CHAPTER

Mathematical Preliminaries

2

As mentioned in Section 1.2, continuum mechanics must be formulated in terms of quantities that are independent of the coordinate system used to describe the problem. This requires the use of a tensor format to express all variables and relations. The main purpose of this chapter is to present the tensor language and theory necessary for the development of our study. The focus is mainly on Cartesian tensors, but will explore non-Cartesian curvilinear coordinates in the last two sections. Both tensor algebra and tensor calculus for field variables will be presented. Considerable time will be devoted to definitions, proper language, and notational issues that will appear many times in the later chapters. Additional related matrix and tensor mathematical topics will also be discussed. Further reading on the general topics of tensors can be found in Goodbody (1982), Simmons (1994), and Itskov (2015).

2.1 INDEX AND DIRECT NOTATION

Continuum mechanics is formulated in terms of many different types of variables: *scalars*, represented by a single value at each point in space (e.g. material density); *vectors*, expressible in terms of three components in a three-dimensional space (e.g. material displacement or velocity); *matrix variables*, which commonly require nine components to quantify (e.g. stress or strain). Other applications incorporate additional quantities that need even more components to characterize. Related operations between these variables should not depend on the coordinate system. Tensor notation and tensor operations offer a unified scheme to formulate all of the required mathematical relations and theories found in continuum mechanics. A scalar method of simply listing every single variable would prove to be totally inefficient for use in this study.

Generally two different tensor notational schemes have been constructed that enable us to list, develop, and manipulate the variables of interest. One method uses *index notation* to identify tensor variables and operations, whereas the second employs a *direct notation* without indices. Index notation generally provides more explicit information on the tensor operations with respect to a given coordinate system used to describe the problem. However, direct notation is a symbolic representational scheme that offers a more shorthand method with somewhat less details on operational information. These concepts will become clearer as we move forward in our study as we employ both notational methods in various places in the text.

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Index notation is a shorthand scheme that allows the representation of a whole set of elements or components by a single symbol with subscripts. For example, the three numbers a_1, a_2, and a_3 are denoted by the symbol a_i, where index i will normally have the range 1–3. In a similar fashion, A_{ij} represents the nine numbers A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, and A_{33}. Although these representations can be written in any manner, it is common to use a scheme related to vector and matrix formats such that

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{2.1.1}$$

In the matrix format, A_{1j} represents the first row, whereas A_{i1} indicates the first column. Other columns and rows are indicated in similar fashion; thus, the first index represents the row, whereas the second denotes the column. Direct notation would simply use a bold character for the vector and matrix definitions in (2.1.1) by writing

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{2.1.2}$$

Since the indices are not included, direct notation provides a less cluttered scheme but it also does not explicitly indicate the specific nature of the representation.

In general, a symbol $A_{ij...k}$ with N distinct indices represents 3^N distinct numbers or components. It should be apparent that a_i and a_j represent the same three numbers, and likewise a_{ij} and a_{mn} signify the same matrix. Addition, subtraction, multiplication, and equality of index symbols are defined in the normal fashion. For example, addition and subtraction are given by

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \quad A_{ij} \pm B_{ij} = \begin{bmatrix} A_{11} \pm B_{11} & A_{12} \pm B_{12} & A_{13} \pm B_{13} \\ A_{21} \pm B_{21} & A_{22} \pm B_{22} & A_{23} \pm B_{23} \\ A_{31} \pm B_{31} & A_{32} \pm B_{32} & A_{33} \pm B_{33} \end{bmatrix} \tag{2.1.3}$$

and scalar multiplication is specified as

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \quad \lambda A_{ij} = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} \end{bmatrix} \tag{2.1.4}$$

The multiplication of two symbols with different indices is called *outer multiplication*, and a simple example is given by

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{2.1.5}$$

The previous operations obey usual commutative, associative, and distributive laws, for example,

$$\begin{aligned}
&a_i + b_i = b_i + a_i \\
&A_{ij} b_k = b_k A_{ij} \\
&a_i + \left(b_i + c_i\right) = \left(a_i + b_i\right) + c_i \\
&a_i \left(B_{jk} c_l\right) = \left(a_i B_{jk}\right) c_l \\
&A_{ij} \left(b_k + c_k\right) = A_{ij} b_k + A_{ij} c_k
\end{aligned} \tag{2.1.6}$$

Note that the simple relations $a_i = b_i$ or $A_{ij} = B_{ij}$ imply that $a_1 = b_1$, $a_2 = b_2$, … and $A_{11} = B_{11}$, $A_{12} = B_{12}$, …. However, relations of the form $a_i = b_j$ or $A_{ij} = B_{kl}$ have ambiguous meaning since the distinct indices on each term are not the same, and thus these types of expressions are to be avoided in this notational scheme. In general, the distinct subscripts on all individual terms in a given equation should match.

2.2 SUMMATION CONVENTION

It will be convenient to adopt the convention that if a subscript appears twice in the same term, then *summation* over that subscript from one to three is implied, for example,

$$\begin{aligned}
A_{ii} &= \sum_{i=1}^{3} A_{ii} = A_{11} + A_{22} + A_{33} \\
A_{ij} b_j &= \sum_{j=1}^{3} A_{ij} b_j = A_{i1} b_1 + A_{i2} b_2 + A_{i3} b_3
\end{aligned} \tag{2.2.1}$$

It should be apparent that $A_{ii} = A_{jj} = A_{kk} = \ldots$, and therefore the *repeated* subscripts or indices are sometimes called *dummy* subscripts. Unspecified indices that are not repeated are called *free* or *distinct* subscripts. The summation convention may be suspended by underlining one of the repeated indices or by writing *no sum*. The use of three or more repeated indices in the same term (e.g. A_{iii} or $A_{iij} B_{ij}$) has ambiguous meaning and is to be avoided. On a given symbol, the process of setting two free indices equal is called *contraction*. For example, A_{ii} is obtained from A_{ij} by contraction on *i* and *j*. The operation of outer multiplication of two indexed symbols followed by contraction with respect to one index from each symbol generates an *inner multiplication*, for example, $A_{ij} B_{jk}$ is an inner product obtained from the outer product $A_{ij} B_{mk}$ by contraction on indices *j* and *m*.

EXAMPLE 2.2.1 INDEX NOTATION EXAMPLES

Consider the following vector and matrix:

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$$

Evaluate the following expressions: $a_i a_i$, $a_i a_j$, A_{ii}, and $A_{ij} a_j$ and specify whether they are scalars, vectors, matrices, etc.

Solution: Using the standard definitions in Sections 2.1 and 2.2 ⇒

$$a_i a_i = 1^2 + 4^2 + 2^2 = 21 \text{ (scalar)}$$

$$a_i a_j = \begin{bmatrix} 1\times1 & 1\times4 & 1\times2 \\ 4\times1 & 4\times4 & 4\times2 \\ 2\times1 & 2\times4 & 2\times2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 16 & 8 \\ 2 & 8 & 4 \end{bmatrix} \text{(matrix)}$$

$$A_{ii} = 1 + 4 + 2 = 7 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 1\times1 + 0\times4 + 2\times2 \\ 2\times1 + 3\times4 + 0\times2 \\ 1\times1 + 4\times4 + 2\times2 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \\ 21 \end{bmatrix} \text{(vector)}$$

It is always a good idea to first review the distinct and summed indices in such expression to first determine the general nature of the expected final evaluation (i.e. scalar, vector, matrix, . . .).

2.3 SYMMETRIC AND ANTISYMMETRIC SYMBOLS

A symbol $A_{ij\ldots m\ldots n\ldots k}$ is said to be *symmetric* with respect to index pair mn if

$$A_{ij\ldots m\ldots n\ldots k} = A_{ij\ldots n\ldots m\ldots k} \tag{2.3.1}$$

whereas it is *antisymmetric* or *skewsymmetric* if

$$A_{ij\ldots m\ldots n\ldots k} = -A_{ij\ldots n\ldots m\ldots k} \tag{2.3.2}$$

Note that if $A_{ij\ldots m\ldots n\ldots k}$ is symmetric in mn, and $B_{pq\ldots m\ldots n\ldots r}$ is antisymmetric in mn, then the product is zero

$$A_{ij\ldots m\ldots n\ldots k} B_{pq\ldots m\ldots n\ldots r} = 0 \tag{2.3.3}$$

A useful identity or decomposition for matrices may be written as

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) = A_{(ij)} + A_{[ij]} \tag{2.3.4}$$

The first term $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$ is symmetric, whereas the second term $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$ is antisymmetric, and thus an arbitrary symbol A_{ij} can always be expressed as the sum of symmetric and antisymmetric pieces. Note that if A_{ij} is symmetric, then it will have only six independent components. On the other hand, if A_{ij} is antisymmetric, then its diagonal terms A_{ii} (no sum on i) must be zero and it will only have three independent components. Since $A_{[ij]}$ has only three independent components, it can be related to a vector quantity with a single index (see Exercise 2.13).

EXAMPLE 2.3.1 DECOMPOSITION EXAMPLE

For the following matrix

$$A_{ij} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

determine the symmetric and antisymmetric decomposition matrices $A_{(ij)}$ and $A_{[ij]}$ specified by relation (2.3.4). Then justify the fact that the product $A_{(ij)}A_{[ij]}$ is zero.

Solution: Using the definitions in (2.3.4) ⇒

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 3/2 \\ 3/2 & 1 & 5/2 \\ 3/2 & 5/2 & 4 \end{bmatrix}$$

$$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$A_{(ij)}A_{[ij]} = A_{(11)}A_{[11]} + A_{(12)}A_{[12]} + A_{(13)}A_{[13]} + A_{(21)}A_{[21]} + A_{(22)}A_{[22]} + A_{(23)}A_{[23]} + \cdots = 0$$

Notice that $A_{(ij)}$ has only six independent components, whereas $A_{[ij]}$ has only three.

2.4 KRONECKER DELTA AND ALTERNATING SYMBOL

A useful special symbol commonly used in index notational schemes is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \,(\text{no sum}) \\ 0 & \text{if } i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4.1}$$

within usual matrix theory, it is observed that this symbol is simply the unit matrix $\boldsymbol{I}$, and we could use direct notation to write

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4.2}$$

Note that the Kronecker delta is a symmetric symbol with respect to its two indices. Particular useful properties of the Kronecker delta include

$$\begin{aligned} &\delta_{ij} = \delta_{ji} \\ &\delta_{ii} = 3, \quad \delta_{\underline{ii}} = 1 \\ &\delta_{ij}a_j = a_i, \quad \delta_{ij}a_i = a_j \\ &\delta_{ij}a_{jk} = a_{ik}, \quad \delta_{jk}a_{ik} = a_{ij} \\ &\delta_{ij}a_{ij} = a_{ii}, \quad \delta_{ij}\delta_{ij} = 3 \end{aligned} \tag{2.4.3}$$

EXAMPLE 2.4.1 KRONECKER DELTA PROPERTY VERIFICATION

Explicitly justify the Kronecker delta property $\delta_{ij}a_j = a_i$.

Solution: Expand the basic relation using summation convention ⇒

$$\begin{aligned}\delta_{ij}a_j &= \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 \\ &= \begin{cases} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = a_1, & i=1 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = a_2, & i=2 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = a_3, & i=3 \end{cases} \\ &= a_i\end{aligned}$$

Note that this relation is often called the *substitution property* associated with the Kronecker delta, and relations (2.4.3)$_{3,4,5}$ further expand on this property. We will make repeated use of such relations in future applications.

Another useful special symbol is the *alternating* or *permutation symbol* defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{otherwise} \end{cases} \tag{2.4.4}$$

Consequently, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$, $\varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \cdots = 0$. Therefore, of the 27 possible terms for the alternating symbol, three are equal to 1, three are equal to -1, and all others are zero. The alternating symbol is antisymmetric with respect to any pair of its indices.

2.5 DETERMINANTS

The alternating symbol proves to be useful in evaluating determinants and vector cross-products. The determinant a_{ij} of an array can be written in two equivalent forms as

$$\det[A_{ij}] = |A_{ij}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3} \tag{2.5.1}$$

where the first index expression represents the row expansion, whereas the second is the column expansion. Note that indices i and j in the expression $\det[A_{ij}]$ are not subscripts in the usual sense defined in Section 2.1, because $\det[A_{ij}]$ denotes a single number through the operations specified in relation (2.5.1).

Note the following properties associated with determinants and the alternating symbol:

$$\begin{aligned}&\text{If} \quad C_{ij} = A_{ik}B_{kj}, \text{or } C_{ij} = A_{ik}B_{jk}, \text{or } C_{ij} = A_{ki}B_{kj} \\ &\quad \text{then} \quad \det[C_{ij}] = \det[A_{ij}]\det[B_{ij}]\end{aligned} \tag{2.5.2}$$

$$\text{If}\quad A_{ik}A_{jk} = \delta_{ij}, \text{or } A_{ki}A_{kj} = \delta_{ij}, \text{then}\quad \det[A_{ij}] = \pm 1 \tag{2.5.3}$$

$$\varepsilon_{ijk}\varepsilon_{pqr}\det[A_{mn}] = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix} \tag{2.5.4}$$

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{2.5.5}$$

$$\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq} \tag{2.5.6}$$

$$\varepsilon_{ijk}\varepsilon_{ijr} = 2\delta_{kr} \tag{2.5.7}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{2.5.8}$$

$$\det[A_{ij}] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}A_{ip}A_{jq}A_{kr} \tag{2.5.9}$$

$$\varepsilon_{ijk}A_{ip}A_{jq}A_{kr} = \varepsilon_{ijk}A_{pi}A_{qj}A_{rk} = \det[A_{mn}]\varepsilon_{pqr} \tag{2.5.10}$$

$$\frac{d}{dt}(\det[A_{ij}]) = \det[A_{ij}]A_{ij}^{-1}\frac{d}{dt}A_{ji} \tag{2.5.11}$$

$$\varepsilon_{ijk}a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{2.5.12}$$

2.6 VECTORS AND COORDINATE FRAMES

Let us first review a few fundamental properties of vectors. Traditionally, a vector is described as a quantity that has both a magnitude and direction, and with respect to a Cartesian coordinate system (frame), these characteristics can be illustrated geometrically by an arrow or directed line segment of particular magnitude and direction as shown in Fig. 2.1.

