CHAPTER

8 General constitutive relations and formulation of nonlinear theories of solids and fluids

We now wish to expand our material modeling and explore several types of nonlinear continuum mechanics theories for both solids and fluids. In order to investigate such theoretical models, we now use finite deformation kinematics and will, in general, incorporate nonlinear constitutive relations. Recall from Chapter 3 that we established many different deformation/strain tensors, and likewise in Chapter 4 we presented several different stress tensors. Thus, we now face a choice in what particular strain measure and stress tensor to use within a given constitutive law. This choice will be guided by the fact that we want our relations to satisfy the principle of objectivity or frame-indifference. This principle was discussed in Sections 2.9, 3.6, 3.14, and 4.9, and thus we already have some guidance from this previous work. We start our presentation with some general principles of constitutive equation development and then move into general simple materials. This will be followed by exploring several particular material theories including nonlinear elasticity, nonlinear and non-Newtonian viscous fluids, and nonlinear viscoelastic materials.

8.1 INTRODUCTION AND GENERAL CONSTITUTIVE AXIOMS

Before going into specific details, we first explore constitutive equations in a much more general way. A constitutive equation can be described as a relation between independent variables and dependent response variables. Independent variables would commonly include the density, material motion, temperature, and their spatial and temporal derivatives and history. Dependent variables would normally be the stress, strain energy, Helmholtz free energy, heat flux, internal energy, and entropy. Thus, we could write a general system of constitutive equations of the form

$$\begin{aligned}
&\boldsymbol{T} = \boldsymbol{T}(\rho, \chi, \theta, \text{and derivatives and history})\\
&U = U(\rho, \chi, \theta, \text{and derivatives and history})\\
&\Psi = \Psi(\rho, \chi, \theta, \text{and derivatives and history})\\
&\boldsymbol{q} = \boldsymbol{q}(\rho, \chi, \theta, \text{and derivatives and history})\\
&\varepsilon = \varepsilon(\rho, \chi, \theta, \text{and derivatives and history})\\
&s = s(\rho, \chi, \theta, \text{and derivatives and history})
\end{aligned} \tag{8.1.1}$$

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Note that spatial derivatives of the motion would lead to the deformation gradient and other strain tensors. This general form provides considerable leeway as to which specific independent variables, derivatives, and history to include in the constitutive model.

Starting with the pioneering work by Noll (1958) and later enhanced and summarized by the classic publications of Truesdell and Toupin (1960) and Truesdell and Noll (1965), several *principles* normally taken as *axioms* have been proposed which provide guidance and restrictions on the development of constitutive relations. The list of axioms that have been developed commonly includes the following:

Consistency Constitutive equations must be consistent or admissible with the principles of balance of mass, momentum, energy, and second law of thermodynamics.
Coordinate Invariance Constitutive equations must be set in tensor language to ensure that they are *invariant* with respect to coordinate transformations.
Just Setting Combined with basic conservation equations, constitutive relations should provide a *unique solution* to a problem with meaningful boundary and initial conditions.
Material Frame-Indifference The material response must be the *same for all observers*. This principle is sometimes known as *material objectivity*.
Equipresence For a given material type, a variable present as an independent variable in one constitutive equation should be present in all, unless there is a contradiction with another constitutive axiom.
Determinism Only the *present and past history* of the independent variables in a constitutive equation can affect the present response, and normally we assume that the material response should exhibit *fading memory* such that recent history has more effect than past history.
Symmetry Constitutive relations must be consistent with known symmetries of the material.
Local Action The material response at point $\boldsymbol{X}$ is only a function of the independent variables (and derivatives and history) in the *neighborhood* of the point.

These concepts act as a *guide* in formulating proper constitutive equations for material behavior, leading to a wide variety of nonlinear, history-dependent relations. The last axiom of local action can be dropped, thus allowing the development *nonlocal continuum models*. Most of our previous work with the classical material theories presented in Chapters 6 and 7 followed these axioms. Due to the limited scope of the text, we will only explore in detail a small variety of nonlinear constitutive models. It should also be mentioned that not all useful constitutive theories satisfy all of these conditions, and thus some of the previous concepts may be relaxed for *special* situations.

EXAMPLE 8.1.1 USE OF SECOND LAW TO MODIFY GENERAL CONSTITUTIVE RELATIONS FOR THERMOVISCOUS FLUIDS

Consider a proposed set of constitutive relations for a general thermoviscous fluid that includes the rate of deformation tensor, mass density, and entropy:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{D}, \frac{1}{\rho}, s\right) \\ \varepsilon &= \varepsilon\left(\boldsymbol{D}, \frac{1}{\rho}, s\right) \\ \boldsymbol{q} &= \boldsymbol{q}\left(\boldsymbol{D}, \frac{1}{\rho}, s\right) \end{aligned} \tag{8.1.2}$$

Note that this set satisfies the axiom of Equipresence. Investigate any limitations coming from application of the second law of thermodynamics.

Solution: Using the second law in the form (5.7.10),

$$\begin{aligned} &\rho(\theta\dot{s} - \dot{\varepsilon}) + T_{ij}D_{ij} - \frac{1}{\theta}(\boldsymbol{q} \cdot \nabla\theta) \geq 0 \Rightarrow \\ &\rho\left(\theta - \frac{\partial \varepsilon}{\partial s}\right)\dot{s} - \rho\frac{\partial \varepsilon}{\partial D_{ij}}\dot{D}_{ij} + \frac{\partial \varepsilon}{\partial(1/\rho)}\frac{\dot{\rho}}{\rho} + T_{ij}D_{ij} - \frac{1}{\theta}(q_k\theta_{,k}) \geq 0 \end{aligned} \tag{8.1.3}$$

The above equation must hold for arbitrary values of $\dot{s}$ and $\dot{D}_{ij}$, and thus we must have

$$\theta = \frac{\partial \varepsilon}{\partial s} \quad \text{and} \quad \frac{\partial \varepsilon}{\partial D_{ij}} = 0$$

This implies that the internal energy ε is independent of $\boldsymbol{D}$ and so θ is also independent of $\boldsymbol{D}$.

Next, defining the *thermodynamic pressure p* by the relation $p = -\dfrac{\partial \varepsilon}{\partial(1/\rho)}$, and using the continuity equation (5.2.4), Eq. (8.1.3) can be expressed as

$$(T_{ij} - p\delta_{ij})D_{ij} - \frac{1}{\theta}(q_k\theta_{,k}) \geq 0 \tag{8.1.4}$$

Now since $\boldsymbol{T}$, p, and $\boldsymbol{q}$ are independent of $\theta_{,k}$, this requires that the heat flux $\boldsymbol{q}$ must vanish identically. Therefore, this class of thermoviscous fluids can only respond *adiabatically*, that is, with no heat flow. This finding is of course troubling since there is no physical reason for this adiabatic response. Furthermore, the fact that the internal energy is independent of the rate of deformation is also unrealistic. We conclude that the problem must lay with our original assumption of the forms of the constitutive laws (8.1.2). Clearly a more general form is needed for a realistic constitutive description of the fluid. Such a more general approach would likely be to include the temperature gradient $\theta_{,k}$ in the list of independent variables in (8.1.2).

More detailed studies on these types of problems are provided in Allen (2016), Chapter 6.

8.2 GENERAL SIMPLE MATERIALS

Several decades ago, Noll (1958) developed a general constitutive theory of *simple materials*. His purpose was to unify and clarify the large variety of constitutive laws that had been previously developed up to that point in time. His approach was part of the rebirth of continuum mechanics, often called *rational mechanics*, and this work served as the foundation for a large part of subsequent research in the field. Noll employed the general axioms of determinism, local action, and material frame-indifference. Starting with determinism, he first established a mechanical theory where the stress at location $\boldsymbol{X}$ and time t was given by the history of material motion $\boldsymbol{x}=\chi(\boldsymbol{X},t)$. Then invoking the concept of local action, he limited the material motion to only the local deformation gradient $\boldsymbol{F}(\boldsymbol{X},t)$, and this leads to the constitutive relation for simple materials:

$$\boldsymbol{T}(\boldsymbol{X},t)=\underset{\tau=-\infty}{\overset{\tau=t}{\mathfrak{J}}}\left(\boldsymbol{F}(\boldsymbol{X},\tau)\right) \tag{8.2.1}$$

where $\mathfrak{J}$ is a tensor-valued *history response functional* which is a function of the deformation gradient function over the history time variable $-\infty\leq\tau\leq t$. Note that if the material is nonhomogeneous, the response functional could depend explicitly on $\boldsymbol{X}$. Using our previous discussions in Chapters 3 and 4, the requirement that form (8.2.1) satisfies material frame-indifference can be written as

$$\boldsymbol{Q}(t)\underset{\tau=-\infty}{\overset{\tau=t}{\mathfrak{J}}}(\boldsymbol{F}(\boldsymbol{X},\tau))\boldsymbol{Q}^T(t)=\underset{\tau=-\infty}{\overset{\tau=t}{\mathfrak{J}}}(\boldsymbol{Q}(\tau)\boldsymbol{F}(\boldsymbol{X},\tau)) \tag{8.2.2}$$

Considering the local deformation relative to location $\boldsymbol{X}$, it can be shown that constitutive form (8.2.1) can be expressed in terms of the relative deformation gradient (see Section 3.15) as

$$\boldsymbol{T}(t)=\underset{s=0}{\overset{s=\infty}{\tilde{\mathfrak{J}}}}\left(\boldsymbol{F}_t(t-s)\right) \tag{8.2.3}$$

Notice that simple materials are not characterized by just material constants nor just material functions, but rather the material functional $\tilde{\mathfrak{J}}$ must be specified. Such a general specification is very challenging and a large body of research has addressed this issue.

Noll also proposed a general *material classification* scheme based on *symmetry groups* which are related to changes in reference configuration that result in equivalent descriptions of the material's constitutive response. Without going into any details of the theory, we can list a few of his conclusions based on the stress constitutive response functional:

- a material is *isotropic* if there is a reference configuration such that proper frame rotations have no effect on the constitutive response functional;
- a material is a *fluid* if it is isotropic and the constitutive response functional is unaffected by any/all density-preserving changes in reference configuration;
- a material is a *solid* if changes in the reference configuration, other than rigid rotations, affect the constitutive response functional.

Allen (2016) provides a more detailed description of these concepts.

In order to apply this type of theoretical constitutive formulation to problem solution, more reduction in the general form (8.2.1) must be made. Such reductions in the response functional can be determined for special material types such as fluids and solids. We will pursue this in the remaining sections of this chapter.

8.3 NONLINEAR FINITE ELASTICITY

We choose the entry point into nonlinear material behavior with the elastic response. For this case, nonlinearity enters the mathematical model though finite deformation relations and the constitutive laws. Unlike linear elasticity from Section 6.2, we will now have to account for differences between current and reference configurations. Because the material is elastic, it will return to its reference configuration (original zero loading configuration) when the loadings are removed. This type of behavior will thus eliminate any history-dependent response in the constitutive law. We will limit our study to include only static mechanical effects and focus on incompressible materials. Our presentation will be brief and more detailed information can be found in Green and Adkins (1970), Ogden (1984), and Holzapfel (2006).

8.3.1 CONSTITUTIVE LAWS AND GENERAL FORMULATION

We now revisit elastic materials but this time will consider nonlinear models under finite deformations. For an elastic solid, the principles of determinism and local action imply that the stress is determined by the *present value of the local deformation gradient tensor*. Thus, the general history functional in relation (8.2.1) would reduce to a simple function

$$\boldsymbol{T}(\boldsymbol{X},t)=\boldsymbol{f}\,(\boldsymbol{F}(\boldsymbol{X},t)) \tag{8.3.1}$$

We could use this simple material starting form and pursue some detailed mathematical steps to get into our final useful constitutive relation. However, a shorter more direct scheme will get us where we want to go and so we pursue that method. The available finite strain deformation tensors previously discussed in Sections 3.5 and 3.6 included $\boldsymbol{F}$, $\boldsymbol{E}$, $\boldsymbol{C}$, and $\boldsymbol{B}$. However, it was only the left Cauchy–Green strain tensor that satisfied objectivity, $\boldsymbol{B}^*=\boldsymbol{QBQ}^{\mathrm{T}}$. Thus, in order to satisfy the axiom of material frame-indifference, we must choose an objective strain tensor for use in the constitutive form. This implies

$$\boldsymbol{T}=\boldsymbol{g}(\boldsymbol{B}) \tag{8.3.2}$$

If we invoke the objectivity test for relation (8.3.2), we find that

$$\boldsymbol{Qg}(\boldsymbol{B})\,\boldsymbol{Q}^T=\boldsymbol{g}(\boldsymbol{QBQ}^T) \tag{8.3.3}$$

and thus conclude that the response function $\boldsymbol{g}$ must be an *isotropic function* of the strain measure $\boldsymbol{B}$. Now from our previous work in Section 2.14, the tensor-valued representation theorem (2.14.3) gives the specific form for the response function, and thus the constitutive law becomes

$$\boldsymbol{T} = \boldsymbol{g}(\boldsymbol{B}) = \phi_0\boldsymbol{I} + \phi_1\boldsymbol{B} + \phi_2\boldsymbol{B}^2 \tag{8.3.4}$$

where the coefficients ϕ_i are functions of the invariants of $\boldsymbol{B}$, $\phi_i = \phi_i(I_B, II_B, III_B)$. Using the Cayley–Hamilton Theorem, see result (2.13.7), we can eliminate the $\boldsymbol{B}^2$ term and write (8.3.4) as

$$\boldsymbol{T} = \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{B} + \alpha_2\boldsymbol{B}^{-1} \tag{8.3.5}$$

where again $\alpha_i = \alpha_i(I_B, II_B, III_B)$.

