CHAPTER

Introduction

1

Continuum mechanics or mechanics of continuous media seeks to develop predictive mathematical models of material behavior relating the applied forces (mechanical and other types) to the material deformation and motion. In this chapter, we begin our study with a brief discussion on a few fundamental issues related to the theories pursued. Some basic material issues along with the continuum hypothesis will be discussed. An explanation for the need of tensors is given in an effort to justify our study and use of this important mathematical formalism. A brief overview of the general structure of continuum mechanics is given to provide an overall perspective of the field that we will explore in detail. The chapter ends with a little history outlining the major developments of this broad and encompassing field of study.

1.1 MATERIALS AND THE CONTINUUM HYPOTHESIS

Continuum mechanics is concerned with general modeling of material behavior that has been commonly observed by the scientific and engineering community. The study will be conducted using *Euclidian three-dimensional space*, and all phenomena will be governed by *Newtonian mechanics*. In pursuing such a study, we should first consider a few fundamental questions such as:

- How do we apply a continuum theory to real materials which are fundamentally not continuous?
- What types of materials are we interested in studying?
- What loading conditions are we concerned in applying to materials?
- What behaviors are we most interested in modeling?
- What are the length and time scales of the problem?

Let us now explore some of these questions and discuss possible answers and their consequences.

First, consider the issues of length and time scales of the problems we might be interested in modeling. In some cases, we may wish to study problems with very small length and time scales, whereas for other studies these scales may be larger by many orders. Terminology for various length scales are commonly referred to as nano (10^{-9} m), micro (10^{-6} m), and macro (10^{-3} to 10^{3} m, visible to the eye). For example, we might be interested in studying the behavior of small structural parts with millimeter length scales or in another problem trying to simulate the geomechanical response of earth layers with kilometer scales. Both length and time scales can have significant impact on

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the choice of the modeling irrespective of whether it be analytical or numerical. As we shall see, many *classical* continuum models of solids and fluids have no built-in length or time scales and thus can mathematically be used for problems of any size. However, more recent and sophisticated continuum models have been developed with one or more internal length scales in an effort to more accurately model micromechanical behaviors of heterogeneous materials. These issues will be further explored in Chapter 9.

Next, consider the fact that all real materials are not spatially continuously distributed. Actual materials are composed of extremely large numbers of atoms and molecules discretely distributed in space. For example, it has been estimated that there are about 10^{22} atoms in a cubic centimeter of many common solids, and a similar number would also estimate the number of molecules per cubic centimeter of water. Typically atoms and molecules are separated by very small distances on the order of 10^{-10} m. So, if we are interested in very small length scales on the order of 10^{-10} m, we may wish to use a modeling scheme that takes into account atomic quantum mechanics and/or discrete molecular theories. Such modeling has been done (Massobrio et al., 2012; Jensen, 2010), but it does not use traditional continuum mechanics, and the results are normally limited to nano length and time scales. In contrast, continuum mechanics is commonly used for predicting *macroscopic* material behavior over length scales of 10^{-6} to 10^{3} m and times scales of 10^{-6} to 10^{6} s. This region is where most engineering applications occur. There are also heterogeneous and composite materials that have considerable microstructures within our length scale of interest (10^{-6} to10^{3} m), thus requiring a modified continuum mechanics modeling approach. We will eventually look at such cases later in the text.