Let $\boldsymbol{a}$ denote an arbitrary vector. The rectangular *projections* of $\boldsymbol{a}$ on the x_1, x_2, and x_3 axes are denoted by a_1, a_2, and a_3, respectively. The *magnitude* or *length* of the vector $\boldsymbol{a}$ is given by

$$|\boldsymbol{a}| = (a_1^2 + a_2^2 + a_3^2)^{1/2} = (a_i a_i)^{1/2} \tag{2.6.1}$$

The cosine of the angle between the vector $\boldsymbol{a}$ and the x_i-axis may be expressed as

$$\cos(\boldsymbol{a}, x_i) = \frac{a_i}{|\boldsymbol{a}|} \tag{2.6.2}$$

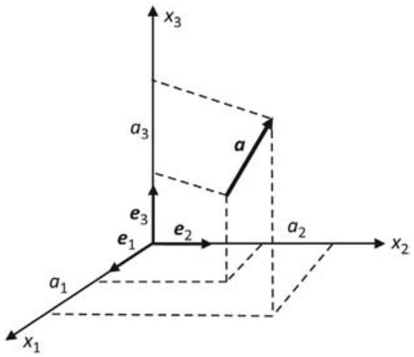

FIGURE 2.1

Vector representation.

Hence if we know the projections of $\boldsymbol{a}$, then we know both the magnitude and direction of the vector.

Eqs. (2.6.1) and (2.6.2) imply that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal (have the same magnitude and direction) if their corresponding projections are equal, $a_i = b_i$. In addition, the *parallelogram rule* for vector addition, $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, follows the indicial notation scheme introduced earlier, that is, $c_i = a_i + b_i$.

We define the three unit vectors along the x_1, x_2, and x_3 axes by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, respectively. This *vector triad* $\{\boldsymbol{e}_i\}$ forms what is called an *orthonormal basis* for the coordinate system. Any and every vector within this system may be expressed in terms of this orthonormal basis by the usual expression

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 = a_i\boldsymbol{e}_i \tag{2.6.3}$$

The numbers a_i are normally called the *Cartesian components* of $\boldsymbol{a}$ in the particular system.

If $\{\boldsymbol{e}_i\}$ forms a *right-handed triad*, then

$$\begin{aligned} \boldsymbol{e}_1 \times \boldsymbol{e}_2 &= \boldsymbol{e}_3 \\ \boldsymbol{e}_2 \times \boldsymbol{e}_3 &= \boldsymbol{e}_1 \\ \boldsymbol{e}_3 \times \boldsymbol{e}_1 &= \boldsymbol{e}_2 \end{aligned} \tag{2.6.4}$$

where the cross operation, $\times$, means the *vector cross product*, which will be defined later. Furthermore, from the definition of $\boldsymbol{e}_i$ and relations (2.6.4), it follows that

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= \delta_{ij} \\ \boldsymbol{e}_i \times \boldsymbol{e}_j &= \varepsilon_{ijk}\boldsymbol{e}_k \end{aligned} \tag{2.6.5}$$

where the dot operation, $\cdot$,indicates the common *dot product* to be defined later.

2.7 CHANGES IN COORDINATE FRAMES: ORTHOGONAL TRANSFORMATIONS

It will be necessary to express continuum field variables and equations in several different coordinate systems. This situation requires the development of particular transformation rules for scalar, vector, matrix, and higher-order variables. This concept is fundamentally connected with the basic definitions of tensor variables and their related tensor transformation laws. At the moment, we will restrict our discussion to transformations only between Cartesian coordinate systems, and thus consider the two systems shown in Fig. 2.2. The two Cartesian frames (x_1, x_2, x_3) and (x_1', x_2', x_3') differ only by orientation, and the unit basis vectors for each frame are $\{\boldsymbol{e}_i\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}_i'\} = \{\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3'\}$.

Let Q_{ij} denote the cosine of the angle between the x_i'-axis and the x_j-axis:

$$Q_{ij} = \cos(x_i', x_j) \tag{2.7.1}$$

Using this definition, the basis vectors in the primed coordinate frame can be easily expressed in terms of those in the unprimed frame by the relations

$$\begin{aligned}\boldsymbol{e}_1' &= Q_{11}\boldsymbol{e}_1 + Q_{12}\boldsymbol{e}_2 + Q_{13}\boldsymbol{e}_3 \\ \boldsymbol{e}_2' &= Q_{21}\boldsymbol{e}_1 + Q_{22}\boldsymbol{e}_2 + Q_{23}\boldsymbol{e}_3 \\ \boldsymbol{e}_3' &= Q_{31}\boldsymbol{e}_1 + Q_{32}\boldsymbol{e}_2 + Q_{33}\boldsymbol{e}_3\end{aligned} \tag{2.7.2}$$

or in index notation

$$\boldsymbol{e}_i' = Q_{ij}\boldsymbol{e}_j \tag{2.7.3}$$

Likewise, the opposite transformation can be written using the same format as

$$\boldsymbol{e}_i = Q_{ji}\boldsymbol{e}_j' \tag{2.7.4}$$

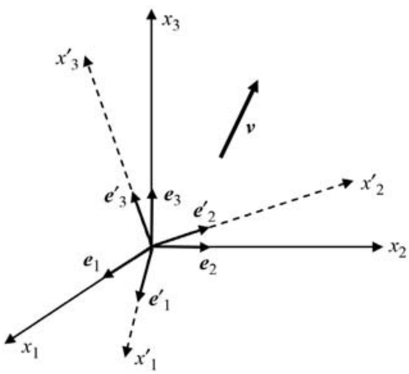

FIGURE 2.2

Change of Cartesian coordinate frames.

Now an arbitrary vector $\boldsymbol{v}$ (see Fig. 2.2) can be written in either of the two coordinate systems as

$$\begin{aligned}\boldsymbol{v} &= v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i \\ &= v_1'\boldsymbol{e}_1' + v_2'\boldsymbol{e}_2' + v_3'\boldsymbol{e}_3' = v_i'\boldsymbol{e}_i'\end{aligned} \tag{2.7.5}$$

Substituting form (2.7.4) into $(2.7.5)_1$ gives

$$\boldsymbol{v} = v_i Q_{ji}\boldsymbol{e}_j'$$

but from $(2.7.5)_2$, $\boldsymbol{v} = v_j'\boldsymbol{e}_j'$, and so after renaming some indices we find

$$v_i' = Q_{ij}v_j \tag{2.7.6}$$

In a similar fashion using (2.7.3) in $(2.7.5)_2$ gives

$$v_i = Q_{ji}v_j' \tag{2.7.7}$$

Relations (2.7.6) and (2.7.7) constitute the transformation laws for the Cartesian components of a vector under a change in rectangular Cartesian coordinate frames. It should be understood that under such transformations, the vector itself is unaltered (retaining original length and orientation) and only its components are changed. Consequently, knowing the components of a vector in one frame, relations (2.7.6) or (2.7.7) can be used to calculate components in any other frame.

The fact that transformations are being made only between orthogonal coordinate systems places some particular restrictions on the transformation or direction cosine matrix Q_{ij}. These can be determined by using (2.7.6) and (2.7.7) together to get

$$v_i = Q_{ji}v_j' = Q_{ji}Q_{jk}v_k \tag{2.7.8}$$

From the properties of the Kronecker delta, this expression can be written as

$$\delta_{ik}v_k = Q_{ji}Q_{jk}v_k \quad \text{or} \quad (Q_{ji}Q_{jk} - \delta_{ik})v_k = 0$$

and since this relation is true for all vectors v_k, the expression in parentheses must be zero giving the result

$$Q_{ji}Q_{jk} = \delta_{ik} \tag{2.7.9}$$

In a similar fashion, relations (2.7.6) and (2.7.7) can be used to eliminate v_i (instead of v'_i) to get

$$Q_{ij}Q_{kj} = \delta_{ik} \tag{2.7.10}$$

Relations (2.7.9) and (2.7.10) comprise the *orthogonality conditions* that Q_{ij} must satisfy. Taking the determinant of either relation and using previous relation (2.5.2) gives another related result

$$\det[Q_{ij}] = \pm 1 \tag{2.7.11}$$

Matrices that satisfy relations (2.7.9)–(2.7.11) are called *orthogonal*, and the transformations given by (2.7.6) and (2.7.7) are therefore referred to as *orthogonal transformations*. An orthogonal matrix (transformation) is said to be *proper* or according to whether $\det[Q_{ij}] = +1$ or -1. We will normally choose proper orthogonal transformations which will maintain right-handed coordinate systems and ensure a few other convenient properties. For the case $\det[Q_{ij}] = +1$, Q_{ij} corresponds to a *rotation*, whereas for $\det[Q_{ij}] = -1$, Q_{ij} corresponds to a *reflection*. The set of all orthogonal transformations constitutes a *group* in the mathematical sense with respect to composition (multiplication).

2.8 CARTESIAN TENSORS AND TRANSFORMATION LAWS

We now proceed to define what is a tensor. As will become evident, we actually cannot precisely define a tensor but rather we describe how a tensor transforms and present some additional general properties. Scalars, vectors, matrices, and higher-order quantities can be represented by a general index notational scheme. Using this approach, all quantities may then be referred to as tensors of different orders. The previously presented transformation properties of a vector can then be used to establish the general transformation properties of these tensors. Restricting the transformations to those only between Cartesian coordinate systems, the general set of transformation relations for various orders can thus be written as

$$\begin{aligned}
a' &= a, && \text{zero order (scalar)}\\
a'_i &= Q_{ip}a_p, && \text{first order (vector)}\\
A'_{ij} &= Q_{ip}Q_{jq}A_{pq}, && \text{second order } (\textit{matrix})\\
A'_{ijk} &= Q_{ip}Q_{jq}Q_{kr}A_{pqr}, && \text{third order}\\
A'_{ijkl} &= Q_{ip}Q_{jq}Q_{kr}Q_{ls}A_{pqrs}, && \text{fourth order}\\
&\;\;\vdots && \quad\vdots\\
A'_{ijk\ldots m} &= Q_{ip}Q_{jq}Q_{kr}\cdots Q_{mt}A_{pqr\ldots t} && \text{general order}
\end{aligned} \tag{2.8.1}$$

Note that according to these definitions, a scalar is a zeroth-order tensor, a vector is a tensor of order 1, and a matrix is a tensor of order 2. Relations (2.8.1) then specify the transformation rules for the components of Cartesian tensors of any order under the rotation Q_{ij}. This transformation theory will prove to be very valuable in determining the displacement, stress, and strain in different coordinate directions. Appendix B contains several different such transformation applications between Cartesian and other curvilinear coordinate systems. It will often be more convenient to write the first- and second-order transformation laws (for vector $\boldsymbol{a}$ and square matrix $\boldsymbol{A}$) in direct notation as

$$\begin{aligned}
\boldsymbol{a}' &= \boldsymbol{Qa} && \text{first order (vector)}\\
\boldsymbol{A}' &= \boldsymbol{QAQ}^T && \text{second order (matrix)}
\end{aligned} \tag{2.8.2}$$

where $\boldsymbol{Q}^T$ is the transpose of $\boldsymbol{Q}$ defined in Section 2.10.

The distinction between the components and the tensor itself should be understood. Recall that a vector $\boldsymbol{v}$ can be expressed as

$$\begin{aligned}\boldsymbol{v} &= v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i \\ &= v_1'\boldsymbol{e}_1' + v_2'\boldsymbol{e}_2' + v_3'\boldsymbol{e}_3' = v_i'\boldsymbol{e}_i'\end{aligned} \tag{2.8.3}$$

In a similar fashion, a second-order tensor $\boldsymbol{A}$ can be written as

$$\begin{aligned}\boldsymbol{A} = &A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 \\ &+A_{21}\boldsymbol{e}_2\boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 \\ &+A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3 \\ &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A_{ij}'\boldsymbol{e}_i'\boldsymbol{e}_j'\end{aligned} \tag{2.8.4}$$

and similar schemes can be used to represent tensors of higher order. The representation used in Eq. (2.8.4) is commonly called *dyadic notation* using dyadic products $\boldsymbol{e}_i\boldsymbol{e}_j$. Another common notation for such products is to use a *tensor product* notation $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Additional information on dyadic notation can be found in Weatherburn (1948) and Chou and Pagano (1967).