Next, if we consider *incompressible materials*, $III_B = \det\boldsymbol{B} = \det(\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}) = (\det\boldsymbol{F})^2 = 1$, and thus $\alpha_i = \alpha_i(I_B, II_B)$. Also since it can be shown (Exercise 2.19) that $I_{B^{-1}} = II_B / III_B$, and so for this case $II_B = I_{B^{-1}}$ and therefore $\alpha_i = \alpha_i(I_B, I_{B^{-1}})$. In general, for incompressible materials, the stress can only be determined up to an arbitrary hydrostatic pressure. This is because any pressure applied to an incompressible material will not produce volumetric change and thus the stress cannot be uniquely determined from the strains. Combining all of this reduces constitutive relation (8.3.5) to

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1\boldsymbol{B} + \alpha_2\boldsymbol{B}^{-1} \tag{8.3.6}$$

We could also proceed to develop the finite elastic constitutive law from strain energy concepts as previously given for the linear elastic case by relation (6.2.6). For the finite strain case, the Cauchy stress can be expressed as (Asaro and Lubarda, 2006)

$$T_{ij} = B_{ik}\frac{\partial U}{\partial B_{kj}} + \frac{\partial U}{\partial B_{ik}}B_{kj} \tag{8.3.7}$$

Choosing the isotropic incompressible case, the strain energy function will only depend on the first two invariants, $U = U(I_B, II_B)$, and thus

$$\frac{\partial U}{\partial B_{ij}} = \frac{\partial U}{\partial I_B}\frac{\partial I_B}{\partial B_{ij}} + \frac{\partial U}{\partial II_B}\frac{\partial II_B}{\partial B_{ij}} \tag{8.3.8}$$

Completing the calculations, we can develop the constitutive law using either the $\boldsymbol{B}^2$ or $\boldsymbol{B}^{-1}$ form as

$$\begin{aligned} T_{ij} + p\delta_{ij} &= 2\left(\frac{\partial U}{\partial I_B} + I_B\frac{\partial U}{\partial II_B}\right)B_{ij} - 2\frac{\partial U}{\partial II_B}B_{ik}B_{kj} \\ T_{ij} + p\delta_{ij} &= 2\frac{\partial U}{\partial I_B}B_{ij} - 2\frac{\partial U}{\partial II_B}B^{-1}_{ij} \end{aligned} \tag{8.3.9}$$

where we have absorbed all hydrostatic terms into p. Forms (8.3.9) represent the *hyperelastic* development and correspond completely with the previous relations using the Cauchy elastic scheme.

Over the years, specific simplified forms of the strain energy function have been developed. For the incompressible case, these have been created starting with the particular general form

$$U = U(I_B, II_B) = \sum_{m,n=0}^{\infty} C_{mn}(I_B - 3)^m (II_B - 3)^n \tag{8.3.10}$$

where C_{mn} are constants, and $C_{00} = 0$. Notice that in the reference configuration, $B_{ij} = \delta_{ij}$ and thus $I_B = II_B = 3$. Thus, this form automatically satisfies the condition that the strain energy vanishes in the reference configuration. Based on this, several specific models have been developed and two of the more common examples are

- *neo-Hookean*: $U = C_{10}(I_B - 3) \Rightarrow \alpha_1 = 2C_{10}, \alpha_2 = 0$;
- *Mooney–Rivlin model*: $U = C_{10}(I_B - 3) + C_{01}(II_B - 3) \Rightarrow \alpha_1 = 2C_{10}, \alpha_2 = -2C_{01}$.

Other forms have also been proposed in the literature (see, e.g. Ogden, 1972; Arruda and Boyce, 1993).

Our study will be limited to only static mechanical behaviors, and thus the stress field will be required to satisfy the equilibrium equations either in terms of the Cauchy stress in the current configuration (5.3.8) or in terms of the Piola–Kirchhoff stress in the reference configuration (5.3.13). Boundary conditions will be basically the same as those used in linear elasticity: boundary specification of either the displacement (motion) or tractions.

8.3.2 PROBLEM SOLUTIONS

In order to illustrate some of the features of nonlinear elasticity, we now explore the solution to several example problems. Many applications of finite elasticity are used to model rubber and polymeric materials which are nearly incompressible. Thus, we will restrict the following examples to this type of material and employ constitutive relation (8.3.6).

EXAMPLE 8.3.1 UNIAXIAL EXTENSION OF AN INCOMPRESSIBLE NONLINEAR ELASTIC BAR

Consider a nonlinear elastic bar under uniform uniaxial loading as shown in Fig. 8.1. Assume that the material is incompressible and that the deformation is uniform and purely extensional. Determine the Cauchy stress field form. Next, eliminate the arbitrary pressure term and find the Cauchy and first Piola–Kirchhoff axial stress in terms of the stretch ratio. Plot and compare these behaviors for the case of neo-Hookean and Mooney–Rivlin constitutive models.

Solution: From Example 3.6.2, extensional motion and its deformation gradient tensor are given by

$$\begin{aligned} x_1 &= \lambda_1 X_1 \\ x_2 &= \lambda_2 X_2 \\ x_3 &= \lambda_3 X_3 \end{aligned} \Rightarrow \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where λ_i are the stretch ratios. Since the deformation is uniform in the X_2 and X_3 directions, $\lambda_2 = \lambda_3$. Assuming sample incompressibility, we must have $\det \boldsymbol{F} = \lambda_1 \lambda_2 \lambda_3 = 1$ and this implies that

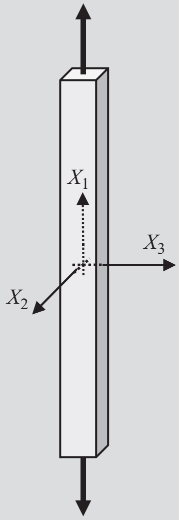


FIGURE 8.1

Uniaxial extension of elastic sample.

$$\lambda_1 = \lambda, \quad \text{and} \quad \lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda}}$$

and thus the entire deformation field is determined by the single stretch ratio parameter λ. Under these conditions, the deformation tensors then become

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

The invariants are given by $I_B = \lambda^2 + 2/\lambda$ and $I_{B^{-1}} = \lambda^{-2} + 2\lambda$. The stresses follow from (8.3.6): $\boldsymbol{T} = -p\boldsymbol{1} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^{-1} \Rightarrow$

$$\begin{aligned} T_{11} &= -p + \alpha_1 \lambda^2 + \alpha_2 \lambda^{-2} \\ T_{22} &= T_{33} = -p + \alpha_1 \frac{1}{\lambda} + \alpha_2 \lambda \\ T_{12} &= T_{23} = T_{31} = 0 \end{aligned} \tag{8.3.11}$$

It is observed that the stresses are all constant (homogeneous) and thus they will automatically satisfy the equations of equilibrium. Furthermore, since the sides of the sample are stress-free, $T_{22} = T_{33} = 0$ everywhere in the sample. Using this fact in relations (8.3.11) allows us to solve for the hydrostatic pressure

$$p = \alpha_1 \frac{1}{\lambda} + \alpha_2 \lambda$$

Substituting this result back into (8.3.11)$_1$, we can solve for the axial stress

$$T_{11} = \left(\lambda^2 - \frac{1}{\lambda}\right)\left(\alpha_1 - \frac{\alpha_2}{\lambda}\right) \tag{8.3.12}$$

The first Piola–Kichhoff stresses are found using relation (4.7.5) with $J = 1$:

$$\boldsymbol{T}^{\text{o}} = \boldsymbol{T}(\boldsymbol{F}^{-1})^T = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & \lambda^{1/2} & 0 \\ 0 & 0 & \lambda^{1/2} \end{bmatrix} = \begin{bmatrix} T_{11}/\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.3.13}$$

Thus, the only nonzero PK1 stress is $T_{11}^{\text{o}} = T_{11}/\lambda$. Note that referring back to Example 3.7.1, Nanson's formula (3.7.3) was used for the extension problem. Applying those results for this case, it is found that the reference cross-section area is related to the current area by the relation $A_{\text{o}} = \lambda A$. It should be pointed out that these results are consistent with the requirement that the total force on the bar cross-section be equal in both the reference and current configurations, that is, $T_{11}^{\text{o}} A_{\text{o}} = T_{11} A$.

Next, we wish to compare the Cauchy and PK1 axial stress predictions for the neo-Hookean and Mooney–Rivlin models. Generated using MATLAB Code C-20, Fig. 8.2 shows plots of each stress for particular choices of material constants. Note that since λ is dimensionless, the stress units in the figure correspond to the units associated with the choice of material constants α_i. Both stresses generally show similar nonlinear behavior versus the stretch λ. However, because of the scaling factor λ, there is considerable difference between the magnitudes of the two stress measures, with $T_{11} > T_{11}^{\text{o}}$.

Finally, we wish to explore some comparisons with experimental data for this extensional deformation problem. Early work by Rivlin and Saunders (1951) provided some basic experimental data on vulcanized rubber material, which is shown in Fig. 8.3A. This data was replotted and curve fit as illustrated in Fig. 8.3B. Using these numerical results (based on data), a Cauchy stress versus stretch plot was constructed and is shown in Fig. 8.3C. For comparison, a Mooney–Rivlin model (with parameters selected to best match with the data prediction) is also shown in this figure. All results were done using MATLAB Code C-22 with the data set shown in Appendix C. Although the Mooney–Rivlin predictions match closely with the data, in general, for other types of deformations, this will not be so, and other more sophisticated models generally result in better comparisons.

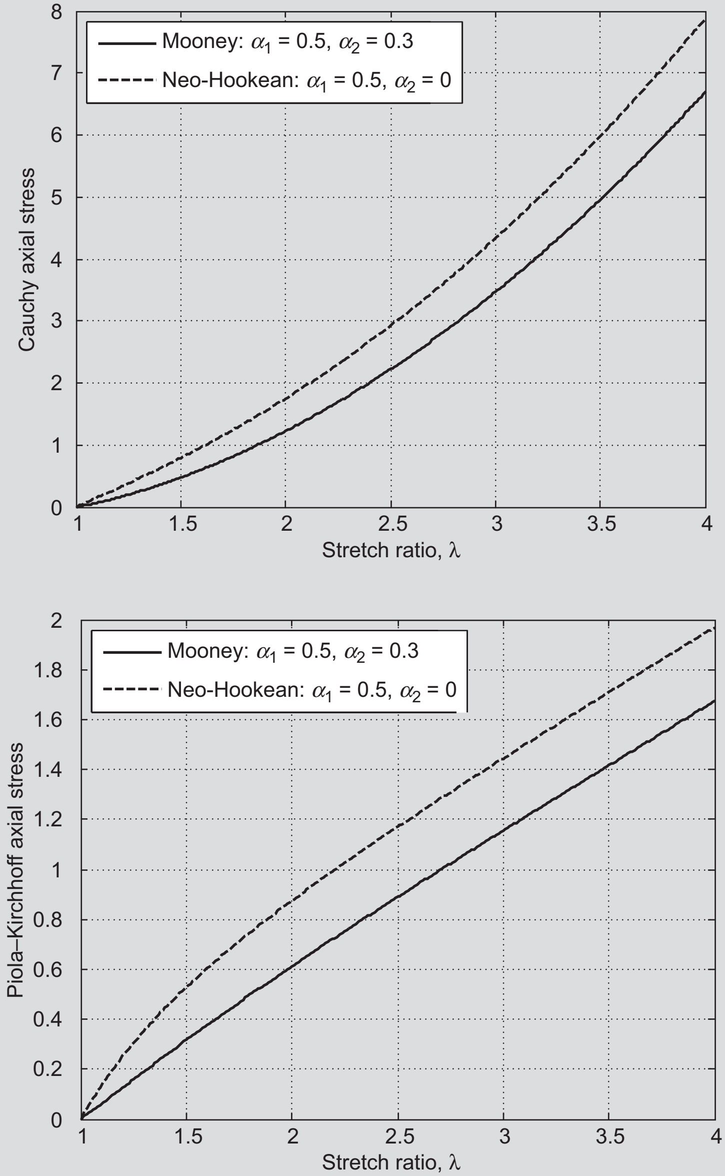


FIGURE 8.2

Comparison of Cauchy and PK1 axial stresses.

