CHAPTER

Kinematics of Motion and Deformation Measures

3

3.1 MATERIAL BODY AND MOTION

We wish to start with a general definition of a *material body* which can describe a broad class of continuum materials. For this purpose, we define a material body B as a continuum set of particles or material points X (see Fig. 3.1). Note that these particles are not discrete mass points as in Newtonian mechanics, but instead they are infinitesimally small portions of a continuous media with definable mass density (see discussions in Section 1.1). For each of these particles, we assign a one-to-one mapping to spatial points $\boldsymbol{X}$ in a three-dimensional Euclidean space that the particles occupy at a given instant of time t_o. This then establishes a particular *configuration* of body B, and we will call this the *reference configuration.* This could be selected as the initial configuration at $t = 0$, or for solids it might be taken as the undeformed configuration. However, the choice of the reference configuration is completely arbitrary.

As the continuum deforms at subsequent times particles move, and this motion can be expressed by the simple relation

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \tag{3.1.1}$$

Thus, particle X at location $\boldsymbol{X}$ in the reference configuration moves to a new location $\boldsymbol{x}$ in the *current configuration* at time t as shown in Fig. 3.1. Note that when $t = t_o$, relation (3.1.1) gives $\boldsymbol{X} = \boldsymbol{\chi}(\boldsymbol{X}, t_0)$. Certain requirements must be made on the allowable types of motion in continuum mechanics. First, it is required that the motion (mapping) be one to one, so that two particles do not end up at the same place at later times. This is often referred to as the *impenetrability of matter*. We also require that the inverse motion exists

$$\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) \tag{3.1.2}$$

and that the motion and its inverse are continuously differentiable functions. Under these conditions, it follows that the *Jacobian determinant* $J = \det(\partial\boldsymbol{x}/\partial\boldsymbol{X})$ cannot vanish, and in fact we will normally assume

$$0 < \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right) < \infty \tag{3.1.3}$$

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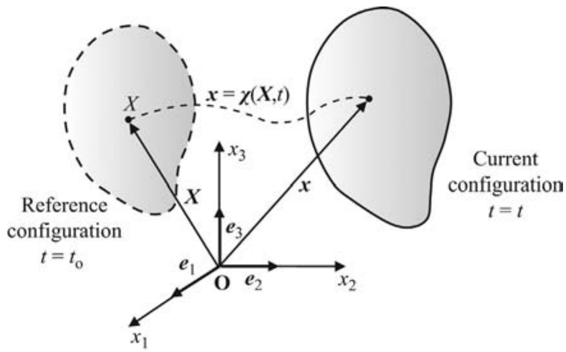

FIGURE 3.1

Continuum motion.

Later in our studies, we will develop some continuum theories for material models that will assume small infinitesimal deformations which will not require the rigor of the two configuration situation shown in Fig. 3.1.

EXAMPLE 3.1.1 SIMPLE SHEARING MOTION

Explore basic features of the following *simple shearing motion* specified in direct vector form as

$$\boldsymbol{x} = \boldsymbol{X} + \gamma X_2 \boldsymbol{e}_1 \tag{3.1.4}$$

where γ is the amount of shear and could be a function of time.

Solution: In component form, this motion is expressed as

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \tag{3.1.5}$$

This motion is confined in the X_1,X_2-plane and is perhaps best illustrated if we consider the deformation of a square element that might be part of a larger body. The reference configuration is shown in Fig. 3.2 as the dotted square with corners *ABCD*. Under the given motion, the reference element shears into a rhombus shape shown by the solid line. Points *A* and *B* do not move because they lie on the line $X_2 = 0$. Points *C* and *D* move only in the X_1-direction in direct proportion to their X_2 coordinate value. We will often use this simple type of deformation field in later chapters to demonstrate particular features of material constitutive laws. Note that for this simple example we can easily determine the inverse relation (3.1.2):

$$\begin{aligned} X_1 &= x_1 - \gamma x_2 \\ X_2 &= x_2 \\ X_3 &= x_3 \end{aligned} \tag{3.1.6}$$

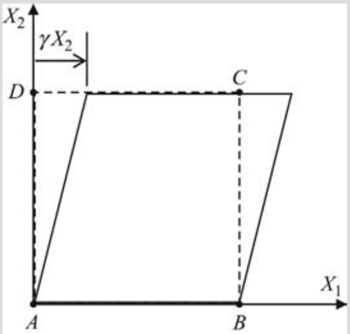

FIGURE 3.2

Simple shearing motion.

EXAMPLE 3.1.2 MATLAB COMPUTER PLOTS OF DEFORMATION/ MOTION

Use MATLAB Code C-3 to make computer plots of the deformation of a unit square and unit circle for the following two-dimensional motion:

$$x_1 = 1.2X_1 + 0.2X_2 + 1.2$$
$$x_2 = 0.2X_1 + 1.2X_2 + 1.2$$
$$x_3 = X_3$$

Solution: Running the code with the given motion yields Fig. 3.3.

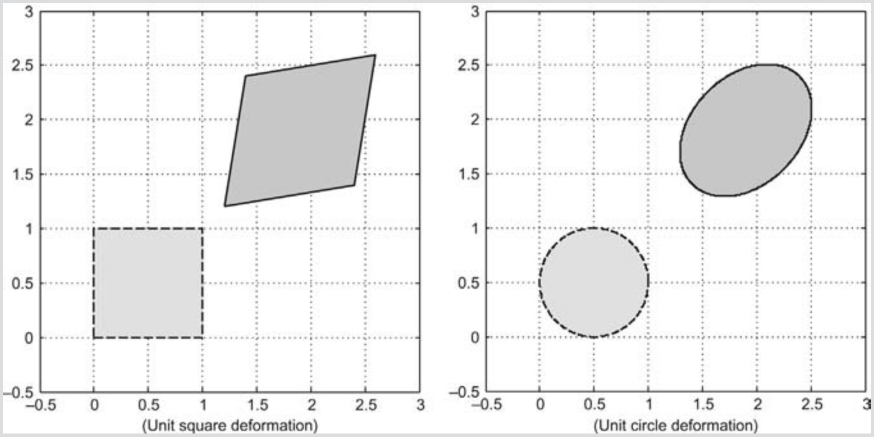

FIGURE 3.3

Computer deformation plots.

3.2 LAGRANGIAN AND EULERIAN DESCRIPTIONS

Motion equation (3.1.1) $x_i = \chi_i\left(X_1, X_2, X_3, t\right)$ can be thought of as a relationship between the *reference* or *material coordinates* X_i and the *current* or *spatial coordinates* x_i. Thus, if we know the material coordinates of a given particle, this relation will allow us to determine the particle's location in the current configuration. Likewise, the inverse equation (3.1.2). $X_i = \chi_i^{-1}\left(x_1, x_2, x_3, t\right)$ gives the opposite relationship, and consequently we can use this form to determine the material coordinates of a particular particle in the current configuration. Example 3.1.1 demonstrates these points for the simple shearing case.

All spatial field variables in continuum mechanics (density, temperature, displacement, strain, stress, etc.) can be described in terms of either the material coordinates X_i or the spatial coordinates x_i. A material coordinate description can be easily transformed into a spatial form or a spatial form transformed into a material description by using (3.1.1) or (3.1.2). Following particles, we can express tensor quantities as functions identified by the material coordinates (X_1,X_2,X_3). Such a description is known as the *Lagrangian*, *material*, or *reference description*. Using the other scheme, we can observe changes at *fixed locations* and thus express tensor quantities as functions of *position coordinates* (x_1,x_2,x_3). Such a description is known as an *Eulerian or spatial description*. Notice that as time progresses, different particles will occupy the same spatial position, and thus a spatial description will not provide specific information about particle properties during the motion. The following example illustrates some of the basic features of each description.

EXAMPLE 3.2.1 EULERIAN AND LAGRANGIAN TEMPERATURE DESCRIPTIONS UNDER SIMPLE SHEARING MOTION

Under simple shearing motion from Example 3.1.1 where $\gamma = \gamma_o t$, consider a continuum with spatial temperature field distribution $\theta = Ax_1 + Btx_2 + Cx_3$, where A, B, and C are constants. Find the material description of the temperature and its time rate of change for particular material particles.

Solution: The temperature is easily converted into a material description by using (3.1.5):

$$\begin{aligned}\theta &= Ax_1 + Btx_2 + Cx_3 \\ &= A(X_1 + \gamma_o tX_2) + BtX_2 + C_3X_3 \\ &= AX_1 + (A\gamma_o + B)tX_2 + C_3X_3\end{aligned}$$

The time rate of change in temperature for a given particle is

$$\left(\frac{\partial\theta}{\partial t}\right)_{X_i-\text{fixed}} = \frac{\partial}{\partial t}\left(AX_1 + (A\gamma_o + B)tX_2 + C_3X_3\right) = (A\gamma_o + B)X_2 = (A\gamma_o + B)x_2$$

3.3 MATERIAL TIME DERIVATIVE

The time rate of change of a tensor quantity following a material particle is known as the *material time derivative* and is commonly denoted by *D*/*Dt*. As we did in the previous example, when the material description of a given tensor field $\boldsymbol{T}$ is used, such a derivative is calculated in the straightforward manner

$$\frac{D\boldsymbol{T}}{Dt}=\frac{\partial}{\partial t}\boldsymbol{T}(X_1,X_2,X_3,t)\Big|_{x_i-fixed} \tag{3.3.1}$$

However, when the spatial description is used for tensor $\boldsymbol{T}$, the time derivative is a bit more complex since the spatial coordinates themselves are now functions of time. This requires use of the chain rule

$$\begin{aligned}\frac{D\boldsymbol{T}}{Dt}&=\frac{\partial}{\partial t}\boldsymbol{T}(x_1,x_2,x_3,t)\Big|_{X_i-fixed}\\&=\frac{\partial\boldsymbol{T}}{\partial x_1}\frac{\partial x_1}{\partial t}+\frac{\partial\boldsymbol{T}}{\partial x_2}\frac{\partial x_2}{\partial t}+\frac{\partial\boldsymbol{T}}{\partial x_3}\frac{\partial x_3}{\partial t}+\left(\frac{\partial\boldsymbol{T}}{\partial t}\right)_{x_i-fixed}\\&=\frac{\partial\boldsymbol{T}}{\partial x_i}\frac{\partial x_i}{\partial t}+\left(\frac{\partial\boldsymbol{T}}{\partial t}\right)_{x_i-fixed}\end{aligned} \tag{3.3.2}$$

Notice that we have used a little hybrid mix of direct and index notation. The quantities $\frac{\partial x_i}{\partial t}$ are taken with X_i fixed, and hence actually represent the velocity v_i of the continuum particle. Consequently, (3.3.2) can be expressed as

$$\frac{D\boldsymbol{T}}{Dt}=\frac{\partial\boldsymbol{T}}{\partial t}+\boldsymbol{T}_{,i}v_i=\frac{\partial\boldsymbol{T}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{T} \tag{3.3.3}$$

EXAMPLE 3.3.1 MATERIAL TIME DERIVATIVE OF EXAMPLE 3.2.1

Consider again the temperature distribution problem given in Example 3.2.1 under simple shearing motion. Using the spatial distribution form $\theta=Ax_1+Btx_2+Cx_3$, calculate the material time derivative of the temperature field.

Solution: We use relation (3.3.3) for the temperature (zeroth-order tensor) field

$$\frac{D\theta}{Dt}=\frac{\partial\theta}{\partial t}+\theta_{,i}v_i=\frac{\partial\theta}{\partial t}+\boldsymbol{v}\cdot\nabla\theta$$

First, we need the velocity field $v_i=\frac{\partial x_i}{\partial t}$. Using the shearing motion relations (3.1.5), we get

$$\begin{aligned}v_1&=\gamma_o X_2=\gamma_o X_2\\v_2&=v_3=0\end{aligned}$$

Next calculating the temperature gradient $\nabla\theta = \left\{ \frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2}, \frac{\partial\theta}{\partial x_3} \right\} = \{A, Bt, C\}$. Using these results,

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \boldsymbol{v} \cdot \nabla\theta = Bx_2 + A\gamma_o x_2 = (A\gamma_o + B)x_2$$

which is the same result we found in the previous example using a different method.

3.4 VELOCITY AND ACCELERATION

As discussed previously, the velocity of a continuum particle is given by its time rate of change of position

$$v_i(\boldsymbol{X},t) = \frac{\partial}{\partial t} x_i(\boldsymbol{X},t)\bigg|_{X_i-\text{fixed}} = \frac{Dx_i}{Dt} \tag{3.4.1}$$

Likewise, the acceleration is the time rate of change of the velocity

$$a_i(\boldsymbol{X},t) = \frac{\partial}{\partial t} v_i(\boldsymbol{X},t)\bigg|_{X_i-\text{fixed}} = \frac{Dv_i}{Dt} \tag{3.4.2}$$

If the velocity is expressed in spatial form $v_i(\boldsymbol{x},t)$, then the acceleration follows from relation (3.3.3):

$$\boldsymbol{a}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla\boldsymbol{v} \tag{3.4.3}$$

or in index notation

$$a_i(\boldsymbol{x},t) = \frac{\partial v_i}{\partial t} + v_j v_{i,j} \tag{3.4.4}$$

Note that $v_{i,j}$ is the spatial *velocity gradient tensor* (see relation (2.17.4)). Thus, if the velocity is expressed in terms of material coordinates, the acceleration is found by simple partial differentiation with respect to time, relation (3.4.2). On the other hand, if the velocity is specified in spatial coordinates, the acceleration is determined by a more complicated procedure given by relations (3.3.3) or (3.4.4).

EXAMPLE 3.4.1 VELOCITY AND ACCELERATION FROM A MATERIAL MOTION

The motion of a continuum is given in material coordinates by

$$x_1 = X_1 + AtX_2$$
$$x_2 = X_2 + Bt^2X_1$$
$$x_3 = X_3$$

where A and B are constants. Determine

(a) the velocity and acceleration field in material coordinates;

(b) the path in the space of a particular particle that was located at $\boldsymbol{X} = \{1,2,4\}$.

Solution: Since the motion is given in material coordinates, we can simply use relations (3.4.1) and (3.4.2) to get

$$v_i(X,t) = \frac{\partial}{\partial t} x_i(X,t)|_{X_i\text{-fixed}} \quad \Rightarrow \quad \begin{aligned} v_1 &= AX_2 \\ v_2 &= 2BtX_1 \\ v_3 &= 0 \end{aligned}$$

$$a_i(X,t) = \frac{\partial}{\partial t} v_i(X,t)|_{X_i\text{-fixed}} \quad \Rightarrow \quad \begin{aligned} a_1 &= 0 \\ a_2 &= 2BX_1 \\ a_3 &= 0 \end{aligned}$$

For the particle located at $\boldsymbol{X} = \{1,2,4\}$, motion now specifies $x_1 = 1 + 2At$, $x_2 = 2 + Bt^2, x_3 = 4$. Since the motion is simply a time-parameterized expression, we can simply eliminate the time from these motion expressions to get

$$x_2 = \frac{B}{4A^2} x_1(x_1 - 2) + \frac{B}{4A^2} + 2, \quad x_3 = 4$$

and so the particle path is a parabolic form in the plane $x_3 = 4$.

EXAMPLE 3.4.2 ACCELERATION FROM SPATIAL VELOCITY FIELD FOR RIGID BODY ROTATION

Determine the acceleration field for a rigid body rotation about the x_3-aixs with a velocity field given by $\boldsymbol{v} = -\omega x_2 \boldsymbol{e}_1 + \omega x_1 \boldsymbol{e}_2 + 0\boldsymbol{e}_3$, where ω is the constant angular velocity.

Solution: First, calculate the velocity gradient tensor

$$v_{i,j} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_{i,j}v_j = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\omega x_2 \\ \omega x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \\ 0 \end{bmatrix}, \quad \frac{\partial v_i}{\partial t} = 0$$

Using (3.4.4)

$$\Rightarrow a_i(\mathbf{x},t) = \frac{\partial v_i}{\partial t} v_j v_{i,j} = -\omega^2 \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

3.5 DISPLACEMENT AND DEFORMATION GRADIENT TENSORS

We now begin our exploration of a few fundamental kinematical variables related to continuum deformation. Fig. 3.4 illustrates again the basic concept of material motion between the reference configuration and the current configuration. Consider two neighboring points $\boldsymbol{P}$ and $\boldsymbol{Q}$ in the reference configuration. Through the general motion $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$, these two points get mapped into $\boldsymbol{P'}$ and $\boldsymbol{Q'}$ in the current configuration. Point $\boldsymbol{P}$ located at $\boldsymbol{X}$ then undergoes a *displacement* $\boldsymbol{u}$ so that it arrives at point $\boldsymbol{P'}$ located at $\boldsymbol{x}$.

