CHAPTER

7 Constitutive relations and formulation of theories involving multiple constitutive fields

7.1 INTRODUCTION

While there is a wide variety of material responses beyond the classical displacement–strain–stress field theories, we will only focus here on three of the more common theoretical models. Our additional fields will include *thermal*, *pore-fluid*, and *electrical effects*, and we will limit the mechanical stress–strain response to small deformation elasticity. This greatly simplifies the presentation and allows us to easily develop the mathematical model and explore some solutions to actual problems. Of course many more theories have been established in the literature that would include a wide variety of additional material fields with more general stress–strain responses including finite deformation.

So, we will first investigate continuum theories for *thermoelastic solids* where our previous linear elastic model described in Section 6.2 will now be coupled with thermal fields of temperature, internal energy, heat flux, and entropy. This will increase the number of unknowns in the problem formulation and thus require development of additional thermodynamic constitutive relations in order to produce closure of the system equations. This theory is very useful in stress analysis of structures and machine parts that undergo significant temperature changes from ambient. If the temperature variation is sufficiently high, the stresses can reach levels that may lead to structural failure, especially for brittle materials.

Our second multiple field theory will be *poroelasticity*, which is an elastic continuum model of a fluid-saturated porous medium. This theory will then bring in additional concepts of porosity and fluid flow, thus requiring additional relations to determine all unknown problem variables. This continuum model has many applications for the response of soil, rock, ceramics, and biological materials.

Finally, we will explore *electro-elasticity or piezoelectric materials* whereby the mechanical linear elasticity fields are coupled with an electric charge field. Again new unknown variables will be created and thus additional electrical constitutive relations will be necessary. Many applications of this theory are found in sensors, actuators, and smart materials.

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7.2 THERMOELASTIC SOLIDS

Thermal effects within an elastic solid produce heat transfer via conduction, and this flow of thermal energy establishes a temperature field within the material. Most solids exhibit a deformation change with temperature variation, and thus the presence of a temperature distribution will generally induce stresses created from boundary or internal constraints. The continuum model will now require the energy equation, and with this coupling the unknowns of internal energy, heat flow, and temperature will now be present. Several new thermal constitutive laws will be needed to relate these new variables with various model unknowns. This will require some additional discussions on a few thermodynamic concepts that we have previously avoided. Starting with general elastic materials, we will develop the basic governing thermoelastic equations for isotropic materials and will investigate a few solutions to problems of engineering interest. As usual, our presentation will be brief, and more detailed information may be found in several studies devoted entirely to the subject such as Boley and Weiner (1960), and Nowinski (1978) and (1986), Kovalenko (1969).

Going back to our discussion on thermodynamics in Sections 5.6 and 5.7, the sum of the internal energy ε and irreversible heat energy $-\theta s$ is known as the *Helmholtz free energy density* Ψ:

$$\Psi = \varepsilon - s\theta \tag{7.2.1}$$

For elastic materials, we can assume that the free energy is only a function of the strain and temperature, $\Psi = \Psi(\varepsilon_{ij}, \theta)$. Note that we expect that the internal energy and entropy would also have the same functional form. Under this condition, the time rate of change of Ψ can be expressed using the chain rule

$$\dot{\Psi} = \left(\frac{\partial \Psi}{\partial \varepsilon_{ij}}\right)_{\theta} \dot{\varepsilon}_{ij} + \left(\frac{\partial \Psi}{\partial \theta}\right)_{\varepsilon_{ij}} \dot{\theta} \tag{7.2.2}$$

The energy equation (5.6.12) can be expressed by

$$\rho\dot{\varepsilon} = \rho(\dot{\Psi} + \dot{s}\theta + s\dot{\theta}) = T_{ij}\dot{\varepsilon}_{ij} - q_{i,i} + \rho h \tag{7.2.3}$$

Solving for $\dot{\Psi}$, we obtain

$$\rho\dot{\Psi} = T_{ij}\dot{\varepsilon}_{ij} - q_{i,i} + \rho h - \rho(\dot{s}\theta + s\dot{\theta}) \tag{7.2.4}$$

Substituting (7.2.2) into (7.2.4) gives

$$\left(\rho\frac{\partial \Psi}{\partial \varepsilon_{ij}} - T_{ij}\right)\dot{\varepsilon}_{ij} + \rho\left(\frac{\partial \Psi}{\partial \theta} + s\right)\dot{\theta} + q_{i,i} - \rho(h - \dot{s}\theta) = 0 \tag{7.2.5}$$

In order for relation (7.25) to be true for all $\dot{\varepsilon}_{ij}$ and $\dot{\theta}$, the following must be true:

$$\begin{aligned} T_{ij} &= \rho\frac{\partial \Psi}{\partial \varepsilon_{ij}} \\ s &= -\frac{\partial \Psi}{\partial \theta} \\ q_{i,i} &= \rho(h - \dot{s}\theta \end{aligned} \tag{7.2.6}$$

Eqs. (7.2.6)$_{1,2}$ are often referred to as the *thermodynamic potential relations*, and (7.2.6)$_1$ should be compared with the nonthermal hyperelastic equation (6.2.6) $T_{ij} = \partial U / \partial \varepsilon_{ij}$. Also, as previously discussed in Section 6.2, for linear elasticity, the mass density is a constant.

The free energy is a thermodynamic potential that is a function of the strain and temperature fields. We can define Ψ_o as *the natural or reference state* with zero strain when the temperature is at θ_o. Using small strain theory and small temperature differences from reference $\Theta = \theta - \theta_o$, we can express the free energy potential per unit volume as a power series in the two variables ε_{ij} and Θ:

$$\rho\Psi(\varepsilon,\Theta) = \rho\Psi_o + \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \beta_{ij}\varepsilon_{ij}\Theta - \frac{c_v}{2\theta_o}\Theta^2 \tag{7.2.7}$$

where C_{ijkl} is a general fourth-order tensor (later shown to be the elasticity tensor) characterizing the mechanical properties of the material, β_{ij} are material constants which account for the coupling of the strain and temperature fields, and c_v is the thermal constant called the *specific heat per unit volume at constant strain*. Note that terms linear in the strain and temperature have been dropped, since they would violate the minimum requirement of the free energy in the natural state.

Using the constitutive from (7.2.7) with properties (7.2.6)$_{1,2}$ yields

$$\begin{aligned} T_{ij} &= C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Theta \\ s &= \beta_{ij}\varepsilon_{ij} + \frac{c_v}{\theta_o}\Theta \end{aligned} \tag{7.2.8}$$

Constitutive law (7.2.8)$_1$ is known as the *Duhamel–Neumann relation* for linear thermoelastic solids, while (7.2.8)$_2$ defines entropy. It can be seen that in the natural state both the stresses and entropy vanish.

Heat transfer in solids normally occurs by means of *conduction*, transferring heat from regions with higher temperature to regions of lower temperature. This process is spontaneous and irreversible and is connected with an increase in entropy. The temperature gradient is thus taken to be a thermodynamic force which causes heat to flow, and thus there must exist a conduction constitutive law which relates the heat flux vector to the temperature gradient. This relation is known as *Fourier's Law* and is given by

$$q_i = -k_{ij}\Theta_{,j}, \quad \boldsymbol{q} = -\boldsymbol{k}\nabla\Theta \tag{7.2.9}$$

where k_{ij} is the *conductivity tensor*. The minus sign is used to indicate heat flow from hot to cold. Using this conduction relation, we can write

$$q_{i,i} - q_{j,j} = 0 \Rightarrow -(k_{ij} - k_{ji})\Theta_{,ij} \Rightarrow k_{ij} = k_{ji} \tag{7.2.10}$$

and thus the conductivity tensor is symmetric. Furthermore, using the relation (5.7.12), we obtain

$$\boldsymbol{q}\cdot\nabla\Theta \le 0 \Rightarrow k_{ij}\Theta_{,i}\Theta_{,j} \ge 0 \tag{7.2.11}$$

and so the conductivity tensor is also positive definite (see Exercise 2.20). This property indicates that every principal minor of $\det(k_{ij})$ is greater than zero, which leads to the result

$$k_{\underline{i}\underline{i}} \ge 0, \quad k_{\underline{i}\underline{i}}k_{\underline{j}\underline{j}} \ge k_{ij}^2 \tag{7.2.12}$$

Using the Fourier conduction law (7.2.9) and $(7.2.8)_2$, in the energy equation $(7.2.6)_3$ gives

$$-k_{ij}\Theta,_{ij} = \rho h - \beta_{ij}\dot{\varepsilon}_{ij}\theta - \frac{c_v}{\theta_o}\dot{\Theta}\theta$$

Linearizing this relation, by letting $\theta = \theta_o$ then produces the *linearized energy equation*

$$k_{ij}\Theta,_{ij} - c_v\dot{\Theta} - \beta_{ij}\theta_o\dot{\varepsilon}_{ij} + \rho h = 0 \tag{7.2.13}$$

The third term $\beta_{ij}\theta_o\dot{\varepsilon}_{ij}$ represents a *coupling term* between the thermal and strain fields. It has been shown that this term is normally quite small and may be neglected for most applications where the strain rates are small (see Boley and Weiner, 1960). Under this assumption, the theory becomes *uncoupled* and the energy equation reduces to the *heat equation*

$$k_{ij}\Theta,_{j} - c_v\dot{\Theta} + \rho h = 0 \tag{7.2.14}$$

which can be solve independently for the temperature distribution.

