APPENDIX

A Basic Field Equations in Cartesian, Cylindrical, and Spherical Coordinates

For convenience, some of the basic three-dimensional field equations of continuum mechanics are listed here for Cartesian, cylindrical, and spherical coordinate systems. This collection will help save time from searching these results in the various chapters of the text. Cylindrical and spherical coordinates previously shown in Figs. 2.6 and 2.7 are related to the basic Cartesian system as shown in Fig. A.1. For convenience, the Cartesian notation x_1, x_2, x_3 is replaced by x, y, z.

A.1 SMALL STRAIN–DISPLACEMENT RELATIONS

CARTESIAN COORDINATES $\boldsymbol{u} = (u, v, w)$

$$\begin{aligned}
&\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \\
&\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
&\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned} \tag{A.1}$$

CYLINDRICAL COORDINATES

$$\begin{aligned}
&\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_z = \frac{\partial u_z}{\partial z} \\
&\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
&\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
&\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned} \tag{A.2}$$

Continuum Mechanics Modeling of Material Behavior. http://dx.doi.org/10.1016/B978-0-12-811474-2.00010-1

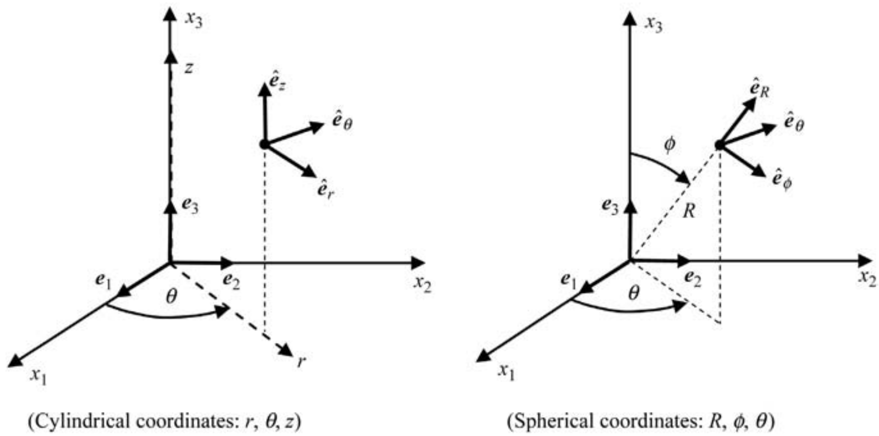

FIGURE A.1

Cylindrical and spherical coordinate systems.

SPHERICAL COORDINATES

$$
\begin{aligned}
&\varepsilon_R = \frac{\partial u_R}{\partial R}, \quad \varepsilon_\phi = \frac{1}{R}\left(u_R + \frac{\partial u_\phi}{\partial \phi}\right)\\
&\varepsilon_\theta = \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi u_R + \cos\phi u_\phi\right)\\
&\varepsilon_{R\phi} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right)\\
&\varepsilon_{\phi\theta} = \frac{1}{2R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \cot\phi u_\theta\right)\\
&\varepsilon_{\theta R} = \frac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right)
\end{aligned}
\tag{A.3}
$$

A.2 DEFORMATION GRADIENT TENSOR FOR FINITE DEFORMATION

CARTESIAN COORDINATES $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$

$$
\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix}
\tag{A.4}
$$

CYLINDRICAL COORDINATES

$$r = r(R,\Theta,Z), \quad \theta = \theta(R,\Theta,Z), \quad z = z(R,\Theta,Z)$$

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix} \tag{A.5}$$

$$r = r(X,Y,Z), \quad \theta = \theta(X,Y,Z), \quad z = z(X,Y,Z)$$

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial X} & \dfrac{\partial r}{\partial Y} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial X} & r\dfrac{\partial \theta}{\partial Y} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{bmatrix} \tag{A.6}$$

SPHERICAL COORDINATES

$$R = R(R_o,\Phi,\Theta), \quad \phi = \phi(R_o,\Phi,\Theta), \quad \theta = \theta(R_o,\Phi,\Theta)$$