We thus can write

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t) \tag{3.5.1}$$

the neighboring point $\boldsymbol{Q}$ then has a similar displacement relation

$$\boldsymbol{x} + d\boldsymbol{x} = \boldsymbol{X} + d\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + d\boldsymbol{X}, \mathrm{t}) \tag{3.5.2}$$

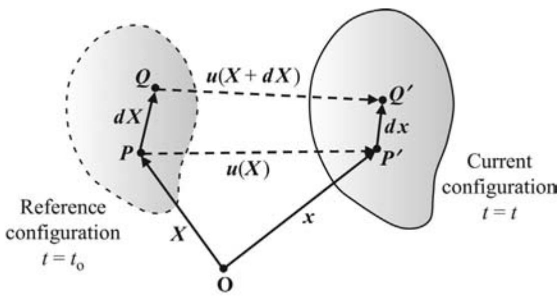

FIGURE 3.4

Displacements in continuum motion.

combining these relations gives

$$
\begin{aligned}
d\boldsymbol{x} &= d\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + d\boldsymbol{X}, t) - \boldsymbol{u}(\boldsymbol{X}, t) \\
&= d\boldsymbol{X} + \frac{\partial \boldsymbol{u}}{\partial \mathbf{X}} d\boldsymbol{X} \\
&= d\boldsymbol{X} + \nabla \boldsymbol{u}\, d\boldsymbol{X}
\end{aligned}
\tag{3.5.3}
$$

The term $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \nabla \boldsymbol{u}$ is the *displacement gradient tensor* with respect to material coordinates and it can be expressed in full matrix form as

$$
\nabla \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}
\tag{3.5.4}
$$

it also follows that

$$
d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}
\tag{3.5.5}
$$

where

$$
\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}, \quad F_{ij} = \frac{\partial x_i}{\partial X_j}
\tag{3.5.6}
$$

is the *deformation gradient tensor*. Note that this tensor is also taken with respect to the material coordinates and can be written out in matrix form

$$
\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix}
\tag{3.5.7}
$$

Combining (3.5.3) and (3.5.5), a simple relationship can easily be derived between the deformation gradient and the displacement gradient

$$
\boldsymbol{F} = \boldsymbol{I} + \nabla \boldsymbol{u}
\tag{3.5.8}
$$

EXAMPLE 3.5.1 DISPLACEMENT AND DEFORMATION GRADIENT TENSORS FOR SIMPLE SHEARING MOTION

For the simple shearing motion given in Example 3.1.1, determine the displacement field, displacement gradient tensor, and the deformation gradient tensor.

Solution: From relation (3.5.1), $\boldsymbol{u}(\boldsymbol{X},t)=\boldsymbol{x}-\boldsymbol{X}$, and so for the motion (3.1.5),

$$\begin{aligned} u_1 &= x_1 - X_1 = \gamma X_2 \\ u_2 &= x_2 - X_2 = 0 \\ u_3 &= x_3 - X_3 = 0 \end{aligned}$$

The displacement gradient tensor follows from $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \nabla\boldsymbol{u} = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The deformation gradient is calculated from $\boldsymbol{F} = \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note that these results satisfy the relation $\boldsymbol{F}=\boldsymbol{I}+\nabla\boldsymbol{u}$.

EXAMPLE 3.5.2 DISPLACEMENTS AND DEFORMATION GRADIENT TENSOR FOR EXTENSIONAL MOTION

Three-dimensional *extensional motion* may be specified by

$$\begin{aligned} x_1 &= \lambda_1 X_1 \\ x_2 &= \lambda_2 X_2 \\ x_3 &= \lambda_3 X_3 \end{aligned} \tag{3.5.9}$$

where λ_i are constants often called the *stretch ratios*. Determine the displacement field, displacement gradient tensor, and the deformation gradient tensor for this motion.

Solution: From relation (3.5.1), $\boldsymbol{u}(\boldsymbol{X},t)=\boldsymbol{x}-\boldsymbol{X}$, and so for the motion (3.5.9),

$$\begin{aligned} u_1 &= x_1 - X_1 = (\lambda_1 - 1)X_1 \\ u_2 &= x_2 - X_2 = (\lambda_2 - 1)X_2 \\ u_3 &= x_3 - X_3 = (\lambda_3 - 1)X_3 \end{aligned}$$

The displacement gradient tensor follows from

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \nabla\boldsymbol{u} = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}$$

The deformation gradient is calculated from $\boldsymbol{F} = \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Note that these results satisfy the relation $\boldsymbol{F}=\boldsymbol{I}+\nabla\boldsymbol{u}$.

Consider next a few comments on some special deformations. A *homogeneous deformation* is one where the deformation gradient tensor is independent of the coordinates. The previous two examples of simple shear and constant extension were both homogeneous deformations, and many experimental testing procedures attempt to maintain this type of deformation within the specimen under study. Note that if we only have rigid body translation, the motion is $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{c}$, where $\boldsymbol{c}$ is the constant translation, and the deformation gradient reduces to the unit tensor $\boldsymbol{F} = \boldsymbol{I}$. It should be apparent that the deformation gradient will not be changed if we add a uniform translation to the motion since the gradient of the scalar addition will always vanish. For the special case of rigid body rotation (about the origin) using a constant rotation tensor $\boldsymbol{R}$, the motion becomes $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{X}$, and thus the deformation gradient is given by $\boldsymbol{F} = \boldsymbol{R}$.

EXAMPLE 3.5.3 DISPLACEMENTS AND DEFORMATION GRADIENT TENSOR FOR NONHOMOGENEOUS DEFORMATION

Consider now a nonhomogeneous deformation case with motion

$$\begin{aligned} x_1 &= AX_1X_2 \\ x_2 &= BX_2^2 \\ x_3 &= CX_3 \end{aligned}$$

where A, B, and C are constants. Determine the displacement field, displacement gradient tensor, and the deformation gradient tensor for this motion.

Solution: From relation (3.5.1), $\boldsymbol{u}(\boldsymbol{X},t) = \boldsymbol{x} - \boldsymbol{X}$, and so for this motion

$$\begin{aligned} u_1 &= x_1 - X_1 = (AX_2 - 1)X_1 \\ u_2 &= x_2 - X_2 = (BX_2 - 1)X_2 \\ u_3 &= x_3 - X_3 = (C - 1)X_3 \end{aligned}$$

The displacement gradient tensor follows from

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \nabla \boldsymbol{u} = \begin{bmatrix} AX_2 - 1 & AX_1 & 0 \\ 0 & 2BX_2 - 1 & 0 \\ 0 & 0 & C - 1 \end{bmatrix}$$

The deformation gradient is calculated from $\boldsymbol{F} = \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} AX_2 & AX_1 & 0 \\ 0 & 2BX_2 & 0 \\ 0 & 0 & C \end{bmatrix}$

Both the displacement gradient and deformation gradient depend on material coordinates and thus the deformation is referred to as nonhomogeneous.

Next let us examine the concept of *sequential deformations* whereby the total deformation is carried out through a sequence of two or more separate steps. Consider the case of a two-step process composed of deformation (1) followed by deformation (2). Using the general result (3.5.6), we can express the first step as $d\boldsymbol{x}^{(1)} = \boldsymbol{F}^{(1)}d\boldsymbol{x}$. Likewise, the second deformation step is then given by $d\boldsymbol{x}^{(2)} = \boldsymbol{F}^{(2)}d\boldsymbol{x}^{(1)} = \boldsymbol{F}^{(2)}\boldsymbol{F}^{(1)}d\boldsymbol{X}$. We can then conclude that for a two-step sequential deformation process of $\boldsymbol{F}^{(1)}$ followed by $\boldsymbol{F}^{(2)}$, the overall deformation gradient is given by the product $\boldsymbol{F}^{(2)}\boldsymbol{F}^{(1)}$. The overall deformation can then be written as

$$d\boldsymbol{x} = \boldsymbol{F}^{(2)}\boldsymbol{F}^{(1)}d\boldsymbol{X} \tag{3.5.10}$$

It is important to note that the product is carried out in reverse order to the deformation steps, and changing this order will result in a totally different overall final deformation. This concept can be shown graphically in Fig. 3.5 which illustrates two 2-step sequential deformations where the order has been switched. It is clearly evident that by changing the sequence order, the final deformations are quite different.

Based on our previous work, we can consider a *spatial deformation gradient tensor* $\boldsymbol{F}^{-1}$:

$$\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}, \quad F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} \tag{3.5.11}$$

that maps the spatial line element $d\boldsymbol{x}$ to the material element $d\boldsymbol{X}$ defined by

$$d\boldsymbol{X} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}d\boldsymbol{x} = \boldsymbol{F}^{-1}d\boldsymbol{x} \tag{3.5.12}$$

The tensor $\boldsymbol{F}^{-1}$ can rightfully be called *inverse of the deformation gradient* $\boldsymbol{F}$ since

$$F_{ij}F_{jk}^{-1} = \frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial x_i}{\partial x_k} = \delta_{ik} \Rightarrow \boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I} \tag{3.5.13}$$

It should be noted that the deformation gradient involves both the spatial coordinates $\boldsymbol{x}$ and the material coordinates through the gradient operation $\partial/\partial\boldsymbol{X}$. The same

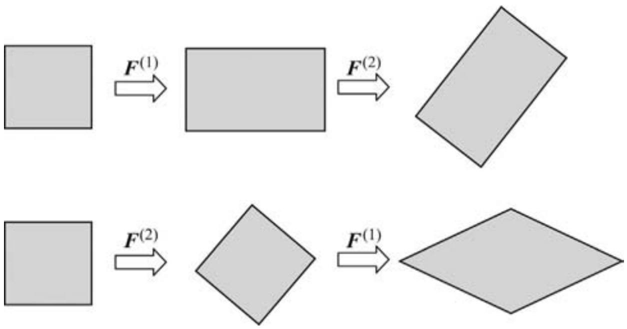

FIGURE 3.5

Two-step sequential deformations with different sequences ($\boldsymbol{F}^{(1)}$—horizontal stretch; $\boldsymbol{F}^{(2)}$ 45°—CCW rotation).

can be said concerning the spatial or inverse deformation gradient $\boldsymbol{F}^{-1}$. Because of this, both $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are called *two-point tensors* being coupled to both configurations. Sometimes, these tensors are written as

$$\begin{aligned}\boldsymbol{F} &= \frac{\partial \boldsymbol{x}}{\partial X_j} \otimes \boldsymbol{E}_j = \frac{\partial x_i}{\partial X_j} \boldsymbol{e}_i \otimes \boldsymbol{E}_j \\ \boldsymbol{F}^{-1} &= \frac{\partial \boldsymbol{X}}{\partial x_j} \otimes \boldsymbol{e}_j = \frac{\partial X_i}{\partial x_j} \boldsymbol{E}_i \otimes \boldsymbol{e}_j\end{aligned} \tag{3.5.14}$$

where $\boldsymbol{e}_i$ are the spatial basis vectors and $\boldsymbol{E}_i$ are the basis vectors in the material configuration.

We next consider how the deformation gradient tensor transforms under the objectivity test established in Section 2.9. The deformation gradients in the two reference frames shown previously in Fig. 2.4 would simply be

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}, \quad \boldsymbol{F}^* = \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{X}}$$

Using the chain rule gives

$$\boldsymbol{F}^* = \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{Q}\boldsymbol{F} \tag{3.5.15}$$

and thus the deformation gradient $\boldsymbol{F}$ satisfies the objectivity test for vectors (2.9.2) but not for second-order tensors as given by (2.9.3). This is expected as $\boldsymbol{F}$ is a function singly of the vector $\boldsymbol{x}$, and the reference position $\boldsymbol{X}$ is to remain the same within the objectivity study. Following analogous steps for the inverse $\boldsymbol{F}^{-1}$:

$$\boldsymbol{F}^{*-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}^*} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}^*} = \boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}} \tag{3.5.16}$$

and this leads to a similar conclusion that $\boldsymbol{F}^{-1}$ does not satisfy the objectivity test for second-order tensors. The deformation gradient and its inverse play key roles in describing the *local* deformation in the neighborhood of a continuum particle; however, we will need to construct additional second-order tensors that more properly describe straining deformation and satisfy the objectivity test.

3.6 LAGRANGIAN AND EULERIAN STRAIN TENSORS

From our previous kinematical discussions, the motion relation $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ can produce movements of continuum particles of unlimited variety and complexity that would include rigid body translation and rotation along with other movements that will cause *relative changes in position* between two or more particles. These relative motions or deformations are what we commonly refer to as *strain*. Rigid body motions will generally be of little interest since they are not related to internal force or stress distributions. Fig. 3.6A illustrates some of these basic concepts graphically

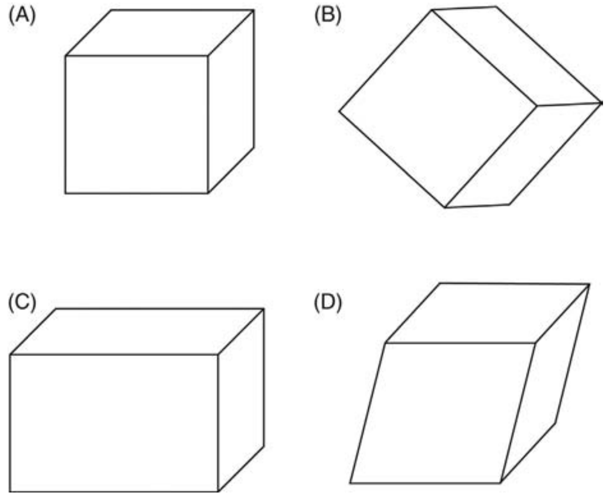

FIGURE 3.6

Examples of motion and deformation: (A) reference element; (B) rigid body motion; (C) extensional deformation; (D) shearing deformation.

by starting with a Cartesian differential element in say the reference configuration. This element can then undergo several different types of motion or deformation. Fig. 3.6B shows the element after rigid body motion including translation and rotation, and no relative distance changes between any particle pairs have occurred. However, in Fig. 3.6C and D extensional and shearing deformations have obviously created relative distance changes among continuum particles, and it is for these cases that we wish to develop quantitative measures.

The major question is then how do we decide on creating such strain measures. This choice is not unique and over the years many schemes have been defined. For modern continuum mechanics theory, we want strain measures that satisfy the principle of objectivity as presented in Section 2.9. Although the deformation gradient plays an important role in the analysis of deformation, it is not suitable by itself to be a measure of strain since it does not satisfy objectivity for second-order tensors. A simple measure is the *distance between two material points* since we have previously shown that this scalar value will be the same for all observers and hence satisfy objectivity.

Thus, we can start in the reference configuration and define the *Lagrangian or Green strain tensor*, by considering the *change in the square of the length of the vector dX* shown in Fig. 3.4.

$$\begin{aligned} |d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 &= (\boldsymbol{F}\,d\boldsymbol{X})\cdot(\boldsymbol{F}\,d\boldsymbol{X}) - d\boldsymbol{X}\cdot d\boldsymbol{X} \\ &= F_{ij}\,dX_j F_{ik}\,dX_k - dX_k\,dX_k \\ &= (F_{ij}F_{ik} - \delta_{jk})dX_j\,dX_k \\ &= d\boldsymbol{X}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right)d\boldsymbol{X} \end{aligned} \tag{3.6.1}$$

Defining the Lagrangian stain tensor $\boldsymbol{E}$ by

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right), \quad E_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) \tag{3.6.2}$$

we then have

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2d\boldsymbol{X}\boldsymbol{E}\,d\boldsymbol{X} = 2E_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j \tag{3.6.3}$$

Using relation (3.5.8), we can express the Lagrangian stain in terms of the displacement gradients

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}[(\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathrm{T}} + (\nabla\boldsymbol{u})^{\mathrm{T}}(\nabla\boldsymbol{u})] \\ E_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \end{aligned} \tag{3.6.4}$$

Notice that in all forms the Lagrangian strain is a symmetric second-order tensor, $E_{ij} = E_{ji}$.

We can repeat the past few steps but instead of using the reference configuration we use the current configuration and define the *Eulerian or Almansi strain tensor*, by considering the *change in the square of the length of the vector d**x*** shown in Fig. 3.4.

$$\begin{aligned} |d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 &= d\boldsymbol{x}\cdot d\boldsymbol{x} - \left(\boldsymbol{F}^{-1}\,d\boldsymbol{x}\right)\cdot\left(\boldsymbol{F}^{-1}\,d\boldsymbol{x}\right) \\ &= d\boldsymbol{x}\left(\boldsymbol{I} - \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right)d\boldsymbol{x} \\ &= (\delta_{jk} - F_{ij}^{-1}F_{ik}^{-1})\,dx_j\,dx_k \end{aligned} \tag{3.6.5}$$

where we have used the spatial deformation gradient tensor defined by (3.5.9). Defining the Eulerian stain tensor $\boldsymbol{e}$ by

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right), \quad e_{ij} = \frac{1}{2}(\delta_{jk} - F_{ij}^{-1}F_{ik}^{-1}) \tag{3.6.6}$$

we then have

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2\,d\boldsymbol{x}\,\boldsymbol{e}\,d\boldsymbol{x} = 2e_{ij}\,dx_i\,dx_j \tag{3.6.7}$$

We can define the spatial or Eulerian displacement gradient tensor by

$$\nabla\boldsymbol{u}^* = \frac{\partial\boldsymbol{u}^*}{\partial\boldsymbol{x}} = \frac{\partial u_i^*}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = u_{i,j}^*\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.6.8}$$

Using the chain rule gives $\nabla\boldsymbol{u} = \nabla\boldsymbol{u}^*\boldsymbol{F}$, and thus $\boldsymbol{F}^{-1} = \boldsymbol{I} - \nabla\boldsymbol{u}^*$. These results allow us to express the Eulerian stain in terms of the spatial displacement gradients

$$\begin{aligned} \boldsymbol{e} &= \frac{1}{2}[(\nabla\boldsymbol{u}^*) + (\nabla\boldsymbol{u}^*)^{\mathrm{T}} + (\nabla\boldsymbol{u}^*)^{\mathrm{T}}(\nabla\boldsymbol{u}^*)] \\ e_{ij} &= \frac{1}{2}\left(u_{i,j}^* + u_{j,i}^* + u_{k,i}^*u_{k,j}^*\right) \end{aligned} \tag{3.6.9}$$

and similar to our previous Lagrangian form, the Eulerian strain is a symmetric second-order tensor, $e_{ij} = e_{ji}$.