For the isotropic material case, the various material tensors reduce to

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ k_{ij} &= k_T\delta_{ij}, \quad \beta_{ij} = (3\lambda + 2\mu)\alpha_T\delta_{ij} \end{aligned} \tag{7.2.15}$$

where λ and μ are the Lamé and shear moduli, respectively, α_T is the coefficient of thermal expansion, and k_T is the thermal conductivity constant. So, the isotropic thermoelastic constitutive relations become

$$\begin{aligned} T_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha_T\delta_{ij}\Theta \\ s &= (3\lambda + 2\mu)\alpha_T\varepsilon_{kk} + \frac{c_v}{\theta_o}\Theta \\ q_i &= -k_T\Theta_{,i} \end{aligned} \tag{7.2.16}$$

Hooke's law $(7.2.16)_1$ can easily be inverted to express the strain in terms of stress

$$\varepsilon_{ij} = \frac{1+\nu}{E}T_{ij} - \frac{\nu}{E}T_{kk}\delta_{ij} + \alpha_T\Theta\delta_{ij} \tag{7.2.17}$$

Notice that Hooke's law in this form implies an additive decomposition of mechanical and thermal strains $\varepsilon_{ij} = \varepsilon_{ij}^{mech} + \varepsilon_{ij}^{thermal}$, which is an expected result from a linear theory.

Our thermodynamic presentation here was only for the linear elastic material model. Other presentations have been given for more general materials including finite deformation and electro-magneto-mechanical materials; see Asaro and Lubarda (2006), Holzapfel (2006), and Bechtel and Lowe (2015) for further details.

7.2.1 GENERAL FORMULATION

Some parts of the ensuing presentation will be identical to the isothermal formulation, whereas other results create new terms or equations. We will pay special attention to these new contributions and recognize them in the field equations and boundary conditions. Following our common pattern, we present a general formulation of isotropic uncoupled linear thermoelasticity necessary for problem solution. Collecting the previous constitutive relations and adding in the other usual relations for small deformation theory then give the following list of governing field equations:

$$\text{Strain–displacement relations:} \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{7.2.18}$$

$$\text{Compatibility relations:} \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{7.2.19}$$

$$\text{Equilibrium equations:} \quad T_{ij,j} + F_i = 0, \quad F_i = \rho b_i \tag{7.2.20}$$

$$\text{Energy equation:} \quad k_T \Theta,_{kk} - c_v \dot{\Theta} + \rho h = 0 \tag{7.2.21}$$

$$\text{Constitutive law:} \quad \begin{aligned} T_{ij} &= (\lambda + \mu)\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha_{\mathrm{T}}\Theta\delta_{ij} \\ \varepsilon_{ij} &= \frac{1+\nu}{E}T_{ij} - \frac{\nu}{E}T_{kk}\delta_{ij} + \alpha_{\mathrm{T}}\Theta\delta_{ij} \end{aligned} \tag{7.2.22}$$

These 16 relations make up the fundamental set of field equations for the 16 unknowns u_i, ε_{ij}, T_{ij}, and Θ. All material constants, body forces, and any thermal source terms are assumed to be given *a priori*. Recall that the compatibility equations are used for the stress formulation in which displacements and strains are eliminated. Notice that these equations are very similar to those in the linear elasticity model given by relations (6.2.28)–(6.2.31). Here, we have temperature terms in Hooke's law and the energy equation. For the uncoupled case, the energy equation can be solved for the temperature distribution *independent of the stress field calculations*. Once determined, the temperatures can then be appropriately placed in the remaining field equations. As discussed in Section 6.2 for isothermal elasticity, the basic problem types and boundary condition forms can be categorized in the same way. However, we must also add thermal boundary conditions of temperature or heat flux specification to solve the heat equation for the temperature distribution.

As previously done for the isothermal case, it will prove to be helpful for problem solution to further reduce the general thermoelastic governing field equations to either a *stress* or *displacement formulation*. Details are similar to those outlined in Section 6.2.2, and we only give the final results

$$T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} + E\alpha_{\mathrm{T}}\left(\frac{1}{1-\nu}\Theta_{,kk}\delta_{ij} + \frac{1}{1+\nu}\Theta_{,ij}\right) = -\frac{\nu}{1-\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i} \tag{7.2.23}$$

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} - (3\lambda + 2\mu)\,\alpha_{\mathrm{T}}\Theta_{,i} + F_i = 0 \tag{7.2.24}$$

where (7.2.23) are the compatibility equations in terms of stress and temperature, whereas (7.2.24) are the equilibrium equations in terms of displacement and temperature. It should be noted that in addition to Hooke's law and the energy equation, the temperature field will also be present in any traction boundary conditions since tractions are related to the stresses. Although other general formulation concepts could be presented, we will end this discussion and move on to the solution of a few basic thermoelastic problems.

7.2.2 PROBLEM SOLUTIONS

We now will explore the solution to some thermoelastic boundary value problems for the two-dimensional uncoupled steady-state case. We shall further assume that body forces and any thermal source terms are zero. The basic two-dimensional formulation follows in a similar fashion as done previously for the isothermal problems in Section 6.2.3.

7.2.2.1 Cartesian coordinate formulation

Again we focus only on the two-dimensional (x,y) *plane stress* case with in-plane stresses being functions of in-plane coordinates:

$$T_{xx} = T_{xx}(x,y), \quad T_{yy} = T_{yy}(x,y), \quad T_{xy} = T_{xy}(x,y), \quad T_{zz} = T_{xz} = T_{yz} = 0 \tag{7.2.25}$$

This particular assumption is most applicable to bodies thin in the out-of-plane direction (see Fig. 6.10) and loaded only with in-plane forces. Likewise, the temperature field is also assumed to be two-dimensional $\Theta = \Theta(x,y)$.

Under these conditions, the strains come from Hooke's law

$$\begin{gathered}\varepsilon_x = \frac{1}{E}(T_{xx} - \nu T_{yy}) + \alpha_{\mathrm{T}}\Theta, \quad \varepsilon_y = \frac{1}{E}(T_{yy} - \nu T_{xx}) + \alpha_{\mathrm{T}}\Theta \\ \varepsilon_{xy} = \frac{1+\nu}{E}T_{xy}, \quad \varepsilon_z = -\frac{\nu}{E}(T_{xx} + T_{yy}) + \alpha_{\mathrm{T}}\Theta, \quad \varepsilon_{xz} = \varepsilon_{yz} = 0\end{gathered} \tag{7.2.26}$$

The equilibrium equations reduce to

$$\begin{gathered}\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0 \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} = 0\end{gathered} \tag{7.2.27}$$

while the strain compatibility relations condense to the single relation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{7.2.28}$$

The displacement equilibrium equations (7.2.24) become

$$\begin{gathered}\mu\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E\alpha_{\mathrm{T}}}{1-\nu}\frac{\partial\Theta}{\partial x} = 0 \\ \mu\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E\alpha_{\mathrm{T}}}{1-\nu}\frac{\partial\Theta}{\partial y} = 0\end{gathered} \tag{7.2.29}$$

and the compatibility relations in terms of stress reduce to

$$\nabla^2(T_{xx} + T_{yy}) + E\alpha_{\mathrm{T}}\nabla^2\Theta = 0 \tag{7.2.30}$$

The tractions can be expressed by

$$
\begin{aligned}
t_x^n &= T_{xx}n_x + T_{xy}n_y = \left[\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial x}+\nu\frac{\partial u}{\partial y}\right)-\frac{E\alpha_{\mathrm{T}}}{1-\nu}\Theta\right]n_x + \left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]n_y \\
t_y^n &= T_{xy}n_x + T_{yy}n_y = \left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]n_x + \left[\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)-\frac{E\alpha_{\mathrm{T}}}{1-\nu}\Theta\right]n_y
\end{aligned}
\tag{7.2.31}
$$

For the steady-state case, the energy or heat equation reduces to the Laplace equation

$$
\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}=\nabla^2\Theta=0 \tag{7.2.32}
$$

It is observed that for thermal plane stress, the temperature effect is equivalent to adding an additional body force $-\frac{E\alpha_{\mathrm{T}}}{1-\nu}\nabla\Theta$ to Navier's equations (7.2.29) and adding a traction term $-\frac{E\alpha_{\mathrm{T}}}{1-\nu}\Theta\boldsymbol{n}$ to the applied boundary tractions (7.2.31). This concept can be generalized to three-dimensional theory.