Second-order tensors can also be defined as *linear vector transformations*. For example, we define the second-order tensor $\boldsymbol{T}$ to have the property that transforms vectors $\boldsymbol{a}$ into other vectors $\boldsymbol{b}$ through the simple inner product operation

$$\boldsymbol{T}\boldsymbol{a} = \boldsymbol{b} \tag{2.8.5}$$

or in index notation

$$T_{ij}a_j = b_i \tag{2.8.6}$$

Relations (2.8.3) and (2.8.4) indicate that any tensor can be expressed in terms of components in any coordinate system, and it is only the components that change under coordinate transformation. For example, the state of stress at a point in a continuum solid will depend on the problem geometry and applied loadings. As will be shown later, these stress components are those of a second-order tensor and will therefore obey transformation law $(2.8.1)_3$. However, although the components of the stress tensor will change with the choice of coordinates, the stress tensor itself will not.

An important property of a tensor is that if we know its components in one coordinate system, then we can find them in any other coordinate frame using the appropriate transformation law. Since the components of Cartesian tensors are representable by indexed symbols, the operations of equality, addition, subtraction, multiplication, etc. are defined in a manner consistent with the indicial notation procedures previously discussed. The terminology tensor without the adjective Cartesian, usually refers to a more general scheme, where the coordinates are not necessarily rectangular Cartesian and the transformations between coordinates are not always orthogonal. Such general tensor theory will be briefly discussed in Section 2.19.

EXAMPLE 2.8.1 TRANSFORMATION EXAMPLES

The components of a first- and second-order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Determine the components of each tensor in a new coordinate system found through a rotation of 60° ($\pi/6$ radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis (see Fig. 2.3).

Solution: The original and primed coordinate systems are shown in Fig. 2.3. The solution starts by determining the rotation matrix for this case

$$Q_{ij} = \begin{bmatrix} \cos 60° & \cos 30° & \cos 90° \\ \cos 150° & \cos 60° & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation for the vector quantity follows from Eq. $(2.8.1)_2$, and using matrix products from the direct notation $(2.8.2)_1$

$$a'_i = Q_{ij}a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2+\sqrt{3} \\ 1-\sqrt{3}/2 \\ 4 \end{bmatrix}$$

and the second-order tensor (matrix) transforms according to $(2.8.1)_3$, and using matrix products from the direct notation $(2.8.2)_2$

$$A'_{ij} = Q_{ip}Q_{jq}A_{pq} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 1+\sqrt{3}/2 \\ \sqrt{3}/4 & 5/4 & 1/2-\sqrt{3} \\ 1/2+\sqrt{3} & 1-\sqrt{3}/2 & 4 \end{bmatrix}$$

While simple transformation cases can be worked out by hand, for more general cases it is much more convenient to use a computational scheme to evaluate the necessary matrix multiplications required in the transformation laws (2.8.1). MATLAB or Mathematica software are ideally suited to carry out such calculations, and an example MATLAB program to evaluate the transformation of second-order tensors is given by the code C-1 in Appendix C.

Using the general transformation relations, several properties can be established as listed in Table 2.1.

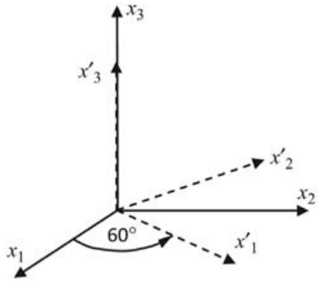

FIGURE 2.3

Coordinate transformation Example 2.8.1.

Table 2.1 Tensor properties

1. The operations addition, scalar multiplication, and subtraction yield Cartesian tensors of the same order
2. If the components of a Cartesian tensor are all zero in one frame, then they are all zero in any frame
3. Two tensors are equal if their components in any one frame are equal
4. If a tensor is symmetric (antisymmetric) in one frame, then it is symmetric (antisymmetric) in any other frame
5. If $A_{ij\ldots k}$ is a tensor of order M and $B_{pq\ldots r}$ is a tensor of order N, then the outer product $A_{ij\ldots k}B_{pq\ldots r}$ is a tensor of order $M + N$
6. If $A_{ij\ldots m\ldots n\ldots k}$ is a tensor of order N, then $A_{ij\ldots m\ldots m\ldots k}$ is a tensor of order $N - 2$
7. If $A_{ij}B_{ij} = c$ = scalar, and if B_{ij} is a second-order tensor, then A_{ij} are the components of a second-order tensor
8. If $A_{ij}b_j = c_i$, and if b_j and c_i are both vectors, then A_{ij} are the components of a second-order tensor

EXAMPLE 2.8.2 TENSOR PROPERTY JUSTIFICATION

Justify tensor property 5 stated in Table 2.1.

Solution: To justify this property, consider the specific outer product case of two second-order tensors $A_{ij}B_{kl}$. Since both tensors must satisfy their own transformation law, we can write

$$A'_{ij}B'_{kl} = Q_{ip}Q_{jq}A_{pq}Q_{kr}Q_{ls}B_{rs}$$

and then by simple rearrangement, we have

$$A'_{ij}B'_{kl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}A_{pq}B_{rs}$$

which for the product term is the standard form for the transformation of fourth-order tensors given by $(2.8.1)_5$.

Next, consider the transformation of the Kronecker delta. Starting with the standard transformation law (2.8.1)$_3$ and then incorporating the substitution property of the Kronecker delta along with the orthogonality relation (2.7.10), we find

$$\delta'_{ij} = Q_{ip}Q_{jq}\delta_{pq} = Q_{ip}Q_{jp} = \delta_{ij} \tag{2.8.7}$$

Therefore, the components of the Kronecker delta are the same in every frame. Tensors with this property are called *isotropic*. We will investigate these types of tensors in more detail in Section 2.15.

Consider next the transformation behavior of the alternating symbol. Again from transformation law (2.8.1)$_4$, we find

$$\varepsilon'_{ijk} = Q_{ip}Q_{jq}Q_{kr}\varepsilon_{pqr}$$

From property (2.5.10) of the alternating symbol, $Q_{ip}Q_{jq}Q_{kr}\varepsilon_{pqr} = \det[Q_{mn}]\varepsilon_{ijk}$, and since from the orthogonality conditions, $\det[Q_{mn}] = \pm 1$, we find $\varepsilon'_{ijk} = \pm\varepsilon_{ijk}$. Choosing only proper orthogonal transformations eliminates the plus/minus ambiguity and then implies that

$$\varepsilon'_{ijk} = \varepsilon_{ijk} \tag{2.8.8}$$

Thus, the alternating symbol also remains the same in all coordinate systems.

2.9 OBJECTIVITY BETWEEN DIFFERENT REFERENCE FRAMES

As previously discussed in Section 1.2, the fundamental relations in continuum mechanics must be formulated in terms of quantities that are independent of the coordinate frame used to describe the problem. So, if two individuals using different coordinate frames observe a common physical event, it should always be possible to state a physical law governing the event so that if the law is true for one observer, it will also be true for the other (once adjusted for the difference in coordinate frame). In this section, we will explore the basic concept of changes in reference frames and how this affects a few specific variables like distance, velocity, and acceleration. Later, we shall explore how additional continuum mechanics tensor variables transform under such reference frame changes, and if these variables transform in a particular way they will be labeled as *objective*. As we develop constitutive equations for material models, we will normally require these relations to contain only objective variables, and this will then place restrictions on general constitutive forms.

Reflecting back on the simple coordinate frame changes in Fig. 2.2, we now consider the more general transformation between two reference frames as shown in Fig. 2.4. Note that the two frames shown now include both time-dependent translational and rotational differences relative to each other. It should be kept in mind that frames of reference are not the same as coordinate systems, as a given observer can choose any coordinate system to observe and analyze events. Using

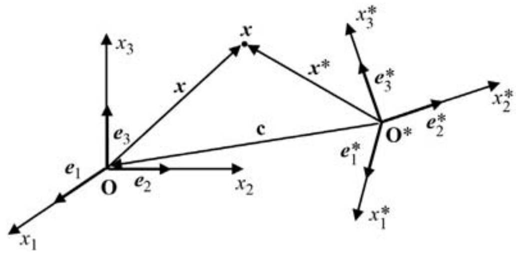

FIGURE 2.4

Objectivity between two reference frames.

Cartesian coordinates, we designate each frame with observers **O** and **O***, and in a Euclidean three-dimensional space and time, each observer can measure relative positions of points in space and intervals of time. With this is mind, we consider an event in **O** as the position/time pair $\{\boldsymbol{x}, t\}$ and in **O*** the event is $\{\boldsymbol{x}^*, t^*\}$. We can assume a simple time shift $t^* - a$, where a is a constant that can be set to zero without loss in generality. Note that the time clocks run at the same rate in each reference frame.

It can be shown that the most general change of frame is given by

$$\boldsymbol{x}^* = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}, \quad x_i^* = c_i(t) + Q_{ij}(t)x_j \tag{2.9.1}$$

where $\boldsymbol{c}(t)$ is the relative displacement between the two reference frames, and $\boldsymbol{Q}(t)$ is a second-order orthogonal tensor accounting for the relative rotational differences. Section 2.7 defined and listed various properties for the rotation tensor $\boldsymbol{Q}(t)$.

Under the frame change given by (2.9.1), let us first consider the displacement between the neighboring points $\boldsymbol{x}$ and $\boldsymbol{y}$. We start with the relations

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x} \\ \boldsymbol{y}^* &= \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{y} \end{aligned}$$

Subtracting these two defines the displacements $\boldsymbol{u}^* = \boldsymbol{y}^* - \boldsymbol{x}^*$ and $\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{x}$ and establishes the transformation relation

$$\boldsymbol{u}^* = \boldsymbol{Q}(t)\boldsymbol{u} \tag{2.9.2}$$

which is exactly the same as the general tensor transformation law for vectors given in Sections 2.7 and 2.8. This is the required transformation behavior for first-order tensors in order to be called objective. Note also that it follows from (2.9.2) that $|\boldsymbol{u}^*| = |\boldsymbol{u}|$ and thus displacement distance or magnitude is invariant as expected for this and all other scalars.

It was pointed out in Section 2.8 that second-order tensors can be defined as linear vector transformations (see Eq. (2.8.5)). Using this definition, we can determine the objective definition that is applied to second-order tensors. Starting with linear vector transformation relation (2.8.5), the second-order tensor $\boldsymbol{T}$ transforms vector $\boldsymbol{a}$ into another vector $\boldsymbol{b}$, through the relation $\boldsymbol{Ta} = \boldsymbol{b}$. Now we wish to have this relation be objective so that $\boldsymbol{T}^*\boldsymbol{a}^* = \boldsymbol{b}^*$. Also each vector should satisfy the objectivity relation (2.9.2), and so $\boldsymbol{a}^* = \boldsymbol{Q}\boldsymbol{a}$, $\boldsymbol{b}^* = \boldsymbol{Q}\boldsymbol{b}$. Combing these results together gives

$$\begin{gathered}\boldsymbol{T}^*\boldsymbol{a}^* = \boldsymbol{b}^* \Rightarrow \boldsymbol{T}^*\boldsymbol{Q}\boldsymbol{a} = \boldsymbol{Q}\boldsymbol{b} \Rightarrow \boldsymbol{T}^*\boldsymbol{Q}\boldsymbol{a} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{a} \\ \therefore \boldsymbol{T}^*\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{T}\end{gathered}$$

post multiplying by $\boldsymbol{Q}^T$ then gives the expected transformation relation for objective second-order tensors

$$\boldsymbol{T}^* = \boldsymbol{Q}(t)\boldsymbol{T}\boldsymbol{Q}^T(t) \tag{2.9.3}$$

Therefore, first- and second-order tensors are called objective or frame indifferent if they satisfy relations (2.9.2) and (2.9.3).

Next, consider the velocity vector transformation behavior. Differentiating (2.9.1) with respect to time gives

$$\begin{aligned}&\boldsymbol{v}^* = \frac{d\boldsymbol{x}^*}{dt} = \dot{\boldsymbol{c}}(t) + \dot{\boldsymbol{Q}}(t)\boldsymbol{x} + \boldsymbol{Q}(t)\dot{\boldsymbol{x}} \Rightarrow \\ &\boldsymbol{v}^* = \dot{\boldsymbol{c}}(t) + \dot{\boldsymbol{Q}}(t)\boldsymbol{x} + \boldsymbol{Q}(t)\boldsymbol{v} = \dot{\boldsymbol{c}}(t) + \boldsymbol{Q}(t)\boldsymbol{v} + \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}^T(t)[\boldsymbol{x}^* - \boldsymbol{c}]\end{aligned} \tag{2.9.4}$$

Note that the quantity $\dot{\boldsymbol{Q}}(t)$ would represent a relative angular velocity between the two reference frames $\mathbf{O}^*$ and $\mathbf{O}$. We thus conclude that the velocity is not in general objective unless $\dot{\boldsymbol{c}}(t) + \dot{\boldsymbol{Q}}(t)\boldsymbol{x}$ were to vanish which could be accomplished by the special case of time-independent changes of observer.

Finally, let us explore the next time derivative to calculate the acceleration

$$\begin{aligned}&\boldsymbol{a}^* = \frac{\mathrm{d}\boldsymbol{v}^*}{\mathrm{d}t} = \ddot{\boldsymbol{c}}(t) + \ddot{\boldsymbol{Q}}(t)\boldsymbol{x} + \dot{\boldsymbol{Q}}(t)\dot{\boldsymbol{x}} + \dot{\boldsymbol{Q}}(t)\dot{\boldsymbol{x}} + \boldsymbol{Q}(t)\ddot{\boldsymbol{x}} \Rightarrow \\ &\boldsymbol{a}^* = \ddot{\boldsymbol{c}}(t) + \ddot{\boldsymbol{Q}}(t)\boldsymbol{x} + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{v} + \boldsymbol{Q}(t)\boldsymbol{a}\end{aligned} \tag{2.9.5}$$

and like the velocity, the acceleration is also not in general objective.