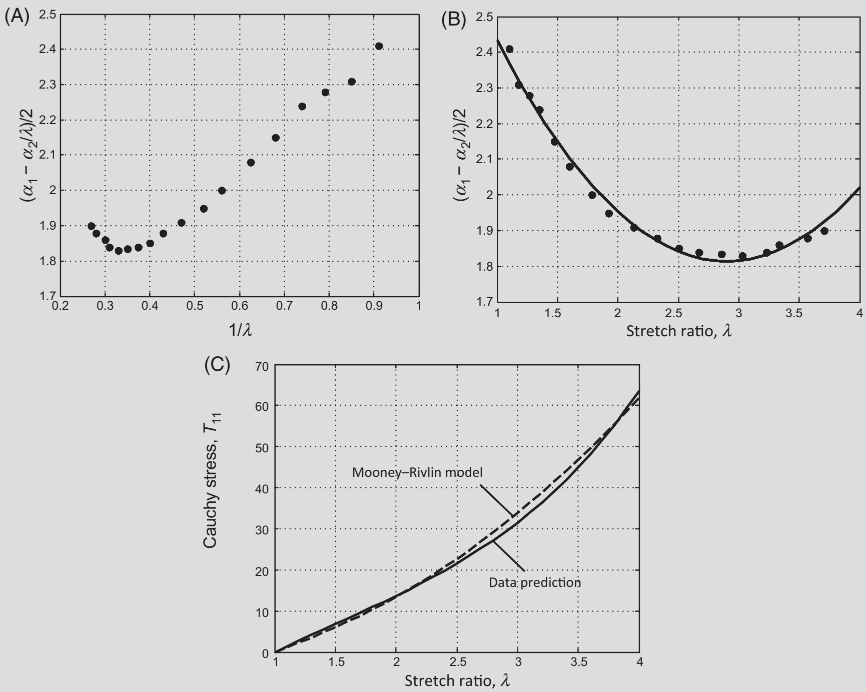


FIGURE 8.3

Comparison of Mooney–Rivlin model with experimental prediction from Rivlin and Saunders (1951): (A) Rivlin and Saunders (1951) data (reprinted with permission of Royal Society); (B) polynomial curve fit to data; (C) comparison of stress predictions.

EXAMPLE 8.3.2 SIMPLE SHEAR DEFORMATION OF A NONLINEAR ELASTIC SAMPLE

A block of a nonlinear incompressible elastic material is to undergo simple shearing deformation as shown in Fig. 8.4. Determine the resulting stress field required to support this deformation and explore other features of the problem which differ from the corresponding linear elastic problem.

Solution: The deformation and several strain tensors were previously given in Example 3.6.2. Collecting this information

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \quad \Rightarrow$$

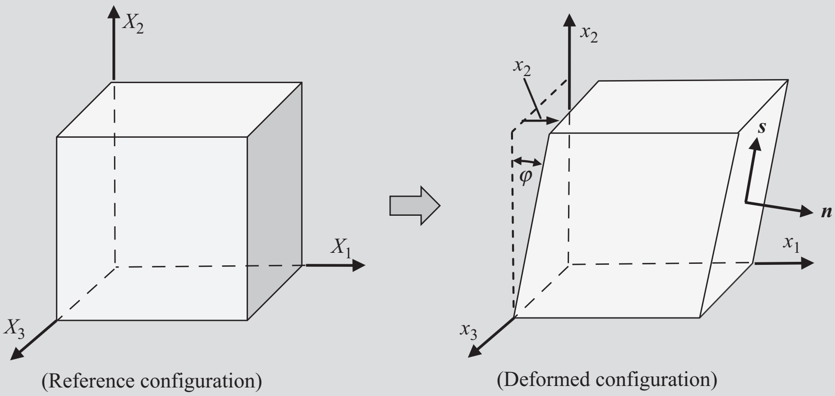


FIGURE 8.4

Simple shear deformation.

$$\boldsymbol{F}=\begin{bmatrix}1&\gamma&0\\0&1&0\\0&0&1\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}1+\gamma^2&\gamma&0\\\gamma&1&0\\0&0&1\end{bmatrix},\quad \boldsymbol{B}^{-1}=\begin{bmatrix}1&-\gamma&0\\-\gamma&1+\gamma^2&0\\0&0&1\end{bmatrix}$$

The invariants are given by $I_B = I_{B^{-1}} = 3+\gamma^2$ and this implies that $\alpha_i = \alpha_i(\gamma^2)$. Note that $\det F = 1$, and hence the deformation is isochoric. The stresses follow from (8.3.6):

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1\boldsymbol{B} + \alpha_2\boldsymbol{B}^{-1} \Rightarrow$$

$$\begin{aligned}T_{11} &= -p+\alpha_1(1+\gamma^2)+\alpha_2\\T_{22} &= -p+\alpha_1+\alpha_2(1+\gamma^2)\\T_{33} &= -p+\alpha_1+\alpha_2\\T_{12} &= (\alpha_1-\alpha_2)\gamma\\T_{23} &= T_{31} = 0\end{aligned} \tag{8.3.14}$$

We can assume a plane stress condition and thus choose $T_{33} = 0$. This allows the determination of the hydrostatic pressure $p = \alpha_1 + \alpha_2$, and thus the nonzero stresses become

$$\begin{aligned}T_{11} &= \alpha_1\gamma^2\\T_{22} &= \alpha_2\gamma^2\\T_{12} &= (\alpha_1-\alpha_2)\gamma\end{aligned} \tag{8.3.15}$$

Notice that this system, either (8.3.14) or (8.3.15) produces the *universal relation* of Rivlin

$$\gamma = \frac{T_{11}-T_{22}}{T_{12}} \tag{8.3.16}$$

which is independent of the material functions α_i. As in the previous example, the stresses are all constant (homogeneous) and thus they will automatically satisfy the equations of equilibrium.

The quantity $\alpha_1 - \alpha_2$ represents the generalized shear modulus for the nonlinear elastic problem, and since $\alpha_i = \alpha_i(\gamma^2)$, we can write a power series expansion for small γ as

$$\alpha_1 - \alpha_2 = \mu + \bar{\mu}\gamma^2 + O(\gamma^4), \quad \gamma \ll 1$$

where μ and $\bar{\mu}$ are constants. This allows the representation of the shear stress as

$$T_{12} = \mu\gamma + \bar{\mu}\gamma^3 + O(\gamma^5) \tag{8.3.17}$$

Thus, the shear stress is an odd function of the shear magnitude γ and the first term corresponds to the linear elastic case. Using a similar analysis, we can show that the normal stresses T_{11} and T_{22} will be even functions of γ with the first-order term $O(\gamma^2)$. Notice that from the linear elastic case, $T_{22} = 0$, but for nonlinear elasticity $T_{22} \neq 0$. Also the case here with $T_{11} \neq T_{22}$ is sometimes called the *Poynting effect*.

Consider next the tractions on the inclined face with normal vector $\boldsymbol{n}$ and tangent vector $\boldsymbol{s}$ as shown in Fig. 8.4. Using the fact that $\sin\varphi = \dfrac{\gamma}{\sqrt{1+\gamma^2}}, \cos\varphi = \dfrac{1}{\sqrt{1+\gamma^2}}$, these unit vectors are easily determined as

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\gamma^2}}(\boldsymbol{e}_1 - \gamma\boldsymbol{e}_2), \quad \boldsymbol{s} = \frac{1}{\sqrt{1+\gamma^2}}(\gamma\boldsymbol{e}_1 + \boldsymbol{e}_2) \tag{8.3.18}$$

The traction vector on this plane is given by

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{1+\gamma^2} \\ -\gamma/\sqrt{1+\gamma^2} \\ 0 \end{bmatrix} = \frac{(T_{11} - \gamma T_{12})}{\sqrt{1+\gamma^2}}\boldsymbol{e}_1 + \frac{(T_{12} - \gamma T_{22})}{\sqrt{1+\gamma^2}}\boldsymbol{e}_2 \tag{8.3.19}$$

and thus the normal and shear traction components are

$$N = \boldsymbol{t}\cdot\boldsymbol{n} = T_{22} - \frac{\gamma T_{12}}{1+\gamma^2}, \quad S = \boldsymbol{t}\cdot\boldsymbol{s} = \frac{T_{12}}{1+\gamma^2} \tag{8.3.20}$$

where we have used relation (8.3.16). Note as $\gamma \to 0$, relations (8.3.20) give $N \to T_{22}$ and $S \to T_{12}$.

EXAMPLE 8.3.3 TORSION OF A NONLINEAR ELASTIC CYLINDER

Consider a nonlinear elastic cylinder of circular section with a radius a in the reference configuration as shown in Fig. 8.5. The cylinder is to undergo torsional deformation and thus using cylindrical coordinates, the deformation can be stated as

$$r = R, \quad \theta = \Theta + \kappa Z, \quad z = Z \tag{8.3.21}$$

where (r, θ, z) are the spatial coordinates, (R, Θ, Z) are the reference coordinates, and κ represents the angle of twist per unit length. The cylinder is assumed to be traction-free along the lateral sides $r = a$. Determine the resulting stress field that corresponds to this deformation.

Solution: For this case, the deformation gradient can be calculated using relation (3.17.10):

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \kappa r \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $\det \boldsymbol{F} = 1$. The left Cauchy–Green strain tensor becomes

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \kappa r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \kappa r & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\kappa^2 r^2 & \kappa r \\ 0 & \kappa r & 1 \end{bmatrix}$$

and its inverse is found to be

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\kappa r \\ 0 & -\kappa r & 1+\kappa^2 r^2 \end{bmatrix}$$

The invariants are given by $I_B = I_{B^{-1}} = 3 + \kappa^2 r^2$ and this implies that $\alpha_i = \alpha_i(\kappa^2 r^2)$, and thus the stresses will only be functions of the radial coordinate. The stresses follow from (8.3.6):

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^{-1} \Rightarrow$$

$$\begin{aligned} T_{rr} &= -p + \alpha_1 + \alpha_2 \\ T_{\theta\theta} &= -p + \alpha_1(1+\kappa^2 r^2) + \alpha_2 \\ T_{zz} &= -p + \alpha_1 + \alpha_2(1+\kappa^2 r^2) \\ T_{\theta z} &= (\alpha_1 - \alpha_2)\kappa r \\ T_{r\theta} &= T_{rz} = 0 \end{aligned} \tag{8.3.22}$$

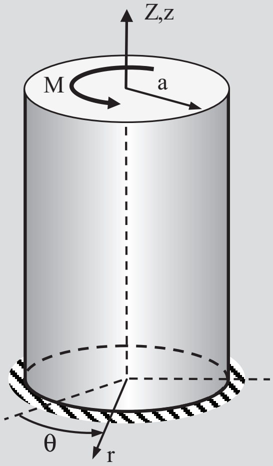


FIGURE 8.5

Torsion of an elastic cylinder.