The problem is then formulated in a two-dimensional region R in the x,y-plane. The energy equation (7.2.32) provides the governing equation for the steady temperature field with boundary conditions specifying either temperature or heat flux over the boundary of R. The displacement formulation is specified by relations (7.2.29) with displacement boundary conditions on u and v. The stress formulation includes the three governing equations (7.2.27) and (7.2.30) with appropriate stress or traction boundary conditions. The solution then involves the determination of the temperature and in-plane displacements, strains, and stresses $\{\Theta, u, v, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}, T_{xx}, T_{yy}, T_{xy}\}$ in R. Out-of-plane strain ε_z can be determined from the in-plane stresses using Hooke's law (7.2.26).

Similar to isothermal elasticity, introducing the Airy stress function approach will greatly help in finding analytical solutions to problems of interest. Thus, we again use the representation

$$
T_{xx}=\frac{\partial^2\phi}{\partial y^2},\quad T_{yy}=\frac{\partial^2\phi}{\partial x^2},\quad T_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{7.2.33}
$$

which satisfies the equilibrium equations identically, and the compatibility relation (7.2.30) becomes

$$
\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}+E\alpha_{\mathrm{T}}\nabla^2\Theta=\nabla^4\phi+E\alpha_{\mathrm{T}}\nabla^2\Theta=0 \tag{7.2.34}
$$

Thus, the plane stress problem of linear thermoelasticity has been reduced to a single equation in terms of the Airy stress function, and this will allow solutions to be found for a large variety of problems. Note that this approach can also be used to formulate and solve the other two-dimensional deformation model *plane strain* (see Sadd, 2014).

7.2.2.2 *Polar coordinate formulation*

It is often desirable to formulate the two-dimensional thermoelastic problem in polar coordinates, thus allowing solutions with radial or angular symmetry to be easily solved. Example 2.18.1 and Appendix A provide help to determine many of the needed relations. For the plane stress case,

$$T_{rr} = T_{rr}(r,\theta), \quad T_{\theta\theta} = T_{\theta\theta}(r,\theta), \quad T_{r\theta} = T_{r\theta}(r,\theta), \quad T_{zz} = T_{rz} = T_{\theta z} = 0 \tag{7.2.35}$$

The in-plane displacements in the radial and tangential directions are given by u_r and u_θ, and all field quantities will depend only on the in-plane coordinates r and θ. The strain displacement relations become

$$\begin{gathered} \varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\ \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \end{gathered} \tag{7.2.36}$$

Hooke's law now reads

$$\begin{gathered} T_{rr} = \frac{E}{1-v^2}[\varepsilon_r + v\varepsilon_\theta - (1+v)\,\alpha_{\mathrm{T}}(\Theta - \Theta_{\mathrm{o}})] \\ T_{\theta\theta} = \frac{E}{1-v^2}[\varepsilon_\theta + v\varepsilon_r - (1+v)\,\alpha_T(\Theta - \Theta_{\mathrm{o}})] \\ T_{r\theta} = \frac{E}{1+v}\varepsilon_{r\theta} \end{gathered} \tag{7.2.37}$$

In the absence of body forces, the equilibrium equations reduce to

$$\begin{gathered} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{2T_{r\theta}}{r} = 0 \end{gathered} \tag{7.2.38}$$

The Airy stress function definition now becomes

$$\begin{gathered} T_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ T_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \\ T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \end{gathered} \tag{7.2.39}$$

which again satisfies (7.2.38) identically. The governing stress function equation is still given by

$$\nabla^4 \phi + E\alpha_T \nabla^2 \Theta = 0 \tag{7.2.40}$$

with the Laplacian and biharmonic operators now of the form

$$\begin{gathered} \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \\ \nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \end{gathered} \tag{7.2.41}$$

This now completes our two-dimensional formulation and we now move on to explore several solutions.

EXAMPLE 7.2.1 THERMAL STRESSES IN A RECTANGULAR STRIP

Consider the thermoelastic problem in a rectangular domain as shown in Fig. 7.1. We will assume that the vertical dimension is unbounded and thus the region may be described as an infinite strip of material. For this problem, assume that the temperature is independent of x and given by $\Theta = T_o \sin\beta y$, where T_o and β are constants. Note that $\nabla^2\Theta = 0$ and thus the given temperature already satisfies the heat equation. If the stresses at $x = \pm a$ are to vanish, determine the thermal plane stress field.

Solution: From the governing equation (7.2.34) $\Rightarrow$

$$\nabla^4\phi = E\alpha_T T_o \beta^2 \sin\beta y \tag{7.2.42}$$

A particular solution to this equation is given by $\phi_p = \dfrac{E\alpha_T T_o}{\beta^2}\sin\beta y$. The homogeneous solution may be found using standard separation of variables using $\phi_h = f(x)\sin\beta y$ which gives the auxiliary ordinary differential equation $f'''' - 2\beta^2 f'' + \beta^4 f = 0$ with general solution

$$f = C_1 \sinh\beta x + C_2 \cosh\beta x + C_3 x \sinh\beta x + C_4 x \cosh\beta x$$

where C_i are constants to be determined. Now since the temperature field was symmetric in x, we expect the stresses to also exhibit the same symmetry. Thus, the stress function must also be symmetric in x and so $C_1 = C_4 = 0$. Combining the particular and homogeneous solutions, the resulting stresses

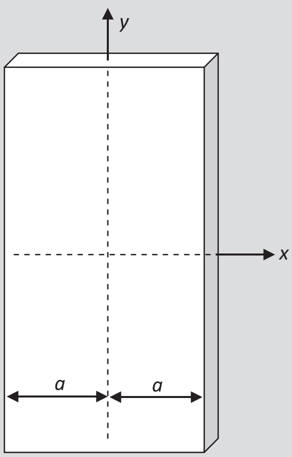

FIGURE 7.1

Thermoelastic rectangular strip problem.

then become

$$T_{xx} = -\beta^2 [C_2 \cosh\beta x + C_3 x \sinh\beta x] \sin\beta y - E\alpha_T T_0 \sin\beta y$$
$$T_{yy} = \beta^2 \left[C_2 \cosh\beta x + C_3 \left(x \sinh\beta x + \frac{2}{\beta} \cosh\beta x \right) \right] \sin\beta y$$
$$T_{xy} = -\beta^2 \left[C_2 \sinh\beta x + C_3 \left(x \cosh\beta x + \frac{1}{\beta} \sinh\beta x \right) \right] \cos\beta y \tag{7.2.43}$$

Applying the stress-free boundary conditions $T_{xx}(\pm a, y) = T_{xy}(\pm a, y) = 0 \Rightarrow$

$$C_2 \cosh\beta a + C_3 a \sinh\beta a = -\frac{E\alpha_T T_o}{\beta^2}$$
$$C_2 \sinh\beta a + C_3 \left(a \cosh\beta a + \frac{1}{\beta} \sinh\beta a \right) = 0$$

Solving for the two remaining constants then gives

$$C_2 = -\frac{E\alpha_T T_o (a\beta \cosh\beta a + \sinh\beta a)}{\beta^2 (a\beta + \sinh\beta a \cosh\beta a)}, \quad C_3 = \frac{E\alpha_T T_o \sinh\beta a}{\beta (a\beta + \sinh\beta a \cosh\beta a} \tag{7.2.44}$$

Note that by using superposition and Fourier methods, we could generate a more general temperature field and the corresponding stress solution.

EXAMPLE 7.2.2 AXISYMMETRIC PROBLEM SOLUTIONS

Determine the axisymmetric stress and displacement solutions for the general thermoelastic plane stress problem where all field quantities are functions only of the radial coordinate r.