EXAMPLE 3.6.1 LAGRANGIAN AND EULERIAN STRAIN TENSORS FOR SIMPLE SHEAR AND EXTENSIONAL DEFORMATIONS

For the previous deformations of simple shear and extension specified in Examples 3.1.1 and 3.5.2, determine the Lagrangian and Eulerian strain tensors.

Solution: For the simple shear case

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \quad \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For the Eulerian forms

$$\begin{aligned} X_1 &= x_1 - \gamma x_2 \\ X_2 &= x_2 \\ X_3 &= x_3 \end{aligned} \quad \Rightarrow \boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$(\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right) = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For the extensional motion example

$$\begin{aligned} x_1 &= \lambda_1 X_1 \\ x_2 &= \lambda_2 X_2 \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow \\ X_3 &= \lambda_3 X_3 \end{aligned}$$

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$E = \frac{1}{2}\left(F^{\mathrm{T}}F - I\right) = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix}$$

and for the Eulerian forms

$$\begin{aligned} &X_1 = x_1 / \lambda_1 \\ &X_2 = x_2 / \lambda_2 \Rightarrow F^{-1} = \frac{\partial F}{\partial x} = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix} \Rightarrow \\ &X_3 = x_3 / \lambda_3 \end{aligned}$$

$$(F^{-1})^T F^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix}\begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix} = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix}$$

$$e = \frac{1}{2}\left(I - \left(F^{-1}\right)^{\mathrm{T}} F^{-1}\right) = \frac{1}{2}\begin{bmatrix} 1 - \lambda_1^{-2} & 0 & 0 \\ 0 & 1 - \lambda_2^{-2} & 0 \\ 0 & 0 & 1 - \lambda_3^{-2} \end{bmatrix}$$

Let us now apply the Polar Decomposition Theorem from Section 2.16. Recall that this mathematical result allows for the multiplicative decomposition of any nonsingular second-order tensor. Since the deformation gradient $\boldsymbol{F}$ is nonsingular (see relation (3.1.3), we can use this theorem to write

$$F = RU = VR \tag{3.6.10}$$

where $\boldsymbol{R}$ is a proper orthogonal tensor and $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric positive definite tensors, such that

$$\begin{aligned} U^2 &= F^{\mathrm{T}} F \\ V^2 &= F F^{\mathrm{T}} \end{aligned} \tag{3.6.11}$$

The proper orthogonal tensor represents the rigid body rotation, and so the tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ must characterize pure strain and are often referred to as the *right and left stretch tensors*. Note that the rotation tensor can be found from one of the relations $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{V}^{-1}\boldsymbol{F}$. Since $\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{\mathrm{T}}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ must have the same principal values $\{\lambda_i\}$ called the *principal stretches*. From another point of view, if we consider the extensional deformation given in Example 3.6.1:

$$U^2 = F^{\mathrm{T}} F = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \Rightarrow U = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = V$$

and thus $\boldsymbol{U}$ and $\boldsymbol{V}$ are directly related to the stretch ratios λ_i from extensional deformation.

Going back to the fundamental definition of the deformation gradient

$$d\boldsymbol{x} = \boldsymbol{F}\,d\boldsymbol{X} = (\boldsymbol{R}\boldsymbol{U})d\boldsymbol{X} = \boldsymbol{R}(\boldsymbol{U}\,d\boldsymbol{X})$$

which can be thought of as a two-step process: pure stretch $\boldsymbol{U}d\boldsymbol{X}$ followed by a rotation $\boldsymbol{R}(\boldsymbol{U}d\boldsymbol{X})$. Likewise, we can use the second form in relation (3.6.10):

$$d\boldsymbol{x} = \boldsymbol{F}\,d\boldsymbol{X} = (\boldsymbol{V}\boldsymbol{R})d\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{R}\,d\boldsymbol{X})$$

which is the two-step process: rotation $\boldsymbol{R}d\boldsymbol{X}$ followed by a stretch $\boldsymbol{V}(\boldsymbol{R}d\boldsymbol{X})$. We thus can see the *sequential or serial mapping* that results from the multiplicative decomposition from the Polar Decomposition Theorem.

Another useful set of strain tensors come from the specification of changes of differential line elements in the reference and spatial configurations. Consider two differential line elements in the reference configuration $d\boldsymbol{X}^{(1)}$ and $d\boldsymbol{X}^{(2)}$ which are mapped into the corresponding elements $d\boldsymbol{x}^{(1)}$ and $d\boldsymbol{x}^{(2)}$ in the current spatial configuration. We can thus write

$$\begin{aligned} d\boldsymbol{x}^{(1)} \cdot d\boldsymbol{x}^{(2)} &= \boldsymbol{F}\,d\boldsymbol{X}^{(1)} \cdot \boldsymbol{F}\,d\boldsymbol{X}^{(2)} \\ &= d\boldsymbol{X}^{(1)}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\right)d\boldsymbol{X}^{(2)} \\ &= d\boldsymbol{X}^{(1)}\boldsymbol{C}\,d\boldsymbol{X}^{(2)} \end{aligned} \tag{3.6.12}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \quad C_{ij} = F_{ki}F_{kj} \tag{3.6.13}$$

is called the *right Cauchy–Green or Green strain tensor*. It can be shown that this tensor is symmetric and positive definite which implies that it has real positive eigenvalues (see Exercise 2.20).

Next, we can reverse the mapping and work from spatial to reference configurations and write

$$\begin{aligned} d\boldsymbol{X}^{(1)} \cdot d\boldsymbol{X}^{(2)} &= \boldsymbol{F}^{-1}\,d\boldsymbol{x}^{(1)} \cdot \boldsymbol{F}^{-1}\,d\boldsymbol{x}^{(2)} \\ &= d\boldsymbol{x}^{(1)}\left((\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{F}^{-1}\right)d\boldsymbol{x}^{(2)} \\ &= d\boldsymbol{x}^{(1)}\boldsymbol{B}^{-1}\,d\boldsymbol{x}^{(2)} \end{aligned} \tag{3.6.14}$$

where $\boldsymbol{B}^{-1} = (\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{F}^{-1}$, and this leads to the definition of $\boldsymbol{B}$ as

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}, \quad B_{ij} = F_{ik}F_{jk} \tag{3.6.15}$$

which is called the *left Cauchy–Green or Finger strain tensor*. Again this tensor is symmetric and positive definite.

Note that $tr\,\boldsymbol{C} = tr\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\right) = tr\left(\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}\right) = tr\,\boldsymbol{B}$ and this result also holds for arbitrary powers of these tensors. Using results (2.11.4) and (2.11.5) implies that the invariants of $\boldsymbol{C}$ and $\boldsymbol{B}$ are the same. Referring back to our definition (3.6.11) of the right and left stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, we can write $\boldsymbol{U}^2 = \boldsymbol{C}$ and $\boldsymbol{V}^2 = \boldsymbol{B}$.

EXAMPLE 3.6.2 RIGHT AND LEFT CAUCHY-GREEN STRAIN TENSORS FOR SIMPLE SHEAR AND EXTENSIONAL DEFORMATIONS

For the previous deformations of simple shear and extension specified in Examples 3.1.1 and 3.5.2, determine the right and left Cauchy–Green strain tensors.

Solution: For the simple shear case

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \quad \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the extensional motion example

$$\begin{aligned} x_1 &= \lambda_1 X_1 \\ x_2 &= \lambda_2 X_2 \Rightarrow \\ x_3 &= \lambda_3 X_3 \end{aligned} \quad \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \boldsymbol{B}$$

EXAMPLE 3.6.3 ROTATION AND RIGHT AND LEFT STRETCH TENSORS

For the following motion, calculate the deformation gradient $\boldsymbol{F}$, the rotation tensor $\boldsymbol{R}$, and the right and left stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$**:**

$$\begin{aligned} x_1 &= X_1 \\ x_2 &= 2X_3 \\ x_3 &= 4X_2 \end{aligned}$$

Solution: Starting with the deformation gradient

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{U}^2 = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow \boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{bmatrix} \Rightarrow \boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that square roots were an easy computation for this simple case.

$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

It is easily checked that the orthogonality condition $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$ is satisfied.

We now wish to develop the relationship between the Lagrangian strain tensor $\boldsymbol{E}$ and the so-called *physical strain components*. It should be kept in mind that our development of the various strains tensors $\boldsymbol{E}$, $\boldsymbol{e}$, $\boldsymbol{C}$, and $\boldsymbol{B}$ were all done for arbitrary finite deformations. Small deformation simplifications will be explored in the next section. The Lagrangian strain components E_{ij} can be related to particular physical extensional and shear strain behaviors. In order to do this, consider the orthogonal triad of vectors $\{d\boldsymbol{X}^{(1)}, d\boldsymbol{X}^{(2)}, d\boldsymbol{X}^{(3)}\}$ located at position $\boldsymbol{X}$ in the reference configuration. Under general motion, these vectors are mapped to another triad $\{d\boldsymbol{x}^{(1)}, d\boldsymbol{x}^{(2)}, d\boldsymbol{x}^{(3)}\}$ which are not necessarily orthogonal (see Fig. 3.7).

We define the *extensional strain components* $\gamma^{(i)}$ as the change in length divided by the original length of each fiber $d\boldsymbol{X}^{(i)}$:

$$\gamma^{(\mathrm{i})} = \frac{\left|d\boldsymbol{x}^{(\mathrm{i})}\right| - \left|d\boldsymbol{X}^{(\mathrm{i})}\right|}{\left|d\boldsymbol{X}^{(\mathrm{i})}\right|} \tag{3.6.16}$$

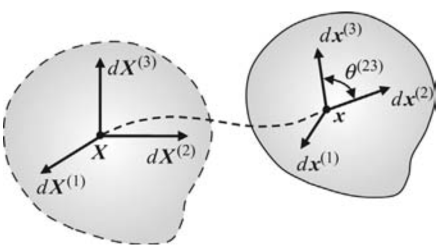

FIGURE 3.7

Deformation of an orthogonal triad.

Likewise, the *shear strain components* $\gamma^{(ij)}$ are the change in angle between fibers $d\boldsymbol{X}^{(i)}$ and $d\boldsymbol{X}^{(j)}$:

$$\gamma^{(ij)} = \frac{\pi}{2} - \theta^{(ij)} \tag{3.6.17}$$

where $\theta^{(ij)}$ is the angle between fibers $d\boldsymbol{x}^{(i)}$ and $d\boldsymbol{x}^{(j)}$ in the current spatial configuration.

Using the basic Lagrangian strain relation (3.6.3), we can write

$$\begin{aligned}
\left|d\boldsymbol{x}^{(i)}\right|^2 - \left|d\boldsymbol{X}^{(i)}\right|^2 &= 2\, d\boldsymbol{X}^{(i)}\boldsymbol{E}\, d\boldsymbol{X}^{(i)} \\
&= 2\left|d\boldsymbol{X}^{(i)}\right|\boldsymbol{e}_i\boldsymbol{E}\left|d\boldsymbol{X}^{(i)}\right|\boldsymbol{e}_i \\
&= 2\boldsymbol{e}_i\boldsymbol{E}\boldsymbol{e}_i\left|d\boldsymbol{X}^{(i)}\right|^2 \\
&= 2E_{i\underline{i}}\left|d\boldsymbol{X}^{(i)}\right|^2 \qquad (\textit{no sum on } i)
\end{aligned} \tag{3.6.18}$$

Relation (3.6.16) can be rewritten as $\left|d\boldsymbol{x}^{(\mathrm{i})}\right| = \left(\gamma^{(\mathrm{i})}+1\right)\left|d\boldsymbol{X}^{(\mathrm{i})}\right|$, and combining this with (3.6.18) gives $\left(\gamma^{(i)}+1\right)^2 = 1+2E_{i\underline{i}}$. Solving for the extensional strain then yields the final result

$$\gamma^{(i)} = \sqrt{1+2E_{i\underline{i}}} - 1 \tag{3.6.19}$$

Next, let us consider the angle changes for the shear strain

$$\sin\gamma^{(ij)} = \cos\theta^{(ij)} = \frac{d\boldsymbol{x}^{(\mathrm{i})}\cdot d\boldsymbol{x}^{(j)}}{\left|d\boldsymbol{x}^{(\mathrm{i})}\right|\left|d\boldsymbol{x}^{(j)}\right|}, \quad i \neq j \tag{3.6.20}$$

From (3.5.4) $d\boldsymbol{x}^{(i)} = \boldsymbol{F}\, d\boldsymbol{X}^{(i)}$ and combining with (3.6.16) and (3.6.2), we can express (3.6.20) as

$$\begin{aligned}
\sin\gamma^{(ij)} &= \frac{\boldsymbol{F}\, d\boldsymbol{X}^{(i)}\cdot \boldsymbol{F}\, d\boldsymbol{X}^{(j)}}{\left(\gamma^{(i)}+1\right)\left|d\boldsymbol{X}^{(\mathrm{i})}\right|\left(\gamma^{(j)}+1\right)\left|d\boldsymbol{X}^{(j)}\right|} \\
&= \frac{d\boldsymbol{X}^{(i)}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\, d\boldsymbol{X}^{(j)}}{\left(\gamma^{(i)}+1\right)\left(\gamma^{(j)}+1\right)\left|d\boldsymbol{X}^{(\boldsymbol{i})}\right|\left|d\boldsymbol{X}^{(j)}\right|} \\
&= \frac{2E_{ij}}{\sqrt{1+2E_{i\underline{i}}}\sqrt{1+2E_{j\underline{j}}}}
\end{aligned} \tag{3.6.21}$$

Relations (3.6.19) and (3.6.21) then provide the relationships for the physical strain components in terms of the Lagrangian strain tensor components.

We can follow similar steps to relate the physical strains in terms of the Eulerian strains, and the results become

$$\begin{aligned}
\gamma^{(i)} &= \frac{1}{\sqrt{1-2e_{i\underline{i}}}} - 1 \\
\sin\gamma^{(ij)} &= \frac{2e_{ij}}{\sqrt{1-2e_{i\underline{i}}}\sqrt{1-2e_{j\underline{j}}}} \quad (i \neq j)
\end{aligned} \tag{3.6.22}$$

EXAMPLE 3.6.4 PHYSICAL STRAIN COMPONENTS FOR SIMPLE SHEAR AND EXTENSIONAL DEFORMATIONS

For the previous deformations of simple shear and extension specified in Examples 3.1.1 and 3.5.2, determine the physical strain components $\gamma^{(i)}$ and $\sin\gamma^{(ij)}$.

Solution: For the simple shear case,

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^{(i)} = \sqrt{1 + 2E_{i\underline{i}}} - 1 = \begin{bmatrix} 0 \\ \sqrt{1+\gamma^2} - 1 \\ 0 \end{bmatrix}$$

$$\sin\gamma^{(ij)} = \frac{2E_{ij}}{\sqrt{1+2E_{i\underline{i}}}\sqrt{1+2E_{j\underline{j}}}} = \begin{bmatrix} 0 \\ \dfrac{\gamma}{\sqrt{1+\gamma^2}} \\ 0 \end{bmatrix} \quad (i \neq j)$$

As expected for simple shear there is no extension for reference fibers along the X_1 and X_3 directions, and no angles changes with X_1,X_3 and X_2,X_3 fiber pairs.

For the extensional motion example,

$$\begin{aligned} x_1 &= \lambda_1 X_1 \\ x_2 &= \lambda_2 X_2 \\ x_3 &= \lambda_3 X_3 \end{aligned} \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix}$$

$$\gamma^{(i)} = \sqrt{1 + 2E_{i\underline{i}}} - 1 = \begin{bmatrix} \lambda_1 - 1 \\ \lambda_2 - 1 \\ \lambda_3 - 1 \end{bmatrix}$$

$$\sin\gamma^{(ij)} = \frac{2E_{ij}}{\sqrt{1+2E_{i\underline{i}}}\sqrt{1+2E_{j\underline{j}}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (i \neq j)$$

As expected for extensional motion, there will be no angle changes for fibers along reference coordinate directions.

We have already seen that our kinematic tensor fields may be expressed in terms of reference or spatial coordinates using the basis vectors $\boldsymbol{E}_i$ or $\boldsymbol{e}_i$. As with the case of the deformation gradient, there are also some two-point tensors associated with both reference frames. These transformations between reference and spatial or current representations are commonly called *push-forward* and *pull-back operations* and would employ using the basic motion relations $\boldsymbol{x}=\chi(\boldsymbol{X},t)$ and $\boldsymbol{X}=\chi^{-1}(\boldsymbol{x},t)$. The push-forward operation transforms a reference tensor field $(\boldsymbol{X},t)$ to the current configuration $(\boldsymbol{x},t)$. An example of this process is the transformation of the reference Lagrangian strain tensor $\boldsymbol{E}$ to the spatial Eulerian–Almansi strain tensor $\boldsymbol{e}$. Starting with the fundamental definition of $\boldsymbol{e}$,

$$\begin{aligned}\boldsymbol{e}&=\frac{1}{2}\left(\boldsymbol{I}-\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right)=\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\frac{1}{2}\left[\boldsymbol{F}^{\mathrm{T}}\left(\boldsymbol{I}-\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right)\boldsymbol{F}\right]\boldsymbol{F}^{-1}\\&=\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}-\boldsymbol{I}\right)\boldsymbol{F}^{-1}=\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{E}\boldsymbol{F}^{-1}\end{aligned}\tag{3.6.23}$$

The pull-back operation is then the inverse of this process, and for the previous example we can easily invert things to get $\boldsymbol{E}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{e}\boldsymbol{F}$.