In order to explore some of these ideas further, consider the following two-dimensional example of a particulate reinforced composite material shown in Fig. 1.1. We simplify the problem by assuming that the particles are all square (dimensions $a \times a$) and equally spaced by the common size dimension a. The particles are embedded in a continuum matrix that holds everything together. The figure sequence (A)–(C) illustrates the regions of increasing particle numbers which also translates to decreasing relative length scale. It should be visually apparent that as we increase the number of

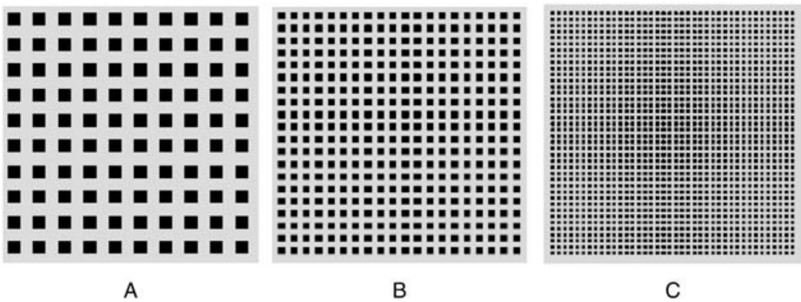

FIGURE 1.1

Example particulate composite samples: (A) 10×10 distribution; (B) 20×20 distribution; (C) 40×40 distribution.

particles in the sample, the overall material seems to become more of a continuum. Thus, while the 10×10 sample shown in Fig. 1.1A might warrant a detailed micromechanical particulate model, the 40×40 sample shown in Fig. 1.1C could be modeled as a continuum with appropriate averaged properties. This idea can be further explored by considering the mass density of the particulate sample as a function of the sample size. To make calculations easier, consider the mass density calculation based on an imaginary square area of increasing size. Starting on a single particle with an $a \times a$ dimension, the area is incrementally increased in each coordinate direction by a. Neglecting the density of the matrix material, the sequential density calculation (mass/area) can easily be made. A plot of the relative density (normalized by the particle density) versus the area dimension (normalized by a) is shown in Fig. 1.2. We observe significant fluctuations in the density for smaller sample areas, but as the area increases, the density variation diminishes and the density itself approaches a constant value (0.25 for this idealized case). Thus, we make the conclusion that as the relative sample size increases (relative length scale decreases), the local variation of density disappears and the material may be thought of as a continuum. The region where this variation becomes negligible is sometimes called the *minimum homogenization volume* or *representative volume element* or *representative elementary volume*. This concept can be extended to atomic, molecular, and other micromechanical mass distributions. We will have more to say about these topics later in the text.

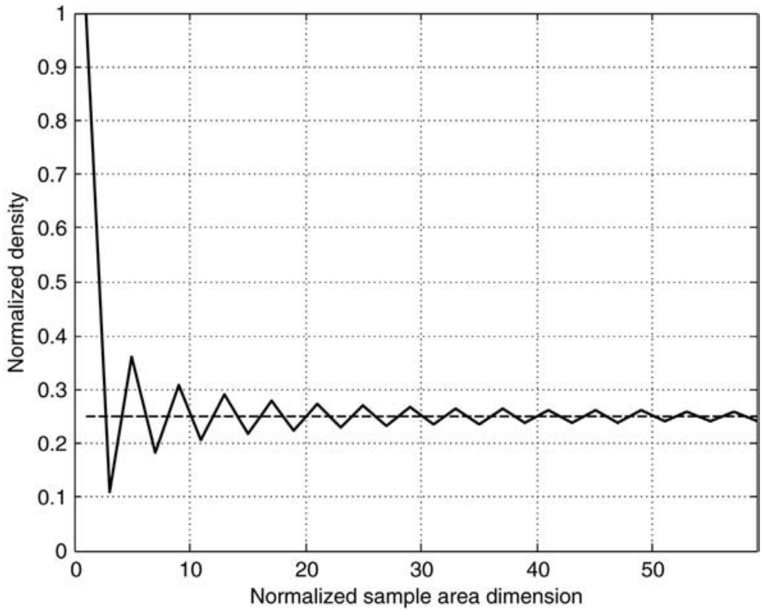

FIGURE 1.2 Relative mass density variation with sample size for particulate composite sample shown in Fig. 1.1.