The equations of equilibrium reduce to

$$\begin{gathered}\frac{\partial T_{rr}}{\partial r}+\frac{T_{rr}-T_{\theta\theta}}{r}=0 \\ \frac{\partial p}{\partial \theta}=\frac{\partial p}{\partial z}=0\end{gathered} \tag{8.3.23}$$

From the last two equations, $p = p(r)$, and thus the first equation becomes

$$-\frac{dp}{dr}+\frac{d}{dr}(\alpha_1+\alpha_2)-\alpha_1\kappa^2 r=0$$

and this can be integrated to give a form for the pressure

$$p(r)=(\alpha_1+\alpha_2)-\int^r \alpha_1\kappa^2 r'\,\mathrm{d}r'+K \tag{8.3.24}$$

where K is the arbitrary constant of integration. Now the traction-free boundary condition on the lateral sides $r = a$ becomes

$$T_{rr}(a)=0 \Rightarrow p(a)=(\alpha_1+\alpha_2)_{r=a} \tag{8.3.25}$$

Using relation (8.3.24) in (8.3.25) gives

$$K=\int^a \int \alpha_1\kappa^2 r'\,\mathrm{d}r' \tag{8.3.26}$$

and thus the pressure relation becomes

$$p(r)=(\alpha_1+\alpha_2)+\int_r^a \alpha_1\kappa^2 r'\,\mathrm{d}r' \tag{8.3.27}$$

The nonzero stresses then become

$$
\begin{aligned}
T_{rr} &= -\int_r^a \alpha_1 \kappa^2 r' \, dr' \\
T_{\theta\theta} &= \alpha_1 \kappa^2 r^2 - \int_r^a \alpha_1 \kappa^2 r' \, dr' \\
T_{zz} &= \alpha_2 \kappa^2 r^2 - \int_r^a \alpha_1 \kappa^2 r' \, dr' \\
T_{\theta z} &= (\alpha_1 - \alpha_2) \kappa r
\end{aligned}
\tag{8.3.28}
$$

The resultant torsional moment M on the free end can be calculated by

$$
M = \int_0^a T_{\theta z} 2\pi r^2 \, dr = 2\pi\kappa \int_0^a (\alpha_1 - \alpha_2) r^3 \, dr \tag{8.3.29}
$$

From (8.3.28), the T_{zz} stress component at any section can be expressed by $T_{zz} = \alpha_2 \kappa^2 r^2 + T_{rr}$, and thus the total normal force on any section is given by

$$
N = \int_0^a T_{zz} 2\pi r \, dr = 2\pi \int_0^a (\alpha_2 \kappa^2 r^3 + r T_{rr}) \, dr \tag{8.3.30}
$$

Now using integration by parts and the equilibrium equation, we can evaluate the integral

$$
\begin{aligned}
\int_0^a r T_{rr} \, dr &= \left[\frac{1}{2} r^2 T_{rr} \right]_0^a - \int_0^a \left(\frac{1}{2} r^2 \frac{dT_{rr}}{dr} \right) dr \\
&= \int_0^a \left(\frac{1}{2} r^2 \frac{T_{rr} - T_{\theta\theta}}{r} \right) dr \\
&= -\frac{1}{2} \int_0^a \alpha_1 \kappa^2 r^3 \, dr
\end{aligned}
$$

and using this result, the normal force becomes

$$
N = 2\pi \int_0^a \alpha_2 \kappa^2 r^3 \, dr - \pi \int_0^a \alpha_1 \kappa^2 r^3 \, dr \tag{8.3.31}
$$

Note that unlike the linear elastic model, the normal force here will not be zero, and in fact most specific nonlinear models would predict $N < 0$.

Given the specific forms of the constitutive function $\alpha_i = \alpha_i(\kappa^2 r^2)$, all stresses, and the twisting moment and normal force can be calculated. Note that for the neo-Hookean model ($\alpha_1 = \text{constant}, \alpha_2 = 0$), and thus the shear stress $T_{\theta z} = \alpha_1 \kappa r$ which is the same form as linear elasticity; however, $N = -\pi \alpha_1 \kappa^2 \dfrac{a^4}{4} \neq 0$.

Some additional nonlinear elastic problems are given in the exercises.

8.4 NONLINEAR VISCOUS FLUIDS

Our previous discussion on incompressible fluid dynamics in Section 6.4 presented the linear viscous or Newtonian model governed by the constitutive relation

$$\boldsymbol{T} + p\boldsymbol{I} = 2\mu_o \boldsymbol{D} \tag{8.4.1}$$

where μ_o is the constant material viscosity parameter and for incompressible fluids the rate of deformation tensor satisfies $\operatorname{tr}\boldsymbol{D} = 0$. The quantity $\boldsymbol{T} - p\boldsymbol{I}$ is often called the *extra stress*. This constitutive law leads to the Navier–Stokes equations and results in accurate predictions for many fluid mechanics problems. Typical fundamental predictions of this model include a linear shear stress–shear strain behavior (6.4.8) and a Hagen–Poiseuille flow with the volumetric flow rate proportional to the pressure gradient (6.4.33). However, there exist many fluids (polymeric melts, paints, gels, colloidal suspensions, biological, and food liquids) with behaviors that do not match these predictions. Because of this, more sophisticated constitutive fluid laws have been developed to provide appropriate theoretical modeling of such flows. These nonclassical fluid models are generally referred to as *non-Newtonian* and they commonly incorporate nonlinear and memory effects in the constitutive relations. Because of the memory effects, these models often include a viscoelastic response. Originally, much of this non-Newtonian study was done under the name of *rheology* defined as *the study of flow and deformation of matter* (see discussion in Section 1.4). We will now present a few basic models focusing on such incompressible fluids. Further details on this topic can be found in Truesdell and Noll (1965), Coleman et al. (1966), Truesdell (1974), and Bird et al. (1987). Although some past simple models only modify the viscosity in relation (8.4.1), we explore more fundamental theories that alter the constitutive relation in much more general and meaningful ways.

8.4.1 REINER–RIVLIN FLUID

Following along a similar approach as that was used for the elastic solids, it would be logical to propose a general nonlinear incompressible fluid model where the stress (less the hydrostatic pressure) would be expressed in terms of a nonlinear function of the rate of deformation tensor:

$$\boldsymbol{T} + p\boldsymbol{I} = \boldsymbol{g}(\boldsymbol{D}) \tag{8.4.2}$$

Note that although the velocity and the velocity gradient are not objective, the rate of deformation does satisfy the objectivity requirement. Since fluids are generally assumed to be isotropic, the material function $\boldsymbol{g}$ must be an isotropic function of the rate of deformation, and so using arguments analogous to those in the previous section

$$\boldsymbol{T} + p\boldsymbol{I} = \beta_1 \boldsymbol{D} + \beta_2 \boldsymbol{D}^2 \tag{8.4.3}$$

where $\beta_i = \beta_i(II_D, III_D)$. Note that for an incompressible fluid, $I_D = 0$. Constitutive form (8.4.3) is commonly called a *Reiner–Rivlin fluid* (R–R fluid), named after their independent development back in the 1950s.

If we consider again the shear flow shown in Fig. 6.22 which is specified by $\boldsymbol{v} = \{v_1(x_2), 0, 0\}$, this yields a rate of deformation tensor and its square as

$$\boldsymbol{D} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}^2 = \frac{1}{4}\begin{bmatrix} \dot{\gamma}^2 & 0 & 0 \\ 0 & \dot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.4.4}$$

where $\dot{\gamma} = \dfrac{\partial v_1}{\partial x_2}$. Using the R–R fluid constitutive law (8.4.3) yields the following stresses:

$$\begin{aligned} &T_{11} = T_{22} = -p + \frac{\beta_2}{4}\dot{\gamma}^2, \quad T_{33} = -p \\ &T_{12} = \frac{\beta_1}{2}\dot{\gamma}, \quad T_{23} = T_{31} = 0 \end{aligned} \tag{8.4.5}$$

where $\beta_i = \beta_i(\dot{\gamma}^2)$. For this case, we note that the traditional form for the viscosity is now given by $\mu = T_{12}/\dot{\gamma} = \beta_1(\dot{\gamma}^2)/2$, which is no longer a constant. Furthermore, while the Newtonian model predicted zero normal stresses, the R–R fluid indicates nonzero T_{11} and T_{22}. Thus, we find *normal stress effects* present in a solely shearing type of flow geometry. These effects will also be present in more complicated flow fields such as in a rotating cylindrical viscometer or in pipe flow. Many of these results match qualitatively with experimental observations of non-Newtonian flows.

However, a major negative issue of our findings in this shear flow example is that $T_{11} = T_{22}$. This result occurs regardless of the material parameters and is thus universal for this flow. The result is unexpected since the two normal stresses act on totally different planes. Sadly this result does not match with experimental results, thereby indicating that our R–R constitutive model is not expected to consistently give quantitative predictions that correspond with real material behavior. We conclude that even though our constitutive creation concepts seemed reasonable, evidently some aspects of the nonlinear fluid behavior were not properly account for. This situation leads us to consider other more general constitutive schemes.

8.4.2 SIMPLE INCOMPRESSIBLE FLUID

Recall Noll's development of the simple material given by relation (8.2.3). For an incompressible fluid, this form would become

$$\boldsymbol{T} + p\boldsymbol{I} = \tilde{\mathfrak{I}}_{s=0}^{s=\infty}\left(\boldsymbol{F}_t(t-s)\right) \tag{8.4.6}$$

It can be shown that this relation is reducible to

$$\boldsymbol{T} + p\boldsymbol{I} = \hat{\mathfrak{I}}_{s=0}^{s=\infty}\left(\boldsymbol{G}_t(t-s)\right) \tag{8.4.7}$$

where $\boldsymbol{G}_t(t-s) = \boldsymbol{C}_t(t-s) - \boldsymbol{I}$ and $\boldsymbol{C}_t(t-s)$ is the relative right Cauchy–Green strain (3.15.6). We assume that when the deformation goes to zero, the stress reduces to just hydrostatic pressure and thus $\overset{s=\infty}{\underset{s=0}{\tilde{\mathfrak{J}}}}(\mathbf{0}) = \overset{s=\infty}{\underset{s=0}{\hat{\mathfrak{J}}}}(\mathbf{0}) = 0$.

Green and Rivlin (1957) proved that this general constitutive functional can be expressed as a series of integrals, thus expressing (8.4.7) in the form

$$\boldsymbol{T} + p\boldsymbol{I} = \int_0^\infty M_1(s)\boldsymbol{G}_t(t-s)\,ds + \int_0^\infty\int_0^\infty M_2(s_1,s_2)\boldsymbol{G}_t(t-s_2)\,ds_1\,ds_2 + \cdots \tag{8.4.8}$$

This integral representation can be truncated to model finite linear and infinitesimal theories, and if we retain only the first term

$$\boldsymbol{T} + p\boldsymbol{I} = \int_0^\infty M_1(s)\boldsymbol{G}_t(t-s)\,ds \tag{8.4.9}$$

which would represent a finite linear viscoelastic fluid. Compare this with the linear integral viscoelastic relation (6.5.38). Although the general formulation (8.4.8) is elegant, it is difficult to apply to specific flow problems, and thus we will not pursue further details of this integral representation. However, we will explore the general simple fluid model as represented by (8.4.7) in more detail later.

8.4.3 RIVLIN–ERICKSEN FLUID

Rivlin and Ericksen (1955) and later Truesdell and Noll (1965) presented a differential rate type representation scheme for nonlinear isotropic materials. We could assume for simple materials that the deformation gradient history $\boldsymbol{F}_t(t-s)$ has constitutive influence on the stress only over a short period of time for small s. Further assuming $\boldsymbol{F}$ has continuous time derivatives, we could expand it in terms of a Taylor series near $s = 0$. For such a case, the response functional $\tilde{\mathfrak{J}}$ reduces to an ordinary nonlinear function $\boldsymbol{f}$, that is, $\boldsymbol{T} = \boldsymbol{f}(\boldsymbol{F}, \dot{\boldsymbol{F}}, \ddot{\boldsymbol{F}}, \dddot{\boldsymbol{F}}, \ldots)$. Such a constitutive case is often referred to as a *material of the differential type.* For the incompressible fluid case, the constitutive form simplifies to

$$\boldsymbol{T} + p\mathbf{I} = \boldsymbol{f}(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \ldots, \boldsymbol{A}_n) \tag{8.4.10}$$

where $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \ldots, \boldsymbol{A}_n$ are the Rivlin–Ericksen tensors previously defined in Section 3.16:

$$\boldsymbol{A}^{(n)} = \frac{D^n}{D\tau^n}[\boldsymbol{C}_t(\tau)]_{\tau=t} = \frac{D^n}{D\tau^n}\left[\boldsymbol{F}_t^{\mathrm{T}}(\tau)\boldsymbol{F}_t(\tau)\right]_{\tau=t} \tag{8.4.11}$$

or

$$\begin{aligned}
\boldsymbol{A}_1 &= \boldsymbol{L} + \boldsymbol{L}^T = 2\boldsymbol{D} \\
\boldsymbol{A}_2 &= \frac{D}{Dt}\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(1)}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}^{(1)} \\
&\vdots \\
\boldsymbol{A}_n &= \frac{D}{Dt}\boldsymbol{A}^{(n-1)} + \boldsymbol{A}^{(n-1)}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}^{(n-1)}
\end{aligned} \tag{8.4.12}$$

Since we are dealing with isotropic materials, the response function must satisfy the usual mathematical isotropic property

$$\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \ldots)\boldsymbol{Q}^T = \boldsymbol{f}(\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}_3\boldsymbol{Q}^T, \ldots, \boldsymbol{Q}\boldsymbol{A}_n\boldsymbol{Q}^T) \tag{8.4.13}$$