Before ending this section, let us explore the objectivity test for the various strain tensors developed. Recall that the fundamental test is based on a change of reference between two frames $\boldsymbol{x}^*$ and $\boldsymbol{x}$, such that $\boldsymbol{x}^*=\boldsymbol{c}(t)+\boldsymbol{Q}(t)\boldsymbol{x}$. Using this, we found earlier that the deformation gradient transformed as $\boldsymbol{F}^*=\boldsymbol{Q}\boldsymbol{F}$, and the inverse followed the relation $\boldsymbol{F}^{*^{-1}}=\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}$. Starting with the Lagrangian strain tensor $\boldsymbol{E}=\frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}-\boldsymbol{I}\right)$, we have

$$\begin{aligned}\boldsymbol{E}^*&=\frac{1}{2}\left(\boldsymbol{F}^{*^{\mathrm{T}}}\boldsymbol{F}^*-\boldsymbol{I}^*\right)=\frac{1}{2}\left((\boldsymbol{Q}\boldsymbol{F})^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{F}-\boldsymbol{I}\right)=\frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{F}-\boldsymbol{I}\right)\\&=\frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}-\boldsymbol{I}\right)=\boldsymbol{E}\end{aligned}\tag{3.6.24}$$

Likewise, the Eulerian strain $\boldsymbol{e}=\frac{1}{2}\left(\boldsymbol{I}-\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\right)$ follows the transformation:

$$\begin{aligned}\boldsymbol{e}^*&=\frac{1}{2}\left(\boldsymbol{I}^*-\left(\boldsymbol{F}^{*^{-1}}\right)^{\mathrm{T}}\boldsymbol{F}^{*^{-1}}\right)=\frac{1}{2}\left(\boldsymbol{I}-\left(\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right)^{\mathrm{T}}\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right)\\&=\frac{1}{2}\left(\boldsymbol{I}-\boldsymbol{Q}(\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}\right)=\boldsymbol{Q}\frac{1}{2}\left(\boldsymbol{I}-(\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{F}^{-1}\right)\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^{\mathrm{T}}\end{aligned}\tag{3.6.25}$$

For the right Cauchy–Green strain tensor $\boldsymbol{C}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$:

$$\boldsymbol{C}^*=\boldsymbol{F}^{*^{\mathrm{T}}}\boldsymbol{F}^*=(\mathbf{QF})^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{F}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{F}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}=\boldsymbol{C}\tag{3.6.26}$$

and finally for the left Cauchy–Green strain $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$:

$$\boldsymbol{B}^*=\boldsymbol{F}^*\boldsymbol{F}^{*^{\mathrm{T}}}=\boldsymbol{Q}\boldsymbol{F}(\mathbf{QF})^{\mathrm{T}}=\boldsymbol{Q}\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{\mathrm{T}}\tag{3.6.27}$$

Thus, we conclude that the Eulerian and left Cauchy–Green tensors are spatially based and satisfy the standard objective relation. However, the Lagrangian and right Cauchy–Green tensors do not satisfy the standard relation, but because they are based in reference coordinates they will transform as a scalar field.

We have now developed several second-order tensors $\boldsymbol{F}$, $\boldsymbol{E}$, $\boldsymbol{e}$, $\boldsymbol{C}$, and $\boldsymbol{B}$ that can be used to characterize finite strain deformations. As previously mentioned, there is no single unique strain tensor, and over the years a sizeable number of strain measures have been constructed. We could continue our study by reviewing some of these additional schemes; however, the current strain tensor set is sufficient for most continuum mechanics theories. Since we wish to complete the basic continuum formulation as soon as possible in order to move into the vast array of constitutive models, we will not further explore other strain measures.

3.7 CHANGES IN LINE, AREA, AND VOLUME ELEMENTS

It will be useful to develop relations for the changes in line, area, and volume elements under the motion from reference to spatial configurations. We simplify things somewhat by choosing a cubical reference element with sides aligned with the reference coordinate frame as shown in Fig. 3.8. Under general motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, this element is mapped to a new element with totally different dimensions and shape in the spatial configuration. This scheme will allow us to determine element changes in the line edge, surface area, and volume. The deformation gradient and other strain tensor components can be used to find these types of changes. We basically seek a relational form that will give the spatial or current value in terms of a simple multiplier times the corresponding reference value.

Starting with line elements, we can use many of the relations from our previous discussion on physical extensional strain components. Starting with (3.6.16) and using (3.6.19) and (3.6.2) gives

$$\left|d\boldsymbol{x}^{(\mathrm{i})}\right| = \left(\gamma^{(i)} + 1\right)\left|d\boldsymbol{X}^{(\mathrm{i})}\right| = \sqrt{1+2E_{i\underline{i}}}\left|d\boldsymbol{X}^{(\mathrm{i})}\right| = \sqrt{F_{ki}F_{k\underline{i}}}\left|d\boldsymbol{X}^{(\mathrm{i})}\right| = \sqrt{C_{i\underline{i}}}\left|d\boldsymbol{X}^{(\mathrm{i})}\right| \tag{3.7.1}$$

Next consider the area behavior of one face of the differential cubical element in Fig. 3.8. In the reference configuration, we choose the differential area $d\boldsymbol{A}$ in the

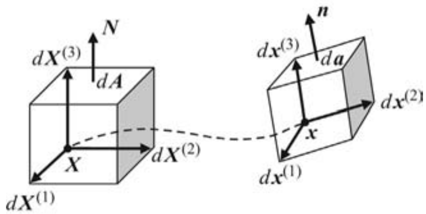

FIGURE 3.8

Line, area, and volume deformation.

$(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)})$-plane. This area is spanned by the vectors $d\boldsymbol{X}^{(1)}$ and $d\boldsymbol{X}^{(2)}$ and thus the area and unit normal vector are given by

$$dA = \left|d\boldsymbol{X}^{(1)} \times d\boldsymbol{X}^{(2)}\right|, \quad \boldsymbol{N} = \frac{d\boldsymbol{X}^{(1)} \times d\boldsymbol{X}^{(2)}}{\left|d\boldsymbol{X}^{(1)} \times d\boldsymbol{X}^{(2)}\right|} \tag{3.7.2}$$

These variables then define what is normally called an *oriented element area in the reference configuration* $d\boldsymbol{A} = \boldsymbol{N}\, dA = d\boldsymbol{X}^{(1)} \times d\boldsymbol{X}^{(2)}$. Under the motion, the reference vectors $d\boldsymbol{X}^{(1)}$ and $d\boldsymbol{X}^{(2)}$ are mapped to the spatial vectors $d\boldsymbol{x}^{(1)}$ and $d\boldsymbol{x}^{(2)}$ in the deformed configuration. The corresponding oriented element area then becomes $d\boldsymbol{a} = \boldsymbol{n}\, da = d\boldsymbol{x}^{(1)} \times d\boldsymbol{x}^{(2)}$. Shifting to index notation

$$\begin{aligned} da_i = n_i\, da &= \varepsilon_{ijk}\, dx_j^{(1)}\, dx_k^{(2)} \\ &= \varepsilon_{ijk} F_{jl}\, dX_l^{(1)} F_{km}\, dX_m^{(2)} \\ &= \varepsilon_{ijk} F_{jl} F_{km}\, dX_l^{(1)}\, dX_m^{(2)} \end{aligned}$$

Multiplying both sides by F_{in} gives

$$da_i F_{in} = \varepsilon_{ijk} F_{in} F_{jl} F_{km}\, dX_l^{(1)}\, dX_m^{(2)}$$

and then using property (2.5.10)

$$da_i F_{in} = (\det \boldsymbol{F})\varepsilon_{nlm}\, dX_l^{(1)}\, dX_m^{(2)} = J(d\boldsymbol{X}^{(1)} \times d\boldsymbol{X}^{(2)})_n = JN_n\, dA$$

where the Jacobian $J = \det \boldsymbol{F}$. Finally, multiplying both sides by F_{nm}^{-1} gives the final result

$$\begin{aligned} da_i F_{in} F_{nm}^{-1} = JF_{nm}^{-1} N_n\, dA &\Rightarrow da_m = JF_{nm}^{-1} N_n\, dA \\ \boldsymbol{n}\, da = J(\boldsymbol{F}^{-1})^{\mathrm{T}} \boldsymbol{N}\, dA& \end{aligned} \tag{3.7.3}$$

This relation is often called *Nanson's formula*.

Our last step in this section is to explore the volume changes that occur during general motion. We again start with the element shown in the reference configuration in Fig. 3.8. The volume of this element is spanned by the vectors $(d\boldsymbol{X}^{(1)}, d\boldsymbol{X}^{(2)}, d\boldsymbol{X}^{(3)})$. It is well known that the volume of such a shape is given by the scalar triple product (2.10.4):

$$dV = d\boldsymbol{X}^{(1)} \cdot d\boldsymbol{X}^{(2)} \times d\boldsymbol{X}^{(2)} = dX^{(1)}\, dX^{(2)}\, dX^{(2)}$$

Likewise, the volume of the element in the deformed configuration is given by $dv = d\boldsymbol{x}^{(1)} \cdot d\boldsymbol{x}^{(2)} \times d\boldsymbol{x}^{(2)}$, and again shifting to index notation

$$\begin{aligned} dv &= \varepsilon_{imp} F_{ij}\, dX_j^{(1)} F_{mn}\, dX_n^{(2)} F_{pq}\, dX_q^{(3)} \\ &= \varepsilon_{imp} F_{ij} F_{mn} F_{pq}\, dX_j^{(1)}\, dX_n^{(2)}\, dX_q^{(3)} \\ &= (\det F)\varepsilon_{jnq}\, dX_j^{(1)}\, dX_n^{(2)}\, dX_q^{(3)} \\ &= J\, \mathrm{d}V \end{aligned} \tag{3.7.4}$$

where we have used similar reduction steps as done with the area analysis.

The *volume dilatation* ϑ is defined by

$$\vartheta = \frac{dv - dv}{dv} = \det \boldsymbol{F} - 1 = J - 1 \tag{3.7.5}$$

Hence, we can say that a motion is *volume preserving* or *isochoric* if det $\boldsymbol{F}$ = 1. It turns out that incompressible materials can only undergo isochoric motions.

EXAMPLE 3.7.1 LINE, AREA, AND VOLUME CHANGES FOR SIMPLE SHEAR AND EXTENSIONAL DEFORMATIONS

Using again our previous deformations of simple shear and extension specified in Examples 3.1.1 and 3.5.2, determine the line, area, and volume changes.

Solution: For the simple shear case,

$$\begin{array}{l} x_1 = X_1 + \gamma X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{array} \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = 1$$

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{F} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left| d\boldsymbol{x}^{(i)} \right| = \sqrt{C_{\underline{ii}}} \left| d\boldsymbol{X}^{(i)} \right| = \begin{bmatrix} 1 \\ \sqrt{1+\gamma^2} \\ 1 \end{bmatrix} \left| d\boldsymbol{X}^{(i)} \right|$$

$$\boldsymbol{n}\, da = J(\boldsymbol{F}^{-1})^{\mathrm{T}} \boldsymbol{N}\, \mathrm{d}A = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dA = \begin{bmatrix} N_1 \\ -\gamma N_1 + N_2 \\ N_3 \end{bmatrix} dA$$

$dv = J\, dV = dV$ ... no volumetric change

For the extensional motion example,

$$\begin{array}{l} x_1 = \lambda_1 X_1 \\ x_2 = \lambda_2 X_2 \Rightarrow \\ x_3 = \lambda_3 X_3 \end{array} \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow \boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix}, \quad J = \lambda_1 \lambda_2 \lambda_3$$

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{F} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$\left| d\boldsymbol{x}^{(i)} \right| = \sqrt{C_{\underline{ii}}} \left| d\boldsymbol{X}^{(i)} \right| = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \left| d\boldsymbol{X}^{(i)} \right|$ ... uniform extensional line changes

$$\boldsymbol{n}\,da = J(\boldsymbol{F}^{-1})^{\mathrm{T}}\boldsymbol{N}\,dA = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dA = \begin{bmatrix} \lambda_1^{-1}N_1 \\ \lambda_2^{-1}N_2 \\ \lambda_3^{-1}N_3 \end{bmatrix} dA$$

$$dv = J\,dV = \lambda_1\lambda_2\lambda_3\,dV$$

3.8 SMALL DEFORMATION KINEMATICS AND STRAIN TENSORS

In many classical continuum mechanics applications, the deformations under study are small and the need to distinguish between reference and spatial configurations is unnecessary. This situation occurs in classical linear elasticity, plasticity, viscoelasticity, and other such theories. Many of these models will be discussed in detail in Chapter 6. This situation greatly simplifies much of our previous kinematical relations, and we now wish to look at the details of these simplifications. In general, our finite strain relations were nonlinear in the displacement gradients, and we expect that for small deformations this nonlinearity will disappear.

With the removal of the difference between reference $\boldsymbol{X}$ coordinates and spatial or current $\boldsymbol{x}$ coordinates, one might think that the deformation gradient tensor becomes the unit tensor, and hence is essentially eliminated from the kinematical study. However, this would eliminate all of the strain tensors previous developed. Thus, we start with relation (3.5.8) $\boldsymbol{F} = \boldsymbol{I} + \nabla\boldsymbol{u}$ under the condition that the displacement gradient tensor is small $\nabla\boldsymbol{u} \ll \boldsymbol{I}$. Likewise, the inverse deformation gradient was given by $\dfrac{\partial u_i}{\partial X_j} = \dfrac{\partial u_i}{\partial x_k}\dfrac{\partial x_k}{\partial X_j} = \dfrac{\partial u_i}{\partial x_k}\left(\dfrac{\partial u_k}{\partial X_j} + \delta_{kj}\right) \approx \dfrac{\partial u_i}{\partial x_k}\delta_{kj} = \dfrac{\partial u_i}{\partial x_j}$, and so for small spatial displacement gradients, $\nabla\boldsymbol{u}^* \ll \boldsymbol{I}$. We now use these criteria to simplify the basic kinematical strain tensors.

Starting with the Lagrangian strain tensor given by (3.6.4),

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

We make the argument that for small deformations, the displacement gradient product $u_{k,i}u_{k,j}$ is small in comparison to the displacement gradient itself, and so it can be dropped from the relation giving

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

In similar fashion, the Eulerian strain given by (3.6.9) will reduce to

$$e_{ij} = \frac{1}{2}\left(u^*_{i,j} + u^*_{j,i}\right)$$

Furthermore, differences between spatial and reference derivatives disappear following the simple analysis:

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \frac{\partial u_i}{\partial x_k}\left(\frac{\partial u_k}{\partial X_j} + \delta_{kj}\right) \approx \frac{\partial u_i}{\partial x_k}\delta_{kj} = \frac{\partial u_i}{\partial x_j}$$

Thus, the Lagrangian and Eulerian strain tensors both reduce to the infinitesimal strain

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3.8.1}$$

and the derivatives in (3.8.1) can be taken with respect to either reference or spatial as they are indistinguishable. This result reduces the right and left Cauchy–Green tensor as

$$\begin{aligned} C_{ij} &= F_{ki}F_{kj} = 2\varepsilon_{ij} + \delta_{ij} \\ B_{ij} &= F_{ik}F_{jk} = 2\varepsilon_{ij} + \delta_{ij} \end{aligned} \tag{3.8.2}$$

For small deformations, since the distinction between reference and spatial position is dropped, the displacement vector is then used to represent position. The velocity and acceleration relations then reduce to

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &= \frac{D\boldsymbol{x}}{Dt} \approx \frac{\partial \boldsymbol{u}}{\partial t} \\ \boldsymbol{a}(\boldsymbol{x},t) &= \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} \approx \frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \end{aligned} \tag{3.8.3}$$

EXAMPLE 3.8.1 PHYSICAL STRAIN COMPONENTS FOR SMALL STRAIN APPROXIMATION

Determine the general physical strain components given by (3.6.19) and (3.6.21) for the case of small strain approximation.

Solution: From (3.6.19) and (3.8.1) $\Rightarrow \gamma^{(i)} = \sqrt{1+2E_{i\underline{i}}} - 1 = \sqrt{1+2\varepsilon_{i\underline{i}}} - 1 \approx \varepsilon_{i\underline{i}}$.
Using (3.6.21) and (3.8.1) $\Rightarrow$

$$\sin\gamma^{(ij)} = \frac{2E_{ij}}{\sqrt{1+2E_{i\underline{i}}}\sqrt{1+2E_{j\underline{j}}}} = \frac{2\varepsilon_{ij}}{\sqrt{1+2\varepsilon_{i\underline{i}}}\sqrt{1+2\varepsilon_{j\underline{j}}}} \approx \frac{2\varepsilon_{ij}}{\left(1+\varepsilon_{i\underline{i}}\right)\left(1+\varepsilon_{j\underline{j}}\right)} \approx \frac{2\varepsilon_{ij}}{1+\varepsilon_{i\underline{i}}+\varepsilon_{j\underline{j}}} \approx 2\varepsilon_{ij} \quad (i \neq j)$$

and for small deformations $\sin\gamma^{(ij)} \approx \gamma^{(ij)} \Rightarrow \gamma^{(ij)} = \dfrac{\pi}{2} - \theta^{(ij)} \approx 2\varepsilon_{ij} \quad (i \neq j)$

Thus, for small deformations, the physical extensional strain is $\varepsilon_{i\underline{i}}$ and the change in the right angle between two orthogonal fibers is $2\varepsilon_{ij}\ (i \neq j)$.