With regard to the question on material types and behaviors, we are interested in modeling solids, fluids, and other materials that have both solid- and fluid-like behaviors together. Studies on solid materials will include both elastic and inelastic responses, whereas fluids will comprise Newtonian and non-Newtonian behaviors. Materials exhibiting both solid- and fluid-like responses will be included in various viscoelastic theories. We wish to establish models that can predict both mechanical and thermomechanical behaviors and this will require the use of the energy equation and second law of thermodynamics. Electromechanical response of linear elastic solids will also be briefly discussed. Loading conditions will then include mechanical, thermal, and electro types. We will also be exploring a few newer continuum theories that have applications to materials with internal microstructure. Our general study will normally include static and dynamic formulations, but detailed applications will generally emphasize static and quasi-static problems. Several of the terms just used have not been properly defined and this will be handled in subsequent chapters.

To complete our discussion in this section, let us refine our concept of the continuum hypothesis. Although most of us have an intuitive idea of a continuum, our scientific study requires we formulate a mathematical basis of this concept. We say that the real number system is a continuum because between any two distinct real numbers, there are an infinite number of other real numbers. Intuitively, we feel that time and space can be represented by real numbers and thus we identify time and space as a multidimensional continuum.

We can extend this concept of continuum to matter and speak of a continuous distribution of matter in space. This idea is illustrated again by considering the concept of *mass density*. If we let the amount of matter be measured by its mass, then consider a certain matter which permeates a particular region of space $\boldsymbol{V}_0$. Let us consider a point P in $\boldsymbol{V}_0$ and a sequence of subspaces $\boldsymbol{V}_0, \boldsymbol{V}_1, \boldsymbol{V}_2$, ... converging on P as shown in Fig. 1.3. Let V_n be the volume of $\boldsymbol{V}_n$ and let M_n be the mass of $\boldsymbol{V}_n$. Then we define the mass density at the point P by

$$\rho(P) = \lim_{\substack{n\to\infty \\ V_n\to\infty}} \frac{M_n}{V_n} \tag{1.1.1}$$

If the mass density is well defined everywhere (at all points P) in $\boldsymbol{V}_0$, the mass is said to be continuously distributed in $\boldsymbol{V}_0$.

A similar scheme can be used to define densities of *force*, *linear and angular momentum*, *energy*, and other variables necessary to model the material. We then can say that a material continuum is one for which all density functions and modeling variables continuously exist in the mathematical sense. The *mechanics* of such materials is *continuum mechanics*. As we shall see, this type of theory is very useful in describing the gross or macroscopic behavior of a large variety of real materials.

Continuum mechanics generally ignores the fine details of the material's microstructure and replaces the discontinuous (likely nano- or microscopic) medium with a hypothetical or model continuum. The resulting model then describes the material behavior under the study using *field quantities* such as *displacements*, *velocities*,

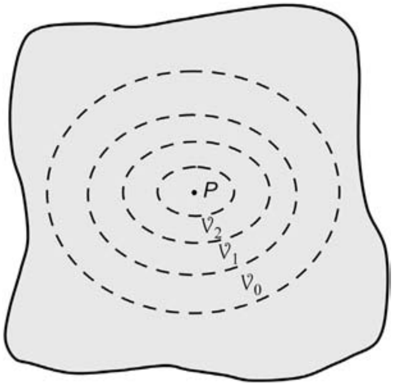

FIGURE 1.3 The concept of continuum mass density.

stresses, etc. which are *piecewise continuous.* The usefulness of continuum mechanics, however, can also be extended to cases which include inhomogeneity and discontinuity such as those used in micromechanical modeling.

1.2 NEED FOR TENSORS

Continuum mechanics is formulated in terms of many different types of variables: *scalars*, represented by a single value at each point in space (e.g. material density); *vectors*, expressible in terms of three components in a three-dimensional space (e.g. material displacement or velocity); *matrix variables*, which commonly require nine components to quantify (e.g. stress or strain). Other applications incorporate additional quantities that require even more components to characterize. Because of this complexity, continuum mechanics makes use of a *tensor formalism* which enables efficient representation of all variables and governing equations using a single standardized scheme. The tensor concept will be defined more precisely in Chapter 2, but for now we can simply say that scalars, vectors, matrices, and other higher-order variables can all be represented by tensors of various orders.