Solution: For this solution class, we choose the following field variable forms:

$$T_{rr} = T_{rr}(r), \quad T_{\theta\theta} = T_{\theta\theta}(r), \quad T_{r\theta} = T_{r\theta}(r), \quad \Theta = \Theta(r), \quad \phi = \phi(r) \tag{7.2.45}$$

Relations (7.2.39) then become

$$T_{rr} = \frac{1}{r}\frac{d\phi}{dr}, \quad T_{\theta\theta} = \frac{d^2\phi}{dr^2} = \frac{d}{dr}(rT_{rr}), \quad T_{r\theta} = 0 \tag{7.2.46}$$

The governing equation in terms of the stress function (7.2.40) simplifies to

$$\frac{1}{r}\frac{d}{dr}\left\{ r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r \frac{d\phi}{dr} \right) \right] \right\} + E\alpha_{\mathrm{T}} \frac{1}{r}\frac{d}{dr}\left(r \frac{d\Theta}{dr} \right) = 0 \tag{7.2.47}$$

This relation can be recast in terms of the radial stress by using (7.2.46)$_1$ giving the result

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2T_{rr}\right)\right]\right\} = -E\alpha_{\mathrm{T}}\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) \tag{7.2.48}$$

and the equation can be directly integrated to give

$$T_{rr} = \frac{C_3}{r^2} + C_2 + \frac{C_1}{4}(2\log r - 1) - \frac{E\alpha_{\mathrm{T}}}{r^2}\int \Theta r\, dr \tag{7.2.49}$$

The constants of integration C_i are normally determined from the boundary conditions, and the temperature Θ appearing in the integral is again the temperature difference from the reference state. Using the boundedness condition, constants C_1 and C_3 must be set to zero for domains that include the origin. Combining this result with (7.2.46)$_2$ gives the general solution for the stress field.

Next, consider the displacement formulation and solution. For the axisymmetric case, $u_r = u(r)$ and $u_\theta = 0$. Going back to the equilibrium equations (7.2.38), it is observed that the second equation vanishes identically. Using Hooke's law and the strain displacement relations in the first equilibrium equation gives

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = (1+\nu)\alpha_{\mathrm{T}}\frac{d\Theta}{dr} \tag{7.2.50}$$

This equation can again be directly integrated giving the displacement solution

$$u = A_1 r + \frac{A_2}{r} + \frac{(1+\nu)\alpha_{\mathrm{T}}}{r}\int \Theta r\, dr \tag{7.2.51}$$

The constants of integration A_i are again determined from the boundary conditions. The general displacement solution (7.2.51) can then be used to determine the strains from relations (7.2.36) and stresses from Hooke's law (7.2.37). It is found that the stresses developed from this displacement solution do not contain the logarithmic term found in relation (7.2.49). Thus, the logarithmic term is considered inconsistent with a continuous single-valued displacement field and is commonly dropped for most but not all problem solutions.

EXAMPLE 7.2.3 CIRCULAR THERMOELASTIC PLATE PROBLEM

Determine the axisymmetric thermal stresses in an annular circular plate as shown in Fig. 7.2. Assume that the inner and outer boundaries are stress-free, while the temperature boundary conditions are $\Theta(r_{\mathrm{i}})=\theta_{\mathrm{i}}, \Theta(r_{\mathrm{o}})=0$. Also examine the case $r_{\mathrm{i}} \to 0$ and determine the solid plate solution with thermal boundary condition $\Theta(r_{\mathrm{o}})=\theta_{\mathrm{o}}$.

Solution: The general stress solution was given in Eq. (7.2.49), and thus (dropping the log-term) we have

$$T_{rr}=\frac{C_3}{r^2}+C_2-\frac{E\alpha_T}{r^2}\int \Theta r\,dr \tag{7.2.52}$$

Using the stress-free boundary conditions $T_{rr}(r_{\mathrm{i}})=T_{rr}(r_{\mathrm{o}})=0$ determines the two constants C_2 and C_3. Incorporating these results, the stresses become

$$\begin{aligned}
T_{rr}&=\frac{E\alpha_T}{r^2}\left\{\frac{r^2-r_{\mathrm{i}}^2}{r_{\mathrm{o}}^2-r_{\mathrm{i}}^2}\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}}\Theta r\,dr-\int_{r_{\mathrm{i}}}^{r}\Theta r\,dr\right\}\\
T_{\theta\theta}&=\frac{E\alpha_T}{r^2}\left\{\frac{r^2+r_{\mathrm{i}}^2}{r_{\mathrm{o}}^2-r_{\mathrm{i}}^2}\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}}\Theta r\,dr+\int_{r_{\mathrm{i}}}^{r}\Theta r\,dr-\Theta r^2\right\}
\end{aligned} \tag{7.2.53}$$

and the corresponding displacement solution is given by

$$u=\frac{\alpha_T}{r}\left\{(1+\nu)\int_{r_i}^{r}\Theta r\,dr+\frac{(1-\nu)r^2+(1+\nu)r_i^2}{r_o^2-r_i^2}\int_{r_i}^{r_o}\Theta r\,dr\right\} \tag{7.2.54}$$

In order to explicitly determine the stress and displacement fields, the temperature distribution must be determined. As mentioned, this

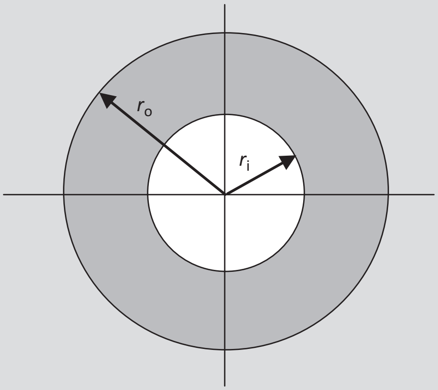

FIGURE 7.2

Annular plate geometry.

is calculated from the energy or heat conduction equation, and for the steady-state case was given by (7.2.32). For the axisymmetric problem, this reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right)=0 \tag{7.2.55}$$

This equation can be integrated directly giving the solution

$$\Theta = A_1 \log r + A_2 \tag{7.2.56}$$

Using the given thermal boundary conditions $\Theta(r_i)=\theta_i, \Theta(r_o)=0$, the temperature solution is obtained as

$$\Theta = \frac{\theta_i}{\log\left(\frac{r_i}{r_o}\right)}\log\left(\frac{r}{r_o}\right) = \frac{\theta_i}{\log\left(\frac{r_o}{r_i}\right)}\log\left(\frac{r_o}{r}\right) \tag{7.2.57}$$

For the case $\theta_i > 0$, this distribution is shown in Fig. 7.3.

Substituting this temperature distribution into the stress solution (7.2.53) gives

$$\begin{aligned} T_{rr} &= \frac{E\alpha_T\theta_i}{2\log(r_o/r_i)}\left\{-\log\left(\frac{r_o}{r}\right) - \frac{r_i^2}{r_o^2-r_i^2}\left(1-\frac{r_o^2}{r^2}\right)\log\left(\frac{r_o}{r_i}\right)\right\} \\ T_{\theta\theta} &= \frac{E\alpha_T\theta_i}{2\log(r_o/r_i)}\left\{1-\log\left(\frac{r_o}{r}\right) - \frac{r_i^2}{r_o^2-r_i^2}\left(1+\frac{r_o^2}{r^2}\right)\log\left(\frac{r_o}{r_i}\right)\right\} \end{aligned} \tag{7.2.58}$$

When $\theta_i > 0$ and $T_{rr} < 0$, the hoop stress $T_{\theta\theta}$ takes on maximum values at the inner and outer boundaries of the plate. For the specific case $r_o/r_i = 3$, the stress distribution is illustrated in Fig. 7.4. MATLAB Code C-17 was used for the calculations and plotting.

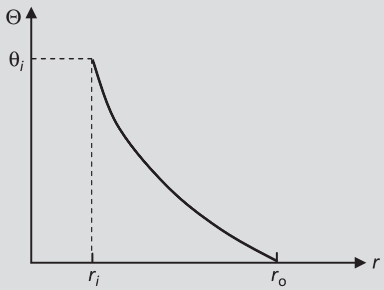

FIGURE 7.3

Temperature distribution in annual plate.

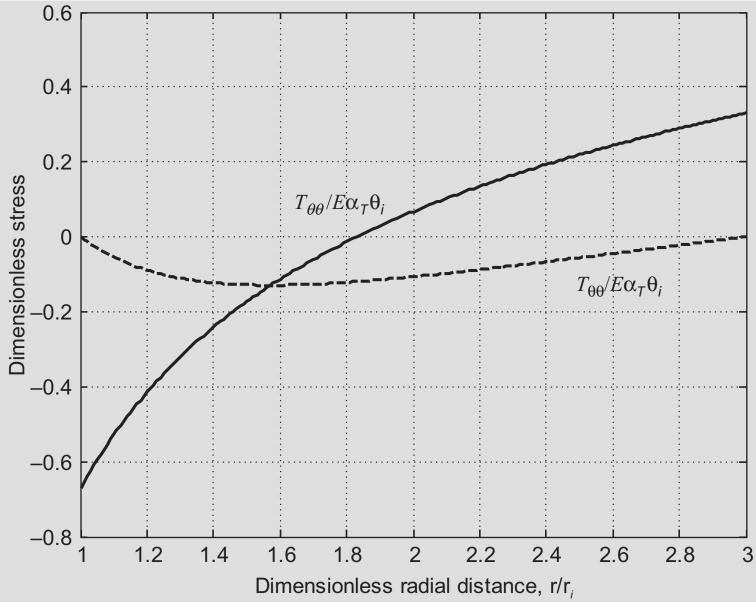

FIGURE 7.4

Thermoelastic stress distribution in annular plate.