In summary, we have seen that under the general change in reference frame given by relation (2.9.1), objective scalars remain the same; objective vectors must satisfy (2.9.2); and objective second-order tensors must obey relation (2.9.3). We have set the stage for more detailed use of the principle of objectivity or frame indifference for many other variables to come in our study. As we have seen already, not all variables in continuum mechanics will be objective.

2.10 VECTOR AND MATRIX ALGEBRA

Continuum mechanics will require the use of many standard algebraic operations among vector, matrix, and tensor variables. These operations include dot and cross products of vectors and numerous matrix/tensor products. All of these operations can be expressed efficiently using compact tensor index or direct notation. We now present many of these operations.

First, consider some particular vector products. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with Cartesian components a_i and b_i, the *scalar* or *dot product* is defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i \tag{2.10.1}$$

Since all indices in this expression are repeated, the quantity must be a scalar, that is, a tensor of order zero. The magnitude of a vector can then be expressed as

$$|\boldsymbol{a}| = (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2} = (a_ia_i)^{1/2} \tag{2.10.2}$$

The *vector* or *cross product* between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk}a_jb_k\boldsymbol{e}_i \tag{2.10.3}$$

where $\boldsymbol{e}_i$ are the unit basis vectors for the coordinate system. Note that the cross product gives a vector resultant whose components are $\varepsilon_{ijk}a_jb_k$. Another common vector product is the *scalar triple product* defined by

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk}a_ib_jc_k \tag{2.10.4}$$

For this triple product, the cross operation must be done first, and the end result is then a scalar which is easily verified by observing the index notation form that has no free indices.

Next, consider some common matrix products. Using the usual direct notation for matrices and vectors, common products between a matrix $\boldsymbol{A} = [\boldsymbol{A}]$ with a vector $\boldsymbol{a}$ can be written as

$$\begin{aligned} \boldsymbol{Aa} &= [\boldsymbol{A}]\{\boldsymbol{a}\} = A_{ij}a_j = a_jA_{ij} \\ \boldsymbol{a}^TA &= \{\boldsymbol{a}\}^T[\boldsymbol{A}] = a_iA_{ij} = A_{ij}a_i \end{aligned} \tag{2.10.5}$$

where $\boldsymbol{a}^T$ denotes the *transpose* and for a vector quantity this simply changes the (3 × 1) column matrix into a (1 × 3) row matrix. Note that each of these products results in a vector resultant. These types of expressions generally involve various inner products within the index notational scheme, and as previously noted once the summation index is properly specified, the order of listing the product terms will not change the result. Considering next the various products among two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, several different combinations are commonly encountered:

$$\begin{gathered} \boldsymbol{AB} = A_{ij}B_{jk} \\ \boldsymbol{AB}^T = A_{ij}B_{kj} \\ \boldsymbol{A}^T\boldsymbol{B} = A_{ji}B_{jk} \\ (\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T = (A_{ij}B_{jk})^T = A_{kj}B_{ji} \\ tr(\boldsymbol{AB}) = A_{ij}B_{ji} \\ tr(\boldsymbol{AB}^T) = tr(\boldsymbol{A}^T\boldsymbol{B}) = A_{ij}B_{ij} \\ \boldsymbol{A}^2 = \boldsymbol{AA} = A_{ij}A_{jk} \\ \boldsymbol{A}^3 = \boldsymbol{AAA} = A_{ij}A_{jk}A_{kl} \end{gathered} \tag{2.10.6}$$

where $\boldsymbol{A}^T$ indicates the *transpose* and *tr* $\boldsymbol{A}$ is the *trace* of the matrix defined by

$$\begin{aligned} &A_{ij}^T = A_{ji}, \quad \boldsymbol{A}^T = A_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &tr\,\boldsymbol{A} = A_{ii} = A_{11} + A_{22} + A_{33} \end{aligned} \tag{2.10.7}$$

Similar to vector products, once the summation index is properly specified, the results in (2.10.6) will not depend on the order of listing the product terms. Note that this does not imply that $\boldsymbol{AB} = \boldsymbol{BA}$, which is certainly not true.

The *inverse* of a second-order tensor (square matrix) $\boldsymbol{A}$ is written as $\boldsymbol{A}^{-1}$ with the following definition properties written in both index and direct notation:

$$\begin{aligned} &A_{ij}^{-1}A_{jk} = A_{ij}A_{jk}^{-1} = \delta_{ik} \\ &\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \end{aligned} \tag{2.10.8}$$

The matrix $\boldsymbol{A}$ is said to be *nonsingular* if it has a nonzero determinant, and if this is the case the matrix inverse is given by the following index notation relation:

$$A_{ij}^{-1} = \frac{1}{2\det[\boldsymbol{A}]}\varepsilon_{ikl}\varepsilon_{jqr}A_{qk}A_{rl} \tag{2.10.9}$$

Note a couple of other useful properties:

$$\begin{aligned} (\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \\ (\boldsymbol{A}^T)^{-1} &= (\boldsymbol{A}^{-1})^T \end{aligned} \tag{2.10.10}$$

2.11 PRINCIPAL VALUES, DIRECTIONS, AND INVARIANTS OF SYMMETRIC SECOND-ORDER TENSORS

It should be apparent from the tensor transformation concepts previously discussed that there might exist particular coordinate systems where the components of a tensor will take on maximum, minimum, or other special values. This concept can be easily visualized by considering the components of a vector shown in Fig. 2.1. If we choose a particular coordinate system that has been rotated so that the x_3-axis lies along the direction of the vector itself, then the vector will have components $\boldsymbol{u} = [0, 0, |\boldsymbol{u}|]$. For this case, two of the components have been reduced to zero while the remaining component becomes the largest possible (the total magnitude).

This situation is most useful for symmetric second-order tensors that will eventually represent the stress and strain at a point in a continuum material. The direction determined by the unit vector $\boldsymbol{n}$ is said to be a *principal direction* or *eigenvector* of the symmetric second-order tensor A_{ij} if there exists a parameter λ such that

$$A_{ij}n_j = \lambda n_i \tag{2.11.1}$$

The parameter λ is called the *principal value* or *eigenvalue* of the tensor. Relation (2.11.1) can be rewritten as

$$(A_{ij} - \lambda\delta_{ij})n_j = 0$$

and this expression is recognized as a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, and n_3. The system possesses a nontrivial solution if and only if the determinant of its coefficient matrix vanishes, that is,

$$\det[A_{ij} - \lambda\delta_{ij}] = 0$$

Expanding the determinant produces a cubic equation in terms of λ:

$$\det[A_{ij} - \lambda\delta_{ij}] = -\lambda^3 + I_A\lambda^2 - II_A\lambda + III_A = 0 \tag{2.11.2}$$

where

$$\begin{aligned} I_A &= A_{ii} = A_{11} + A_{22} + A_{33} \\ II_A &= \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \\ III_A &= \det[A_{ij}] \end{aligned} \tag{2.11.3}$$

The scalars I_A, II_A, and III_A are called the *fundamental invariants* of the tensor A_{ij} and the relation (2.11.2) is known as the *characteristic equation*. As expected by their name, the three invariants do not change value under coordinate transformation.

Another alternate set of invariants of a tensor that is sometimes used is given by

$$I_1 = tr\,\boldsymbol{A}, \quad I_2 = tr\,\boldsymbol{A}^2, \quad I_3 = tr\,\boldsymbol{A}^3 \tag{2.11.4}$$

These two sets of invariants are not independent and are related by the relations

$$I_1 = I_A, \quad I_2 = I_A^2 - 2II_A, \quad I_3 = I_A^3 - 3II_AI_A + 3III_A \tag{2.11.5}$$

The roots of the characteristic equation determine the allowable values for λ, and each of these may be back-substituted into relation (2.11.1) to solve for the associated principal direction $\boldsymbol{n}$. Normally, the principal directions are normalized so that $n_1^2 + n_2^2 + n_3^2 = 1$, thus adding another needed equation to the system to be solved (see Example 2.11.1).

Under the condition that the components A_{ij} are real, it can be shown that all three roots λ_1, λ_2, and λ_3 of the cubic equation (2.11.2) must be real. Furthermore, if these roots are distinct, the principal directions associated with each principal value will be orthogonal. Thus, we can conclude that every symmetric second-order tensor has at least three mutually perpendicular principal directions and at most three distinct principal values which are the roots of the characteristic equation. Denoting the principal directions $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$, and $\boldsymbol{n}^{(3)}$ corresponding to the principal values λ_1, λ_2, and λ_3, three possibilities arise:

1. All three principal values distinct; thus, the three corresponding principal directions are unique (except for sense).

2. Two principal values equal ($\lambda_1 \neq \lambda_2 = \lambda_3$); the principal direction $\boldsymbol{n}^{(1)}$ is unique (except for sense) and every direction perpendicular to $\boldsymbol{n}^{(1)}$ is a principal direction associated with λ_2 and λ_3.
3. All three principal values equal; every direction is principal and the tensor is isotropic, as per discussion in the previous section.

Therefore, according to what we have presented, it is always possible to identify a right-handed Cartesian coordinate system with axes that lie along the principal directions of any given symmetric second-order tensor. Such axes are called the *principal axes* of the tensor. For this case, the basis vectors are actually the unit principal directions [$\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$, $\boldsymbol{n}^{(3)}$], and it can be shown from (2.8.3) that with respect to principal axes the tensor components can be calculated using the relation

$$A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j = \boldsymbol{n}^{(i)} \cdot \boldsymbol{A}\boldsymbol{n}^{(j)} \tag{2.11.6}$$

which gives

$$\begin{aligned}
A_{11} &= \boldsymbol{n}^{(1)} \cdot \boldsymbol{A}\boldsymbol{n}^{(1)} = \boldsymbol{n}^{(1)} \cdot (\lambda_1 \boldsymbol{n}^{(1)}) = \lambda_1 \\
A_{22} &= \boldsymbol{n}^{(2)} \cdot \boldsymbol{A}\boldsymbol{n}^{(2)} = \boldsymbol{n}^{(2)} \cdot (\lambda_2 \boldsymbol{n}^{(2)}) = \lambda_2 \\
A_{33} &= \boldsymbol{n}^{(3)} \cdot \boldsymbol{A}\boldsymbol{n}^{(3)} = \boldsymbol{n}^{(3)} \cdot (\lambda_3 \boldsymbol{n}^{(3)}) = \lambda_3 \\
A_{12} &= \boldsymbol{n}^{(1)} \cdot \boldsymbol{A}\boldsymbol{n}^{(2)} = \boldsymbol{n}^{(1)} \cdot (\lambda_2 \boldsymbol{n}^{(2)}) = 0 = A_{21} \\
A_{23} &= \boldsymbol{n}^{(2)} \cdot \boldsymbol{A}\boldsymbol{n}^{(3)} = \boldsymbol{n}^{(2)} \cdot (\lambda_3 \boldsymbol{n}^{(3)}) = 0 = A_{32} \\
A_{31} &= \boldsymbol{n}^{(3)} \cdot \boldsymbol{A}\boldsymbol{n}^{(1)} = \boldsymbol{n}^{(3)} \cdot (\lambda_1 \boldsymbol{n}^{(1)}) = 0 = A_{13}
\end{aligned}$$

Collecting these results indicates that with respect to the principal axes, the tensor reduces to the diagonal form:

$$A_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.11.7}$$

Note that the fundamental invariants defined by relations (2.11.3) can be expressed in terms of the principal values as

$$\begin{aligned}
I_A &= \lambda_1 + \lambda_2 + \lambda_3 \\
II_A &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\
III_A &= \lambda_1\lambda_2\lambda_3
\end{aligned} \tag{2.11.8}$$

The eigenvalues of a symmetric second-order tensor have important *extremal properties*. If we consider an arbitrary unit vector $\boldsymbol{e}_1 = \alpha\boldsymbol{n}^{(1)} + \beta\boldsymbol{n}^{(2)} + \gamma\boldsymbol{n}^{(3)}$, then $\alpha^2 + \beta^2 + \gamma^2 = 1$. Using relation (2.11.6) again

$$A_{11} = \boldsymbol{e}_1 \cdot \boldsymbol{A}\boldsymbol{e}_1 = \lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\gamma^2$$

Without loss in generality, let $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and then since $\alpha^2 + \beta^2 + \gamma^2 = 1$, we can write

$$\lambda_1 = \lambda_1(\alpha^2 + \beta^2 + \gamma^2) \geq \lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\gamma^2$$

thus proving that $\lambda_1 \geq A_{11}$. Likewise

$$\lambda_3 = \lambda_3(\alpha^2 + \beta^2 + \gamma^2) \leq \lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\gamma^2$$

and so $\lambda_3 \leq A_{11}$.

Thus, we can conclude that the maximum (minimum) principal value of $\boldsymbol{A}$ is the maximum (minimum) value of any diagonal element of the second-order tensor in any coordinate frame. Or stated in another way: if we arbitrarily rank the principal values such that $\lambda_1 > \lambda_2 > \lambda_3$, then λ_1 will be the largest of all possible diagonal elements, whereas λ_3 will be the smallest diagonal element possible. Several of these concepts will be applied in our subsequent studies as we seek the largest stress or strain components in a continuum material.