3.9 PRINCIPAL AXES FOR STRAIN TENSORS

From our previous developments, the Lagrangian strain $\boldsymbol{E}$, the Eulerian strain $\boldsymbol{e}$, the right and left Cauchy–Green strains $\boldsymbol{C}$ and $\boldsymbol{B}$, and the infinitesimal strain $\boldsymbol{\varepsilon}$ were all symmetric second-order tensors. For each of these, we can therefore apply our previous principal value theory from Section 2.11.

Focusing our attention to the Lagrangian strain tensor defined by relation (3.6.2) $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right)$, we can then conclude that there will exist at least three principal directions and at most three principal values for the tensor $\boldsymbol{E}$. The principal values (eigenvalues) E_1, E_2, and E_3 are referred to as *principal strains* and come from the roots of the characteristic equation

$$E^3 - I_E E^2 + II_E E - III_E = 0 \tag{3.9.1}$$

where the fundamental invariants of $\boldsymbol{E}$ are given by

$$\begin{aligned}
I_E &= E_{ii} = E_{11} + E_{22} + E_{33} \\
II_E &= \frac{1}{2}\left(E_{ii}E_{jj} - E_{ij}E_{ij}\right) = \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix} \\
III_E &= \det \boldsymbol{E}
\end{aligned} \tag{3.9.2}$$

Using the special principal coordinate system for strain, we can conclude that tensor $\boldsymbol{E}$ written as a matrix will take the diagonal form

$$E_{ij} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \tag{3.9.3}$$

and using this representation, the invariants can be expressed in simpler form using the principal values:

$$\begin{aligned}
I_E &= E_1 + E_2 + E_3 \\
II_E &= E_1E_2 + E_2E_3 + E_3E_1 \\
III_E &= E_1E_2E_3
\end{aligned} \tag{3.9.4}$$

Obviously, all of this theoretical structure would hold for any of the other symmetric strain tensors. Note that for the infinitesimal strain case, the absence of the off-diagonal terms in the principal coordinate system implies zero angle change deformation (see Example 3.8.1).

3.10 SPHERICAL AND DEVIATORIC STRAIN TENSORS

Again from our previous developments in Section 2.12, any of our strain tensors $\boldsymbol{E}$, $\boldsymbol{e}$, $\boldsymbol{C}$, $\boldsymbol{B}$, and $\boldsymbol{\varepsilon}$ can be decomposed into spherical and deviatoric parts. For the infinitesimal strain $\boldsymbol{\varepsilon}$, the spherical part is given by

$$\tilde{\varepsilon}_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{3.10.1}$$

while the deviatoric part is specified as

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{3.10.2}$$

Note again that the sum of these two parts then gives the original tensor ε_{ij}.

For the infinitesimal strain, the spherical part is associated with *volumetric deformation* while the deviatoric part measures *shape or shear changes*. It can be shown that the principal directions of the strain deviator are the same as those of the strain tensor, and the deviator principal values are $\{\varepsilon_1 + p, \varepsilon_2 + p, \varepsilon_3 + p\}$, where $p = \varepsilon_{kk}/3$. Also note that $I_{\hat{\varepsilon}} = 0$, and spherical strain tensor is isotropic.

3.11 STRAIN COMPATIBILITY

Strain compatibility is a very interesting topic as it generates additional continuum field equations that come from unexpected places. While most of our field equations originate from Euclidean geometry, force analysis, balance or conservation principles, and thermodynamic theories, strain compatibility arises from a more abstract mathematical concept. The idea starts with a careful review of the various strain tensor forms that have been previously developed. If the motions or displacements are known, then the strain tensors, $\boldsymbol{E}$, $\boldsymbol{e}$, $\boldsymbol{C}$, $\boldsymbol{B}$, and $\boldsymbol{\varepsilon}$ can be determined by differentiation (assuming the motions/displacements are differentiable functions). However, the question also comes up of determining the displacements when the strains are given. Because our previous strain definitions (3.6.4), (3.6.9), (3.6.13), (3.6.15), and (3.8.1) all yield symmetric second-order tensors, there are *six* independent partial differential equations for *three* unknown displacement components. Such a system of equations is *over-determined* and will not have a *single-valued* solution for the displacements. One would expect, however, that a solution would exist if the strains satisfy certain additional conditions, and these are called *integrability* or *compatibility conditions*.

We will now only pursue in detail the infinitesimal strain $\boldsymbol{\varepsilon}$ case which has the linear relation form with the displacement gradients $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Writing these equations out in a scalar format yields the six *strain–displacement* relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \Rightarrow \begin{array}{ll} \varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}, & \varepsilon_{12} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) \\ \varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2}, & \varepsilon_{23} = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \varepsilon_{33} = \dfrac{\partial u_3}{\partial x_3}, & \varepsilon_{31} = \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3}\right) \end{array} \tag{3.11.1}$$

Thus, if we specify continuous, single-valued displacements u_1, u_2, and u_3, then through differentiation the resulting strain field can be determined and will be equally well-behaved. However, the converse is not necessarily true, that is, given the six strain components, integration of the strain–displacement relations (3.11.1) will not necessarily produce continuous, single-valued displacements. This should not be totally surprising since we are trying to solve six equations for only three unknown displacement components. In order to ensure continuous, single-valued displacements, the strains must satisfy additional relations.

Before proceeding with the mathematics to develop these equations, it is instructive to consider a geometric interpretation of this concept (Sadd, 2014). A two-dimensional example is shown in Fig. 3.9A whereby a continuum is first divided up into a series of elements. For simple visualization, consider only four such elements. In the undeformed configuration shown in Fig. 3.9B, these elements of course fit together perfectly. Next let us arbitrarily specify the strain of each of the four elements and attempt to reconstruct the continuum back together. For Fig. 3.9C, the elements have been carefully strained taking into consideration neighboring elements so that the system fits together, thus yielding continuous, single-valued displacements. However, for Fig. 3.9D, the

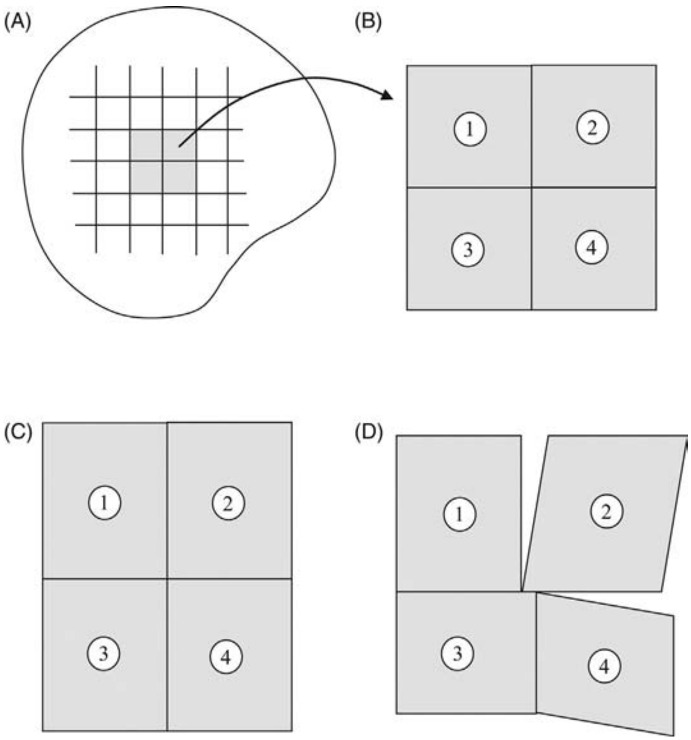

FIGURE 3.9

Physical interpretation of strain compatibility: (A) discretized continuum; (B) undeformed configuration; (C) deformed configuration continuous displacements; (D) deformed configuration discontinuous displacements.

elements have been individually deformed regardless of neighboring deformations. It is observed for this case that the system will not fit together without voids and gaps, and this situation produces a discontinuous displacement field. So we again conclude that the strain components must be somehow related to yield continuous, single-valued displacements. We now pursue these particular relations.

The development process is based on eliminating the displacements from the strain–displacement relations. Working in index notation, we start by differentiating (3.11.1) twice with respect to x_k and x_l:

$$\varepsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

Through simple interchange of subscripts, we can generate the following additional relations:

$$\begin{aligned}
\varepsilon_{kl,ij} &= \frac{1}{2}(u_{k,lij} + u_{l,kij}) \\
\varepsilon_{jl,ik} &= \frac{1}{2}(u_{j,lik} + u_{l,jik}) \\
\varepsilon_{ik,jl} &= \frac{1}{2}(u_{i,kjl} + u_{k,ijl})
\end{aligned}$$

Under the assumption of continuous displacements, we can interchange the order of differentiation on u_i and the displacements can be eliminated from the preceding set to get

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{3.11.2}$$

These are called the well-known *Saint Venant compatibility equations*. Although the system would generally have 81 individual equations, most are either simple identities or repetitions and only six are meaningful. It turns out that one can set $k = l$ without loss in generality, and this leads to determine the six nonzero relations given by

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \\
&\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} \\
&\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = 2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} \\
&\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) \\
&\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right) \\
&\frac{\partial^2 \varepsilon_{33}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right)
\end{aligned} \tag{3.11.3}$$

It should be noted that not all six of these equations are independent, and it can be shown that they are equivalent to three independent fourth-order relations (see Exercise 3.22). However, in applications, it is usually more convenient to use the second-order equations (3.11.3). Most applications of these equations are found in linear elasticity for the case where the displacements are not included in the basic formulation. We will discuss this situation later in Chapter 6. Note that all linear strain fields will satisfy these equations and thus are automatically compatible.

Using relation (2.5.5), we can show that the compatibility relations (3.11.2) with $l = k$ can be expressed by

$$\eta_{ij} = \varepsilon_{ikl}\varepsilon_{jmp}\epsilon_{lp,km} = 0 \tag{3.11.4}$$

where η_{ij} is sometimes referred to as the *incompatibility tensor* (Asaro and Lubarda, 2006). Note the notation is a little less than ideal where $\varepsilon_{...}$ is the alternating symbol while $\epsilon_{..}$ is the small strain tensor. This form can also be written in vector notation as

$$\nabla \times \boldsymbol{\epsilon} \times \nabla = 0 \tag{3.11.5}$$

In the previous development of the compatibility relations, we assumed that the displacements were continuous and thus the resulting equations (3.11.3) are actually only a necessary condition. It is not very difficult to prove that they are also sufficient (Sadd, 2014), but with the caveat that the region must be simply connected. Thus, the compatibility equations (3.11.3) are necessary and sufficient conditions for single-valued continuous displacements in simply connected domains. Such domains refer to regions of space for which all simple closed curves drawn in the region can be continuously shrunk to a point without going outside the region. Domains not having this property are called *multiply connected*. For two-dimensional regions, the presence of one or more holes will make the region multiply connected.

Note that these compatibility conditions do not ensure uniqueness of the displacements; and in actuality the displacements are not unique since we can always superpose a rigid body motion which changes the displacements of points but has no effect on the strains. If the displacements are explicitly included in the problem formulation, then necessarily the solution will produce single-valued displacements and there is no need to use the compatibility equations. For multiply-connected regions, additional conditions must be satisfied by the strains for single-valued displacements (see Fung, 1965).

Compatibility for the nonlinear finite strain case is much more difficult to handle since the relations are nonlinear partial differential equations. For this case, it is necessary to use general tensor theory, and the interested reader is referred to Truesdell and Toupin (1960) or Malvern (1969).

EXAMPLE 3.11.1 COMPATIBLE AND INCOMPATIBLE INFINITESIMAL STRAIN TENSORS

Show that the strain field (a) is compatible, whereas strain (b) is not.

$$\text{a.}\quad \varepsilon_{ij}=\begin{bmatrix} Ax_2^2 & 0 & 0 \\ 0 & -Ax_1^2 & 0 \\ 0 & 0 & Bx_3 \end{bmatrix},\quad \text{b.}\quad \varepsilon_{ij}=\begin{bmatrix} 0 & Ax_1x_2 & 0 \\ Ax_1x_2 & 0 & 0 \\ 0 & 0 & Bx_3 \end{bmatrix},\quad (A \text{ and } B \text{ small constants})$$

Solution:

(a) From the first compatibility relation $\dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\dfrac{\partial^2\varepsilon_{21}}{\partial x_1^2}=2\dfrac{\partial^2\varepsilon_{12}}{\partial x_1\,\partial x_2}\Rightarrow$

$2A-2A=0$. All other equations give 0 = 0, and so this strain is compatible.

(b) From the first compatibility relation $\dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\dfrac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\dfrac{\partial^2\varepsilon_{12}}{\partial x_1\,\partial x_2}\Rightarrow$

$0+0=2A$, which is not satisfied, and so the strain is not compatible.

EXAMPLE 3.11.2 DETERMINATION OF STRAINS FROM DISPLACEMENTS, AND DISPLACEMENTS FROM STRAINS FOR SMALL DEFORMATION THEORY

(a) For small deformation theory, find the strains from the given displacement field $u_1=Ax_1^2+Bx_1x_2^2,\ u_2=Cx_1x_2^2,\ u_3=0$, with A, B, and C constants.

(b) Given the following strain field, $\varepsilon_{ij}=\begin{bmatrix} Ax_1^2 & 0 & 0 \\ 0 & Bx_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with A and B constants. Determine the corresponding displacements assuming they only depend on x_1 and x_2, and that $u_3=0$ (two-dimensional problem).

Solution: (a) To get strains, simply use (3.11.1) and differentiate the displacements

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}=2Ax_1+Bx_2^2,\quad \varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)=\frac{1}{2}\left(2Bx_1x_2+Cx_2^2\right)$$

$$\varepsilon_{22}=\frac{\partial u_2}{\partial x_2}=2Cx_1x_2,\quad \varepsilon_{23}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)=0$$

$$\varepsilon_{33}=\frac{\partial u_3}{\partial x_3}=0,\quad \varepsilon_{31}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)=0$$

Solution: (b) We have to integrate the strain–displacement relations to determine displacements

$\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1} = Ax_1^2 \Rightarrow u_1 = \dfrac{A}{3}x_1^3 + f(x_2)$, integration creates arbitrary function $f(x_2)$

$\varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2} = Bx_2^2 \Rightarrow u_2 = \dfrac{B}{3}x_2^3 + g(x_1)$, integration creates arbitrary function $g(x_1)$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0 \Rightarrow \frac{\mathrm{d}f(x_2)}{\mathrm{d}x_2} + \frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1} = 0 \Rightarrow \frac{\mathrm{d}f(x_2)}{\mathrm{d}x_2} = -\frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1} = \text{constant}$$

The last relation must equal a constant since x_1 and x_2 are independent variables.

$\therefore f(x_2) = ax_2 + b, g(x_1) = -ax_1 + c$, with a, b, and c all arbitrary constants.

Thus, the displacements are given by $u_1 = \dfrac{A}{3}x_1^3 + ax_2 + b, \quad u_2 = \dfrac{B}{2}x_2^2 - ax_1 + c,$

Note that portions from $f(x_2)$ and $g(x_1)$ are actually rigid body motion terms.

3.12 ROTATION TENSOR

Let us now explore the kinematical concept of continuum rotation. We will look at this special type of motion from both small and large deformation theory. Starting first with small deformations, relation (3.6.3) may be written as

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2\varepsilon_{ij}\, dX_i\, dX_j \tag{3.12.1}$$

and so a body moves rigidly (preserving relative distances) if and only if $\varepsilon_{ij} = 0$. This fact implies that

$$u_{i,j} = -u_{j,i} \tag{3.12.2}$$

Next combining previous relations (3.5.5) and (3.5.8) yields

$$dx_i = F_{ij}\, dX_j = \left(\delta_{ij} + u_{i,j}\right) dX_j \tag{3.12.3}$$

Expanding the displacement gradient into symmetric and antisymmetric parts as per relation (2.3.4)

$$u_{i,j} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{3.12.4}$$

and defining

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$
$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{3.12.5}$$

we can write

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij} \tag{3.12.6}$$

The symmetric part of $u_{i,j}$ is the infinitesimal strain tensor ε_{ij} previously defined by (3.8.1), whereas the antisymmetric part ω_{ij} is called the *infinitesimal rotation tensor*. In light of relation (3.12.2), the skew-symmetric tensor ω_{ij} represents kinematical motion without strain and is thus associated with rigid body rotational motion. Combining (3.12.3) and (3.12.6) allows us to write

$$d\boldsymbol{x} = (\boldsymbol{I} + \omega + \varepsilon)\, d\boldsymbol{X} \tag{3.12.7}$$

We can then conclude that every infinitesimal deformation consists locally of the *sum of a translation, rotation*, and *pure strain.* The order of these individual motions is arbitrary but the decomposition is unique.