Another important point is that in order to develop a set of general laws and principles, the fundamental relations in continuum mechanics must be formulated in terms of quantities *that are independent of the coordinate frame used to describe the problem*. Thus, if two individuals using different coordinate frames observe a common physical event (see Fig. 1.4), it should always be possible to state a physical law governing the event, such that if the law is true for one observer, it will also be true for the other (once adjusted for the difference in coordinate frame). This concept is normally referred to as the principle of *objectivity*, *frame indifference*, or *isotropy of space*. This situation will require all relations to be written in an appropriate *tensor format* so as to guarantee the proper invariance. Thus, the form of any field equation

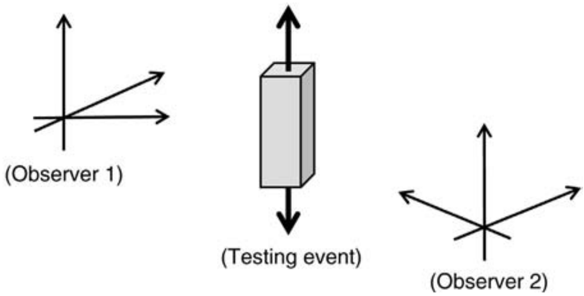

FIGURE 1.4 The concept of principle of objectivity or frame indifference.

can have general validity in any reference frame only if every term in the equation has the same tensorial characteristics. If this condition is not satisfied, a simple change in the coordinates will destroy the form of the relationship, thus indicating that the original form was only fortuitous and/or accidental.

For these reasons, use of tensors is essential for the formulation and solution of field equations in continuum mechanics. Consequently, the knowledge of tensor theory is required for this study, and we will devote appropriate detailed coverage of this topic in Chapter 2. In his text, Fung (1994) makes the insightful and poignant statement with regard to tensors: “A beautiful story needs a beautiful language to tell. Tensor is the language of mechanics.”

1.3 STRUCTURE OF THE STUDY

Continuum mechanics may be generally considered as a four-part structure leading to a final model of particular materials as shown in Fig. 1.5. Kinematics deals with the geometrical relations between material motion, strain, displacement, and various rate-dependent variables. These topics will be covered in detail in Chapter 3. Additional fundamental relations include the concepts of force, traction, and stress at continuum points, and these are presented in Chapter 4. General balance principles involve the continuum interpretation of conservation of mass, momentum, and energy along with some fundamental thermodynamic laws, and these are given in Chapter 5. All of these are basic relations common to all continua irrespective of their material properties.

Constitutive relations characterize a particular material's macroscopic response to applied mechanical, thermal, or other types of loadings. Such relations are based on the material's internal constitution and commonly result in idealized material models such as *linear elastic solids* or *linear viscous fluids*. These models do not come directly from general principles but rather are often developed from observed experimental data. Combining constitutive laws with the general principle relations creates a closed system of field equations that contains sufficient number of equations

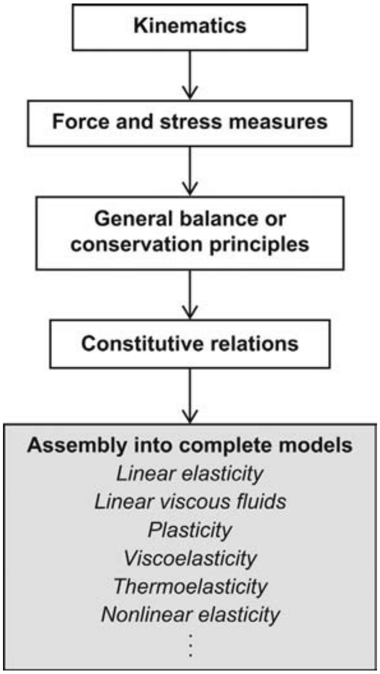

FIGURE 1.5 Structure of continuum mechanics.

to solve for all of the model unknowns.. It should be noted that solutions to particular problems within a given model require appropriate boundary and/or initial conditions.