EXAMPLE 2.11.1 PRINCIPAL VALUE PROBLEM

Determine the invariants, and principal values and directions of the following symmetric second-order tensor:

$$A_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The invariants follow from relations (2.11.3)

$$I_A = a_{ii} = -1 - 1 + 1 = -1$$
$$II_A = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 1 - 1 = -2$$
$$III_A = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

The characteristic equation then becomes

$$\det[A_{ij} - \lambda\delta_{ij}] = -\lambda^3 - \lambda^2 + 2\lambda = 0$$
$$\Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda - 1)(\lambda + 2) = 0$$
$$\therefore \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = -2$$

Thus, for this case, all principal values are distinct.

For the $\lambda_1 = 1$ root, Eq. (2.11.1) gives the system

$$-2n_1^{(1)} + n_2^{(1)} = 0$$
$$n_1^{(1)} - 2n_2^{(1)} = 0$$
$$0 = 0$$

which gives a solution $n_1^{(1)} = n_2^{(1)} = 0$ and leaves $n_3^{(1)}$ undefined. Adding the normalization equation $n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1$ implies that $n_3^{(1)} = 1$, and thus $\boldsymbol{n}^{(1)} = \pm\boldsymbol{e}_3$. In a similar fashion, the other two principal directions are found to be $\boldsymbol{n}^{(2)} = \pm\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2), \boldsymbol{n}^{(3)} = \pm\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \boldsymbol{e}_2)$. It is easily verified that these directions are mutually orthogonal. Fig. 2.5 illustrates their directions with respect to the given coordinate system and this establishes the right-handed principal coordinate axes (x_1', x_2', x_3'). For this case, the transformation matrix Q_{ij} defined by (2.7.1) becomes

$$Q_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

Notice the eigenvectors actually form the rows of the Q-matrix.

Using this in the transformation law $(2.8.1)_3$, the components of the given second-order tensor become

$$A_{ij}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This result then validates the general theory given by relation (2.11.7) indicating that the tensor should take on diagonal form with the principal values as the elements.

Only simple second-order tensors will lead to a characteristic equation that is factorable, thus allowing an easy solution via hand calculation. Most of the other cases will normally develop a general cubic equation and thus a more complicated system to solve for the principal directions. Again particular routines within the MATLAB package offer convenient tools to solve these more general problems. Code C2 in Appendix C provides a simple code to determine the principal values and directions for symmetric second-order tensors.

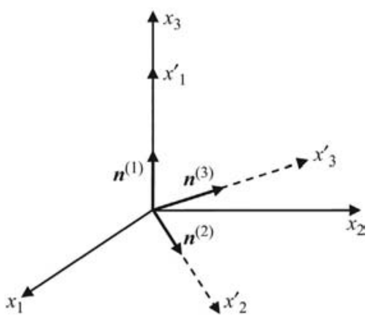

FIGURE 2.5

Principal axes for Example 2.11.1.

2.12 SPHERICAL AND DEVIATORIC SECOND-ORDER TENSORS

In particular applications, it is often convenient to decompose second-order tensors into two parts called *spherical* and *deviatoric tensors*. Considering a general second-order tensor A_{ij}, the spherical part is defined by

$$\tilde{A}_{ij} = \frac{1}{3} A_{kk} \delta_{ij} \tag{2.12.1}$$

while the deviatoric part is specified as

$$\hat{A}_{ij} = A_{ij} - \frac{1}{3} A_{kk} \delta_{ij} \tag{2.12.2}$$

Note that the sum of the two parts then gives the original tensor A_{ij}

$$A_{ij} = \tilde{A}_{ij} + \hat{A}_{ij} \tag{2.12.3}$$

The spherical part is an isotropic tensor, being the same in all coordinate systems (as per discussion in Section 2.8). It can be shown that the principal directions of the deviatoric tensor are the same as those of the original tensor itself and that $tr\hat{\mathbf{A}} = \hat{A}_{kk} = 0$. These definitions will prove to be useful in later sections of the text dealing with strain, stress, and constitutive relations.

EXAMPLE 2.12.1 SPHERICAL AND DEVIATORIC DECOMPOSITION

Determine the spherical and deviatoric parts of the given second-order tensor:

$$A_{ij} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 0 & 2 \end{bmatrix}$$

Solution: From definitions (2.12.1) and (2.12.2)

$$\tilde{A}_{ij} = \frac{1}{3} A_{kk} \delta_{ij} = \frac{1}{3}(6)\delta_{ij} = 2\delta_{ij} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \frac{1}{3} A_{kk} \delta_{ij} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

A simple check will verify that $A_{ij} = \tilde{A}_{ij} + \hat{A}_{ij}$ and $tr\,\hat{A} = 0$.

2.13 CAYLEY–HAMILTON THEOREM AND MATRIX POLYNOMIALS

The Cayley–Hamilton theorem states that a square matrix or second-order tensor satisfies its own characteristic equation. So, for application to a symmetric second-order tensor $\boldsymbol{A}$, we can write

$$\boldsymbol{A}^3 - I_A \boldsymbol{A}^2 + II_A \boldsymbol{A} - III_A \boldsymbol{I} = 0 \tag{2.13.1}$$

where scalars I_A, II_A, and III_A are the fundamental invariants of tensor $\boldsymbol{A}$ previously defined in (2.11.3).

Clearly result (2.13.1) then allows one to express $\boldsymbol{A}^3$ in terms of $\boldsymbol{A}^2$ and $\boldsymbol{A}$:

$$\boldsymbol{A}^3 = I_A \boldsymbol{A}^2 - II_A \boldsymbol{A} + III_A \boldsymbol{I} \tag{2.13.2}$$

Multiplying (2.13.2) by $\boldsymbol{A}$ gives an expression for $\boldsymbol{A}^4$, and back-substituting this with (2.13.2) to eliminate $\boldsymbol{A}^3$ gives a relationship for $\boldsymbol{A}^4$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^2$:

$$\boldsymbol{A}^4 = (I_A^2 - II_A)\boldsymbol{A}^2 + (III_A - I_A II_A)\boldsymbol{A} + I_A III_A \boldsymbol{I} \tag{2.13.3}$$

We can then continue this process to get

$$\begin{aligned}\boldsymbol{A}^5 = {}&(I_A^3 - 2I_A II_A + III_A)\boldsymbol{A}^2 + (I_A III_A - I_A^2 II_A + II_A^2)\boldsymbol{A} \\ &+(I_A^2 III_A - II_A III_A)\boldsymbol{I}\end{aligned} \tag{2.13.4}$$

It then follows that by using this procedure, one can express any power of $\boldsymbol{A}$ greater than or equal to 3 in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^2$. Consequently, any *polynomial representation* of $\boldsymbol{A}$

$$f(\boldsymbol{A}) = c_0 \boldsymbol{I} + c_1 \boldsymbol{A} + c_2 \boldsymbol{A}^2 + c_3 \boldsymbol{A}^3 + \cdots + c_n \boldsymbol{A}^n \tag{2.13.5}$$

where c_i are *constants*, can be expressed as

$$f(\boldsymbol{A}) = k_0 \boldsymbol{I} + k_1 \boldsymbol{A} + k_2 \boldsymbol{A}^2 \tag{2.13.6}$$

where k_i are now polynomial functions of the invariants I_A, II_A, and III_A. Result (2.13.6) is useful in developing nonlinear constitutive equations in continuum mechanics, and we will put this into play in later chapters. On a related topic, the Cayley–Hamilton theorem is also useful to express $\boldsymbol{A}^2$ in terms of $\boldsymbol{A}$ its inverse $\boldsymbol{A}^{-1}$ through the relation

$$\boldsymbol{A}^2 = -II_A \boldsymbol{I} + I_A \boldsymbol{A} + III_A \boldsymbol{A}^{-1} \tag{2.13.7}$$

Using this result, representation relation (2.13.6) can be expressed in the alternative form

$$f(\boldsymbol{A}) = K_0 \boldsymbol{I} + K_1 \boldsymbol{A} + K_2 \boldsymbol{A}^{-1} \tag{2.13.8}$$

2.14 REPRESENTATION THEOREMS

Many applications in continuum mechanics associated with constitutive equation development require representation of *scalar- and tensor-valued functions*. These particular functions involve tensor arguments. In general, these relations should be *form invariant*, that is, if the coordinate system is changed, then with the appropriate tensorial changes of each tensor in the equation, the basic form of the relation should remain invariant. Several useful representation theorems are hereby listed.

Scalar-Valued Theorem

A scalar-valued function $f(\boldsymbol{A})$ of a symmetric second-order tensor ($\boldsymbol{A} = \boldsymbol{A}^T$), is form invariant, that is, $f(\boldsymbol{A}) = f(\boldsymbol{QAQ}^T)$, if

$$f(\boldsymbol{A}) = f(I_A, II_A, III_A) \tag{2.14.1}$$

Note that if f is also to be a *linear function*, then form (2.14.1) must reduce to

$$f(\boldsymbol{A}) = c_0 I_A = c_0(\text{tr}\,\boldsymbol{A}) = c_0 A_{ii} \tag{2.14.2}$$

with c_0 being an arbitrary constant.

Tensor-Valued Theorem

A symmetric tensor-valued function $\boldsymbol{f}(\boldsymbol{A})$ of a symmetric second-order tensor ($\boldsymbol{A} = \boldsymbol{A}^T$) is form invariant, that is, $\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{A})\boldsymbol{Q}^T = \boldsymbol{f}(\boldsymbol{QAQ}^T)$, if

$$\boldsymbol{f}(\boldsymbol{A}) = c_0\boldsymbol{I} + c_1\boldsymbol{A} + c_2\boldsymbol{A}^2 \tag{2.14.3}$$

where $c_i = c_i(I_A, II_A, III_A)$. Note that if $\boldsymbol{f}$ is *linear* in $\boldsymbol{A}$, then form (2.14.3) must reduce to

$$\boldsymbol{f}(\boldsymbol{A}) = c_0 I_A \boldsymbol{I} + c_1\boldsymbol{A} \tag{2.14.4}$$

Relations (2.14.3) and (2.14.4) are useful in elasticity theories where the stress tensor is a function of the current value of the strain tensor.

Tensor-Valued Theorem with Two Arguments

A symmetric tensor-valued function $\boldsymbol{f}(\boldsymbol{A},\boldsymbol{B})$ of two symmetric second-order tensors ($\boldsymbol{A} = \boldsymbol{A}^T$, $\boldsymbol{B} = \boldsymbol{B}^T$) is form invariant, that is, $\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{Q}^T = \boldsymbol{f}(\boldsymbol{QAQ}^T, \boldsymbol{QBQ}^T)$, if

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{A},\boldsymbol{B}) = {} & c_0\boldsymbol{I} + c_1\boldsymbol{A} + c_2\boldsymbol{A}^2 + c_3\boldsymbol{B} + c_4\boldsymbol{B}^2 \\ & + c_5(\boldsymbol{AB} + \boldsymbol{BA}) + c_6(\boldsymbol{A}^2\boldsymbol{B} + \boldsymbol{BA}^2) \\ & + c_7(\boldsymbol{AB}^2 + \boldsymbol{B}^2\boldsymbol{A}) + c_8(\boldsymbol{A}^2\boldsymbol{B}^2 + \boldsymbol{B}^2\boldsymbol{A}^2) \end{aligned} \tag{2.14.5}$$

where the coefficients c_i are functions of the invariants: $tr\boldsymbol{A}$, $tr\boldsymbol{A}^2$, $tr\boldsymbol{A}^3$, $tr\boldsymbol{B}$, $tr\boldsymbol{B}^2$, $tr\boldsymbol{B}^3$, $tr(\boldsymbol{AB})$, $tr(\boldsymbol{A}^2\boldsymbol{B})$, $tr(\boldsymbol{AB}^2)$, $tr(\boldsymbol{A}^2\boldsymbol{B}^2)$.

2.15 ISOTROPIC TENSORS

As pointed out in Section 2.8, a tensor is called *isotropic if its components are the same in all coordinate frames.* In that section, we showed that the Kronecker delta and the alternating symbol had such properties. We now wish to explore a more complete collection of general forms of isotropic tensors of various orders. For a general tensor $\boldsymbol{A}$ to be isotropic, it must satisfy the transformation relation

$$A_{ijk\ldots m} = Q_{ip}Q_{jq}Q_{kr}\cdots Q_{mt}A_{pqr\ldots t} \tag{2.15.1}$$

Using this result, we can construct general three-dimensional forms for orders up to 4 as listed in Table 2.2. Notice that the fourth-order case involves products of the Kronecker delta which is an expected result.

Table 2.2 General forms of isotropic tensors

Zeroth order	All tensors (scalars)
First order	Zero or null vector
Second order	$C\delta_{ij}$, for any constant C
Third order	$C\varepsilon_{ijk}$, for any constant C
Fourth order	$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$, for any constants α, β, and γ

2.16 POLAR DECOMPOSITION THEOREM

A useful multiplicative decomposition in tensor theory is known as the *Polar Decomposition Theorem*, which states that any nonsingular second-order tensor $\boldsymbol{A}$ may be uniquely written as the product

$$\boldsymbol{A} = \boldsymbol{RU} = \boldsymbol{VR}, \quad A_{ij} = R_{ik}U_{kj} = V_{ik}R_{kj} \tag{2.16.1}$$

where $\boldsymbol{R}$ is a proper orthogonal tensor, and $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric positive definite tensors (see definition in Exercise 2.20). It can be shown that

$$\begin{aligned} \boldsymbol{U}^2 &= \boldsymbol{A}^T\boldsymbol{A}, \quad U_{ik}U_{kj} = A_{ki}A_{kj} \\ \boldsymbol{V}^2 &= \boldsymbol{AA}^T, \quad V_{ik}V_{kj} = A_{ik}A_{jk} \end{aligned} \tag{2.16.2}$$

This result will be useful to decompose particular deformation tensors.