From the discussion in Section 2.3 and Exercise 2.13, the rotation tensor ω_{ij} can be associated with an axial or dual vector ω_i defined by

$$\omega_i = -\frac{1}{2}\varepsilon_{ijk}\omega_{jk} = \frac{1}{2}\varepsilon_{ijk}u_{k,j} \Rightarrow \boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{u} \tag{3.12.8}$$

where $\boldsymbol{\omega}$ is called the *rotation vector* with components

$$\omega_1 = \omega_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$
$$\omega_2 = \omega_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \tag{3.12.9}$$
$$\omega_3 = \omega_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

For the large or finite strain case, we have already (Section 3.6) presented most of the relations associated with the rotation tensor. For example, starting with the basic relation $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$, we used the Polar Decomposition Theorem to write

$$d\boldsymbol{x} = \boldsymbol{R}\boldsymbol{U}\, d\boldsymbol{X} = \boldsymbol{V}\boldsymbol{R}\, d\boldsymbol{X} \tag{3.12.10}$$

The orthogonal tensor $\boldsymbol{R}$ represented the rotation tensor, whereas $\boldsymbol{U}$ and $\boldsymbol{V}$ were the right and left stretch tensors that encompass pure strain. Relation (3.12.10) can be thought of as a two-step process of pure stretch $\boldsymbol{U}\,\mathrm{d}\boldsymbol{X}$ followed by a rotation $\boldsymbol{R}(\boldsymbol{U}\,\mathrm{d}\boldsymbol{X})$; or rotation $\boldsymbol{R}\,\mathrm{d}\boldsymbol{X}$ followed by a stretch $\boldsymbol{V}(\boldsymbol{R}\,\mathrm{d}\boldsymbol{X})$. This format then implies a *sequential or serial mapping* that results from the *multiplicative decomposition* of the deformation into pure strain and rigid body motion. This of course is totally different than the additive decomposition for the small deformation case (3.12.7).

EXAMPLE 3.12.1 SEPARATION OF STRAIN AND ROTATION TENSORS FOR SMALL DEFORMATION

Determine the strain and rotation tensors for the following small deformation case with a displacement field given by

$$u_1 = Ax_1^2x_2 - \omega_0 x_2 + u_0,\quad u_2 = Ax_1x_2^2 + \omega_0 x_1 + v_0,\quad u_3 = Bx_2^2x_3,\quad A, B, \omega_0, u_0, v_0 \text{ all constants}$$

Solution:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \begin{bmatrix} 2Ax_1x_2 & A(x_1^2 + x_2^2)/2 & 0 \\ A(x_1^2 + x_2^2)/2 & 2Ax_1x_2 & 2Bx_2x_3 \\ 0 & 2Bx_2x_3 & Bx_2^2 \end{bmatrix}$$

$$\begin{aligned}\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) &= \begin{bmatrix} 0 & A(x_1^2 - x_2^2)/2 - \omega_0 & 0 \\ -A(x_1^2 - x_2^2)/2 + \omega_0 & 0 & -2Bx_2x_3 \\ 0 & 2Bx_2x_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & A(x_1^2 - x_2^2)/2 & 0 \\ -A(x_1^2 - x_2^2)/2 & 0 & -2Bx_2x_3 \\ 0 & 2Bx_2x_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

Note that the second term in the rotation tensor is actually a rigid body rotation about the x_3-axis. An example of calculating the rotation and strain tensors for a finite deformation was previously given in Example 3.6.3.

3.13 RATE OF STRAIN TENSORS

Since many materials have rate-dependent behaviors, we will be interested in extending our analysis into time rates of change of various kinematical tensors. This will bring in added complications, as we have already seen in Section 3.3. There we presented both a material time derivative holding particles constant, and a spatial time derivate holding spatial coordinates constant. Thus, we will have to keep track of the proper kinematical representation that we are interested in calculating the time rate of change.

We start with the material time derivative of the differential element $d\boldsymbol{x}$:

$$\begin{aligned}\frac{D}{Dt}(d\boldsymbol{x}) &= \frac{D}{Dt}(\boldsymbol{x}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}, t) - \boldsymbol{x}(\boldsymbol{X}, t)) \\ &= \frac{D}{Dt}\boldsymbol{x}(\boldsymbol{X} + d\boldsymbol{X}, t) - \frac{D}{Dt}\boldsymbol{x}(\boldsymbol{X}, t) \\ &= \boldsymbol{v}(\boldsymbol{X} + d\boldsymbol{X}, t) - \boldsymbol{v}(\boldsymbol{X}, t) = \nabla\boldsymbol{v}(\boldsymbol{X}, t)\, d\boldsymbol{X}\end{aligned} \tag{3.13.1}$$

where $\boldsymbol{v}(\boldsymbol{X}, t)$ is the velocity vector in material coordinates, and $\nabla\boldsymbol{v}(\boldsymbol{X},t)=\dfrac{\partial\boldsymbol{v}}{\partial\boldsymbol{X}}$ is the *velocity gradient tensor with respect to the Lagrangian description*. Likewise, we could use the Eulerian or spatial description and rewrite (3.13.1) as

$$\frac{D}{Dt}(d\boldsymbol{x})=\nabla\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x} \tag{3.13.2}$$

where $\nabla\boldsymbol{v}(\boldsymbol{x},t)=\dfrac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}$ is the *spatial velocity gradient tensor*.

Using the spatial description, it is common to define $\boldsymbol{L}(\boldsymbol{x},t)=\nabla\boldsymbol{v}(\boldsymbol{x},t)=\dfrac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}$ and decompose the velocity gradient into symmetric and antisymmetric parts as

$$\boldsymbol{L}(\boldsymbol{x},t)=\nabla\boldsymbol{v}(\boldsymbol{x},t)=\boldsymbol{D}+\boldsymbol{W} \tag{3.13.3}$$

where

$$\begin{aligned}\boldsymbol{D}&=\frac{1}{2}\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^{\mathrm{T}}\right)\\ \boldsymbol{W}&=\frac{1}{2}\left(\nabla\boldsymbol{v}-(\nabla\boldsymbol{v})^{\mathrm{T}}\right)\end{aligned} \tag{3.13.4}$$

$\boldsymbol{D}$ is called the *rate of deformation tensor* and $\boldsymbol{W}$ is known as the *spin tensor*. In hydrodynamics, the quantity $2\boldsymbol{W}$ is commonly referred to as the *vorticity tensor*. Recall our previous discussions on the dual vector associated with antisymmetric tensors. Using this theory, we can define the dual vector $\boldsymbol{w}$ to the spin tensor $\boldsymbol{W}$ by the standard relations

$$w_i=-\frac{1}{2}\varepsilon_{ijk}W_{jk}=\frac{1}{2}\varepsilon_{ijk}v_{k,j}=\frac{1}{2}\varepsilon_{ijk}L_{kj} \tag{3.13.5}$$

where $\boldsymbol{w}$ is called the *angular velocity vector*. Again in classical fluid dynamics, $2\boldsymbol{w}$ is often referred to as the *vorticity vector*. Note that the discussion here makes no assumption on the magnitudes of the velocity gradient components, and so all results have no smallness restrictions.

Next let us consider the material time derivative of the squared length $(ds)^2$:

$$\begin{aligned}\frac{D}{Dt}[(ds)^2]&=\frac{D}{Dt}[d\boldsymbol{x}\cdot d\boldsymbol{x}]=2\,d\boldsymbol{x}\cdot\frac{D}{Dt}(d\boldsymbol{x})\\ &=2\,d\boldsymbol{x}\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x}=2\,d\boldsymbol{x}(\boldsymbol{D}+\boldsymbol{W})\,d\boldsymbol{x}\\ &=2\,d\boldsymbol{x}\,\boldsymbol{D}\,d\boldsymbol{x}=2D_{ij}\,dx_i\,dx_j\end{aligned} \tag{3.13.6}$$

where we have used (3.13.2) and (3.13.3), and the spin tensor $\boldsymbol{W}$ drops out since it is antisymmetric. Note that relation (3.13.6) indicates that only the rate of deformation tensor governs the material time derivative of $(ds)^2$.

Let us move on to time rates of several of our strain tensors. Consider first the material time derivatives of the deformation gradient and its inverse:

$$\begin{aligned}&\dot{F}_{ij}=\frac{D}{Dt}\frac{\partial x_i}{\partial X_j}=\frac{\partial}{\partial X_j}\frac{Dx_i}{Dt}=\frac{\partial v_i}{\partial X_j}=\frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}=L_{ik}F_{kj}\\ &\text{or }\ \dot{\boldsymbol{F}}=\boldsymbol{L}\boldsymbol{F}\end{aligned} \tag{3.13.7}$$

$$\begin{gathered}
\frac{D}{Dt}\left(F_{ik}F_{kj}^{-1}=\delta_{ij}\right)\Rightarrow F_{ik}\dot{F}_{kj}^{-1}+\dot{F}_{ik}F_{kj}^{-1}=0\Rightarrow F_{ik}\dot{F}_{kj}^{-1}=-\dot{F}_{ik}F_{kj}^{-1}\Rightarrow \\
F_{li}^{-1}\left(F_{ik}\dot{F}_{kj}^{-1}=-\dot{F}_{ik}F_{kj}^{-1}\right)\Rightarrow \dot{F}_{lj}^{-1}=-F_{li}^{-1}L_{im}F_{mk}F_{kj}^{-1}=-F_{li}^{-1}L_{ij} \\
\text{or } \dot{\boldsymbol{F}}^{-1}=-\boldsymbol{F}^{-1}\boldsymbol{L}
\end{gathered} \tag{3.13.8}$$

Using property (2.5.11), we can determine the time rate of the Jacobian J:

$$\begin{gathered}
\frac{D}{Dt}J=\frac{D}{Dt}\left(\det F_{ij}\right)=\det(\boldsymbol{F})F_{ij}^{-1}\dot{F}_{ji}=JF_{ij}^{-1}L_{jk}F_{ki}=JL_{kk} \\
\text{or } \dot{J}=Jv_{k,k}=J\,div\,\boldsymbol{v}
\end{gathered} \tag{3.13.9}$$

Since the Jacobian provides a local measure of the volume change, (3.13.9) would give the time rate of volume change.

Next, consider the material time derivative of the Lagrangian strain tensor $\boldsymbol{E}$:

$$\begin{aligned}
\dot{\boldsymbol{E}}&=\frac{D}{Dt}\left[\frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}-\boldsymbol{I}\right)\right]=\frac{1}{2}\left(\dot{\mathbf{F}}^{\mathrm{T}}\boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}}\dot{\mathbf{F}}\right) \\
&=\frac{1}{2}\left((\boldsymbol{LF})^{\mathrm{T}}\boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}}\boldsymbol{LF}\right)=\frac{1}{2}\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})\boldsymbol{F}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{DF}
\end{aligned} \tag{3.13.10}$$

where we have used (3.13.7). This result clearly indicates that the rate of deformation tensor $\boldsymbol{D}$ is not the same as time rate of change of $\boldsymbol{E}$. However, for the infinitesimal strain case with $\boldsymbol{F}=\boldsymbol{I}+\nabla\boldsymbol{u}\approx\boldsymbol{I}$:

$$\dot{\boldsymbol{\varepsilon}}=\boldsymbol{D} \tag{3.13.11}$$

Continuing on with time rates, consider next the right Cauchy–Green strain $\boldsymbol{C}$**:**

$$\boldsymbol{C}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}=2\boldsymbol{E}+\boldsymbol{I}\Rightarrow\dot{\mathbf{C}}=2\dot{\mathbf{E}}=2\boldsymbol{F}^{\mathrm{T}}\boldsymbol{DF} \tag{3.13.12}$$

and finally the left Cauchy–Green stain $\boldsymbol{B}$

$$\begin{aligned}
\dot{\mathbf{B}}&=\frac{D}{Dt}\left(\boldsymbol{FF}^{\mathrm{T}}\right)=\dot{\mathbf{F}}\boldsymbol{F}^{\mathrm{T}}+\boldsymbol{F}\dot{\mathbf{F}}^{\mathrm{T}}=\boldsymbol{LFF}^{\mathrm{T}}+\boldsymbol{F}(\boldsymbol{LF})^{\mathrm{T}} \\
&=\boldsymbol{LFF}^{\mathrm{T}}+\boldsymbol{FF}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}=\boldsymbol{LB}+\boldsymbol{BL}^{\mathrm{T}}
\end{aligned} \tag{3.13.13}$$

EXAMPLE 3.13.1 RATE EXAMPLES

A continuum material undergoes the following time-dependent motion:

$$\begin{aligned}
x_1&=2tX_1+4t^2X_2 \\
x_2&=4tX_2 \\
x_3&=X_3
\end{aligned}$$

Determine the following rate variables: $\boldsymbol{v}(\boldsymbol{X},t)$, $\boldsymbol{v}(\boldsymbol{x},t)$, $\boldsymbol{L}$, $\boldsymbol{D}$, $\boldsymbol{W}$, $\boldsymbol{w}$, $\dot{\boldsymbol{E}}$, $\dot{J}$.

Solution: The inverse motion is easily found $X_1=(x_1-tx_2)/2t, X_2=x_2/4t$, $X_3=x_3$ and the deformation gradient becomes $\boldsymbol{F}=\begin{bmatrix}2t & 4t^2 & 0\\ 0 & 4t & 0\\ 0 & 0 & 1\end{bmatrix}\Rightarrow$ $J=\det\boldsymbol{F}=8t^2$.

The velocity in material coordinates is $\boldsymbol{v}(\boldsymbol{X},t) = \dfrac{D}{Dt}\boldsymbol{x}(\boldsymbol{X},t) = \begin{bmatrix} 2X_1 + 8tX_2 \\ 4X_2 \\ 0 \end{bmatrix}$

and using the inverse motion results, $\boldsymbol{v}(\boldsymbol{x},t) = \begin{bmatrix} (x_1 - tx_2)/t + 2x_2 \\ x_2/t \\ 0 \end{bmatrix}$

$= \begin{bmatrix} x_1/t + x_2 \\ x_2/t \\ 0 \end{bmatrix}$

The velocity gradient tensor follows from the basic definition $\boldsymbol{L} = \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}$
$= \begin{bmatrix} 1/t & 1 & 0 \\ 0 & 1/t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

The rate of deformation and spin tensors are thus $\boldsymbol{D} = \dfrac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\right)$

$= \begin{bmatrix} 1/t & 1/2 & 0 \\ 1/2 & 1/t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{W} = \dfrac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

For the angular velocity vector,

$$w_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk} = -\frac{1}{2}(\varepsilon_{i12}W_{12} + \varepsilon_{i21}W_{21}) = -\frac{1}{4}(\varepsilon_{i12} - \varepsilon_{i21}) = -\frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\boldsymbol{e}_3$$

The rate of strain tensor is given by

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{F} = \begin{bmatrix} 2t & 0 & 0 \\ 4t^2 & 4t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/t & 1/2 & 0 \\ 1/2 & 1/t & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2t & 4t^2 & 0 \\ 0 & 4t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & t & 0 \\ 6t & 2t^2+4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2t & 4t^2 & 0 \\ 0 & 4t & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4t & 12t^2 & 0 \\ 12t^2 & 32t^3 + 16t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and finally the Jacobian rate $\dot{J} = Jv_{k,k} = JL_{kk} = 8t^2(2/t) = 16t$.

Before completing this section, let us check on the objectivity properties of some of the deformation rate tensors we have developed. Recall for a second-order tensor $\boldsymbol{A}$, the objectivity test expects that under change in reference frame $\boldsymbol{x}^* = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}$, the tensor will obey the transformation relation $\boldsymbol{A}^* = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{\mathrm{T}}$. First, consider the spatial velocity gradient $\boldsymbol{L}(\boldsymbol{x},t)=\dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}$. From (3.13.7) and using the transformation properties of $\boldsymbol{F}$,

$$\begin{aligned}\boldsymbol{L}^* &= \dot{\boldsymbol{F}}^*\boldsymbol{F}^{*-1} = (\dot{\boldsymbol{Q}}\boldsymbol{F}+\boldsymbol{Q}\dot{\boldsymbol{F}})(\boldsymbol{Q}\boldsymbol{F})^{-1}\\ &= (\dot{\boldsymbol{Q}}\boldsymbol{F}+\boldsymbol{Q}\dot{\boldsymbol{F}})\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}\\ &= \dot{\boldsymbol{Q}}\boldsymbol{F}\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}\boldsymbol{Q}^{\mathrm{T}}\\ &= \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}}\end{aligned} \tag{3.13.14}$$

Next, consider the rate of deformation tensor $\boldsymbol{D}$:

$$\begin{aligned}\boldsymbol{D}^* &= \frac{1}{2}\left(\boldsymbol{L}^*+\boldsymbol{L}^{*\mathrm{T}}\right)\\ &= \frac{1}{2}\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}}+(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}})^{\mathrm{T}}\right)\\ &= \frac{1}{2}\left((\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\dot{\boldsymbol{Q}}^{\mathrm{T}})+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}\right)\\ &= \frac{1}{2}\left(\overset{\cdot}{\overline{(\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}})}}+\boldsymbol{Q}(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}})\boldsymbol{Q}^{\mathrm{T}}\right)=\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}\end{aligned} \tag{3.13.15}$$

and then the spin tensor $\boldsymbol{W}$:

$$\begin{aligned}\boldsymbol{W}^* &= \frac{1}{2}\left(\boldsymbol{L}^*-\boldsymbol{L}^{*\mathrm{T}}\right)\\ &= \frac{1}{2}\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}}-(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}})^{\mathrm{T}}\right)\\ &= \frac{1}{2}\left((\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}-\boldsymbol{Q}\dot{\boldsymbol{Q}}^{\mathrm{T}})+\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{\mathrm{T}}-\boldsymbol{Q}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}\right)\\ &= \frac{1}{2}\left(2\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}(\boldsymbol{L}-\boldsymbol{L}^{\mathrm{T}})\boldsymbol{Q}^{\mathrm{T}}\right)=\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^{\mathrm{T}}\end{aligned} \tag{3.13.16}$$

Finally, let us check on the strain rate tensor $\dot{\boldsymbol{E}}$:

$$\begin{aligned}\dot{\boldsymbol{E}}^* &= \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{D}^*\boldsymbol{F}^*\\ &= (\boldsymbol{Q}\boldsymbol{F})^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{Q}\boldsymbol{F})\\ &= \boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{F}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{F}=\dot{\boldsymbol{E}}\end{aligned} \tag{3.13.17}$$

We thus conclude that only the rate of deformation tensor $\boldsymbol{D}$ satisfies the standard objective relation. However, as we pointed out in Section 3.6, the Lagrangian strain is based on reference coordinates and thus will transform as a scalar field.