This text will place emphasis on exploring a very wide collection of constitutive relations and developing the corresponding material model formulations. Such material behavior models will include classical linear theories of elasticity, fluid mechanics, viscoelasticity, and plasticity. Additional linear theories including multiple constitutive fields such as poro, thermo, and electro will also be developed. Nonlinear theories of solids and fluids including finite elasticity, nonlinear viscous fluids, and nonlinear viscoelastic materials will be presented. Finally, several relatively new continuum theories based on incorporation of material microstructure will be presented including: micropolar elasticity, elasticity with voids, nonlocal higher gradient elasticity, fabric tensor theories for granular materials, and damage mechanics.

1.4 A LITTLE HISTORY

Before embarking on the theoretical aspects of continuum mechanics, a very brief historical presentation of the subject will be given. This account is intended to provide a short overview of some of the major developments over the last few centuries, with emphasis on the twentieth century. Much more detailed historical information may be found in Maugin (2013), Truesdell and Toupin (1960), Truesdell and Noll (1965), Truesdell (1966), Soutas-Little (2011), Doraiswamy (2017), Walters (2017), and Tanner and Walters (1998).

Solid mechanics is perhaps one of the oldest branches of physical sciences, and many would say that it can be traced back to Archimedes (250 BC) who claimed that he could move the earth if he were given a proper place to stand and a lever long enough. Many prominent and brilliant scientists and mathematicians have made numerous contributions to continuum mechanics. Studies began with *rigid solids* which were Euclidean bodies that do not undergo deformation. Through human curiosity and societal needs to successfully build things, the study of material behavior has been explored for countless centuries. Early development of quantitative material behavior relations began in the 17th century with Galileo Galilei (nature of the resistance of solids), Robert Hooke (the linear elastic response), and of course Isaac Newton's *Principia*. The following century brought forward considerable important work including that of Leonhard Euler and Joseph-Louis Lagrange on deformation and strain. In the 19th century, Claude-Louis Navier presented results on the general equations of equilibrium and Augustin-Louis Cauchy published his work on the concept of stress at a point. During this time period, considerable work was developed on linear elasticity by Siméon Denis Poisson, Gabriel Léon Jean Baptiste Lamé, Albert Green, Adhémar Jean Claude Barré de Saint-Venant, George Airy, and others. Toward the latter half of the 1800s, Gabrio Piola, Gustav Kirchhoff, and Joseph Valentin Boussinesq initiated studies on the theory of finite deformations, and James Clerk Maxwell, William Lord Kelvin, and Ludwig Boltzmann proposed early viscoelastic constitutive material relations.

In regard to fluid mechanics, early work can be traced back to Leonardo da Vinci and others in the 16th century. Later in the 18th century, Leonhard Euler developed the concept of continuity of flow, and Daniel Bernoulli introduced the term *hydrodynamics* and formulated *Bernoulli's Principle*. Inviscid fluid mechanics was established, and work dealing with fluid friction was beginning. The next century brought the work of Navier and George Stokes together to formulate the famous governing relations for linearly viscous fluids—the *Navier–Stokes equations*. Even though these governing relations used a linear relation between stress and deformation rate, the equations are nonlinear and thus they proved to be very challenging to solve at this point in history. In the early part of the 20th century, Ludwig Prandtl provided important work to relate inviscid and viscous flows, and later developed important *boundary layer theory*.