2.17 CALCULUS OF CARTESIAN FIELD TENSORS

Most variables used in continuum mechanics are field variables that depend on the spatial coordinates used to formulate the problem under study. In addition, for time-dependent problems, these variables could also have temporal variation; however, at the moment, we will hold off on including time dependency. Thus, with respect to a Cartesian coordinate system, our scalar, vector, matrix, and other general tensor variables will be functions of the spatial coordinates (x_1, x_2, x_3). Since many of our equations will involve differential and integral operations, it is necessary to have an understanding of the calculus of Cartesian tensor fields. The field concept for tensor components can be expressed as

$$\begin{aligned} &a = a(x_1, x_2, x_3) = a(x_m) = a(\boldsymbol{x}) \\ &a_i = a_i(x_1, x_2, x_3) = a_i(x_m) = \boldsymbol{a}(\boldsymbol{x}) \\ &A_{ij} = A_{ij}(x_1, x_2, x_3) = A_{ij}(x_m) = \boldsymbol{A}(\boldsymbol{x}) \\ &\vdots \end{aligned} \tag{2.17.1}$$

The transformation law for the components of a Cartesian tensor field is then

$$A'_{ij\cdots k}(x'_m) = Q_{ip}Q_{jq}\cdots Q_{ks}A_{ij\cdots k}(x_m) \tag{2.17.2}$$

where $x'_m = Q_{mn}x_n$.

It is convenient to introduce the *comma notation* for partial differentiation

$$a_{,i} = \frac{\partial}{\partial x_i}a, \quad a_{i,j} = \frac{\partial}{\partial x_j}a_i, \quad A_{ij,k} = \frac{\partial}{\partial x_k}A_{ij}, \ldots \tag{2.17.3}$$

If the differentiation index is distinct as shown in the examples in relation (2.17.3), the order of the tensor will be increased by 1. For example, the derivative operation on a vector $a_{i,j}$ produces a second-order tensor or matrix given by

$$a_{i,j} = \begin{bmatrix} \dfrac{\partial a_1}{\partial x_1} & \dfrac{\partial a_1}{\partial x_2} & \dfrac{\partial a_1}{\partial x_3} \\ \dfrac{\partial a_2}{\partial x_1} & \dfrac{\partial a_2}{\partial x_2} & \dfrac{\partial a_2}{\partial x_3} \\ \dfrac{\partial a_3}{\partial x_1} & \dfrac{\partial a_3}{\partial x_2} & \dfrac{\partial a_3}{\partial x_3} \end{bmatrix} \tag{2.17.4}$$

We can formally prove this general property for the case of differentiating a second-order tensor. Starting with the standard transformation relation

$$A'_{ij} = Q_{ip}Q_{jq}A_{pq}$$

differentiate with respect to x'_k:

$$A'_{ij,k} = \frac{\partial}{\partial x'_k}(Q_{ip}Q_{jq}A_{pq}) = Q_{ip}Q_{jq}\frac{\partial A_{pq}}{\partial x'_k} = Q_{ip}Q_{jq}\frac{\partial A_{pq}}{\partial x_r}\frac{\partial x_r}{\partial x'_k}$$

Note we used the fact that the transformation matrices are constants and employed the chain rule. Now since $x_r = Q_{sr}x'_s$, $\dfrac{\partial x_r}{\partial x'_k} = Q_{sr}\dfrac{\partial x'_s}{\partial x'_k} = Q_{sr}\delta_{sk} = Q_{kr}$. Putting all of this together gives

$$A'_{ij,k} = Q_{ip}Q_{jq}Q_{kr}A_{pq,r}$$

and thus $A_{ij,k}$ must be a third-order tensor since it satisfies the standard transformation law.

Consider next the *directional derivative* of a scalar field function f with respect to direction s:

$$\frac{df}{ds} = \frac{\partial f}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial f}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial f}{\partial x_3}\frac{dx_3}{ds} \tag{2.17.5}$$

Noting that the unit vector in direction s can be written as

$$\boldsymbol{n} = \frac{dx_1}{ds}\boldsymbol{e}_1 + \frac{dx_2}{ds}\boldsymbol{e}_2 + \frac{dx_3}{ds}\boldsymbol{e}_3 \tag{2.17.6}$$

and so the directional derivative can be expressed as the following scalar product:

$$\frac{df}{ds} = \boldsymbol{n}\cdot\nabla f \tag{2.17.7}$$

where ∇f is called the *gradient* of the scalar function f and is defined by

$$\nabla f = \text{grad}\, f = \boldsymbol{e}_1\frac{\partial f}{\partial x_1} + \boldsymbol{e}_2\frac{\partial f}{\partial x_2} + \boldsymbol{e}_3\frac{\partial f}{\partial x_3} \tag{2.17.8}$$

and the symbolic vector differential operator ∇ called the *del operator* is

$$\nabla = \boldsymbol{e}_1\frac{\partial}{\partial x_1} + \boldsymbol{e}_2\frac{\partial}{\partial x_2} + \boldsymbol{e}_3\frac{\partial}{\partial x_3} \tag{2.17.9}$$

These and other useful operations can be expressed in Cartesian tensor notation. Given the scalar field ϕ and vector field $\boldsymbol{u}$, the following common differential operations can then be written in index notation:

$$\begin{aligned}
\text{Gradient of a scalar:}&\quad \nabla\phi = \text{grad}\,\phi = \phi_{,i}\boldsymbol{e}_i\\
\text{Gradient of a vector:}&\quad \nabla\boldsymbol{u} = \text{grad}\,\boldsymbol{u} = u_{i,j}\boldsymbol{e}_i\boldsymbol{e}_j\\
\text{Laplacian of a scalar:}&\quad \nabla^2\phi = \nabla\cdot\nabla\phi = \phi_{,ii}\\
\text{Divergence of a vector:}&\quad \nabla\cdot\boldsymbol{u} = \text{div}\,\boldsymbol{u} = u_{i,i}\\
\text{Curl of a vector:}&\quad \nabla\times\boldsymbol{u} = \text{curl}\,\boldsymbol{u} = \varepsilon_{ijk}u_{k,j}\boldsymbol{e}_i\\
\text{Laplacian of a vector:}&\quad \nabla^2\boldsymbol{u} = u_{i,kk}\boldsymbol{e}_i
\end{aligned} \tag{2.17.10}$$

These operations will appear in many places in continuum mechanics.

If ϕ and ψ are scalar fields and $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector fields, several useful identities exist:

$$\begin{gathered}
\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)\\
\nabla^2(\phi\psi) = (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi\\
\nabla\cdot(\phi\boldsymbol{u}) = \nabla\phi\cdot\boldsymbol{u} + \phi(\nabla\cdot\boldsymbol{u})\\
\nabla\times(\phi\boldsymbol{u}) = \nabla\phi\times\boldsymbol{u} + \phi(\nabla\times\boldsymbol{u})\\
\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{v})\\
\nabla\times\nabla\phi = 0\\
\nabla\cdot\nabla\phi = \nabla^2\phi\\
\nabla\cdot\nabla\times\boldsymbol{u} = 0\\
\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}\\
\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla\boldsymbol{u}
\end{gathered} \tag{2.17.11}$$

Each of these identities can be easily justified using index notation from definition relations (2.17.10).

Next consider some results from vector/tensor integral calculus. We will simply list some theorems that will have later use in the development of continuum mechanics theory.

Divergence or Gauss Theorem

Let S be a piecewise continuous surface bounding the region of space V. If a vector field $\boldsymbol{u}$ is continuous and has continuous first derivatives in V, then

$$\iint_S \boldsymbol{u}\cdot\boldsymbol{n}\,dS = \iiint_V \nabla\cdot\boldsymbol{u}\,dV \tag{2.17.12}$$

where $\boldsymbol{n}$ is the outer unit normal vector to surface S. This result is also true for tensors of any order, that is,

$$\iint_S A_{ij\ldots k}n_k\,dS = \iiint_V A_{ij\ldots k,k}\,dV \tag{2.17.13}$$

Stokes Theorem

Let S be an open two-sided surface bounded by a piecewise continuous, simple closed curve C. If $\boldsymbol{u}$ is continuous and has continuous first derivatives on S, then

$$\oint_C \boldsymbol{u}\cdot d\boldsymbol{r} = \iint_S (\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}\,dS \tag{2.17.14}$$

where the positive sense for the line integral is for the region S to lie to the left as one traverses curve C, and $\boldsymbol{n}$ is the unit normal vector to S. Again, this result is also valid for tensors of arbitrary order, and so

$$\oint_C A_{ij\ldots k}\,dx_k = \iint_S \varepsilon_{rsk}A_{ij\ldots k,s}n_r\,dS \tag{2.17.15}$$

It can be shown that both Divergence and Stokes Theorems can be generalized so that the dot product in (2.17.12) and (2.17.14) can be replaced with a cross product.

Localization Theorem

Let $F_{ij\ldots k}$ be a continuous tensor field of any order defined in an arbitrary region V. If the integral of $F_{ij\ldots k}$ over V vanishes, then F_{ijk} must vanish in V, that is,

$$\iiint_V F_{ij\ldots k}\,dV = 0 \Rightarrow F_{ij\ldots k} = 0 \in V \tag{2.17.16}$$

This result will be repeatedly used in Chapter 5 in going through the conversion process from integral to differential forms related to conservation laws.

Further information on vector differential and integral calculus can be found in the text by Hildebrand (1976) and Kreyszig (2011).

2.18 ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

In many applications, continuum mechanics is applied to problems that have curved domain and boundary geometry commonly including circular, cylindrical, and spherical surfaces. For such applications, it is highly desirable to use a curvilinear coordinate system to formulate and solve such problems. This requires redevelopment of some previous results in orthogonal curvilinear coordinates. Before pursuing these general steps, we will review the two most common curvilinear systems, *cylindrical* and *spherical coordinates*. The cylindrical coordinate system shown in Fig. 2.6 uses (r, θ, z) coordinates to describe spatial geometry. Relations between the Cartesian and cylindrical systems are given by

$$\begin{gathered} x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z \\ r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}\frac{x_2}{x_1}, \quad z = x_3 \end{gathered} \tag{2.18.1}$$

The spherical coordinate system shown in Fig. 2.7 uses (R, ϕ, θ) coordinates to describe geometry, and the relations between Cartesian and spherical coordinates are

$$\begin{gathered} x_1 = R\cos\theta\sin\phi, \quad x_2 = R\sin\theta\sin\phi, \quad x_3 = R\cos\phi \\ R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \phi = \cos^{-1}\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad \theta = \tan^{-1}\frac{x_2}{x_1} \end{gathered} \tag{2.18.2}$$

The unit basis vectors for each of these curvilinear systems are illustrated in Figs. 2.6 and 2.7, and these represent unit tangent vectors along each of the three orthogonal coordinate curves.

Although primary use of curvilinear systems will employ cylindrical and spherical coordinates, we briefly present a general discussion valid for arbitrary coordinate systems. Consider the general case where three orthogonal curvilinear coordinates

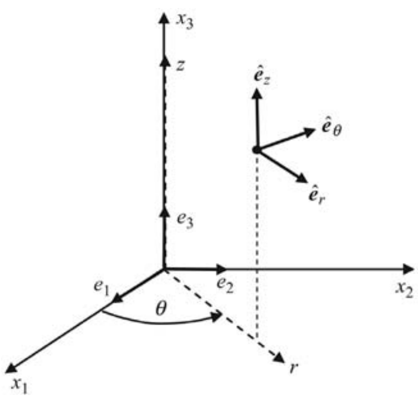

FIGURE 2.6

Cylindrical coordinate system.