We can use this constitutive scheme including various orders n and can then apply appropriate representation theorems from Section 2.14 to express the specific constitutive forms. Truncation at zero order gives $\boldsymbol{T} = -p\boldsymbol{I}$ (inviscid model), while truncation at first order yields the Newtonian case $\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_o\boldsymbol{D}$. The results for order $n = 2$ (sometimes called *complexity* 2) in general give

$$\begin{aligned}\boldsymbol{T} &= -p\boldsymbol{I} + \boldsymbol{f}(\boldsymbol{A}_1, \boldsymbol{A}_2) \\ &= -p\boldsymbol{I} + \mu_1\boldsymbol{A}_1 + \mu_2\boldsymbol{A}_1^2 + \mu_3\boldsymbol{A}_2 + \mu_4\boldsymbol{A}_2^2 \\ &\quad + \mu_5(\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \mu_6(\boldsymbol{A}_1\boldsymbol{A}_2^2 + \boldsymbol{A}_2^2\boldsymbol{A}_1) \\ &\quad + \mu_7(\boldsymbol{A}_1^2\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1^2) + \mu_8(\boldsymbol{A}_1^2\boldsymbol{A}_2^2 + \boldsymbol{A}_2^2\boldsymbol{A}_1^2)\end{aligned} \tag{8.4.14}$$

where $\mu_i = \mu_i(tr\,\boldsymbol{A}_1^2, tr\,\boldsymbol{A}_1^3, tr\,\boldsymbol{A}_2, tr\,\boldsymbol{A}_2^2, tr\,\boldsymbol{A}_2^3, tr(\boldsymbol{A}_1\boldsymbol{A}_2), tr(\boldsymbol{A}_1^2\boldsymbol{A}_2), tr(\boldsymbol{A}_1\boldsymbol{A}_2^2), tr(\boldsymbol{A}_1^2\boldsymbol{A}_2^2))$

Note that (8.4.14) includes the strain rate terms of third and fourth order, for example, $\boldsymbol{A}_1\boldsymbol{A}_2^2$ and $\boldsymbol{A}_1^2\boldsymbol{A}_2^2$. This complexity of the general $n = 2$ case is often simplified by considering only the second-order terms

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu_1\boldsymbol{A}_1 + \mu_2\boldsymbol{A}_1^2 + \mu_3\boldsymbol{A}_2 \tag{8.4.15}$$

where $\mu_i = \mu_i(II_D, III_D)$. This constitutive model is normally called a *second-order fluid*.

EXAMPLE 8.4.1 STEADY SIMPLE SHEARING FLOW FOR FLUID MODELS

Consider steady simple shearing flow with a velocity field $\boldsymbol{v} = \{\kappa x_2, 0, 0\}$, where κ is the rate of shear. Determine the stress fields for the Newtonian, Reiner–Rivlin, and second-order fluid models, and compare the results.

Solution: For this flow field, the various required kinematical tensors are given by

$$\begin{aligned}&\boldsymbol{D} = \frac{1}{2}\boldsymbol{A}_1 = \frac{\kappa}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}^2 = \frac{\kappa^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\boldsymbol{A}_2 = 2\kappa^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_1^2 = \kappa^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{8.4.16}$$

For the Newtonian case, $\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_o\boldsymbol{D}$, and the stresses become

$$\begin{aligned}&T_{11} = T_{22} = T_{33} = -p \\ &T_{12} = \mu_o\kappa, \quad T_{23} = T_{31} = 0\end{aligned} \tag{8.4.17}$$

For the R–R fluid, $\boldsymbol{T} = -p\boldsymbol{I} + \beta_1\boldsymbol{D} + \beta_2\boldsymbol{D}^2$, and the stresses are given by

$$T_{11} = T_{22} = -p + \frac{\beta_2}{4}\kappa^2, \quad T_{33} = -p$$
$$T_{12} = \frac{\beta_1}{2}\kappa, \quad T_{23} = T_{31} = 0 \tag{8.4.18}$$

Finally, the second-order fluid model $\boldsymbol{T} = -p\boldsymbol{I} + \mu_1\boldsymbol{A}_1 + \mu_2\boldsymbol{A}_1^2 + \mu_3\boldsymbol{A}_2$ gives

$$\begin{aligned} T_{11} &= -p + \mu_2\kappa^2 \\ T_{22} &= -p + \mu_2\kappa^2 + 2\mu_3\kappa^2 \\ T_{33} &= -p \\ T_{12} &= \mu_1\kappa, \quad T_{23} = T_{31} = 0 \end{aligned} \tag{8.4.19}$$

We observe that both nonlinear theories predict normal stress effects under the shear flow. As previously mentioned, the R–R model predicts $T_{11} = T_{22}$, whereas second-order fluid results give the more realist case of $T_{11} \neq T_{22}$. While the Newtonian model indicates constant viscosity, both nonlinear models predict a variable viscosity coefficient, T_{12}/κ.

8.4.4 VISCOMETRIC FLOWS OF INCOMPRESSIBLE SIMPLE FLUIDS

Coleman et al. (1966) and Truesdell (1974) summarize considerable research on non-Newtonian fluid flow. They present a detailed look at viscometric flows of simple fluids, and this work has led to very useful schemes to characterize such fluids. A *viscometric flow* (sometimes called a *laminar shear flow*) is generally defined by a velocity field of the form

$$v_1 = 0, \quad v_2 = \kappa x_1, \quad v_3 = 0 \tag{8.4.20}$$

where κ is the rate of shear. We have of course already used this type of deformation in our previous examples and have discussed that this flow is commonly used as an approximation of the velocity in various types of viscometers (devices used to measure fluid properties). We now wish to explore more details about this general type of deformation and look at its application for general non-Newtonian memory fluids.

We will generalize the situation by considering a local flow field form (8.4.20) not only in the Cartesian system, but also in other orthogonal systems that might involve curvilinear flow. Using some of our past kinematics from Chapter 3, a viscometric flow (8.4.20) will yield the following deformation tensors:

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ \kappa t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{F}_t(t-s) = \begin{bmatrix} 1 & 0 & 0 \\ -\kappa s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{C}_t(t-s) = \begin{bmatrix} 1+\kappa^2 s^2 & -\kappa s & 0 \\ -\kappa s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.4.21}$$

and the Rivlin–Ericksen tensors

$$\boldsymbol{A}_1 = \kappa\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_2 = 2\kappa^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_n = 0 \quad (n \geq 3) \tag{8.4.22}$$

Note that for this case

$$C_t(t-s) = I - sA_1 + \frac{s^2}{2}A_2 \tag{8.4.23}$$

In terms of a matrix $\boldsymbol{N}$,

$$\boldsymbol{N} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.4.24}$$

we can express the Rivlin–Ericksen tensors as

$$\begin{aligned} \boldsymbol{A}_1 &= \kappa(\boldsymbol{N} + \boldsymbol{N}^T) \\ \boldsymbol{A}_2 &= 2\kappa^2 \boldsymbol{N}^T\boldsymbol{N} \end{aligned} \tag{8.4.25}$$

Relation (8.4.23) indicates that for viscometric flow $\boldsymbol{C}_t(t-s)$ depends only on the two Rivlin–Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. For such a case, it has been shown that the general simple fluid constitutive relation (8.4.7) reduces to

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{f}(\boldsymbol{A}_1, \boldsymbol{A}_2) \tag{8.4.26}$$

where the general functional has reduced to a simple function $\boldsymbol{f}$. This of course is a major simplification. Now using the representation form (8.4.14) along with relations (8.4.25) yields

$$\boldsymbol{T} = -p\boldsymbol{I} + \tau(\kappa)(\boldsymbol{N} + \boldsymbol{N}^T) + \sigma_1(\kappa)\boldsymbol{N}^T\boldsymbol{N} + \sigma_2(\kappa)\boldsymbol{N}\boldsymbol{N}^T \tag{8.4.27}$$

where the coefficients $\tau(\kappa), \sigma_1(\kappa), \sigma_2(\kappa)$ are referred to as the *viscometric functions* defined by

$$\begin{aligned} \tau(\kappa) &= T_{12} && \ldots \text{ shear stress function} \\ \sigma_1(\kappa) &= T_{11} - T_{33} && \ldots \text{ first normal stress function} \\ \sigma_2(\kappa) &= T_{22} - T_{33} && \ldots \text{ second normal stress function} \end{aligned} \tag{8.4.28}$$

From constitutive law (8.4.27), we conclude that these three viscometric functions fully characterized the fluid under viscometric flow. The behavior of these viscometric functions with the rate of shear κ can be determined by experimentation, and typical qualitative results are shown in Fig. 8.6.

Note that the normal stress functions are sometimes defined in a slightly different way. Our scheme (8.4.28) is the form originally defined by Truesdell and Noll (1965). Written in terms of the original parameters μ_i in relation (8.4.14), these functions may be expressed as

$$\begin{aligned} \tau(\kappa) &= \mu_1\kappa + 2\mu_5\kappa^3 + 4\mu_6\kappa^5 \\ \sigma_1(\kappa) &= \mu_2\kappa^2 + 2\mu_3\kappa^2 + 4\mu_4\kappa^4 + 4\mu_7\kappa^4 + 8\mu_8\kappa^6 \\ \sigma_2(\kappa) &= \mu_2\kappa^2 \end{aligned} \tag{8.4.29}$$

Since it can be shown that the material parameters μ_i will be even functions of κ, the shear stress function will be an odd function of κ, whereas the two normal stress functions will be even functions of this variable. For the case where rate of shear $\kappa \to 0$, all

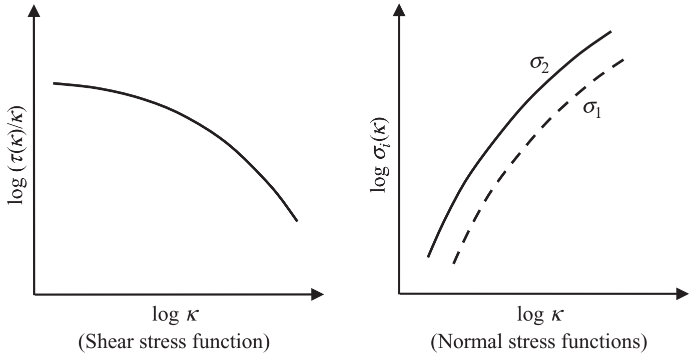


FIGURE 8.6

Typical behaviors of viscometric functions.

viscometric functions should vanish $\tau(0)=\sigma_1(0)=\sigma_2(0)=0$. Based on these arguments, the viscometric functions could be expanded in a Taylor series for small κ:

$$\begin{aligned}\tau(\kappa)&=\mu_o\kappa+a_1\kappa^3+a_2\kappa^5+\cdots\\ \sigma_1(\kappa)&=b_1\kappa^2+b_2\kappa^4+\cdots\\ \sigma_2(\kappa)&=c_1\kappa^2+c_2\kappa^4+\cdots\end{aligned}\tag{8.4.30}$$

for constants a_i, b_i, and c_i. Note that first-order results in κ give Newtonian behavior with $\tau=T_{12}\approx\mu_o\kappa$ and $\sigma_1=\sigma_2=0$. The next order of behavior would give the beginnings of non-Newtonian response, and for small κ these effects would be $O(\kappa^2)$ in the normal stresses and $O(\kappa^3)$ in the shear stress.

The individual stresses coming from relation (8.4.27) are given by

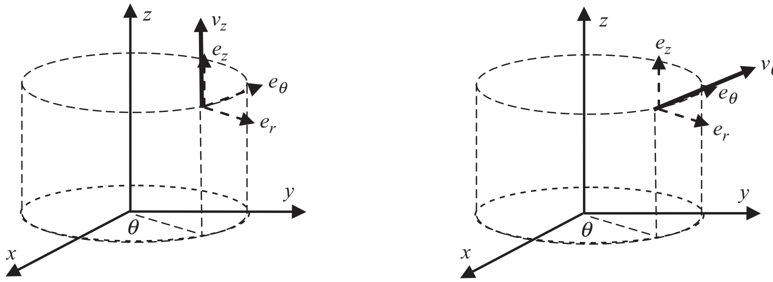
$$\begin{aligned}&T_{11}=-p+\sigma_1(\kappa)\\ &T_{22}=-p+\sigma_2(\kappa)\\ &T_{33}=-p\\ &T_{12}=\tau(\kappa),\quad T_{23}=T_{31}=0\end{aligned}\tag{8.4.31}$$

Stresses from these rectilinear viscometric flows are thus constants and will automatically satisfy the equations of motion.