The 19th century led to the development of the science of thermodynamics, which had its origins previously connected with heat engines. During the 1800s, scientists like Nicolas Léonard Carnot, James Prescott Joule, William John Rankine, and Rudof Clausius (initial definition of entropy) made fundamental contributions in this field, and in 1850 Lord Kelvin first labeled the study thermodynamics. Later in this century, Maxwell made studies of statistical thermodynamics and Josiah Willard Gibbs first defined *free energy* and *enthalpy*. During this time period, Jean-Baptiste Joseph Fourier made fundamental contributions in heat transfer.

In 1929, Eugene Bingham helped to create a branch of physics called *rheology* that specifically dealt with problems pertaining to the behavior of unusual liquids and solids. After consulting a language professor, Bingham took "rheo" from Greek, meaning "everything flows" and "-ology" meaning "the study of". That same year the *Society of Rheology* was formed eventually leading to a specialized journal presenting research on the subject matter. Rheology is classically defined as the study of flow of matter, primarily in a liquid state, but also as soft solids under conditions in which they respond with some combination of elastic, plastic, and viscous deformation behaviors. Rheology is obviously closely connected with continuum mechanics and many would categorize it as a subarea of study within the overall structure.

In general, most work before the 20th century was concerned with linear continuum theories of solids and fluids. By the turn of the 20th century, there was general acknowledgement of the existence of materials which could not be classified as Hookean solids or Newtonian fluids, and research began to come forward to describe behavior of such materials. Starting in the 1930s, rheologists Bingham and Markus Reiner began to explore some of these more complicated behaviors. In the 1940s and 50s, a great resurgence began that led to the development of much more general and mathematical material behavior formulations. Major contributions were made by Ronald Rivlin, James Gardner Oldroyd, and others.

During the 1950s, Clifford Truesdell established himself as the godfather of modern continuum mechanics or as he coined it *rational mechanics*. He began his brilliant work with a 1952 review article (later reprinted in 1966) that set down the modern fundamentals of elasticity and fluid mechanics (Truesdell, 1966). Among his numerous contributions were two monumental published works *Classical Field Theories* (1960) and *Nonlinear Field Theories* (1965) which developed a rigorous mathematical approach to continuum mechanics. He also initiated a new journal devoted to this field. Through this work, he stimulated many others to follow this mathematical path. Jerald Ericksen, Walter Noll, Richard Toupin, Bernard Coleman, and others made important contributions to the subject during the 1960s and 70s. Extensive research and publications on nonlinear theories of elasticity, viscoelasticity, plasticity, and non-Newtonian fluids occurred during this time period. Also, this era ushered in computational finite and boundary element methods that greatly aided solution capabilities for the application of continuum theories to problems of engineering interest.

The latter half of the 20th century produced a wide variety of work looking to extend traditional continuum theories to model behavior of multiphase composites, graded materials, granular substances, cellular and porous solids, materials with microstructure, coupled multifield problems, etc. For example, Ahmed Cemal Eringen developed theoretical foundations of micropolar and nonlocal theories of continuous media. Raymond Mindlin and Elias Aifantis also explored such developments by looking at continuously distributed microstructure and strain gradient theories of material behavior. Bernard Budiansky and Mark Kachanov studied elastic materials with distributed cracks, and Stephen Cowin and Jace Nuziato looked at materials with distributed voids. Following the previous fundamental work on soil mechanics by Karl von Terzaghi, Maurice Biot developed the theory of poroelasticity that incorporates the combined response of a fluid-saturated porous elastic material. Many other interesting new continuum theories of this type have been developed over the last few decades, and such work continues to the present day.

For well over a century, the scientific engineering community has successfully employed continuum mechanics to simulate the mechanical, thermal, and electrical/magnetic behavior of a broad class of materials under a variety of loading conditions. These theories have also provided necessary tools for the construction of *computational schemes* based on *finite differences* and *finite and boundary element methods*. By any measure, continuum mechanics has made significant contributions to our formulation and solution of engineering problems and has provided significant help in the design of safe and efficient structures and systems.

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