3.14 OBJECTIVE TIME DERIVATIVES

Recall that in Section 2.9, we explored the principle of objectivity or frame indifference and indicated that we wish to have our general tensor variables and relations satisfy this principle. This can be again summarized that under the change in reference frame $\boldsymbol{x}^* = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}$, objective first- and second -order tensors $\boldsymbol{a}$ and $\boldsymbol{A}$ must satisfy the relations

$$\begin{aligned} \boldsymbol{a}^* &= \boldsymbol{Q}\boldsymbol{a} \\ \boldsymbol{A}^* &= \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{\mathrm{T}} \end{aligned} \tag{3.14.1}$$

In that section, we found that the velocity vector is in general not objective, since it transforms as

$$\boldsymbol{v}^* = \dot{\boldsymbol{c}} + \dot{\boldsymbol{Q}}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{v} \tag{3.14.2}$$

Since the velocity is part of the material time derivative as per relation (3.3.3), this derivative is not objective or frame indifferent. Because several rate-dependent constitutive relations use such time derivatives, this will produce an unpleasant situation. In order to correct this problem, various *objective time derivatives* have been developed.

Let us first consider the material time derivative of an objective vector $\boldsymbol{a}$:

$$\dot{\boldsymbol{a}}^* = \boldsymbol{Q}\dot{\boldsymbol{a}} + \dot{\boldsymbol{Q}}\boldsymbol{a} \tag{3.14.3}$$

Clearly $\dot{\boldsymbol{a}}$ is no longer objective because of the extra term $\dot{\boldsymbol{Q}}\boldsymbol{a}$ in relation (3.14.3). We now wish to modify the ordinary material time derivative to another form which will satisfy the objectivity test. Recall from the previous relation (3.13.16) that the spin tensor satisfied the relation $\boldsymbol{W}^* = \boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^{\mathrm{T}} + \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}$ and $\boldsymbol{W} = -\boldsymbol{W}^{\mathrm{T}}$. Now from (3.14.1) and (3.14.3),

$$\begin{gathered} \dot{\boldsymbol{a}}^* = \boldsymbol{Q}\dot{\boldsymbol{a}} + \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{a}^* = \\ \boldsymbol{Q}\dot{\boldsymbol{a}} + (\boldsymbol{W}^* - \boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^{\mathrm{T}})\boldsymbol{a}^* \Rightarrow \dot{\boldsymbol{a}}^* - \boldsymbol{W}^*\boldsymbol{a}^* = \boldsymbol{Q}(\dot{\boldsymbol{a}} - \boldsymbol{W}\boldsymbol{a}) \end{gathered} \tag{3.14.4}$$

and thus a modified time derivative form for vectors which satisfies objectivity can be written as

$$\overset{\circ}{\boldsymbol{a}} = \dot{\boldsymbol{a}} - \boldsymbol{W}\boldsymbol{a} \tag{3.14.5}$$

Going through a similar process for second-order tensors $\boldsymbol{A}$ gives the objective time derivative

$$\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{W}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{W} \tag{3.14.6}$$

Relations (3.14.5) and (3.14.6) are known as *Zarembra–Jaumann objective time rates*. These may be thought of as the rate of change as seen by an observer who is rotating with the media. Several other such rates have been defined in the literature most of which are special forms of the so-called *Lie derivative*. We will not pursue any further relations of this type here, and the interested reader can find more information in Haupt (2002) and Holzapfel (2006).

3.15 CURRENT CONFIGURATION AS REFERENCE CONFIGURATION

Recall back in Section 3.1, we indicated that the choice of the reference configuration was completely arbitrary. We now wish to redevelop some of the basic kinematics for the case where *the current configuration is used as the reference configuration*. This concept provides a convenient method to determine the *past history of the deformation*, and this is an important ingredient in some constitutive equations to be developed later. Let x_i and ξ_i be the positions at times t and τ of particle X which was at position X_α in the reference configuration.

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t), \quad \boldsymbol{\xi} = \boldsymbol{\chi}(\boldsymbol{X}, \tau) \tag{3.15.1}$$

Using the standard definition, the deformation gradients at times t and τ are given by

$$F_{i\alpha}(t) = \frac{\partial x_i}{\partial X_\alpha}, \quad F_{i\alpha}(\tau) = \frac{\partial \xi_i}{\partial X_\alpha} \tag{3.15.2}$$

Now, if instead, we take the current configuration at time t to be the reference configuration, then we can write the *relative motion* as

$$\boldsymbol{\xi} = \boldsymbol{\chi}(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), \tau) = \boldsymbol{\xi}(\boldsymbol{x}, t, \tau) \tag{3.15.3}$$

and the *relative deformation gradient tensor* becomes

$$\begin{aligned} F_t(\tau)_{ij} &= \frac{\partial \xi_i}{\partial x_j} = F_{i\alpha}(\tau) F^{-1}_{\alpha j}(t) \\ \boldsymbol{F}_t(\tau) &= \boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t) \end{aligned} \tag{3.15.4}$$

For this case, the right and left Cauchy–Green strain tensors are then written as

$$\begin{aligned} C_t(\tau)_{ij} &= F_t(\tau)_{ki} F_t(\tau)_{kj}, \quad \boldsymbol{C}_t(\tau) = \boldsymbol{F}_t(\tau)^{\mathrm{T}} \boldsymbol{F}_t(\tau) \\ B_t(\tau)_{ij} &= F_t(\tau)_{ik} F_t(\tau)_{jk}, \quad \boldsymbol{B}_t(\tau) = \boldsymbol{F}_t(\tau) \boldsymbol{F}_t(\tau)^{\mathrm{T}} \end{aligned} \tag{3.15.5}$$

Note that

$$\begin{aligned} \boldsymbol{C}_t(\tau) &= \boldsymbol{F}_t(\tau)^{\mathrm{T}} \boldsymbol{F}_t(\tau) \\ &= \left(\boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t)\right)^{\mathrm{T}} \boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t) \\ &= \boldsymbol{F}^{-1}(t)^{\mathrm{T}} \boldsymbol{F}(\tau)^{\mathrm{T}} \boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t) \\ &= \boldsymbol{F}^{-1}(t)^{\mathrm{T}} \boldsymbol{C}(\tau)\boldsymbol{F}^{-1}(t) \end{aligned} \tag{3.15.6}$$

Applying our objectivity test to relation (3.15.6) reveals that

$$\boldsymbol{C}_t^*(\tau) = \boldsymbol{Q}\boldsymbol{C}_t(\tau)\boldsymbol{Q}^{\mathrm{T}} \tag{3.15.7}$$

and thus the relative right Cauchy–Green strain tensor $\boldsymbol{C}_t(\tau)$ is objective, but recall that $\boldsymbol{C}(t)$ was not. It can be shown that for the left Cauchy–Green strain tensor both $\boldsymbol{B}(t)$ and $\boldsymbol{B}_t(\tau)$ are objective.

Now if time τ is in the range $-\infty < \tau < t$, then $\boldsymbol{C}_t(\tau)$ represents the entire past history of the deformation. This concept is also commonly written in an alternative form by letting $\tau = t - s$, and thus $\boldsymbol{C}_t(t-s)$ is the past strain history for s ranging $0 < s < \infty$. Other various strain and rate of strain tensors follow similar definitions with respect to this referencing scheme. We will put some of these tensors to use later in our studies on constitutive equation theories in Chapter 8.

EXAMPLE 3.15.1 RELATIVE DEFORMATION IN SIMPLE SHEAR AND EXTENSION

For the following time-dependent motions of simple shear and extension, determine the relative deformation gradient and relative right and left Cauchy–Green strain tensors:

Simple shear	Extension motion
$x_1 = X_1 + \gamma(t)X_2$	$x_1 = \lambda_1(t)X_1$
$x_2 = X_2$	$x_2 = \lambda_2(t)X_2$
$x_3 = X_3$	$x_3 = \lambda_3(t)X_3$

Solution: For the simple shear case,

$$\begin{array}{l} x_1 = X_1 + \gamma(t)X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{array} \Rightarrow \boldsymbol{F}(t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1 & \gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{F}^{-1}(t) = \begin{bmatrix} 1 & -\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{F}_t(\tau) = \boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t) = \begin{bmatrix} 1 & \gamma(\tau) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma(\tau)-\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{C}_t(\tau) = \boldsymbol{F}_t(\tau)^{\mathrm{T}}\boldsymbol{F}_t(\tau) &= \begin{bmatrix} 1 & 0 & 0 \\ \gamma(\tau)-\gamma(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma(\tau)-\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \gamma(\tau)-\gamma(t) & 0 \\ \gamma(\tau)-\gamma(t) & 1+[\gamma(\tau)-\gamma(t)]^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{B}_t(\tau) = \boldsymbol{F}_t(\tau)\boldsymbol{F}_t(\tau)^{\mathrm{T}} &= \begin{bmatrix} 1 & \gamma(\tau)-\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \gamma(\tau)-\gamma(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+[\gamma(\tau)-\gamma(t)]^2 & \gamma(\tau)-\gamma(t) & 0 \\ \gamma(\tau)-\gamma(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For the extensional deformation,

$$\begin{aligned} x_1 &= \lambda_1(t)X_1 \\ x_2 &= \lambda_2(t)X_2 \\ x_3 &= \lambda_3(t)X_3 \end{aligned} \quad \Rightarrow \boldsymbol{F}(t) = \begin{bmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{bmatrix}$$

$$\Rightarrow \boldsymbol{F}^{-1}(t) = \begin{bmatrix} \lambda_1^{-1}(t) & 0 & 0 \\ 0 & \lambda_2^{-1}(t) & 0 \\ 0 & 0 & \lambda_3^{-1}(t) \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{F}_t(\tau) = \boldsymbol{F}(\tau)\boldsymbol{F}^{-1}(t) &= \begin{bmatrix} \lambda_1(\tau) & 0 & 0 \\ 0 & \lambda_2(\tau) & 0 \\ 0 & 0 & \lambda_3(\tau) \end{bmatrix} \begin{bmatrix} \lambda_1^{-1}(t) & 0 & 0 \\ 0 & \lambda_2^{-1}(t) & 0 \\ 0 & 0 & \lambda_3^{-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\lambda_1(\tau)}{\lambda_1(t)} & 0 & 0 \\ 0 & \dfrac{\lambda_2(\tau)}{\lambda_2(t)} & 0 \\ 0 & 0 & \dfrac{\lambda_3(\tau)}{\lambda_3(t)} \end{bmatrix} \end{aligned}$$

$$\boldsymbol{C}_t(\tau) = \boldsymbol{F}_t(\tau)^{\mathrm{T}}\boldsymbol{F}_t(\tau) = \begin{bmatrix} \left(\dfrac{\lambda_1(\tau)}{\lambda_1(t)}\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{\lambda_2(\tau)}{\lambda_2(t)}\right)^2 & 0 \\ 0 & 0 & \left(\dfrac{\lambda_3(\tau)}{\lambda_3(t)}\right)^2 \end{bmatrix} = \boldsymbol{B}_t(\tau)$$

3.16 RIVLIN–ERICKSEN TENSORS

Nonlinear rate-dependent constitutive equations often use functional forms that incorporate higher-order time derivatives of strain tensors. These forms are constructed to be objective strain rates and are called *Rivlin–Ericksen tensors* $\boldsymbol{A}^{(n)}$. Truesdell and Noll (1965) defined these tensors by the relation

$$\boldsymbol{A}^{(n)} = \frac{D^n}{D\tau^n}[\boldsymbol{C}_t(\tau)]_{\tau=t} \tag{3.16.1}$$

and another equivalent form often given (Malvern, 1969; Eringen, 1967) is related to the expression

$$\frac{D^n}{Dt^n}\left[(ds)^2\right] = d\boldsymbol{x}\,\boldsymbol{A}^{(n)}\,d\boldsymbol{x} \tag{3.16.2}$$

which can be compared with (3.13.6) for the case with $n = 1$. These forms produce the set of tensors

$$\begin{aligned} \boldsymbol{A}^{(1)} &= \boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}} = 2\boldsymbol{D} \\ \boldsymbol{A}^{(2)} &= \frac{D}{Dt}\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(1)}\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A}^{(1)} \\ &\vdots \\ \boldsymbol{A}^{(n)} &= \frac{D}{Dt}\boldsymbol{A}^{(n-1)} + \boldsymbol{A}^{(n-1)}\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A}^{(n-1)} \end{aligned} \tag{3.16.3}$$

Note from (3.6.12), $(ds)^2 = d\boldsymbol{X}\,\boldsymbol{C}\,d\boldsymbol{X}$, and so

$$\frac{D^n}{Dt^n}\left[(ds)^2\right] = d\boldsymbol{X}\frac{D^n\boldsymbol{C}}{Dt^n}d\boldsymbol{X} \tag{3.16.4}$$

Now from definition (3.16.2),

$$\begin{aligned} \frac{D^n}{Dt^n}\left[(ds)^2\right] &= d\boldsymbol{x}\,\boldsymbol{A}^{(n)}\,d\boldsymbol{x} = (d\boldsymbol{X}\,\boldsymbol{F}^T)\boldsymbol{A}^{(n)}(\boldsymbol{F}\,\mathrm{d}\boldsymbol{X}) \\ &= d\boldsymbol{X}(\boldsymbol{F}^T\boldsymbol{A}^{(n)}\boldsymbol{F})\,d\boldsymbol{X} \end{aligned} \tag{3.16.5}$$

Comparing results (3.16.4) and (3.16.5) implies

$$\frac{D^n\boldsymbol{C}}{Dt^n} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{A}^{(n)}\boldsymbol{F} \quad \text{or} \quad \boldsymbol{A}^{(n)} = (\boldsymbol{F}^{\mathrm{T}})^{-1}\frac{D^n\boldsymbol{C}}{Dt^n}\boldsymbol{F}^{-1} \tag{3.16.6}$$

which matches with form (3.16.1).

3.17 CURVILINEAR CYLINDRICAL AND SPHERICAL COORDINATE RELATIONS

In order to formulate and solve many continuum mechanics problems, it is necessary to use curvilinear coordinates typically cylindrical or spherical systems. We now wish to develop a few of the kinematical expressions for these coordinate systems. Starting first with small deformation theory, the strain–displacement relation was given by equation (3.8.1):

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right] \tag{3.17.1}$$

The desired curvilinear relations can be determined using the appropriate forms for the displacement gradient term $\nabla\boldsymbol{u}$.

The cylindrical coordinate system previously defined in Fig. 2.6 establishes new components for the displacement vector and strain tensor

$$\begin{aligned} \boldsymbol{u} &= u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z \\ \boldsymbol{\varepsilon} &= \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix} \end{aligned} \tag{3.17.2}$$

Notice that the symmetry of the strain tensor is preserved in this orthogonal curvilinear system. Using results (2.18.18) and (2.18.10), the derivative operation $\nabla \boldsymbol{u}$ in cylindrical coordinates can be expressed by

$$\begin{aligned}\nabla \boldsymbol{u} = & \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_z \\ & + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z \\ & + \frac{\partial u_r}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z\end{aligned} \tag{3.17.3}$$

Placing this result into the strain–displacement form (3.17.1) gives the desired relations in cylindrical coordinates. The individual scalar equations are given by

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\ \varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \varepsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\end{aligned} \tag{3.17.4}$$

For spherical coordinates defined by Fig. 2.7, the displacement vector and strain tensor can be written as

$$\begin{aligned}\boldsymbol{u} &= u_R\boldsymbol{e}_R + u_\phi\boldsymbol{e}_\phi + u_\theta\boldsymbol{e}_\theta \\ \boldsymbol{\varepsilon} &= \begin{bmatrix} \varepsilon_{RR} & \varepsilon_{R\phi} & \varepsilon_{R\theta} \\ \varepsilon_{R\phi} & \varepsilon_{\phi\phi} & \varepsilon_{\phi\theta} \\ \varepsilon_{R\theta} & \varepsilon_{\phi\theta} & \varepsilon_{\theta\theta} \end{bmatrix}\end{aligned} \tag{3.17.5}$$

Following identical procedures as used for the cylindrical equation development, the strain–displacement relations for spherical coordinates become

$$\begin{aligned}\varepsilon_{RR} &= \frac{\partial u_R}{\partial R}, \quad \varepsilon_{\phi\phi} = \frac{1}{R}\left(u_R + \frac{\partial u_\phi}{\partial \phi}\right) \\ \varepsilon_{\theta\theta} &= \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi u_R + \cos\phi u_\phi\right) \\ \varepsilon_{R\phi} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right) \\ \varepsilon_{\phi\theta} &= \frac{1}{2R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \cot\phi u_\theta\right) \\ \varepsilon_{\theta R} &= \frac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right)\end{aligned} \tag{3.17.6}$$

It is observed that these relations in curvilinear systems contain additional terms that do not include derivatives of individual displacement components. For example, in spherical coordinates, a simple uniform radial displacement u_R will give rise to transverse extensional strains $\varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \dfrac{u_R}{R}$. This deformation can be simulated by blowing up a spherical balloon and observing the separation of points on the balloon's surface. Such terms were not found in the Cartesian forms, and their appearance is related to the curvature of the spatial coordinate system. Clearly, the curvilinear forms (3.17.4) and (3.17.6) are more complicated than the corresponding Cartesian relations (3.11.1). However, for particular problems, the curvilinear relations when combined with other field equations will allow analytical solutions to be developed that could not be found using a Cartesian formulation.