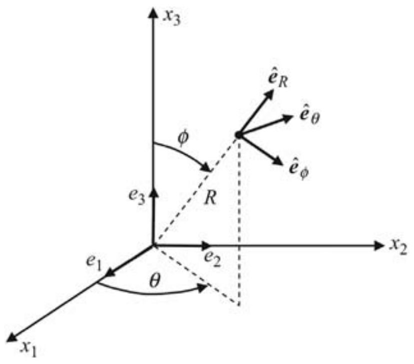

FIGURE 2.7

Spherical coordinate system.

are denoted by ξ^1, ξ^2, and ξ^3 whereas the Cartesian coordinates are defined by x^1, x^2, and x^3 (see Fig. 2.8). We assume there exists invertible coordinate transformations between these systems specified by

$$\xi^m = \xi^m(x^1, x^2, x^3), \quad x^m = x^m(\xi^1, \xi^2, \xi^3) \tag{2.18.3}$$

In the curvilinear system, an arbitrary differential length in space can be expressed by

$$(ds)^2 = (h_1\, d\xi^1)^2 + (h_2\, d\xi^2)^2 + (h_3\, d\xi^3)^2 \tag{2.18.4}$$

where h_1, h_2, and h_3 are called *scale factors* that are in general nonnegative functions of position. Let $\boldsymbol{e}_k$ be the fixed Cartesian basis vectors and $\hat{\boldsymbol{e}}_k$ the curvilinear basis (see Fig. 2.8). Using similar concepts from the transformations discussed in Section 2.8, the curvilinear basis can be expressed in terms of the Cartesian basis as

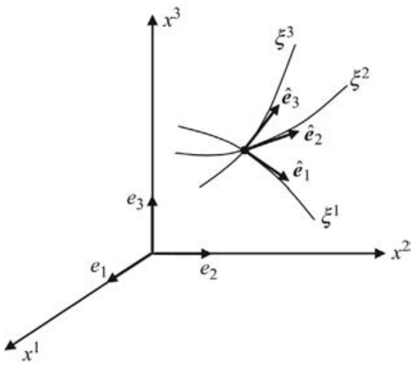

FIGURE 2.8

General curvilinear coordinates.

$$
\begin{aligned}
\hat{\boldsymbol{e}}_1 &= \frac{dx^k}{ds_1}\boldsymbol{e}_k = \frac{1}{h_1}\frac{\partial x^k}{\partial \xi^1}\boldsymbol{e}_k \\
\hat{\boldsymbol{e}}_2 &= \frac{dx^k}{ds_2}\boldsymbol{e}_k = \frac{1}{h_2}\frac{\partial x^k}{\partial \xi^2}\boldsymbol{e}_k \\
\hat{\boldsymbol{e}}_3 &= \frac{dx^k}{ds_3}\boldsymbol{e}_k = \frac{1}{h_3}\frac{\partial x^k}{\partial \xi^3}\boldsymbol{e}_k
\end{aligned}
\tag{2.18.5}
$$

where we have used (2.18.4). Using the fact that $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$, relation (2.18.5) gives

$$
\begin{aligned}
(h_1)^2 &= \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} \\
(h_2)^2 &= \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} \\
(h_3)^2 &= \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3}
\end{aligned}
\tag{2.18.6}
$$

It follows from (2.18.5) that the quantity

$$
Q_r^k = \frac{1}{h_r}\frac{\partial x^k}{\partial \xi^r}(\text{no sum on } r)
\tag{2.18.7}
$$

represents the transformation tensor giving the curvilinear basis in terms of the Cartesian basis. This concept is similar to the transformation tensor Q_{ij} defined by (2.7.1) that was used between Cartesian systems.

The *physical components* of a vector or tensor are simply the components in a local set of Cartesian axes tangent to the curvilinear coordinate curves at any point in space. Thus, using transformation relation (2.18.7), the physical components of a tensor $\boldsymbol{A}$ in a general curvilinear system are given by

$$
A_{\langle ij\ldots k\rangle} = Q_i^p Q_j^q \cdots Q_k^s A_{pq\ldots s}
\tag{2.18.8}
$$

where $A_{pq\ldots s}$ are the components in a fixed Cartesian frame. Note that the tensor can be expressed in either system as

$$
\begin{aligned}
\boldsymbol{A} &= A_{ij\ldots k}\boldsymbol{e}_i\boldsymbol{e}_j \cdots \boldsymbol{e}_k \\
&= A_{\langle ij\ldots k\rangle}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j \cdots \hat{\boldsymbol{e}}_k
\end{aligned}
\tag{2.18.9}
$$

Since many applications involve differentiation of tensors, we must consider the differentiation of the curvilinear basis vectors. The Cartesian basis system $\boldsymbol{e}_k$ is fixed in orientation and therefore $\dfrac{\partial \boldsymbol{e}_k}{\partial x^j} = \dfrac{\partial \boldsymbol{e}_k}{\partial \xi^j} = 0$. However, derivatives of the curvilinear basis will not in general vanish, and differentiation of relations (2.18.5) gives the following results

$$
\begin{aligned}
&\frac{\partial \hat{\boldsymbol{e}}_m}{\partial \xi^m} = -\frac{1}{h_n}\frac{\partial h_m}{\partial \xi^n}\hat{\boldsymbol{e}}_n - \frac{1}{h_r}\frac{\partial h_m}{\partial \xi^r}\hat{\boldsymbol{e}}_r, \quad m \neq n \neq r \\
&\frac{\partial \hat{\boldsymbol{e}}_m}{\partial \xi^n} = \frac{1}{h_m}\frac{\partial h_n}{\partial \xi^m}\hat{\boldsymbol{e}}_n, \quad m \neq n, \quad \text{no sum on repeated indices}
\end{aligned}
\tag{2.18.10}
$$

Using these results, the derivative of any tensor can be evaluated. Consider the first-order case of the derivative of a vector $\boldsymbol{u}$:

$$\frac{\partial}{\partial \xi^n}\boldsymbol{u} = \frac{\partial}{\partial \xi^n}(u_{\langle m\rangle}\hat{\boldsymbol{e}}_m) = \frac{\partial u_{\langle m\rangle}}{\partial \xi^n}\hat{\boldsymbol{e}}_m + u_{\langle m\rangle}\frac{\partial \hat{\boldsymbol{e}}_m}{\partial \xi^n} \tag{2.18.11}$$

The last term can be evaluated using (2.18.10), and thus the derivative of $\boldsymbol{u}$ can be expressed in terms of curvilinear components. Following a similar scheme, the derivative of a second-order tensor $\boldsymbol{A}$ is given by

$$\frac{\partial}{\partial \xi^r}\boldsymbol{A} = \frac{\partial}{\partial \xi^r}(A_{\langle mn\rangle}\hat{\boldsymbol{e}}_m\hat{\boldsymbol{e}}_n) = \frac{\partial A_{\langle mn\rangle}}{\partial \xi^r}\hat{\boldsymbol{e}}_m\hat{\boldsymbol{e}}_n + A_{\langle mn\rangle}\frac{\partial \hat{\boldsymbol{e}}_m}{\partial \xi^r}\hat{\boldsymbol{e}}_n + A_{\langle mn\rangle}\hat{\boldsymbol{e}}_m\frac{\partial \hat{\boldsymbol{e}}_n}{\partial \xi^r} \tag{2.18.12}$$

Analogous patterns follow for derivatives of higher-order tensors.

All vector differential operators of gradient, divergence, curl, etc. can be expressed in any general curvilinear system using these techniques. For example, the vector differential operator previously defined in Cartesian coordinates in (2.17.8) is given by

$$\nabla = \hat{\boldsymbol{e}}_1\frac{1}{h_1}\frac{\partial}{\partial \xi^1} + \hat{\boldsymbol{e}}_2\frac{1}{h_2}\frac{\partial}{\partial \xi^2} + \hat{\boldsymbol{e}}_3\frac{1}{h_3}\frac{\partial}{\partial \xi^3} = \sum_i \hat{\boldsymbol{e}}_i\frac{1}{h_i}\frac{\partial}{\partial \xi^i} \tag{2.18.13}$$

This then leads to the construction of the other common forms

$$\text{Gradient of a scalar:}\quad \nabla f = \hat{\boldsymbol{e}}_1\frac{1}{h_1}\frac{\partial f}{\partial \xi^1} + \hat{\boldsymbol{e}}_2\frac{1}{h_2}\frac{\partial f}{\partial \xi^2} + \hat{\boldsymbol{e}}_3\frac{1}{h_3}\frac{\partial f}{\partial \xi^3} = \sum_i \hat{\boldsymbol{e}}_i\frac{1}{h_i}\frac{\partial f}{\partial \xi^i} \tag{2.18.14}$$

$$\text{Divergence of a vector:}\quad \nabla\cdot\boldsymbol{u} = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial \xi^i}\left(\frac{h_1h_2h_3}{h_i}u_{\langle i\rangle}\right) \tag{2.18.15}$$

$$\text{Laplacian of a scalar:}\quad \nabla^2\phi = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial \xi^i}\left(\frac{h_1h_2h_3}{(h_i)^2}\frac{\partial \phi}{\partial \xi^i}\right) \tag{2.18.16}$$

$$\text{Curl of a vector:}\quad \nabla\times\boldsymbol{u} = \sum_i\sum_j\sum_k \frac{\varepsilon_{ijk}}{h_jh_k}\frac{\partial}{\partial \xi^j}(u_{\langle k\rangle}h_k)\hat{\boldsymbol{e}}_i \tag{2.18.17}$$

$$\text{Gradient of a vector:}\quad \nabla\boldsymbol{u} = \sum_i\sum_j \frac{\hat{\boldsymbol{e}}_i}{h_i}\left(\frac{\partial u_{\langle j\rangle}}{\partial \xi^i}\hat{\boldsymbol{e}}_j + u_{\langle j\rangle}\frac{\partial \hat{\boldsymbol{e}}_j}{\partial \xi^i}\right) \tag{2.18.18}$$

$$\text{Laplacian of a vector:}\quad \nabla^2 u = \left(\sum_i \frac{\hat{\boldsymbol{e}}_i}{h_i}\frac{\partial}{\partial \xi^i}\right)\cdot\left(\sum_j\sum_k \frac{\hat{\boldsymbol{e}}_k}{h_k}\left[\frac{\partial u_{\langle j\rangle}}{\partial \xi^k}\hat{\boldsymbol{e}}_j + u_{\langle j\rangle}\frac{\partial \hat{\boldsymbol{e}}_j}{\partial \xi^k}\right]\right) \tag{2.18.19}$$

It should be noted that these curvilinear forms are significantly different from those previously given in relations (2.17.10) for Cartesian coordinates. Curvilinear systems will add additional terms not found in rectangular coordinates. Other operations on higher-order tensors can be developed in a similar fashion (see Malvern 1969,

Appendix II). Specific transformation relations and field equations in cylindrical and spherical coordinate systems are given in Appendices A and B. Further discussion of these results will be taken up in later chapters.

EXAMPLE 2.18.1 POLAR COORDINATES

Determine the various vector differential operators for a two-dimensional polar coordinate system as shown in Fig. 2.9.

Solution: The differential length relation (2.18.4) for this case can be written as

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2$$

and thus $h_1 = 1$ and $h_2 = r$. From relations (2.18.5) or simply using the geometry shown in Fig. 2.9,

$$\begin{aligned}\hat{\boldsymbol{e}}_r &= \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_2\\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2\end{aligned}$$

and so

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\theta} = \hat{\boldsymbol{e}}_\theta, \quad \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\hat{\boldsymbol{e}}_r, \quad \frac{\partial\hat{\boldsymbol{e}}_r}{\partial r} = \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial r} = 0$$

The basic vector differential operations then follow from relations (2.18.13)–(2.18.19) to be

$$\begin{aligned}
\nabla &= \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\\
\nabla f &= \hat{\boldsymbol{e}}_r\frac{\partial f}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta}\\
\nabla\cdot\boldsymbol{u} &= \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\\
\nabla\times\boldsymbol{u} &= \left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z\\
\nabla\boldsymbol{u} &= \frac{\partial u_r}{\partial r}\hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_\theta + \frac{\partial u_\theta}{\partial r}\hat{\boldsymbol{e}}_\theta\hat{\boldsymbol{e}}_r + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right)\hat{\boldsymbol{e}}_\theta\hat{\boldsymbol{e}}_\theta\\
\nabla^2\boldsymbol{u} &= \left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r^2}\right)\hat{\boldsymbol{e}}_r + \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\hat{\boldsymbol{e}}_\theta
\end{aligned} \tag{2.18.20}$$

where $\boldsymbol{u} = u_r\hat{\boldsymbol{e}}_r + u_\theta\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\theta$. Notice that the Laplacian of a vector does not simply pass through and operate on each of the individual components as in the Cartesian case. Additional terms are generated due to the curvature of the polar coordinate system. Similar relations can be developed for cylindrical and spherical coordinate systems (see Exercises 2.26 and 2.27).

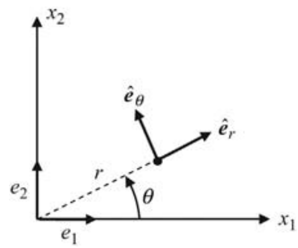

FIGURE 2.9

Polar coordinate system.

2.19 GENERAL TENSORS

Rather than using the concepts scale factors and physical components presented in the previous section, we now present a more general scheme to deal with curvilinear coordinate systems. With the exception of Section 2.18, our previous discussions were valid only for Cartesian coordinates. Transformation laws given in relations (2.8.1) were valid only between such reference frames. Another more fundamental scheme to handle continuum mechanics formulation in curvilinear coordinate systems is to employ *general tensor theory* that is not restricted to only Cartesian systems. A very brief introduction into general tensors and their application for equations in continuum mechanics will now be presented.

We start with the general transformation relations (2.18.3) between rectangular coordinates x^i and orthogonal curvilinear coordinates ξ^i:

$$\xi^m = \xi^m(x^1, x^2, x^3), \quad x^m = x^m(\xi^1, \xi^2, \xi^3) \tag{2.19.1}$$

where the use of both subscripts and superscripts will now be used. Fig. 2.10 illustrates the two coordinate systems, and the position vector $\boldsymbol{r}$ in the Cartesian coordinate system is given by

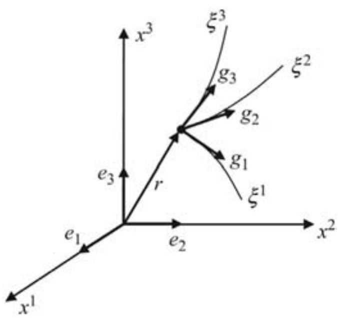

FIGURE 2.10

Cartesian and curvilinear coordinates.

$$\boldsymbol{r} = x^k \boldsymbol{e}_k \tag{2.19.2}$$

where now summations are carried from subscripts to superscripts.