Finally, by allowing the inner radius r_i to zero, we obtain the solution for a solid circular plate. For this case, the constant C_3 in solution (7.2.52) must be set to zero for finite stresses at the origin. The resulting stress field for zero boundary tractions becomes

$$\begin{aligned} T_{rr} &= E\alpha_T \left\{ \frac{1}{r_o^2} \int_0^{r_o} \Theta r\,dr - \frac{1}{r^2} \int_0^{r} \Theta r\,dr \right\} \\ T_{\theta\theta} &= E\alpha_T \left\{ \frac{1}{r_o^2} \int_0^{r_o} \Theta r\,dr + \frac{1}{r^2} \int_0^{r} \Theta r\,dr - \Theta \right\} \end{aligned} \tag{7.2.59}$$

The behavior of the integral term $\frac{1}{r^2}\int_0^r \Theta r\,dr$ at the origin can be investigated using l'Hospital's rule, and it can be shown that $\lim_{r\to 0}\left(\frac{1}{r^2}\int_0^r \Theta r\,dr\right) = \frac{1}{2}\Theta(0)$, thus indicating bounded behavior at the plate's center. Using temperature boundary condition $\Theta(r_o) = \theta_o$, the general solution (7.2.55) predicts a uniform temperature $\Theta = \theta_o$ throughout the entire plate. For this case, relations (7.2.58) give $T_{rr} = T_{\theta\theta} = 0$, and thus the plate is stress-free. This particular result verifies the fact that a steady temperature distribution in a simply connected region with zero traction boundary conditions gives rise to zero stresses.

7.3 POROELASTICITY

Considerable applications of continuum mechanics have been made for porous materials containing an internal connected network of pore space that contains a viscous pore fluid. Examples of such materials include soils, rock, foams, powders, ceramics, and biological substances. Fig. 6.3 illustrated a few such material types and a more detailed photograph of a sponge sample is shown in Fig. 7.5. The usual density functions used in classical continuum mechanics must be modified to account for such microstructures. In order to apply continuum mechanics to such materials, it is normally assumed that the pore structures are uniformly distributed within the material and occur at length scales at least a few orders of magnitude less than that of the basic problem. This will then allow many of the usual classical methods of formulation to be extended to such materials. This concept is related to our discussion back in Section 1.1 on the definition of the representative volume element.

We consider the case of a porous isotropic linear elastic material in which pore fluid is freely allowed to move within the completely interconnected porous microstructure. The pore fluid motion is governed by diffusion mechanisms and these will be coupled with deformation within the elastic skeleton. For example, a consolidation of the solid skeleton will produce a rise in fluid pore pressure; likewise, a rise in pore pressure will produce material dilatational deformation. As done in the previous thermoelastic presentation, we will focus our attention on how this coupling integrates into the various model field equations. We will also explore a few basic solutions to problems of interest.

Pioneering work in formulating this theory was initiated by Terzaghi (1943), but it was Biot (collection of papers edited by Tolstoy, 1992) who over a period of 30 years

FIGURE 7.5

Sponge sample as an example of poroelastic material.

developed most of the fundamental details of poroelasticity. A few other good sources for further more detailed information can be found in Wang (2000) and Cheng (2016).

7.3.1 CONSTITUTIVE LAWS AND GENERAL FORMULATION

The theory assumes a coherent elastic solid skeleton and a freely moving pore fluid such that the solid and fluid phases are fully connected and uniformly distributed. This allows the construction of a continuum mechanics theory relating the various field quantities of interest within a governing set of coupled field equations. The pore fluid motion is governed by diffusion mechanisms and these will be coupled with linear elastic deformation behaviors of the solid skeleton. As previously mentioned, a consolidation of the solid skeleton will produce a rise in fluid pore pressure, which in turn will produce material dilation. Therefore, the total medium stress will be a combination of that within the solid and fluid portions as illustrated in Fig. 7.6. It is instructive to point out that there exists an analogy between our previous thermoelasticity model and that of poroelasticity, and Norris (1992) presents a detailed account of this fact. Temperature corresponds to pore fluid pressure, heat flux corresponds to fluid flow, and entropy corresponds to fluid mass.

New parameters for poroelasticity include a *specific discharge vector* q_i, which relates the relative fluid motion with respect to the solid, and is defined as the rate of fluid volume crossing a unit area specified by a normal in the x_i direction. Also the change in fluid content within the porous material is specified by the quantity ζ which represents the *increment in fluid volume per unit material volume.* For no source terms, the fluid mass balance relation can then be expressed as

$$\frac{\partial \zeta}{\partial t} = -q_{i,i} \tag{7.3.1}$$

Note that positive ζ corresponds to an increase in the fluid in the porous solid. Neglecting body forces, the fluid transport within the interstitial space is governed by the *Darcy law*

$$q_i = -\kappa p_{,i} \tag{7.3.2}$$

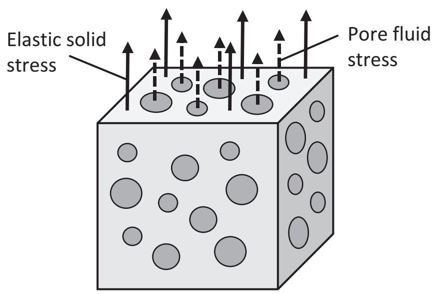

FIGURE 7.6

Dual stress distribution in poroelastic media.

where κ is the *permeability coefficient* (generally a function of the pore geometry and fluid viscosity) and p is the *fluid pore pressure*. Relation (7.3.2) is analogous to Fourier's law of heat conduction (7.2.9) and can be derived from the Navier–Stokes equations (6.4.13) by dropping the inertia terms. Combining relations (7.3.1) and (7.3.2) gives

$$\frac{\partial \zeta}{\partial t} = \kappa \nabla^2 p \tag{7.3.3}$$

Incorporating this fluid diffusion model within an isotropic elastic theory, Biot proposed a coupled linear constitutive law of the form

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2\mu} T_{ij} - \left(\frac{1}{6\mu} - \frac{1}{9k} \right) T_{kk} \delta_{ij} + \frac{1}{3H} p \delta_{ij} \\ \zeta &= \frac{1}{3H} T_{kk} + \frac{1}{R} p \end{aligned} \tag{7.3.4}$$

where μ and k are, respectively, the usual shear and bulk elastic moduli of the *drained elastic skeleton*, and the additional material constants H and R characterize the coupling between the solid and fluid stress and strain fields. Stress–strain relation $(7.3.4)_1$ can be inverted and recast as

$$T_{ij} + \alpha p \delta_{ij} = (\lambda + \mu) \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{7.3.5}$$

where λ is the usual Lamé's constant of the drained elastic skeleton and $\alpha = k/H$ and is commonly referred to as the *Biot coefficient of effective stress*. Note that all drained elastic constants satisfy relations presented back in Section 6.2.1. In this form, the constitutive relation is similar to that of a drained elastic solid with the left-hand side often referred to as an *effective stress*. It should be observed that by dropping the pore pressure term, relations $(7.3.4)_1$ and (7.3.5) reduce to the classical isotropic elasticity constitutive law (6.2.31). Constitutive relations (7.3.4) can be easily decomposed into volumetric and deviatoric equations as

$$\begin{aligned} \varepsilon_{kk} &= -\frac{P}{k} + \frac{p}{H} \\ \hat{\varepsilon}_{ij} &= \frac{1}{2\mu} \hat{T}_{ij} \\ \zeta &= -\frac{P}{H} + \frac{p}{R}, \quad P = -\frac{1}{3} T_{kk} \end{aligned} \tag{7.3.6}$$

In order to understand the behavior of fluid-filled porous materials, it is important to consider its special response during *undrained* and *drained conditions*. These two situations represent limiting behaviors with undrained conditions corresponding to the case where the fluid is trapped in the porous solid such that $\zeta = 0$, while the drained case corresponds to zero pore pressure, $p = 0$. It can be shown from Eqs. (7.3.6) that under undrained conditions the volumetric response can be written as

$$\varepsilon_{kk} = \varepsilon = -\frac{P}{K_u}, \quad \text{where } K_u = k\left[1 + \frac{kR}{H^2 - kR}\right], P = -\frac{1}{3} T_{kk} \tag{7.3.7}$$

Note that K_u may be thought of as the *undrained bulk modulus* of the material. For the drained case, the volumetric relationship from (7.3.6) becomes

$$\varepsilon_{kk} = \varepsilon = -\frac{P}{k} \tag{7.3.8}$$

Thus, under either drained or undrained conditions, the poroelastic model produces an elastic response with the undrained case having a higher volumetric stiffness ($K_u > k$). Substituting relation (7.3.8), with $p = 0$, yields

$$\zeta = \alpha\varepsilon \tag{7.3.9}$$

It can be observed that α may be interpreted as the ratio of fluid volume change in a material element to the volume change of the element itself under drained conditions. In regard to the time-dependent nature of the problem, see (7.3.3), the case of rapid loadings would correspond to the situation where pore fluid would have little time to flow, thus indicating a more undrained condition. On the other hand, for very slow loading situations, the pore pressure will have sufficient time to equilibrate with the boundary pressure, and if this boundary value is zero, then this case will correspond to a drained condition.

