APPENDIX

Transformation of Field Variables Between Cartesian, Cylindrical, and Spherical Components

B

This appendix contains some three-dimensional transformation relations between displacement and stress components in Cartesian, cylindrical and spherical coordinates. The coordinate systems where previously shown in Fig. A.1 and the related stress components are re-illustrated in Fig. B.1. These results follow from the general transformation laws (2.8.1) and (4.3.7). Note that the stress results and can also be applied for any second order tensor including strain transformation.

B.1 CYLINDRICAL COMPONENTS FROM CARTESIAN

The transformation matrix for this case is given by

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.1}$$

Displacement Transformation:

$$\begin{aligned} u_r &= u\cos\theta + v\sin\theta \\ u_\theta &= -u\sin\theta + v\cos\theta \\ u_z &= w \end{aligned} \tag{B.2}$$

Stress Transformation:

$$\begin{aligned} T_{rr} &= T_{xx}\cos^2\theta + T_{yy}\sin^2\theta + 2T_{xy}\sin\theta\cos\theta \\ T_{\theta\theta} &= T_{xx}\sin^2\theta + T_{yy}\cos^2\theta - 2T_{xy}\sin\theta\cos\theta \\ T_{zz} &= T_{zz} \\ T_{r\theta} &= -T_{xx}\sin\theta\cos\theta + T_{yy}\sin\theta\cos\theta + T_{xy}(\cos^2\theta - \sin^2\theta) \\ T_{\theta z} &= T_{yz}\cos\theta - T_{zx}\sin\theta \\ T_{zr} &= T_{yz}\sin\theta + T_{zx}\cos\theta \end{aligned} \tag{B.3}$$

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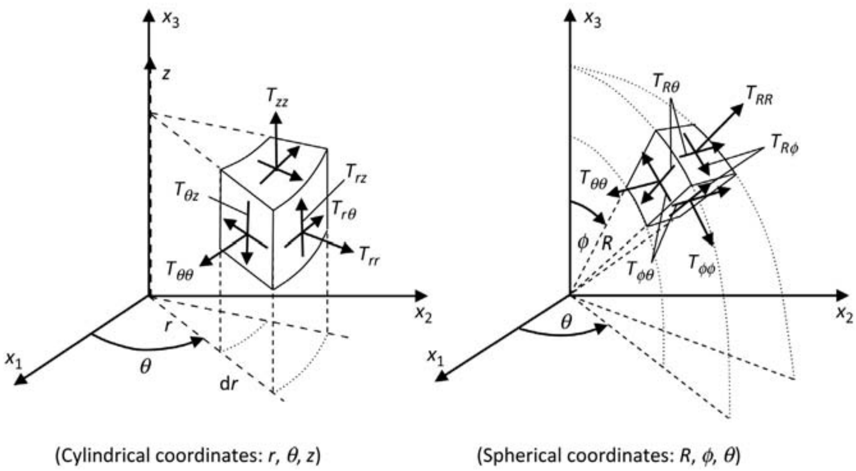

FIGURE B.1

Stress components in cylindrical and spherical coordinates.

B.2 SPHERICAL COMPONENTS FROM CYLINDRICAL

The transformation matrix from cylindrical to spherical coordinates is given by

$$\boldsymbol{Q} = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \end{bmatrix} \tag{B.4}$$

Displacement Transformation:

$$\begin{aligned} u_R &= u_r \sin\phi + u_z \cos\phi \\ u_\phi &= u_r \cos\phi - u_z \sin\phi \\ u_\theta &= u_\theta \end{aligned} \tag{B.5}$$

Stress Transformation:

$$\begin{aligned} T_{RR} &= T_{rr}\sin^2\phi + T_{zz}\cos^2\phi + 2T_{rz}\sin\phi\cos\phi \\ T_{\phi\phi} &= T_{rr}\cos^2\phi + T_{zz}\sin^2\phi - 2T_{rz}\sin\phi\cos\phi \\ T_{\theta\theta} &= T_{\theta\theta} \\ T_{R\phi} &= (T_{rr} - T_{zz})\sin\phi\cos\phi - T_{rz}(\sin^2\phi - \cos^2\phi) \\ T_{\phi\theta} &= T_{r\theta}\cos\phi - T_{\theta z}\sin\phi \\ T_{\theta R} &= T_{r\theta}\sin\phi + T_{\theta z}\cos\phi \end{aligned} \tag{B.6}$$

B.3 SPHERICAL COMPONENTS FROM CARTESIAN

The transformation matrix from Cartesian to spherical coordinates can be obtained by combining the previous transformations given by (B.1) and (B.4); in the proper order of (B.1) first. Tracing back through tensor transformation theory, this is accomplished by the simple matrix multiplication.

$$\boldsymbol{Q} = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \tag{B.7}$$

Displacement Transformation:

$$\begin{aligned} u_R &= u\sin\phi\cos\theta + v\sin\phi\sin\theta + w\cos\phi \\ u_\phi &= u\cos\phi\cos\theta + v\cos\phi\sin\theta - w\sin\phi \\ u_\theta &= -u\sin\theta + v\cos\theta \end{aligned} \tag{B.8}$$

Stress Transformation:

$$\begin{aligned} T_{RR} &= T_{xx}\sin^2\phi\cos^2\theta + T_{yy}\sin^2\phi\sin^2\theta + T_{zz}\cos^2\phi \\ &\quad + 2T_{xy}\sin^2\phi\sin\theta\cos\theta + 2T_{yz}\sin\phi\cos\phi\sin\theta + 2T_{zx}\sin\phi\cos\phi\cos\theta \\ T_{\phi\phi} &= T_{xx}\cos^2\phi\cos^2\theta + T_{yy}\cos^2\phi\sin^2\theta + T_{zz}\sin^2\phi \\ &\quad + 2T_{xy}\cos^2\phi\sin\theta\cos\theta - 2T_{yz}\sin\phi\cos\phi\sin\theta - 2T_{zx}\sin\phi\cos\phi\cos\theta \\ T_{\theta\theta} &= T_{xx}\sin^2\theta + T_{yy}\cos^2\theta - 2T_{xy}\sin\theta\cos\theta \\ T_{R\phi} &= T_{xx}\sin\phi\cos\phi\cos^2\theta + T_{yy}\sin\phi\cos\phi\sin^2\theta - T_{zz}\sin\phi\cos\phi \\ &\quad + 2T_{xy}\sin\phi\cos\phi\sin\theta\cos\theta - T_{yz}(\sin^2\phi - \cos^2\phi)\sin\theta \\ &\quad - T_{zx}(\sin^2\phi - \cos^2\phi)\cos\theta \\ T_{\phi\theta} &= -T_{xx}\cos\phi\sin\theta\cos\theta + T_{yy}\cos\phi\sin\theta\cos\theta + T_{xy}\cos\phi(\cos^2\theta - \sin^2\theta) \\ &\quad - T_{yz}\sin\phi\cos\theta + T_{zx}\sin\phi\sin\theta \\ T_{\theta R} &= -T_{xx}\sin\phi\sin\theta\cos\theta + T_{yy}\sin\phi\sin\theta\cos\theta + T_{xy}\sin\phi(\cos^2\theta - \sin^2\theta) \\ &\quad + T_{yz}\cos\phi\cos\theta - T_{zx}\cos\phi\sin\theta \end{aligned} \tag{B.9}$$

Inverse transformations of these results can be computed by formally inverting the system equations or redeveloping the results using tensor transformation theory. For example, going from cylindrical to Cartesian would be accomplished by using the inverse of (B.1):

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.10}$$