As previously mentioned, we are also interested in applying this viscometric flow geometry to other orthogonal flow fields. Using standard coordinate frame transformation, we can easily justify that several other flow geometries are also viscometric. For example, consider the two flow cases (Hagen–Poiseuille and circular Couette) shown in Fig. 8.7. For the Hagen–Poiseulle flow, the velocity field $\boldsymbol{v}=v_z(r)\boldsymbol{e}_z$ in the cylindrical coordinate system can be transformed into the standard Cartesian form by the particular orthogonal transformation

$$\boldsymbol{v}^*=\boldsymbol{Q}\boldsymbol{v}=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}0\\0\\v_z(r)\end{bmatrix}=\begin{bmatrix}0\\v_z(r)\\0\end{bmatrix}\tag{8.4.32}$$

and the new right-handed basis vectors become $\boldsymbol{n}_i=\{\boldsymbol{e}_r,\boldsymbol{e}_z,-\boldsymbol{e}_\theta\}$.



(Hagen–Poiseuille flow)
$v_r = v_\theta = 0,\ v_z = v_z(r)$

(Circular Couette flow)
$v_r = 0,\ v_\theta = r\omega(r),\ v_z = 0$

FIGURE 8.7

Viscometric curvilinear flows.

Likewise, the circular Couette flow $\boldsymbol{v} = r\omega(r)\boldsymbol{e}_\theta$ can also be transformed into the standard Cartesian form using a different $\boldsymbol{Q}$ matrix as

$$\boldsymbol{v}^* = \boldsymbol{Q}\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r\omega(r) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ r\omega(r) \\ 0 \end{bmatrix} \tag{8.4.33}$$

and the new basis vectors become $\boldsymbol{n}_i = \{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$. Note that for this case, the transformation matrix is simply the unit tensor. So, we have shown that the two curvilinear flows are actually viscometric flows with respect to different local orthogonal coordinates. This result allows various flows in standard viscometers to be used to characterize the various viscometric functions. Truesdell and Noll (1965) and Coleman et al. (1966) discuss several such experimental schemes.

EXAMPLE 8.4.2 CIRCULAR COUETTE FLOW

Circular Couette flow $v_r = 0, v_\theta = r\omega(r), v_z = 0$ often exists in the region between two infinitely long concentric cylinders turning with different angular velocities as shown in Fig. 8.8. For this flow geometry using cylindrical coordinates, determine the two Rivlin–Ericksen tensors and the corresponding stress field and the velocity distribution v_θ.

Solution: In order to use our existing viscometric flow theory, we must use the transformation (8.4.33) and adjust our basis vectors $\boldsymbol{n}_i$ such that $\boldsymbol{n}_i = \{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$. Note that for this case, it is just a unity transformation. Using these cylindrical coordinates,

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{A}_1 = \frac{1}{2}\begin{bmatrix} 0 & \kappa(r) & 0 \\ \kappa(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 2\kappa^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.4.34}$$

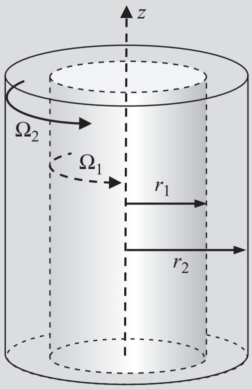


FIGURE 8.8

Circular couette flow geometry.

where $\kappa(r) = r\dfrac{d\omega}{dr}$. Note the identical form as (8.4.22). Using (8.4.27) in the cylindrical system

$$\begin{aligned} T_{r\theta} &= \tau(\kappa) \\ T_{rr} &= -p + \sigma_1(\kappa) \\ T_{\theta\theta} &= -p + \sigma_2(\kappa) \\ T_{zz} &= -p \\ T_{\theta z} &= T_{zr} = 0 \end{aligned} \tag{8.4.35}$$

We assume that the three viscometric functions are known and then wish to determine the velocity distribution $v_\theta = r\omega(r)$ and the stress components. In cylindrical coordinates, the equations of motion reduce to

$$\begin{aligned} -\frac{dp}{dr} + \frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} &= -\rho r\omega^2 \\ \frac{1}{r}\frac{d}{dr}(r^2 T_{r\theta}) &= 0 \end{aligned} \tag{8.4.36}$$

Relation $(8.4.36)_2$ is easily integrated giving $T_{r\theta} = C/r^2$, where C is the constant of integration. Then applying the torque M per unit height condition $M = (2\pi r T_{r\theta})r$ gives that $C = M/2\pi$ and thus the shear stress is determined:

$$T_{r\theta} = \tau(\kappa(r)) = \frac{M}{2\pi r^2} \tag{8.4.37}$$

Following Coleman et al. (1966), we now define the inverse function λ such that $\kappa(r) = \lambda(S)$ with $S(r) = \tau(\kappa) = M/2\pi r^2$. Since we have assumed that $\tau(\kappa)$ is known, we likewise assume that λ is also known. Next, consider the expression

$$\frac{d\omega}{dr} = \frac{d\omega}{dS}\frac{dS}{dr} = \frac{d\omega}{dS}\left(-\frac{2M}{2\pi r^3}\right) = -\frac{2}{r}S\frac{d\omega}{dS} \tag{8.4.38}$$

and thus

$$\kappa(r) = \lambda(S) = r\frac{d\omega}{dr} = -2S\frac{d\omega}{dS} \Rightarrow d\omega = -\frac{\lambda(S)}{2S}dS \tag{8.4.39}$$

Then integrating this previous result from inner radius r_1 to variable radius r gives

$$\omega(r) = \Omega_1 - \frac{1}{2}\int_{M/2\pi r_1^2}^{M/2\pi r^2}\frac{\lambda(S)}{S}dS \tag{8.4.40}$$

where Ω_1 is the angular velocity of the inner cylinder. With $\lambda(S)$ known, (8.4.40) then gives $\omega(r)$ and thus the velocity $v_\theta = r\omega(r)$ can be determined.

The first equation of motion $(8.4.36)_1$ can be integrated (see Coleman et al., 1966 for details). This leads to an expression for T_{rr} which can be used to determine the difference between normal tractions on the inner and outer cylinders:

$$T_{rr}(r_2) - T_{rr}(r_1) = \int_{r_1}^{r_2}\left\{\frac{1}{r}\left[\hat{\sigma}_2\left(\frac{M}{2\pi r^2}\right) - \hat{\sigma}_1\left(\frac{M}{2\pi r^2}\right)\right] - \rho r\omega^2\right\}dr \tag{8.4.41}$$

where $\hat{\sigma}_1(S) = \sigma_1(\lambda(S))$ and $\hat{\sigma}_2(S) = \sigma_2(\lambda(S))$ are the *modified normal stress functions*. Since experimental techniques can collect data on normal tractions on the inner and outer cylinders, relation (8.4.41) provides a scheme to determine the difference between these modified normal stress functions $\hat{\sigma}_2 - \hat{\sigma}_1$.

For Newtonian flows, $\sigma_1 = \sigma_2 = 0$, while the Reiner–Rivlin fluid predicted $\sigma_1 = \sigma_2$. For either of these cases, (8.4.41) implies $T_{rr}(r_2) - T_{rr}(r_1) < 0$ or expressed in terms of a wall pressure $(p = -T_{rr})$ we find $p(r_2) > p(r_1)$. Thus, the wall pressure is higher on the outer cylinder and this would correspond with a free surface which would slope upwards as r increases as shown in Fig. 8.9A. Now for the non-Newtonian case, it has been found for most fluids that $\sigma_2 > \sigma_1$ as shown in Fig. 8.6. Thus, for this case applying similar arguments related to relation (8.4.41), we conclude that it may be possible for $T_{rr}(r_2) - T_{rr}(r_1) > 0$ implying that $p(r_2) < p(r_1)$. With higher pressure on the inner cylinder, the free surface profile would now slope opposite to the Newtonian flow as shown in Fig. 8.9B. This non-Newtonian behavior has often been observed and is commonly called the *rod climbing effect*. A simple web search will show several photographs and videos of such behavior for real fluids.

Additional features and analysis of this flow example along with several other curvilinear flows are presented in Coleman et al. (1966).

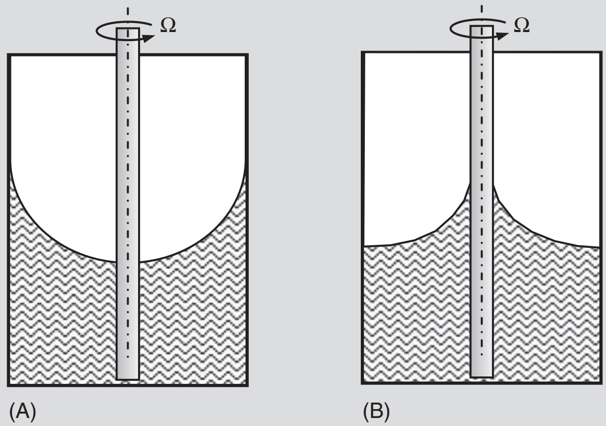


FIGURE 8.9

Typical free surface variations in circular Couette flow (Ω_1=Ω, Ω_2=0): (A) Newtonian fluid—no normal stresses and free surface shape from centrifugal force alone; (B) non-Newtonian fluid—normal stress effects modify free surface shape.

Many more specific constitutive relations for non-Newtonian flow have been presented in the literature. Oldroyd (1984) has developed an extensive body of work using a *convective coordinate* approach whereby the coordinates are embedded in the material as it undergoes deformation and flow. Studies by Lodge (1964) and Bird et al. (1987) provide more details on Oldroyd's approach and several other flow models. We will not pursue these models here, and instead move on to investigate nonlinear viscoelastic behavior in the final section of this chapter.

8.5 NONLINEAR INTEGRAL VISCOELASTIC CONSTITUTIVE MODELS

Although we have already seen some nonlinear viscoelastic constitutive relations in the previous section dealing with non-Newtonian viscous fluids, we now wish to further explore this type of response for general materials including both solids and fluids. Over the years, numerous single integral constitutive relations have been developed for nonlinear viscoelastic behavior. These approaches likely originated from the linear viscoelastic integral forms that were developed decades earlier (Section 6.5). Also, it should be noted that a multiple integral form came out of the general simple fluid model discussed in Section 8.4.2. Here, we will focus on just a couple of the more common integral models, and Bird et al. (1987) provide a nice summary of many other constitutive models of this type.

8.5.1 INTEGRAL MODELS USING A SINGLE DEFORMATION TENSOR

Recall that a single integral memory constitutive equation was previously given by relation (8.4.9). This form was the result of a truncation of the multiple integral series representation of the general simple fluid model of Noll. This relation can be rewritten in a somewhat more general expression as

$$\boldsymbol{T} + p\boldsymbol{I} = \int_0^\infty \mu(I_i, t, s)\tilde{\boldsymbol{E}}_t(t-s)\,ds \tag{8.5.1}$$

where $\tilde{\boldsymbol{E}}_t(t-s)$ is some relative strain tensor, often incorporating $\boldsymbol{C}_t(t-s)$, $\boldsymbol{C}_t^{-1}(t-s)$, or $\boldsymbol{G}_t(t-s)$, and $\mu(I_i, t, s)$ is a material memory function that could depend on particular invariants and time constants. These single-integral models have been presented by Lodge (1964), Bird et al. (1987), and others. Normally $\mu(I_i, t, s)$ is a decreasing function of the elapsed time and often its form is expressed by

$$\mu(I_i, t, s) = \sum_i \frac{\mu_o(I_i)}{\tau_i^2} e^{-s/\tau_i} \tag{8.5.2}$$

Models like these can actually be connected to *convected Maxwell models* somewhat similar to Example 6.5.6.

Consider the case with $\tilde{\boldsymbol{E}}_t(t-s) = \boldsymbol{C}_t^{-1}(t-s)$ under shearing flow deformation $v_1 = 0, v_2 = \kappa x_1, v_3 = 0$. This gives

$$\boldsymbol{C}_t^{-1}(t-s) = \begin{bmatrix} 1 & \kappa s & 0 \\ \kappa s & 1+\kappa^2 s^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} T_{11} = T_{33} = -p + \int_0^\infty \mu(s)\,ds \\ T_{22} = -p + \int_0^\infty (1+\kappa^2 s^2)\mu(s)\,ds \\ T_{12} = \kappa \int_0^\infty s\mu(s)\,ds \\ T_{23} = T_{31} = 0 \end{array} \tag{8.5.3}$$

and hence for this case the viscometric functions become

$$\begin{aligned} \tau(\kappa) &= T_{12} = \kappa \int_0^\infty s\mu(s)\,ds \\ \sigma_1(\kappa) &= T_{11} - T_{33} = 0 \\ \sigma_2(\kappa) &= T_{22} - T_{33} = \kappa^2 \int_0^\infty s^2 \mu(s)\,ds \end{aligned} \tag{8.5.4}$$

Because the first normal stress function vanishes for all choices of the material function μ, this model will, in general, not match with experimental observations.