Linear isotropic poroelasticity is thus described by the coupled constitutive laws (7.3.4) for the solid skeleton and fluid, the Darcy relation (7.3.2), and the continuity equation (7.3.1). We also need the small deformation strain–displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{7.3.10}$$

and limiting the discussion to quasi-static problems, we also include the equations of equilibrium

$$T_{ij,j} + F_i = 0 \tag{7.3.11}$$

A set of five material constants (μ, k, R, H, and κ) are needed to fully characterize the material, and it is common in the literature to use physically meaningful material constants (e.g. K_u and α) in place of R and H. Mechanical (tractions or displacements) and fluid drainage boundary conditions are required for solution to specific problems.

7.3.2 PROBLEM SOLUTIONS

The solution to problems in linear poroelasticity closely follows the procedures in linear elasticity presented in Sections 6.2.2 and 6.2.3. Stress and displacement formulations can be developed. For the stress formulation following the usual procedures, we can generate the Beltrami–Michell equations

$$T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} + \frac{(1-2\nu)\alpha}{1-\nu}\left(\delta_{ij}p_{,kk} + \frac{1-\nu}{1+\nu}p_{,ij}\right) = -\frac{\nu}{1-\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i} \tag{7.3.12}$$

and for the displacement formulation Navier's equations (equilibrium in terms of displacements) would become

$$\mu u_{i,kk} + (\lambda+\mu)u_{k,ki} - \alpha p_{,i} + F_i = 0 \tag{7.3.13}$$

As with the thermoelastic case, we again see that the coupling term $-\alpha p_{,i}$ acts as an added body force in the momentum balance relation. To help with problem solution, the common elasticity schemes of stress functions and displacement potentials along with two-dimensional models of plane strain and plane stress can be used.

EXAMPLE 7.3.1 ONE DIMENSIONAL CONSOLIDATION

Consider the one-dimensional poroelastic deformation of a material layer in the domain $0 \leq x \leq h; -\infty \leq y, z \leq \infty$ as shown in Fig. 7.7. The layer rests on a rigid impermeable base and its top surface has zero pore pressure and carries a suddenly applied uniform normal loading, p_o. Determine the time-dependent vertical displacement.

Solution: We assume for this problem deformations only in the x-direction where the only nonzero strain will be ε_x. Under such uniaxial strain, the stress field reduces to

$$\begin{aligned} T_{xx} &= (\lambda + 3\mu)\varepsilon_x - \alpha p \\ T_{yy} &= T_{zz} = (\lambda + \mu)\varepsilon_x - \alpha p \end{aligned} \tag{7.3.14}$$

while the equilibrium equations become

$$\frac{\partial T_{xx}}{\partial x} = 0 \tag{7.3.15}$$

Thus, the stress T_{xx} does not depend on x. Now since the problem loading is constant in time, this implies that the stress will also be independent of time, and thus will be constant. Boundary condition $T_{xx}(0,t) = -p_o H(t)$, thus gives $T_{xx} = -p_o$.

Navier's equations (7.3.13) reduce to the single relation

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial p}{\partial x} = 0 \tag{7.3.16}$$

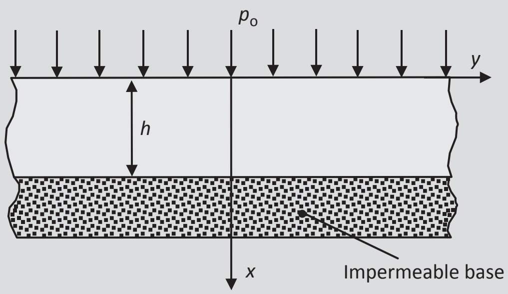

FIGURE 7.7

One-dimensional consolidation problem.

Since T_{xx} is constant, the diffusion equations (7.3.3) and $(7.3.4)_2$ give

$$\frac{1}{R}\frac{\partial p}{\partial t}-\kappa\frac{\partial^2 p}{\partial x^2}=0 \tag{7.3.17}$$

and this simplification removes the coupling between pore pressure and stress and thus provides a single equation to solve for the pore pressure p. The boundary conditions for this problem are

$$p(0,t)=0,\quad \frac{\partial p(h,t)}{\partial x}=0 \tag{7.3.18}$$

while the initial condition is taken as an undrained situation and hence using $(7.3.6)_3$, we get

$$p(x,0)=\frac{R}{3H}p_{\mathrm{o}} \tag{7.3.19}$$

The solution to Eq. (7.3.17) can be found by standard separation of variables (see Wang, 2000 for details) and is given by

$$p(x,t)=\frac{Rp_{\mathrm{o}}}{3H}\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi\hat{x}}{2}\right)\mathrm{e}^{-n^2\pi^2\tau} \tag{7.3.20}$$

where we have introduced dimensionless distance $\hat{x}=x/h$ and dimensionless time $\tau=\kappa t/4Rh^2$.

The displacement is found from integrating Eq. (7.3.16), and Wang gives the result of the displacement of the top of the layer as

$$u(0,t)=K_1\left[1+K_2\sum_{n=1,3,5,\ldots}^{\infty}\frac{8}{n^2\pi^2}\left(1-\mathrm{e}^{-n^2\pi^2\tau}\right)\right] \tag{7.3.21}$$

with $K_1=\dfrac{p_{\mathrm{o}}h(1-2\nu_{\mathrm{u}})}{2\mu(1-\nu_{\mathrm{u}})}, K_2=\dfrac{\nu_{\mathrm{u}}-\nu}{(1-\nu_{\mathrm{u}})(1-2\nu_{\mathrm{u}})}$, and the drained and undrained Poisson's ratios given by

$$\nu=\frac{3k-2\mu}{2(3k+\mu)},\quad \nu_u=\frac{3K_{\mathrm{u}}-2\mu}{2(3K_{\mathrm{u}}+\mu)} \tag{7.3.22}$$

The time behavior of the pore pressure is shown in Fig. 7.8 for three different locations within the layer. At all locations, the pore pressure starts at the same value and exhibits a monotonic decrease to zero. Fig. 7.9 illustrates the surface displacement $u(0,t)/K_1$ versus time response for the case with $K_2=1$. Although it is difficult to see from the graph, there exists an initial undrained elastic displacement followed by a time-dependent component. The layer deforms initially with an elastic response determined by the undrained Poisson's ratio ν_u, and as time progresses it consolidates into another elastic condition characterized by the drained value ν. All calculations and plotting were done using MATLAB Code C-18.

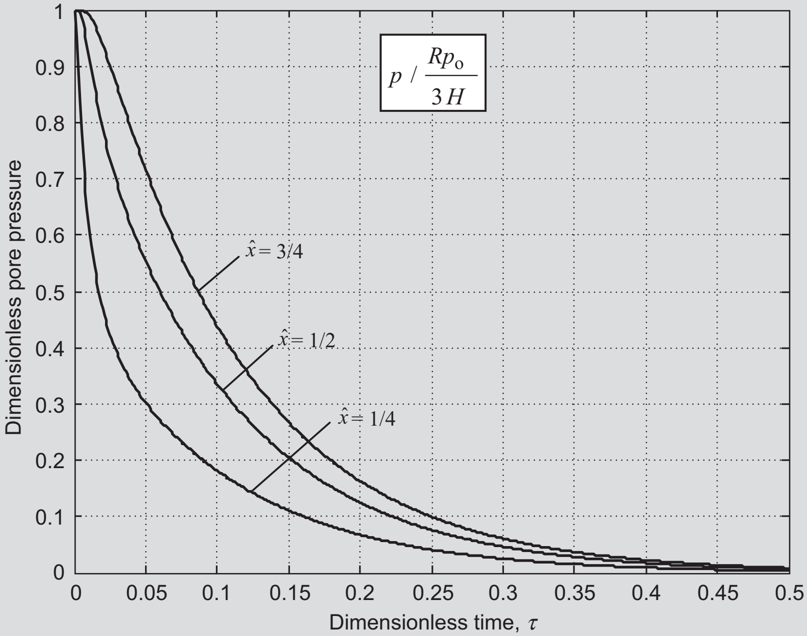

FIGURE 7.8

Pore pressure behavior for one-dimensional consolidation problem.

