{2}{n}\right)^n$.

5. Let $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$. Suppose $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Then what is the Cartesian product $A \times B$? How many elements in $2^{A\times B}$? Produce four sub-collections $\mathcal{A} \subset 2^{A\times B}$ that are sigma-algebras.

6. Prove or disprove:
   (a) If $A \subset X$ and $B \subset Y$ then $A \times B \subset X \times Y$.
   (b) $(A \cup B) \times C = (A \cup C) \times (B \cup C)$.