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial R}{\partial R_o} & \dfrac{1}{R_o}\dfrac{\partial R}{\partial \Phi} & \dfrac{1}{R_o \sin\Phi}\dfrac{\partial R}{\partial \Theta} \\ R\dfrac{\partial \phi}{\partial R_o} & \dfrac{R}{R_o}\dfrac{\partial \phi}{\partial \Phi} & \dfrac{R}{R_o \sin\Phi}\dfrac{\partial \phi}{\partial \Theta} \\ R\sin\phi\dfrac{\partial \theta}{\partial R_o} & \dfrac{R\sin\phi}{R_o}\dfrac{\partial \theta}{\partial \Phi} & \dfrac{R\sin\phi}{R_o \sin\Phi}\dfrac{\partial \theta}{\partial \Theta} \end{bmatrix} \tag{A.7}$$

A.3 EQUATIONS OF MOTION (CONSERVATION OF LINEAR MOMENTUM)

CARTESIAN COORDINATES

$$\begin{aligned} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} + F_x &= \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} + F_y &= \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} + F_z &= \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) \end{aligned} \tag{A.8}$$

CYLINDRICAL COORDINATES

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta}+\frac{\partial T_{rz}}{\partial z}+\frac{1}{r}(T_{rr}-T_{\theta\theta})+F_r=\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)\\
&\frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{\partial T_{\theta z}}{\partial z}+\frac{2}{r}T_{r\theta}+F_\theta=\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)\\
&\frac{\partial T_{rz}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta}+\frac{\partial T_{zz}}{\partial z}+\frac{1}{r}T_{rz}+F_z=\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)
\end{aligned} \tag{A.9}$$

SPHERICAL COORDINATES

$$\begin{aligned}
&\frac{\partial T_{RR}}{\partial R}+\frac{1}{R}\frac{\partial T_{R\varphi}}{\partial \varphi}+\frac{1}{R\sin\varphi}\frac{\partial T_{R\theta}}{\partial \theta}+\frac{1}{R}(2T_{RR}-T_{\phi\phi}-T_{\theta\theta}+T_{R\phi}\cot\phi)+F_R\\
&=\rho\left(\frac{\partial v_R}{\partial t}+v_R\frac{\partial v_R}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_R}{\partial \theta}-\frac{v_\phi^2+v_\theta^2}{R}\right)\\
&\frac{\partial T_{R\varphi}}{\partial R}+\frac{1}{R}\frac{\partial T_{\varphi\varphi}}{\partial \varphi}+\frac{1}{R\sin\varphi}\frac{\partial T_{\varphi\theta}}{\partial \theta}+\frac{1}{R}[(T_{\phi\phi}-T_{\theta\theta})\cot\phi+3T_{R\phi}]+F_\phi\\
&=\rho\left(\frac{\partial v_\phi}{\partial t}+v_R\frac{\partial v_\phi}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_\phi}{\partial \theta}+\frac{v_R v_\phi}{R}-\frac{v_\theta^2\cot\phi}{R}\right)\\
&\frac{\partial T_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial T_{\varphi\theta}}{\partial \varphi}+\frac{1}{R\sin\varphi}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{1}{R}(2T_{\phi\theta}\cot\phi+3T_{R\theta})+F_\theta\\
&=\rho\left(\frac{\partial v_\theta}{\partial t}+v_R\frac{\partial v_\theta}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_\theta}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\theta v_R}{R}+\frac{v_\phi v_\theta\cot\phi}{R}\right)
\end{aligned} \tag{A.10}$$

A.4 HOOKE'S LAW

CARTESIAN COORDINATES

$$\begin{aligned}
&T_{xx}=\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)+2\mu\varepsilon_x && \varepsilon_x=\frac{1}{E}\left[T_{xx}-\nu(T_{yy}+T_{zz})\right]\\
&T_{yy}=\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)+2\mu\varepsilon_y && \varepsilon_y=\frac{1}{E}\left[T_{yy}-\nu(T_{zz}+T_{xx})\right]\\
&T_{zz}=\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)+2\mu\varepsilon_z && \varepsilon_z=\frac{1}{E}\left[T_{zz}-\nu(T_{xx}+T_{yy})\right]\\
&T_{xy}=2\mu\varepsilon_{xy} && \varepsilon_{xy}=\frac{1+\nu}{E}T_{xy}\\
&T_{yz}=2\mu\varepsilon_{yz} && \varepsilon_{yz}=\frac{1+\nu}{E}T_{yz}\\
&T_{zx}=2\mu\varepsilon_{zx} && \varepsilon_{zx}=\frac{1+\nu}{E}T_{zx}
\end{aligned} \tag{A.11}$$