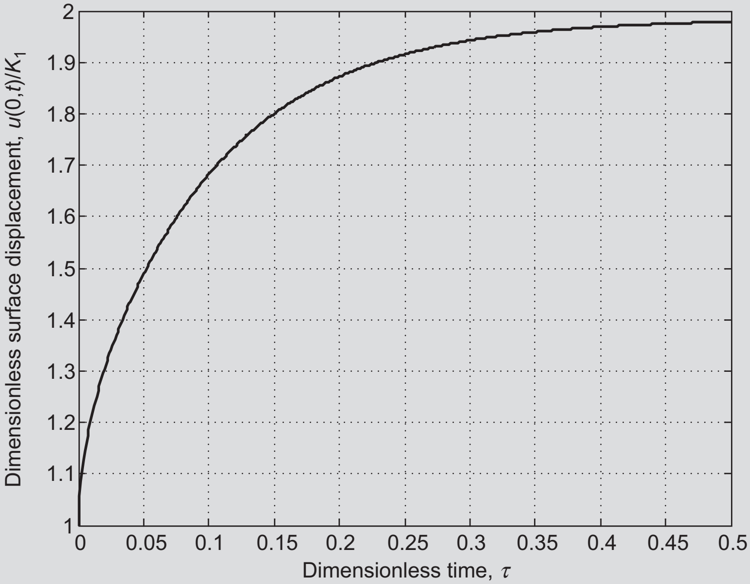

FIGURE 7.9

Surface displacement behavior for one-dimensional consolidation problem ($K_2 = 1$).

EXAMPLE 7.3.2 MANDEL'S CONSOLIDATION PROBLEM

Consider the problem originally presented by Mandel (1953) of a long rectangular cross-section poroelastic medium sandwiched between two impermeable plates as shown in Fig. 7.10. A compressive force $F = 2p_o aH(t)$ is suddenly applied to the material as shown. The right and left boundaries are assumed to be stress-free and drained. The geometry is such that the deformation is only in the x,y-plane and the pore fluid moves only in the x-direction. Thus, the stress components and pore pressure are functions of x and t. Determine the pore pressure distribution and show that it results in a nonmonotonic response.

Solution: The problem may be modeled using a *two-dimensional plane strain* formulation where the displacements are of the form $u = u(x, y), v = v(x, y), w = 0$. From the two-dimensional equilibrium equations, we find that $\dfrac{\partial T_{xx}}{\partial x} = 0$ which implies that T_{xx} is a constant, but since it is to vanish at $x = \pm a$, $T_{xx} = 0$.

For the plane strain case, the poroelastic Beltrami–Michell compatibility equations (see elasticity discussion in Section 6.2.3) reduce to

$$\nabla^2 (T_{xx} + T_{yy} + 2\eta p) = 0 \tag{7.3.23}$$

where $\eta = \dfrac{1-2\nu}{2(1-\nu)}\alpha$. Since T_{xx} vanishes and T_{yy} and p depend only on x and t, (7.3.23) implies

$$T_{yy} + 2\eta p = g(t) \tag{7.3.24}$$

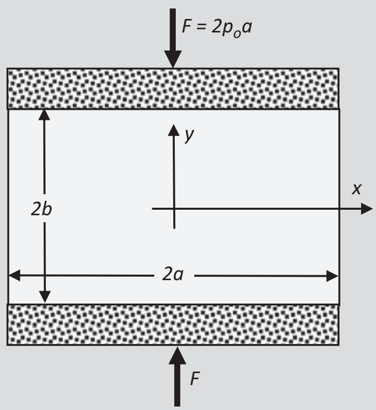

FIGURE 7.10

Mandel's consolidation problem geometry.

where the arbitrary function of time $g(t) = T_{yy}(a,t)$, since the pore pressure vanishes a $x = a$.

For plane strain, relation (7.3.4)$_2$ can be written as

$$\zeta = \frac{\alpha(1+\nu)}{3K}(T_{xx} + T_{yy} + A_1 p) \tag{7.3.25}$$

with $A_1 = \dfrac{3}{B(1+\nu_u)}$.

In general, using the Beltrami–Michell compatibility equation (7.3.12) for the poroelastic case, one can show that

$$\frac{\partial \zeta}{\partial t} = c\nabla^2 \zeta \tag{7.3.26}$$

where $c = \dfrac{2\kappa\mu(1-\nu)(\nu_u - \nu)}{\alpha^2(1-2\nu)^2(1-\nu_u)}$. Note that definitions of the Poisson's ratios were given previously in relations (7.3.22). Diffusion equation (7.3.26) for ζ becomes uncoupled which is not the case for the pore pressure equation.

Using results (7.3.23) and (7.3.24) with $T_{xx} = 0$ gives the result

$$\frac{\partial}{\partial t}(T_{yy} + A_1 p) = c\frac{\partial^2}{\partial x^2}(T_{yy} + A_1 p) \tag{7.3.27}$$

Result (7.3.27) can be solved using Laplace transforms (see Wang, 2000 for details). The pore pressure terms can be eliminated using the Beltrami–Michell relation (7.3.23) and the inverse Laplace transform is handled by some basic contour integration. The final result for the pore pressure is given by

$$p(x,t) = \frac{2p_o}{A_1}\sum_{n=1}^{\infty}\frac{\sin\lambda_n}{\lambda_n - \sin\lambda_n \cos\lambda_n}\left(\cos\frac{\lambda_n x}{a} - \cos\lambda_n\right)e^{-\lambda_n^2 ct/a^2} \tag{7.3.28}$$

where λ_n are the roots of the equation $\dfrac{\tan\lambda}{\lambda} = \dfrac{3}{2\eta B(1+\nu_u)} = \dfrac{1-\nu}{\nu_u - \nu}$. Using MATLAB Code C-19 for the calculations and plotting, Fig. 7.11 illustrates the pore pressure distribution along the positive x-axis for several different values of dimensionless times $\tau = ct/a^2$. For small times, the pressure is not completely monotonic in time. This behavior is normally explained by the fact that there is a contraction at the drained edges of the sample and this will induce pore pressure buildup in the interior. See Cheng and Detournay (1988) for more complete solution details and interpretation.

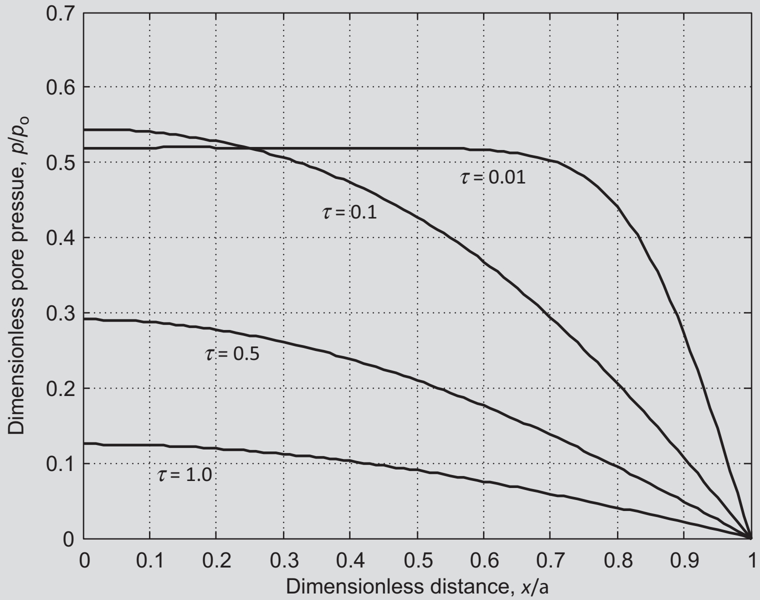

FIGURE 7.11

Pore pressure profiles for the Mandel problem with poroelastic constants $\nu = 0.2$, $\nu_u = 0.5$ and $B = 1.0$.

7.4 ELECTROELASTICITY

As our final multicontinuum theory we now wish to explore *electroelasticity* whereby the basic linear elasticity model will be coupled with an electrical field. The basic issue is that when an electric field is applied to some materials, they experience a deformation or strain. Such behavior is due to *piezoelectric or electrostrictive effects*. Electrostriction is a property of dielectric materials and is caused by a small displacement of ions in the crystal lattice exposed to an external electric field. Such displacements accumulate in the material and result in an overall strain in the direction of the field. The piezoelectric effect is the electric charge that accumulates in certain solid materials such as crystals, certain ceramics, and biological matter in response to applied mechanical stress. The piezoelectric effect is a *reversible process* in that materials exhibiting the internal generation of electrical charge resulting from an applied mechanical force also exhibit the reverse effect of the internal generation of a mechanical strain resulting from an applied electrical field. Both the piezoelectric or electrostrictive effects were discovered over a century ago and thus much research and application has been done on such materials. A large variety of both natural and synthetic piezoelectric crystalline and ceramic materials exist. Applications in this field include sensors, actuators, motors, smart materials, and many new microscale devices. We will only present a brief overview of the continuum mechanics modeling of such materials, and more detailed presentations can be found in Eringen and Maugin (1990), Kuang (2014), Peric (2004), and Bechtel and Lowe (2015).