In regard to finite deformation theory, curvilinear forms are more challenging to generate, and we will only give a few of the relations for cylindrical coordinates. If we choose cylindrical coordinates for both the reference (R, Θ, Z) and current configurations (r, θ, z), the motion can be expressed by

$$\begin{aligned} r &= r(R,\Theta,Z,t) \\ \theta &= \theta(R,\Theta,Z,t) \\ z &= z(R,\Theta,Z,t) \end{aligned} \tag{3.17.7}$$

Using the results in either Sections 2.18 or 2.19, we can write

$$\begin{aligned} d\boldsymbol{x} &= dr\boldsymbol{e}_r + r\,d\theta\boldsymbol{e}_\theta + dz\boldsymbol{e}_z \\ d\boldsymbol{X} &= dR\boldsymbol{E}_R + R\,d\Theta\boldsymbol{E}_\Theta + dZ\boldsymbol{E}_Z \end{aligned} \tag{3.17.8}$$

where we are using $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ as the basis in the current configuration and $(\boldsymbol{E}_R, \boldsymbol{E}_\Theta, \boldsymbol{E}_Z)$ in the reference configuration. Employing the usual relation $d\boldsymbol{x} = \boldsymbol{F}\,d\boldsymbol{X}$, we find

$$\begin{aligned} dr\boldsymbol{e}_r + r\,d\theta\boldsymbol{e}_\theta + dz\boldsymbol{e}_z &= \boldsymbol{F}(dR\boldsymbol{E}_R + R\,d\Theta\boldsymbol{E}_\Theta + dZ\boldsymbol{E}_Z) \Rightarrow \\ dr &= dR(\boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_R) + R\,d\Theta(\boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_\Theta) + dZ(\boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_Z) \\ r\,d\theta &= dR(\boldsymbol{e}_\theta\boldsymbol{F}\boldsymbol{E}_R) + R\,d\Theta(\boldsymbol{e}_\theta\boldsymbol{F}\boldsymbol{E}_\Theta) + dZ(\boldsymbol{e}_\theta\boldsymbol{F}\boldsymbol{E}_Z) \\ dz &= dR(\boldsymbol{e}_z\boldsymbol{F}\boldsymbol{E}_R) + R\,d\Theta(\boldsymbol{e}_z\boldsymbol{F}\boldsymbol{E}_\Theta) + dZ(\boldsymbol{e}_z\boldsymbol{F}\boldsymbol{E}_Z) \end{aligned} \tag{3.17.9}$$

Thus, $\boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_R = F_{11} = \dfrac{\partial r}{\partial R}, \boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_\Theta = F_{12} = \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta}, \boldsymbol{e}_r\boldsymbol{F}\boldsymbol{E}_Z = F_{13} = \dfrac{\partial r}{\partial Z}, \ldots$, and so the final form for the deformation gradient tensor is

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix} \tag{3.17.10}$$

If we were to use Cartesian coordinates for reference (X, Y, Z) and cylindrical coordinates for the current configuration (r, θ, z), the deformation gradient would become

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial X} & \dfrac{\partial r}{\partial Y} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial X} & r\dfrac{\partial \theta}{\partial Y} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{bmatrix} \tag{3.17.11}$$

Knowing these forms for the deformation gradient allows for the calculation of the right and left Cauchy–Green strain tensors. Lai et al. (2010) provided further details on these calculations. Appendices A and B also give additional summary details on curvilinear forms for a variety of continuum mechanics relations and variables.

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EXERCISES

3.1 Use MATLAB Code C-3 to make two-dimensional deformation plots of reference unit squares and circles for the following material motions:

(a) $\begin{aligned} x_1 &= (\sqrt{3}/2)X_1 + 0.5X_2 + 1.5 \\ x_2 &= -0.5X_1 + (\sqrt{3}/2)X_2 + 1.5 \end{aligned}$ (b) $\begin{aligned} x_1 &= 2X_1 - 0.5X_2 + 1.5 \\ x_2 &= 0.5X_1 + 1.2X_2 + 1.5 \end{aligned}$ (c) $\begin{aligned} x_1 &= 2X_1 + 1.5X_2 + 2 \\ x_2 &= 0.5X_1 + 2 \end{aligned}$

3.2 For the following temperature and motion fields, determine the material time derivative of the temperature

(a) $\begin{array}{l} \theta = x_1 + 2x_2 + 4tx_3 \\ x_1 = 2tX_1 + 3tX_2 \\ x_2 = X_1 + 2tX_2 \\ x_3 = X_3 \end{array}$ (b) $\begin{array}{l} \theta = 2tx_1 + 4x_2 + x_3 \\ x_1 = X_1 + 3tX_2 \\ x_2 = 2tX_1 + 3t^2X_2 \\ x_3 = X_3 \end{array}$ (c) $\begin{array}{l} \theta = x_1 + 4t^2x_2 + x_3 \\ x_1 = X_1 + 2X_2 \\ x_2 = 2X_1 + 3tX_2 \\ x_3 = 2X_3 \end{array}$

3.3 For the following motions, determine the velocity and acceleration fields in material and spatial coordinates:

(a) $\begin{array}{l} x_1 = X_1 + 3t^2X_2 \\ x_2 = X_1 + 2tX_2 \\ x_3 = X_3 \end{array}$ (b) $\begin{array}{l} x_1 = 2tX_1 + 4X_2 \\ x_2 = X_1 + 2t^3X_2 \\ x_3 = X_3 \end{array}$ (c) $\begin{array}{l} x_1 = tX_1 + 3X_2 \\ x_2 = 2t^2X_1 + X_2 \\ x_3 = X_3 \end{array}$

3.4 Calculate the acceleration from the following spatial velocity fields:

(a) $\boldsymbol{v} = 2x_1x_2\boldsymbol{e}_1 + 3x_2\boldsymbol{e}_2 + \boldsymbol{e}_3$

(b) $\boldsymbol{v} = 4x_1t\boldsymbol{e}_1 + 2x_2\boldsymbol{e}_2 + 2t^2\boldsymbol{e}_3$

(c) $\boldsymbol{v} = 2x_1t\boldsymbol{e}_1 + 4x_2t^2\boldsymbol{e}_2 + 3\boldsymbol{e}_3$

3.5 Determine the displacement field, displacement gradient, and deformation gradient tensors for the following material motions:

(a) $\begin{array}{l} x_1 = X_1 + 3X_2 \\ x_2 = 2X_1 + 4X_2 \\ x_3 = 4X_1X_3 \end{array}$ (b) $\begin{array}{l} x_1 = X_1 + 3X_2X_3 \\ x_2 = 2X_1 + 4X_2 + X_3 \\ x_3 = 6X_1X_2 \end{array}$ (c) $\begin{array}{l} x_1 = 3X_1X_3 + 2X_2 \\ x_2 = 2X_1 + 4X_2X_3 \\ x_3 = 4X_1 + 3X_3 \end{array}$

3.6 Consider the deformations of simple shear and two-dimensional extension given by the two deformation gradients

$\boldsymbol{F}^{(1)} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{F}^{(2)} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using the sequential deformation concept, determine the overall relation between spatial and reference differential elements for the cases $d\boldsymbol{x} = \boldsymbol{F}^{(1)}\boldsymbol{F}^{(2)}\,d\boldsymbol{X}$ and $d\boldsymbol{x} = \boldsymbol{F}^{(2)}\boldsymbol{F}^{(1)}\,d\boldsymbol{X}$.

3.7 For the following motions, determine the deformation gradient $\boldsymbol{F}$. Next, calculate the inverse motion $\boldsymbol{X} = \chi^{-1}(\boldsymbol{x},t)$, and then using $\boldsymbol{F}^{-1} = \dfrac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}$ find the inverse deformation gradient tensor. Finally, verify that $\boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I}$.

(a) $\begin{array}{l} x_1 = X_1 + 2X_2 \\ x_2 = 2X_1 + X_2 \\ x_3 = X_3 \end{array}$ (b) $\begin{array}{l} x_1 = 4X_1 + X_2 \\ x_2 = 4X_2 + X_3 \\ x_3 = X_3 \end{array}$ (c) $\begin{array}{l} x_1 = X_1 + X_2 \\ x_2 = 2X_1 + 4X_2 \\ x_3 = 4X_1 + 3X_3 \end{array}$

3.8 *Rigid body motion* preserves the relative distances between particles in the body. Show that a body moves rigidly if $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \boldsymbol{I}$, and hence det $\boldsymbol{F} = 1$. Also justify that such a motion must be of the form $\boldsymbol{x} = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t)$ where $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{I}$.

3.9 Using the sequential deformation concept, consider the case where the deformation gradient can be decomposed into volumetric and distortional parts such that $\boldsymbol{F} = \boldsymbol{F}_v\boldsymbol{F}_d$. Since there is no volume change for the distortional deformation, $\det \boldsymbol{F}_d = 1$. Show that this condition is satisfied by the form $\boldsymbol{F}_d = J^{-1/3}\boldsymbol{F}$, and that $J = \det \boldsymbol{F}_v$.

3.10 Determine the Lagrangian, Eulerian, and right and left Cauchy–Green strain tensors for the following motions: (a) $x_1 = X_1 + 3X_2$, $x_2 = 2X_1 + 4X_2$, $x_3 = X_3$ (b) $x_1 = X_1 + 3X_2$, $x_2 = 2X_1 + 4X_3$, $x_3 = X_3$

3.11 Explicitly verify that the invariants of the right and left Cauchy–Green strain tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ are the same.

3.12 Using the Polar Decomposition Theorem, explicitly justify relations (3.6.11) and $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{V}^{-1}\boldsymbol{F}$, $\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{\mathrm{T}}$. Also justify that $\boldsymbol{U}$ and $\boldsymbol{V}$ must have the same principal values.

3.13 For the following motions, calculate the deformation gradient $\boldsymbol{F}$, the rotation tensor $\boldsymbol{R}$, and the right and left stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$. For each case, also verify that $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$:

$$\text{(a)}\quad \begin{aligned} x_1 &= 2X_1 \\ x_2 &= 4X_3 \\ x_3 &= 4X_2 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_1 &= 2X_3 \\ x_2 &= 4X_1 \\ x_3 &= 4X_2 \end{aligned} \qquad \text{(c)}\quad \begin{aligned} x_1 &= 4X_1 \\ x_2 &= 4X_2 \\ x_3 &= 2X_3 \end{aligned}$$

3.14 For the motions previously given in Exercise 3.10, determine the physical strain components $\gamma^{(i)}$ and $\sin\gamma^{(ij)}$.

3.15 For a unit length reference fiber $d\boldsymbol{X} = \dfrac{1}{\sqrt{2}}(\boldsymbol{E}_1 + \boldsymbol{E}_2)$, determine the fiber's new length and orientation after simple shearing and extensional deformation given in Examples 3.5.1 and 3.5.2.

3.16 Develop relations (3.6.22) for the physical strains in terms of the Eulerian strains.

3.17 As per the relations developed in Section 3.7, determine the line, area, and volume changes for the motions given in Exercise 3.10.

3.18 Assuming small deformations for the following set of displacements, calculate the strain tensor $\boldsymbol{\varepsilon}$ and the rotation tensor $\boldsymbol{\omega}$: (a) $\boldsymbol{u} = x_1x_2\boldsymbol{e}_1 + 3x_2\boldsymbol{e}_2 + x_3^2\boldsymbol{e}_3$

(b) $\boldsymbol{u} = 2x_1x_3^2\boldsymbol{e}_1 + 2x_1x_2\boldsymbol{e}_2 + 2x_3^3\boldsymbol{e}_3$ (c) $\boldsymbol{u} = 2\dfrac{x_1}{x_2}\boldsymbol{e}_1 + 4x_2x_3\boldsymbol{e}_2 + 3x_1\boldsymbol{e}_3$

3.19 Show that for small strain, the volume dilatation defined by relation (3.7.5) may be written as $\vartheta = \dfrac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V} = \varepsilon_{kk} = I_\varepsilon$.

3.20 Justify that the first invariant of the deviatoric small strain tensor is zero. In light of the results from Exercise 3.19, what does the vanishing of the dilatation imply?

3.21 For small strain, show that the principal directions of the strain deviator are the same as those of the strain tensor, and the deviator principal values are $\{\varepsilon_1 - \varepsilon_{kk}/3,\ \varepsilon_2 - \varepsilon_{kk}/3,\ \varepsilon_3 - \varepsilon_{kk}/3\}$.

3.22 Using relation (2.5.5), show that the compatibility relations (3.11.2) with $l = k$ can be expressed by $\eta_{ij} = \varepsilon_{ikl}\varepsilon_{jmp}\varepsilon_{lp,km} = 0$, which can also be written in vector notation as $\nabla \times \boldsymbol{e} \times \nabla = \mathbf{0}$.

3.23 Show that the six compatibility equations (3.11.3) may also be represented by the three independent fourth-order equations

$$\frac{\partial^4 \varepsilon_{11}}{\partial x_2^2 \partial x_3^2} = \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$
$$\frac{\partial^4 \varepsilon_{22}}{\partial x_3^2 \partial x_1^2} = \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right)$$
$$\frac{\partial^4 \varepsilon_{33}}{\partial x_1^2 \partial x_2^2} = \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right)$$

3.24 Determine if the following strain fields are compatible (A and B constants):

$$\text{(a)}\quad \varepsilon_{11} = Ax_2x_3,\quad \varepsilon_{22} = -Ax_1x_3,\quad \varepsilon_{33} = Bx_3,\quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\text{(b)}\quad \varepsilon_{11} = Ax_2^3,\quad \varepsilon_{22} = Ax_1^3,\quad \varepsilon_{12} = \frac{3}{2}Ax_1x_2(x_1 + x_2),\quad \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\text{(c)}\quad \varepsilon_{12} = A\frac{x_1}{x_1^2 + x_2^2},\quad \varepsilon_{13} = -A\frac{x_2}{x_1^2 + x_2^2},\quad \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = 0$$

3.25 Consider a two-dimensional small deformation case with strains given by $\varepsilon_{11} = Ax_1, \varepsilon_{22} = Ax_2, \varepsilon_{12} = Bx_1$. Following the integration scheme demonstrated in Example 3.11.2, determine the in-plane displacements u_1 and u_2.

3.26 Show for the two-dimensional small deformation case, the displacements $u_1 = -\omega_o x_2 + u_o, u_2 = \omega_o x_1 + v_o$ (ω_o, u_o, v_o all constants) represent rigid body motion and will yield zero strains.

3.27 Determine the following rate variables: $\boldsymbol{v}(\boldsymbol{X},t)$, $\boldsymbol{v}(\boldsymbol{x},t)$, $\boldsymbol{L}$, $\boldsymbol{D}$, $\boldsymbol{W}$, $\boldsymbol{w}$, $\dot{\boldsymbol{E}}$, $\dot{J}$ for the following material motions:

$$\text{(a)}\quad \begin{aligned} x_1 &= 2t^2X_1 + 4tX_2 \\ x_2 &= 4X_2 \\ x_3 &= 2tX_3 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_1 &= 2tX_1 + 4t^2X_2 \\ x_2 &= 4X_2 - 2tX_1 \\ x_3 &= X_3 \end{aligned} \qquad \text{(c)}\quad \begin{aligned} x_1 &= 2tX_1 + 4X_2 \\ x_2 &= 4t^2X_2 - 2tX_1 \\ x_3 &= 6t^2X_3 \end{aligned}$$

3.28 Justify that the objective time derivative for a second-order tensor is given by (3.14.6)

$$\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{W}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{W}$$

3.29 Apply the objectivity test to the relative right Cauchy–Green strain tensor to show that

$$\boldsymbol{C}_t^*(\tau) = \boldsymbol{Q}\boldsymbol{C}_t(\tau)\boldsymbol{Q}^{\mathrm{T}}$$

Next explore the corresponding result for the case of the left Cauchy–Green strain $\boldsymbol{B}_t(\tau)$.

3.30 For the following time-dependent motions, determine the relative deformation gradient and the relative right and left Cauchy–Green strain tensors

$$\text{(a)}\ \begin{array}{l} x_1 = 2X_1 + 4tX_2 \\ x_2 = 4X_2 \\ x_3 = 2tX_3 \end{array} \quad \text{(b)}\ \begin{array}{l} x_1 = 2tX_1 + X_2 \\ x_2 = 4X_2 - 2tX_1 \\ x_3 = X_3 \end{array} \quad \text{(c)}\ \begin{array}{l} x_1 = 2tX_1 + 4X_2 \\ x_2 = -4X_2 + 2tX_1 \\ x_3 = 6t^2X_3 \end{array}$$

3.31 Using Cartesian coordinates for reference and cylindrical coordinates for the current configuration, consider the finite deformation defined by

$$r^2 = 2AX, \quad \theta = BY, \quad z = \frac{Z}{AB}$$

where A and B are arbitrary constants. First determine the deformation geometry of where a reference rectangular domain: $X_1 \le X \le X_2, Y_1 \le Y \le Y_2$ gets mapped into the current configuration. Next, using the appropriate form from Section 3.17, determine the deformation gradient for this case and show that the motion is isochoric.