CYLINDRICAL COORDINATES

$$
\begin{aligned}
T_{rr} &= \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2\mu\varepsilon_r \\
T_{\theta\theta} &= \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2\mu\varepsilon_\theta \\
T_{zz} &= \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2\mu\varepsilon_z \\
T_{r\theta} &= 2\mu\varepsilon_{r\theta} \\
T_{\theta z} &= 2\mu\varepsilon_{\theta z} \\
T_{zr} &= 2\mu\varepsilon_{zr}
\end{aligned}
\qquad
\begin{aligned}
\varepsilon_r &= \frac{1}{E}\left[T_r - \nu(T_\theta + T_z)\right] \\
\varepsilon_\theta &= \frac{1}{E}\left[T_\theta - \nu(T_z + T_r)\right] \\
\varepsilon_z &= \frac{1}{E}\left[T_z - \nu(T_r + T_\theta)\right] \\
\varepsilon_{r\theta} &= \frac{1+\nu}{E}T_{r\theta} \\
\nu_{\theta z} &= \frac{1+\nu}{E}T_{\theta z} \\
\varepsilon_{zr} &= \frac{1+\nu}{E}T_{zr}
\end{aligned}
\tag{A.12}
$$

SPHERICAL COORDINATES

$$
\begin{aligned}
T_{RR} &= \lambda(\varepsilon_R + \varepsilon_\phi + \varepsilon_\theta) + 2\mu\varepsilon_R \\
T_{\phi\phi} &= \lambda(\varepsilon_R + \varepsilon_\phi + \varepsilon_\theta) + 2\mu\varepsilon_\phi \\
T_{\theta\theta} &= \lambda(\varepsilon_R + \varepsilon_\phi + \varepsilon_\theta) + 2\mu\varepsilon_\theta \\
T_{R\phi} &= 2\mu\varepsilon_{R\phi} \\
T_{\phi\theta} &= 2\mu\varepsilon_{\phi\theta} \\
T_{\theta R} &= 2\mu\varepsilon_{\theta R}
\end{aligned}
\qquad
\begin{aligned}
\varepsilon_R &= \frac{1}{E}\left[T_{RR} - \nu(T_{\phi\phi} + T_{\theta\theta})\right] \\
\varepsilon_\phi &= \frac{1}{E}\left[T_{\phi\phi} - \nu(T_{\theta\theta} + T_{RR})\right] \\
\varepsilon_\theta &= \frac{1}{E}\left[T_{\theta\theta} - \nu(T_{RR} + T_{\phi\phi})\right] \\
\varepsilon_{R\phi} &= \frac{1+\nu}{E}T_{R\phi} \\
\varepsilon_{\phi\theta} &= \frac{1+\nu}{E}T_{\phi\theta} \\
\varepsilon_{\theta R} &= \frac{1+\nu}{E}T_{\theta R}
\end{aligned}
\tag{A.13}
$$

A.5 EQUILIBRIUM EQUATIONS IN TERMS OF DISPLACEMENTS (NAVIER'S EQUATIONS)

CARTESIAN COORDINATES

$$
\begin{aligned}
\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_x &= 0 \\
\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_y &= 0 \\
\mu\nabla^2 w + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_z &= 0
\end{aligned}
\tag{A.14}
$$

CYLINDRICAL COORDINATES

$$\begin{aligned}
&\mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + (\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_r = 0\\
&\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) + (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_\theta = 0\\
&\mu\nabla^2 u_z + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_z = 0
\end{aligned} \tag{A.15}$$

SPHERICAL COORDINATES

$$\begin{aligned}
&\mu\left(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{2u_\phi \cot\phi}{R^2} - \frac{2}{R^2\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right)\\
&+(\lambda+\mu)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + F_R = 0\\
&\mu\left(\nabla^2 u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2\sin^2\phi} - \frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial u_\theta}{\partial \theta}\right)\\
&+(\lambda+\mu)\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + F_\phi = 0\\
&\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{R^2\sin^2\phi} + \frac{2}{R^2\sin^2\phi}\frac{\partial u_R}{\partial \theta} + \frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial u_\phi}{\partial \theta}\right)\\
&+(\lambda+\mu)\frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + F_\theta = 0
\end{aligned} \tag{A.16}$$