7.4.1 CONSTITUTIVE LAWS AND GENERAL FORMULATION

Similar to our previous theories in this chapter, we now present a linear theory of electroelasticity for small deformations under quasi-static conditions with no body forces. We further assume that there are no magnetic fields, free currents, or electric charges. From Maxwell–Faraday theory of electrostatics with no mechanical deformation, we can define a linear electrical constitutive relation

$$D_i = \breve{\varepsilon}_{ik} E_k \tag{7.4.1}$$

where D_i is the *electric displacement*, $\breve{\varepsilon}_{ik}$ is the *dielectric permittivity material tensor*, and E_k is the *electric field strength*. Furthermore, these fields must satisfy

$$\begin{aligned} &D_{i,i} = 0, \quad \nabla \cdot \boldsymbol{D} = 0 \\ &\varepsilon_{ijk} E_{k,j} = 0, \quad \nabla \times \boldsymbol{E} = 0 \Rightarrow \boldsymbol{E} = -\nabla \phi^c \end{aligned} \tag{7.4.2}$$

where ϕ^c is the electric potential function.

Introducing the mechanical deformation, Hooke's law then couples the stress–strain with the electric field, and vice versa. The most common way to determine these coupled constitutive relations is to use energy schemes previously discussed. Following Pak (1992), the internal energy ε stored in a linear electroelastic material can be written as

$$\varepsilon = \frac{1}{2} T_{ij} \varepsilon_{ij} + \frac{1}{2} E_i D_i \tag{7.4.3}$$

If we wish to derive the governing equations in terms of E_i instead of D_i, we need to introduce another thermodynamic potential. This can be done by defining an *electric enthalpy density*

$$h = \varepsilon - E_i D_i \tag{7.4.4}$$

This form then becomes the correct potential when the mechanical displacement and the electrical potential are taken to be the independent variables.

For linear electroelastic materials, the electric enthalpy can be expressed by

$$h = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \frac{1}{2} \breve{\varepsilon}_{ij} E_i E_j - e_{ikl} \varepsilon_{kl} E_i \tag{7.4.5}$$

where e_{ikl} is the *piezoelectric material tensor*. Using the usual thermodynamic procedures, the constitutive relations come from the potential relations

$$\begin{aligned} T_{ij} &= \frac{\partial h}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} - e_{kij} E_k \\ D_i &= -\frac{\partial h}{\partial E_i} = \breve{\varepsilon}_{ik} E_k + e_{ijk} \varepsilon_{jk} \end{aligned} \tag{7.4.6}$$

Kuang, (2014) shows that actually several different forms of these couple equations can be derived. Typical boundary conditions for this problem are

$$\begin{aligned} T_{ij}n_j &= t_i \quad \text{(specified traction)} \\ D_i n_i &= -q_s \quad \text{(specified surface charge)} \end{aligned} \tag{7.4.7}$$

In addition to the symmetries for C_{ijkl} from Section 6.2.1, the electromaterial tensors generally satisfy the conditions $\breve{\varepsilon}_{ij} = \breve{\varepsilon}_{ji}, e_{ijk} = e_{ikj}$. Thus, for the general problem, there exists 21 elastic constants, 18 piezoelectric constants, and 6 permittivity constants. However, for most applications, there is considerable symmetry and this number of independent material constants reduce to more reasonable numbers.

Most applications of electroelasticity involve anisotropic crystal, ceramic, and composite materials and thus require the anisotropic constitutive forms in (7.4.6). As used previously for linear elasticity, we can use the Voigt notational scheme to express these electroconstitutive relations. Kuang (2014) provides a specific example for a transversely isotropic ceramic (such as lead zirconate titanate, PZT). For the case where x_3 is the electrical poling direction and axis of symmetry, x_1,x_2 is the isotropic plane and relations (7.4.6) would take the form

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ \cdot & C_{11} & C_{13} & 0 & 0 & 0 \\ \cdot & \cdot & C_{33} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & C_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{44} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{7.4.8}$$

with $C_{66} = (C_{11} - C_{22})/2$.

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \breve{\varepsilon}_{11} & 0 & 0 \\ 0 & \breve{\varepsilon}_{22} & 0 \\ 0 & 0 & \breve{\varepsilon}_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} \tag{7.4.9}$$

For two-dimensional plane strain or plane stress problems, considerable reduction in this large system will occur and a variety of analytical solutions for the stress, strain, displacement, and electric fields have been developed. However, since the solution methods are lengthy and detailed, we will not go into this material. We thus end our presentation on this interesting multifield coupled continuum mechanics topic.

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EXERCISES

7.1 Starting from the energy relation (7.2.6)$_3$, explicitly develop the linearize energy equation (7.2.13).

7.2 Invert Hooke's law (7.2.16)$_1$ to express the strain in terms of stress

$$\varepsilon_{ij} = \frac{1+\nu}{E} T_{ij} - \frac{\nu}{E} T_{kk}\delta_{ij} + \alpha_T \Theta \delta_{ij}$$

7.3 Develop Navier's thermoelastic equilibrium equations in terms of the displacements (7.2.24)

$$\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} - (3\lambda + 2\mu)\alpha_T \Theta_{,i} + F_i = 0$$

7.4 For an anisotropic solid, Fourier's conduction law was given by $q_i = -k_{ij}\Theta,_j$ and it was shown that the conductivity tensor was symmetric, $k_{ij} = k_{ji}$. Show that for a two-dimensional problem in the x,y-plane, the heat conduction equation for uncoupled steady-state conditions becomes

$$k_{xx}\frac{\partial^2\Theta}{\partial x^2}+2k_{xy}\frac{\partial^2\Theta}{\partial x\partial y}+k_{yy}\frac{\partial^2\Theta}{\partial y^2}=0$$

7.5 Justify the plane stress thermoelastic relations (7.2.29) and (7.2.30).

7.6 For the plane stress case, invert relations (7.2.26) and determine the nonzero stress forms in terms of the strains and thus verify the traction relations given by (7.2.31).

7.7 For Example 7.2.1, show that the stresses are given by relation (7.2.43) and also verify that under the zero traction boundary conditions, constants C_2 and C_3 are expressed by (7.2.44).

7.8 Using the general displacement solution (7.2.51), solve the thermoelastic problem of a solid circular elastic plate $0 \le r \le a$ with a totally restrained boundary edge at $r = a$. Next show that for the case of a uniform temperature distribution, displacements and hence stresses will be zero.

7.9 Consider the axisymmetric plane stress problem of a solid circular plate of radius a with a *constant internal heat generation* specified by h_o. The steady-state conduction equation thus becomes

$$\frac{\partial^2\Theta}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta}{\partial r}+h_o=0$$

Determine the temperature distribution for the case with boundary condition $\Theta(\boldsymbol{a}) = \theta_o$ and then show that the resulting thermal stresses for the case with zero boundary stress are given by

$$T_{rr}=\frac{E\alpha_T h_o}{16}(r^2-a^2),\quad T_{\theta\theta}=\frac{E\alpha_T h_o}{16}(3r^2-a^2)$$

Such solutions are useful to determine the thermal stresses in structures made of radioactive materials.

7.10 Invert the poroelastic constitutive law $(7.3.4)_1$ to develop the form (7.3.5).

7.11 Verify the undrained volumetric response (7.3.7).

7.12 Develop the Beltrami–Michell equation (7.3.12) for the general poroelastic case.

7.13 Using the Beltrami–Michell equation (7.3.12), verify the result

$$\nabla^2(T_{kk}+4\eta^* p)=-\frac{1+\nu}{1-\nu}F_{k,k}$$

where η^* is defined in terms of α and ν.

7.14 Develop Navier's equation (7.3.13) for the general poroelastic case.

7.15 Justify for the two-dimensional plane strain case (defined in Example 7.3.2) that the Beltrami–Michell compatibility equation for poroelastic materials is given by (7.3.23).

7.16 Combine constitutive relations (7.4.8) and (7.4.9) into a single matrix equation of the form7.17

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = [9\times 9] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

7.17 Consider the electroelastic material given by Eqs. (7.4.8) and (7.4.9). For case of x_2, x_3 plane strain with $\varepsilon_{11} = \varepsilon_{13} = \varepsilon_{12} = 0$ and $E_1 = 0$, determine the reduced forms of these relations. The result should reduce to a single 5 × 5 system as per the form from Exercise 7.16.