Furthermore, if the material function is taken as only the first term in relation (8.5.2) and no invariant dependency is included, the simplified expression becomes $\mu(I_i, t, s) = \dfrac{\mu_o}{\tau_1^2} e^{-s/\tau_1}$, and the stresses and viscometric functions reduce to

$$\begin{array}{l} T_{11} = T_{33} = -p + \dfrac{\mu_o}{\tau_1} \\ T_{22} = -p + \dfrac{\mu_o}{\tau_1}(1 + 2\kappa^2 \tau_1^2) \\ T_{12} = \mu_o \kappa \\ T_{23} = T_{31} = 0 \end{array} \Rightarrow \begin{array}{l} \tau(\kappa) = T_{12} = \mu_o \kappa \\ \sigma_1(\kappa) = T_{11} - T_{33} = 0 \\ \sigma_2(\kappa) = T_{22} - T_{33} = 2\kappa^2 \mu_o \tau_1 \end{array} \tag{8.5.5}$$

Note for this simplified case, the shear stress and shear stress function become the Newtonian values. Many other specific forms of constitutive relation (8.5.1) are given in Bird et al. (1987).

8.5.2 K-BKZ INTEGRAL MODELS

A remarkable integral constitutive law was co-developed and first presented by Kaye (1962) and Bernstein et al. (1963). The constitutive law has been known as the *K-BKZ theory for incompressible nonlinear viscoelastic materials*. Over the last 50 plus years, this constitutive relation has proved to be very useful in predicting experimental observations and has resulted in several hundred publications (see recent review by Mitsoulis, 2013). The basic constitutive assumptions coming from the BKZ group are that the material is incompressible, with no preferred reference configuration. The existence of a stored energy function is retained under the idea that the material wants to return to its past shapes; however, it does this by having a stronger desire to return to shapes in its immediate history than those in its more distant history. Thus, we have a fading memory built into the constitutive concept. The basic theory was initially developed under isothermal conditions, and we will limit our discussion to this case. However, the theory has now been extended to problems containing temperature variation.

Although construction of the K-BKZ theory could be made using some of the fundamentals previously described, we take a somewhat different path. This constitutive concept for incompressible viscoelastic materials could be thought of as coming from a further generalization our previous relation (8.5.1):

$$\boldsymbol{T} + p\boldsymbol{I} = \int_0^\infty m(s)\boldsymbol{H}(\boldsymbol{C}_t(t-s))\,ds \tag{8.5.6}$$

where $m(s)$ is a material function of the elapsed time and $\boldsymbol{H}(\boldsymbol{C}_t(t-s))$ is an isotropic tensor-valued function of the objective relative right Cauchy–Green strain tensor. Similar to the finite elastic case in Section 8.3.1, we can employ the standard representation theory and express $\boldsymbol{H}$ as a polynomial in terms of $\boldsymbol{C}$ and $\boldsymbol{C}^{-1}$. Thus, rewrite (8.5.6) as

$$\boldsymbol{T} + p\boldsymbol{I} = \int_0^\infty m(s)[\phi_1(I_1,I_2)\boldsymbol{C}_t(t-s) + \phi_2(I_1,I_2)\boldsymbol{C}_t^{-1}(t-s)]\,ds \tag{8.5.7}$$

and the invariants are simply the traces of the two strain tensors. Thus, we have developed an *integral constitutive form that uses two deformation tensors*.

The K-BKZ theory is more commonly expressed by

$$\boldsymbol{T} + p\boldsymbol{I} = 2\int_0^\infty \left[\frac{\partial U}{\partial I_1}\boldsymbol{C}_t^{-1}(t-s) - \frac{\partial U}{\partial I_2}\boldsymbol{C}_t(t-s)\right] ds \tag{8.5.8}$$

where $U = U(s, I_1, I_2)$ with

$$I_1 = \operatorname{tr}\boldsymbol{C}_t^{-1}(t-s), \quad I_2 = \operatorname{tr}\boldsymbol{C}_t(t-s) \tag{8.5.9}$$

This form should be compared with relation $(8.3.9)_2$ used for the nonlinear elastic constitutive law. It is noted that U plays the role of an elastic stored energy function similar to our constitutive work for elastic materials. However, U is also a relaxing time-dependent potential function with nonequilibrium properties related through the invariants I_1 and I_2 and the elapsed time s. Thus, we can also interpret U as a *memory function*. These issues represent fundamental aspects of this elastic fluid theory. Several variants of the K-BKZ relation have been given in the literature and many of these have been presented in the review article by Mitsoulis (2013).

Specific forms for the material potential function U have been proposed, and one of the more original forms was given by Zapas (1966):

$$\begin{aligned} U = &-\frac{\alpha'(t)}{2}(I_1-3)^2 - \frac{9}{2}\beta'(t)\log\left(\frac{I_1+I_2+3}{9}\right) \\ &-24(\beta'(t)-c'(t))\log\left(\frac{I_1+15}{I_2+15}\right) - c'(t)(I_1-3) \end{aligned} \tag{8.5.10}$$

where $\alpha(t), \beta(t), c(t)$ are time functions that are positive and monotonically decreasing. Other proposed forms commonly include *separable formulations* where $U(s, I_1, I_2) = M(s)W(I_1, I_2)$.

We now explore a couple of examples that apply this theory to some standard flow/deformation problems.

EXAMPLE 8.5.1 STEADY SIMPLE SHEARING OF A K-BKZ VISCOELASTIC FLUID

Consider the steady shearing flow deformation $v_1 = 0, v_2 = \kappa x_1, v_3 = 0$ of a K-BKZ viscoelastic fluid. Determine the resulting stress components and the viscometric functions.

Solution: For this volume-preserving flow, the strain tensors have been computed in previous examples:

$$\boldsymbol{C}_t^{-1}(t-s) = \begin{bmatrix} 1 & \kappa s & 0 \\ \kappa s & 1+\kappa^2 s^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{C}_t(t-s) = \begin{bmatrix} 1+\kappa^2 s^2 & -\kappa s & 0 \\ -\kappa s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.5.11}$$

and thus $I_1 = I_2 = 3 + \kappa^2 s^2$. The K-BKZ constitutive relation (8.5.8) gives

$$\begin{aligned} T_{11} &= -p + 2\int_0^\infty \left[\frac{\partial U}{\partial I_1} - \frac{\partial U}{\partial I_2}(1+\kappa^2 s^2)\right] ds \\ T_{22} &= -p + 2\int_0^\infty \left[\frac{\partial U}{\partial I_1}(1+\kappa^2 s^2) - \frac{\partial U}{\partial I_2}\right] ds \\ T_{33} &= -p + 2\int_0^\infty \left[\frac{\partial U}{\partial I_1} - \frac{\partial U}{\partial I_2}\right] ds \\ T_{12} &= 2\kappa\int_0^\infty s\left[\frac{\partial U}{\partial I_1} + \frac{\partial U}{\partial I_2}\right] ds, \quad T_{23} = T_{31} = 0 \end{aligned} \tag{8.5.12}$$

The viscometric functions follow from (8.4.28):

$$
\begin{aligned}
\tau(\kappa) &= T_{12} = 2\kappa\int_0^\infty s\left[\frac{\partial U}{\partial I_1}+\frac{\partial U}{\partial I_2}\right]ds = \int_0^\infty \frac{\partial U}{\partial(\kappa s)}ds \\
\sigma_1(\kappa) &= T_{11}-T_{33} = -2\kappa^2\int_0^\infty s^2\frac{\partial U}{\partial I_2}ds \\
\sigma_2(\kappa) &= T_{22}-T_{33} = 2\kappa^2\int_0^\infty s^2\frac{\partial U}{\partial I_1}ds
\end{aligned}
\tag{8.5.13}
$$

where we have used the chain rule to get $\frac{\partial U}{\partial I_1}+\frac{\partial U}{\partial I_2} = \frac{\partial U}{\partial(\kappa s)}\frac{\partial(\kappa s)}{\partial I_1}+\frac{\partial U}{\partial(\kappa s)}\frac{\partial(\kappa s)}{\partial I_2}$ $= \frac{1}{\kappa s}\frac{\partial U}{\partial(\kappa s)}$. Since these stresses are all homogeneous and time independent (steady flow), they will identically satisfy the equations of motion.

EXAMPLE 8.5.2 UNIAXIAL EXTENSION OF AN INCOMPRESSIBLE NONLINEAR K-BKZ VISCOELASTIC MATERIAL

Determine the stress components in an incompressible K-BKZ viscoelastic material that is subjected to a uniaxial deformation in the x_1-direction.

Solution: Using the previous results in Example 3.15.1, the relative isochoric motion is given by

$$
\xi_1(\tau) = \frac{\lambda(\tau)}{\lambda(t)}x_1(t), \quad \xi_2(\tau) = \sqrt{\frac{\lambda(t)}{\lambda(\tau)}}x_2(t), \quad \xi_3(\tau) = \sqrt{\frac{\lambda(t)}{\lambda(\tau)}}x_3(t) \tag{8.5.14}
$$

where λ is the primary stretch ratio. Thus, the deformation gradient and right Cauchy–Green strain tensors become

$$
\boldsymbol{F}_t(\tau) = \frac{\partial\xi}{\partial\boldsymbol{x}} = \begin{bmatrix} \frac{\lambda(\tau)}{\lambda(t)} & 0 & 0 \\ 0 & \sqrt{\frac{\lambda(t)}{\lambda(\tau)}} & 0 \\ 0 & 0 & \sqrt{\frac{\lambda(t)}{\lambda(\tau)}} \end{bmatrix} \tag{8.5.15}
$$

$$
\boldsymbol{C}_t(\tau) = \begin{bmatrix} \left(\frac{\lambda(\tau)}{\lambda(t)}\right)^2 & 0 & 0 \\ 0 & \frac{\lambda(t)}{\lambda(\tau)} & 0 \\ 0 & 0 & \frac{\lambda(t)}{\lambda(\tau)} \end{bmatrix}, \quad \boldsymbol{C}_t^{-1}(\tau) = \begin{bmatrix} \left(\frac{\lambda(t)}{\lambda(\tau)}\right)^2 & 0 & 0 \\ 0 & \frac{\lambda(\tau)}{\lambda(t)} & 0 \\ 0 & 0 & \frac{\lambda(\tau)}{\lambda(t)} \end{bmatrix} \tag{8.5.16}
$$

Thus, the invariants become $I_1 = \frac{\lambda^2(t)}{\lambda^2(\tau)} + 2\frac{\lambda(\tau)}{\lambda(t)}, I_2 = \frac{\lambda^2(\tau)}{\lambda^2(t)} + 2\frac{\lambda(t)}{\lambda(\tau)}$.

The stresses follow from (8.5.8):

$$\begin{aligned}
T_{11} &= -p + 2\int_{-\infty}^{t}\left[\frac{\partial U}{\partial I_1}\frac{\lambda^2(t)}{\lambda^2(\tau)} - \frac{\partial U}{\partial I_2}\frac{\lambda^2(\tau)}{\lambda^2(t)}\right]d\tau \\
T_{22} = T_{33} &= -p + 2\int_{-\infty}^{t}\left[\frac{\partial U}{\partial I_1}\frac{\lambda(\tau)}{\lambda(t)} - \frac{\partial U}{\partial I_2}\frac{\lambda(t)}{\lambda(\tau)}\right]d\tau \\
T_{12} = T_{23} &= T_{31} = 0
\end{aligned} \tag{8.5.17}$$

We can define a stress difference as

$$\sigma \equiv T_{11} - T_{22} = T_{11} - T_{33} = \int_{-\infty}^{t}\left[\frac{\lambda^2(t)}{\lambda^2(\tau)} - \frac{\lambda(\tau)}{\lambda(t)}\right]h\left(\frac{\lambda(t)}{\lambda(\tau)}, t-\tau\right)\mathrm{d}\tau \tag{8.5.18}$$

where

$$h\left(\frac{\lambda(t)}{\lambda(\tau)}, t-\tau\right) = 2\left[\frac{\partial U}{\partial I_1} + \frac{\lambda(\tau)}{\lambda(t)}\frac{\partial U}{\partial I_2}\right] \tag{8.5.19}$$

For a single-step stress relaxation, with $\lambda(t)=1, t<0$ and $\lambda(t)=\lambda=$constant, $t \geq 0$, relation (8.5.18) gives

$$\sigma = \left(\lambda^2 - \frac{1}{\lambda}\right)H(\lambda, t) \tag{8.5.20}$$

where

$$H(\lambda, t) = \int_t^{\infty} h(\lambda, \xi)\,d\xi \Rightarrow h(\lambda, t) = -\frac{\partial H(\lambda, t)}{\partial t} \tag{8.5.21}$$

Relations (8.5.20) and (8.5.21) then imply that data from a stress relaxation experiment allow the determination of the material functions H and h and thus allow the calculation of the stress response to any other uniaxial deformation history.

Although many other additional nonlinear continuum mechanics theories including plastic and viscoplastic response could be presented, in order to keep the text of reasonable length, we end our discussion of this general topic.

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EXERCISES

8.1 Starting with the general nonlinear elastic constitutive form (8.3.1) $\boldsymbol{T} = \boldsymbol{f}(\boldsymbol{F})$, first show that frame-indifference would imply $\boldsymbol{QTQ}^T = \boldsymbol{f}(\boldsymbol{QF})$. Next consider the Polar Decomposition Theorem for the deformation gradient (3.6.10), $\boldsymbol{F} = \boldsymbol{RU}$ where $\boldsymbol{R}$ is the rotation tensor and $\boldsymbol{U}$ is the right stretch tensor. Choosing $\boldsymbol{Q} = \boldsymbol{R}^T$ show that $\boldsymbol{T} = \boldsymbol{Rf}(\boldsymbol{U})\boldsymbol{R}^T$. Finally, show that the constitutive form can be expressed in terms of the second Piola–Kirchhoff stress and the right Cauchy–Green strain tensor $\boldsymbol{S} = \boldsymbol{h}(\boldsymbol{C})$ for an appropriately defined function $\boldsymbol{h}$. Note this additional form is often referred to as the *reference constitutive relation*.

8.2 Verify the constitutive form $(8.3.9)_1$.

8.3 Explicitly determine the Cauchy and PK1 axial stress predictions for the neo-Hookean and Mooney–Rivlin models in Example 8.3.1.

8.4 Determine the second Piola–Kirchhoff stress tensor PK2 for the extensional deformation problem in Example 8.3.1.

8.5 The finite elastic uniaxial extension data from Rivlin and Saunders (1951) is again shown in the figure. Similar to Example 8.3.1, we wish to curve-fit the data this time using a linear line as shown. Clearly this linear approximation is only valid for extension ratios $1 \le \lambda \le 2.5$. First determine the linear relation to properly fit the data. Next using the MATLAB Code C-21 in Appendix C, insert the fitting relation to plot the stress versus extension ratio for the data relation. Finally, using a trial-and-error scheme, select the α_1 and α_2 parameters to make the same stress plots for the Mooney and neo-Hookean models that will closely match with the data predictions. It is suggested that you start with $\alpha_1 = 3, \alpha_2 = 0.5$.

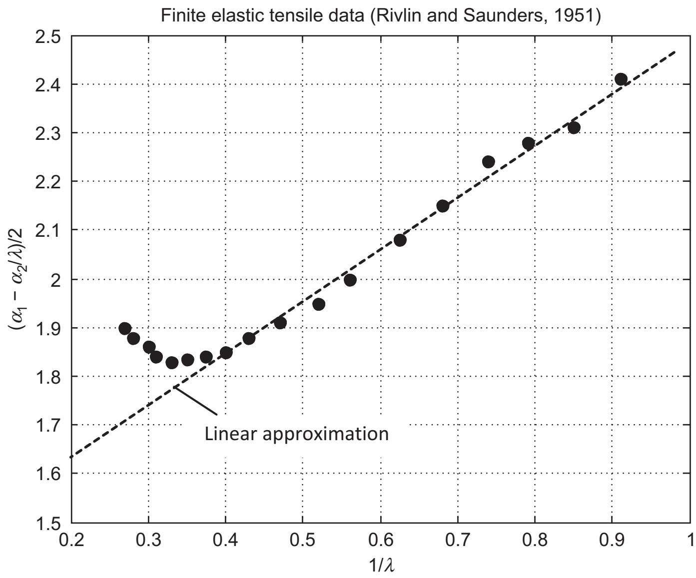

Figure Exercise 8.5

8.6 Verify result (8.3.20) in Example 8.3.2.

8.7 For the simple shear deformation problem of Example 8.3.2, determine the Piola–Kirchhoff stress tensors PK1 and PK2.

8.8 For the simple shear deformation problem of Example 8.3.2, if the strain energy form (8.3.10) is given by $U = C_{10}(I_B - 3) + C_{20}(I_B - 3)^2$, determine the stress field. Using unit values for C_{10} and C_{20}, plot the response T_{12} versus γ and compare with corresponding neo-Hookean response.

8.9 For the torsional Example 8.3.3, evaluate all stresses, the twisting moment, and normal axial force for the neo-Hookean solid. Compare these values with corresponding predictions from linear elasticity.

8.10 Consider the nonlinear elastic problem that includes the torsion, extension, and inflation of an incompressible solid cylinder:

$$r = \lambda_r R, \quad \theta = \Theta + \kappa Z, \quad z = \lambda_z Z$$

where $\lambda_r, \kappa, \lambda_z$ are constants satisfying the incompressibility constraint $\lambda_r^2 \lambda_z = 1$.
Following the solution steps used in Example 8.3.3, determine the stresses, twisting moment, and normal axial stress.

8.11 An incompressible elastic sheet is to be stretched uniformly in the x_1,x_2-plane by forces F_1 and F_2 as shown. Assume that the deformation is given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \mu X_3$$

where the stretches $\lambda_1, \lambda_2, \mu$ are all constants. First determine the incompressibility constraint, and then calculate all Cartesian stresses. Finally, using the traction boundary conditions $F_1 = T_{11} 2hb, F_2 = T_{22} 2ha$, determine the specific forms of the two material functions α_1 and α_2.

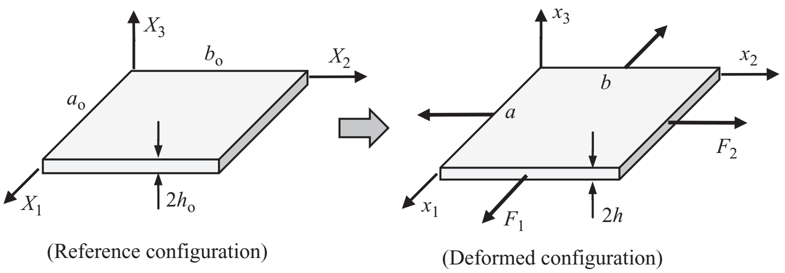

Figure Exercise 8.11

8.12 A fluid moves with the velocity field $v_1 = kx_1, v_2 = -kx_2, v_3 = 0$. Verify that this motion is isochoric. Next determine the rate of deformation tensor and the stress field using the Reiner–Rivlin fluid model.

8.13 Under steady simple shear flow, the Reiner–Rivlin fluid stresses were given by

$$T_{11} + p = T_{22} + p = \frac{\beta_2}{4}\kappa^2, \quad T_{12} = \frac{\beta_1}{2}\kappa$$

where $\beta_i = \beta_i(\kappa^2)$. Choose the specific forms $\beta_1 / 2 = \mu_o + \mu_1\kappa^2$ and $\beta_2 / 4 = \eta_o + \eta_1\kappa^2$ where μ_i and η_i are constants. Calculate the stress components and make log–log plots of $(T_{11} + p)/\eta_o$ and T_{12}/μ_o versus κ for the case of $\mu_1/\mu_o = -0.2$ and $\eta_1/\eta_o = -0.1$.

8.14 Determine the viscometric functions $\tau(\kappa), \sigma_1(\kappa), \sigma_2(\kappa)$ for the second-order fluid model under simple shearing flow in Example 8.4.1.

8.15 For the velocity field specified in Exercise 8.12, determine the rate of deformation tensor and the two nonzero Rivlin–Ericksen tensors. Next, calculate the stresses predicted by the second-order fluid model.

8.16 Starting with representation (8.4.14) and using the forms (8.4.25) for the Rivlin–Ericksen tensors, formally derive the viscometric flow representation (8.4.27) with the definitions of the viscometric functions given by (8.4.29).

8.17 A fluid of *grade three* that satisfies the axiom of consistency has been found to be

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu_1 \boldsymbol{A}_1 + \mu_2 \boldsymbol{A}_1^2 + \mu_3 \boldsymbol{A}_2 + \beta_3 (\mathrm{tr}\, \boldsymbol{A}_1^2)\boldsymbol{A}_1$$

Determine the stress field for a flow $v_1 = 2kx_1, \quad v_2 = -kx_2, \quad v_3 = -kx_3$.

8.18 For the general constitutive form (8.5.1), consider the case with a general material function $\mu = \mu(s)$ and $\tilde{\boldsymbol{E}}_t(t-s) = \boldsymbol{G}_t(t-s) = \boldsymbol{C}_t(t-s) - \boldsymbol{I}$. Assuming the material to be under shearing flow deformation $v_1 = 0, v_2 = \kappa x_1, v_3 = 0$, determine the stresses and the viscometric functions $\tau(\kappa), \sigma_1(\kappa), \sigma_2(\kappa)$.

8.19 Using the material memory function form $\mu(I_i, t, s) = \frac{\mu_o}{\tau_1^2} \mathrm{e}^{-s/\tau_1}$, explicitly develop the stresses given by equations (8.5.5).

8.20 Show that the shear viscometric function for the K-BKZ theory under shear flow in Example 8.5.1 is given by

$$\tau(\kappa) = T_{12} = 2\kappa \int_0^\infty s \left[\frac{\partial U}{\partial I_1} + \frac{\partial U}{\partial I_2} \right] \mathrm{d}s = \int_0^\infty \frac{\partial U}{\partial(\kappa s)} \mathrm{d}s$$

8.21 For the case of shear flow $v_1 = 0, v_2 = \kappa x_1, v_3 = 0$, show that the proposed form for the K-BKZ material potential function U given by relation (8.5.10) will reduce to

$$U = -\frac{\alpha'(t)}{2}\kappa^4 s^4 - \frac{9}{2}\beta'(t)\log\left(1 + \frac{2}{9}\kappa^2 s^2\right) - c'(t)\kappa^2 s^2$$

8.22 For the shear flow case, use the results from Exercises 8.20 and 8.21 to show that the shear viscometric function is given by

$$\tau(\kappa) = T_{12} = -2\kappa\int_0^\infty \left[\alpha'(s)\kappa^2 s^2 + \frac{9\beta'(s)}{9+2\kappa^2 s^2} + c'(s)\right] s\,\mathrm{d}s$$

8.23 Using the results of Exercise 8.22, the apparent viscosity is given by

$$\eta(\kappa) = \left|\frac{T_{12}}{\kappa}\right| = \left|2\int_0^\infty \left[\alpha'(s)\kappa^2 s^2 + \frac{9\beta'(s)}{9+2\kappa^2 s^2} + c'(s)\right] s\,\mathrm{d}s\right|$$

Choosing exponential forms for the material functions

$$\alpha(t) = A_o e^{-a_o t}, \quad \beta(t) = B_o e^{-b_o t}, \quad c(t) = C_o e^{-c_o t}$$

show that the viscosity is given by

$$\eta(\kappa) = 2\left[\frac{6A_o}{a_o^3}\kappa^2 + \frac{9B_o b_o}{2\kappa^2} g\left(\frac{3b_o}{\kappa\sqrt{2}}\right) + \frac{C_o}{c_o}\right]$$

where g is an auxiliary function associated with the sine and cosine integrals

$$g(z) = \int_0^\infty \frac{t\,\mathrm{e}^{-zt}}{1+t^2}dt$$

8.24 Verify the computations for the kinematical tensors in relations (8.5.14)–(8.5.16) in uniaxial extension Example 8.5.2.

8.25 Verify the analysis steps in relations (8.5.18)–(8.5.21) for the uniaxial extension Example 8.5.2.

8.26 Consider a K-BKZ material under *biaxial deformation* in the x_1,x_2-plane

$$\xi_1(\tau) = \frac{\lambda_1(\tau)}{\lambda_1(t)}x_1(t), \quad \xi_2(\tau) = \frac{\lambda_2(\tau)}{\lambda_2(t)}x_2(t), \quad \xi_3(\tau) = \frac{\lambda_3(\tau)}{\lambda_3(t)}x_3(t)$$

(a) Show that if the deformation is to be isochoric $\dfrac{\lambda_3(\tau)}{\lambda_3(t)} = \dfrac{\lambda_1(t)}{\lambda_1(\tau)}\dfrac{\lambda_2(t)}{\lambda_2(\tau)}$, and thus we can eliminate the stretch in the x_3-direction.

(b) Determine the kinematical deformation tensors $\boldsymbol{F}_t(\tau), \boldsymbol{C}_t(\tau), \boldsymbol{C}_t^{-1}(\tau)$, and the two invariants $I_1 = \operatorname{tr}\boldsymbol{C}_t^{-1}(\tau), I_2 = \operatorname{tr}\boldsymbol{C}_t(\tau)$.

(c) Finally, calculate the stress field associated with this deformation.