The *covariant base vectors* $\boldsymbol{g}_k$, tangent to the curvilinear system, then follows to be

$$\boldsymbol{g}_k = \frac{\partial x^m}{\partial \xi^k} \boldsymbol{e}_m \tag{2.19.3}$$

It should be noted that these base vectors are *not unit vectors*, but they will be orthogonal for our case, since we are considering only orthogonal curvilinear coordinate systems.

From relation (2.19.3), it follows that

$$\boldsymbol{e}_m = \frac{\partial \xi^k}{\partial x^m} \boldsymbol{g}_k \tag{2.19.4}$$

and thus the differential position vector $d\boldsymbol{r}$ can be written as

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \xi^k} d\xi^k = \boldsymbol{g}_k \, d\xi^k \tag{2.19.5}$$

This allows the calculation of the square of the arc length (see Eq. (2.18.4)):

$$(ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \boldsymbol{g}_k \cdot \boldsymbol{g}_m \, d\xi^k \, d\xi^m = g_{km} \, d\xi^k \, d\xi^m \tag{2.19.6}$$

where g_{ij} is called the *covariant metric tensor* defined by

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \frac{\partial x^m}{\partial \xi^i} \frac{\partial x^n}{\partial \xi^j} \delta_{mn} \tag{2.19.7}$$

The terminology metric tensor follows from the fact that g_{ij} provides a means to calculate lengths and angles in space. Since the coordinates are orthogonal, $g_{ij} = 0$ for $i \neq j$, and in general the components of the metric tensor are functions of the spatial coordinates. We thus could express (2.19.6) as

$$(ds)^2 = g_{11}(d\xi^1)^2 + g_{22}(d\xi^2)^2 + g_{33}(d\xi^3)^2 \tag{2.19.8}$$

note the similarity with the previous relation (2.18.4).

Another set of base vectors $\boldsymbol{g}^k$ can also be defined called the *reciprocal base vectors* and may be found from the solution to the equations

$$\boldsymbol{g}^k \cdot \boldsymbol{g}_m = \delta^k_m \tag{2.19.9}$$

where δ^k_m is the Kronecker delta with upper and lower indices. It can be shown that the reciprocal base vectors can be written as

$$\boldsymbol{g}^k = g^{km} g_m \tag{2.19.10}$$

where g^{km} is the contravariant metric tensor given by

$$g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j = \frac{\partial \xi^i}{\partial x^m} \frac{\partial \xi^j}{\partial x^n} \delta^{mn} \tag{2.19.11}$$

with properties

$$g^{ij} g_{jk} = \delta^i_k \tag{2.19.12}$$

Note that with orthogonal coordinates $g^{11} = 1/g_{11}, g^{22} = 1/g_{22}, g^{33} = 1/g_{33}$.

Now any vector $\boldsymbol{u}$ (or tensor) may be decomposed into its *covariant components* u_k or *contravariant components* u^k by the expressions

$$\boldsymbol{u} = u_k \boldsymbol{g}^k = u^k \boldsymbol{g}_k \tag{2.19.13}$$

The covariant and contravariant components are not identical, and since the base vectors are not unit vectors these components will not necessarily have the same physical dimensions (units) as the vector $\boldsymbol{u}$ itself. Using the metric tensors, one can raise and lower indices as follows:

$$u_k = g_{km} u^m, \quad u^k = g^{km} u_m \tag{2.19.14}$$

By decomposing a vector $\boldsymbol{u}$ into its components along *unit vectors* $\hat{\boldsymbol{e}}_k$ which lie along the coordinate curves, one obtains the *physical components*, $u^{\langle k \rangle}$ of the vector, that is,

$$\boldsymbol{u} = u^{\langle k \rangle} \hat{\boldsymbol{e}}_k \tag{2.19.15}$$

where the unit vectors $\hat{\boldsymbol{e}}_k$ are defined by

$$\hat{\boldsymbol{e}}_k = \frac{\boldsymbol{g}_k}{(g_{kk})^{1/2}} \, (\text{no sum on } k) \tag{2.19.16}$$

Next considering spatial derivatives of tensor fields within this more general scheme, we follow similar steps as done in the previous section. As before, both covariant and contravariant base vectors vary with position in space, so their spatial derivatives will not be zero. One can show that

$$\frac{\partial \boldsymbol{g}_k}{\partial \xi^j} = \Gamma^k_{ij} \boldsymbol{g}_k, \quad \frac{\partial \boldsymbol{g}^k}{\partial \xi^j} = -\Gamma^i_{kj} \boldsymbol{g}^k \tag{2.19.17}$$

where Γ^i_{kj} are known as the *Christoffel symbols of the second kind* given by

$$\Gamma^i_{kj} = \Gamma^i_{jk} = \frac{\partial^2 x^m}{\partial \xi^k \, \partial \xi^j} \frac{\partial \xi^i}{\partial x^m} = \frac{1}{2}(g_{km,j} + g_{jm,k} - g_{kj,m}) g^{mi} \tag{2.19.18}$$

We then can define the *covariant derivative* of a contravariant vector as

$$u^i_{;j} = u^i_{,j} + \Gamma^i_{kj} u^k, \quad \text{where } u^i_{,j} = \frac{\partial u^i}{\partial \xi^j} \tag{2.19.19}$$

while the covariant derivative of a covariant vector is given by

$$u_{i;j} = u_{i,j} - \Gamma^k_{ij} u_k \tag{2.19.20}$$

For the special case of *transformations only between Cartesian frames*, relation (2.19.1) becomes

$$\xi^i = Q^i_j x^j + a^i \tag{2.19.21}$$

where Q^i_j is the constant rotation matrix defined by (2.7.1) with orthogonal property $Q^i_k Q^j_k = \delta^{ij}$, and a^i is a constant translation vector. For this case,

$$\frac{\partial \xi^i}{\partial x^k} = Q^i_k \Rightarrow g^{ij} = Q^i_m Q^j_n \delta^{mn} = \delta^{ij}, \quad \Gamma^i_{jk} = 0 \tag{2.19.22}$$

thus, all Christoffel symbols vanish for the case of Cartesian coordinates, and the covariant derivatives reduce to the ordinary partial derivatives.

For higher-order tensors, the covariant derivative is defined in an analogous manner. For example, for second-order tensors A^{ij} and A_{ij}, we have

$$\begin{aligned} A^{ij}_{;k} &= A^{ij}_{,k} + \Gamma^i_{km} A^{mj} + \Gamma^j_{km} A^{im} \\ A_{ij;k} &= A_{ij,k} - \Gamma^m_{ik} A_{mj} - \Gamma^m_{jk} A_{im} \end{aligned} \tag{2.19.23}$$

Details on the derivation of these derivative expressions can be found in Malvern (1969). Using relations (2.19.19) and (2.19.20) allows development of the standard vector differential operators in curvilinear coordinate systems previously present in Section 2.18. Other differential relations involving second-order tensors follow from relations (2.19.23). The following example illustrates many of the basic applications for the specific case of a polar coordinate curvilinear system.

EXAMPLE 2.19.1 POLAR COORDINATES

For a two-dimensional polar coordinate system as shown in Fig. 2.9, determine

(a) covariant and reciprocal base vectors;
(b) covariant and contravariant metric tensors;
(c) relations between the physical, covariant, and contravariant components of a vector $\boldsymbol{u}$;
(d) Christoffel symbols;
(e) gradient of vector $\boldsymbol{u}$ and express it in physical components.

Solution: For polar coordinates with $\xi^1 = r$ and $\xi^2 = \theta$, the transformation relations (2.19.1) take the form

$$\begin{aligned} &\xi^1 = \sqrt{(x^1)^2 + (x^2)^2}, \quad \xi^2 = \tan^{-1}(x^2/x^1) \\ &x^1 = \xi^1 \cos\xi^2, \quad x^2 = \xi^1 \sin\xi^2 \end{aligned}$$

(a) From relation (2.19.3)

$$\boldsymbol{g}_1 = \frac{\partial x^m}{\partial \xi^1}\boldsymbol{e}_m = \cos\xi^2\boldsymbol{e}_1 + \sin\xi^2\boldsymbol{e}_2$$
$$\boldsymbol{g}_2 = \frac{\partial x^m}{\partial \xi^2}\boldsymbol{e}_m = -\xi^1\sin\xi^2\boldsymbol{e}_1 + \xi^1\cos\xi^2\boldsymbol{e}_2$$

Using (2.19.9) $\boldsymbol{g}^k \cdot \boldsymbol{g}_m = \delta^k_m \Rightarrow$

$$\boldsymbol{g}^1 \cdot \boldsymbol{g}_1 = 1 \Rightarrow \boldsymbol{g}^1 = \boldsymbol{g}_1 = \cos\xi^2\boldsymbol{e}_1 + \sin\xi^2\boldsymbol{e}_2$$
$$\boldsymbol{g}^2 \cdot \boldsymbol{g}_2 = 1 \Rightarrow \boldsymbol{g}^2 = \frac{1}{(\xi^1)^2}\boldsymbol{g}_2 = \frac{1}{(\xi^1)}\left(-\sin\xi^2\boldsymbol{e}_1 + \cos\xi^2\boldsymbol{e}_2\right)$$

(b) Eqs. (2.19.7) and (2.19.12), then give

$$g_{11} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_1 = 1 = 1/g^{11}$$
$$g_{22} = \boldsymbol{g}_2 \cdot \boldsymbol{g}_2 = (\xi^1)^2 = 1/g^{22}$$
$$g_{12} = g_{21} = 0, \quad g^{12} = g^{21} = 0$$

(c) Using (2.19.16) $\hat{\boldsymbol{e}}_k = \dfrac{\boldsymbol{g}_k}{(g_{kk})^{1/2}}$ (no sum on k) $\Rightarrow$

$$\hat{\boldsymbol{e}}_1 = \frac{\boldsymbol{g}_1}{(g_{11})^{1/2}} = \boldsymbol{g}_1 = \boldsymbol{g}^1, \quad \hat{\boldsymbol{e}}_2 = \frac{\boldsymbol{g}_2}{(g_{22})^{1/2}} = \frac{1}{\xi^1}\boldsymbol{g}_2 = \xi^1\boldsymbol{g}^2$$

(d) The Christoffel symbols follow from (2.19.18) $\Gamma^i_{kj} = \dfrac{1}{2}\left(g_{km,j} + g_{jm,k} - g_{kj,m}\right)$ $\Rightarrow$

$$\Gamma^1_{22} = \frac{1}{2}\left(g_{21,2} + g_{21,2} - g_{22,1}\right)g^{11} = -\xi^1$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\left(g_{22,1} + g_{12,2} - g_{21,2}\right)g^{22} = \frac{1}{\xi^1}$$

and all others are zero, for example

$$\Gamma^1_{11} = \frac{1}{2}\left(g_{11,1} + g_{11,1} - g_{11,1}\right)g^{11} = 0$$
$$\Gamma^2_{11} = \frac{1}{2}\left(g_{11,1} + g_{11,1} - g_{11,1}\right)g^{22} = 0$$

. . .

(e) Using the contravariant form from (2.19.13), a vector $\boldsymbol{u}$ can be expressed by $\boldsymbol{u}=u^k\boldsymbol{g}_k$, and so the gradient operation is expressed using (2.19.19)

$$\operatorname{grad}\boldsymbol{u} = u^i_{;j}\boldsymbol{g}_i\boldsymbol{g}^j = \left(u^i_{,j} + \Gamma^i_{kj}u^k\right)\boldsymbol{g}_i\boldsymbol{g}^j = \left(\frac{\partial u^i}{\partial \xi^j} + \Gamma^i_{kj}u^k\right)\boldsymbol{g}_i\boldsymbol{g}^j$$
$$= \frac{\partial u^1}{\partial \xi^1}\boldsymbol{g}_1\boldsymbol{g}^1 + \left(\frac{\partial u^1}{\partial \xi^2} + \Gamma^1_{22}u^2\right)\boldsymbol{g}_1\boldsymbol{g}^2 + \left(\frac{\partial u^2}{\partial \xi^1} + \Gamma^2_{21}u^2\right)\boldsymbol{g}_2\boldsymbol{g}^1 + \left(\frac{\partial u^2}{\partial \xi^2} + \Gamma^2_{12}u^1\right)\boldsymbol{g}_2\boldsymbol{g}^2$$
$$= \frac{\partial u^1}{\partial \xi^1}\boldsymbol{g}_1\boldsymbol{g}^1 + \left(\frac{\partial u^1}{\partial \xi^2} - \xi^1u^2\right)\boldsymbol{g}_1\boldsymbol{g}^2 + \left(\frac{\partial u^2}{\partial \xi^1} + \frac{1}{\xi^1}u^2\right)\boldsymbol{g}_2\boldsymbol{g}^1 + \left(\frac{\partial u^2}{\partial \xi^2} + \frac{1}{\xi^1}u^1\right)\boldsymbol{g}_2\boldsymbol{g}^2$$

Using part (c), we can easily translate these results in terms of the physical components as

$$\begin{aligned}
\operatorname{grad}\boldsymbol{u} &= \frac{\partial u^1}{\partial \xi^1}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + \left(\frac{1}{\xi^1}\frac{\partial u^1}{\partial \xi^2} - u^2\right)\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + \left(\xi^1\frac{\partial u^2}{\partial \xi^1} + u^2\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1 + \left(\frac{\partial u^2}{\partial \xi^2} + \frac{1}{\xi^1}u^1\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 \\
&= \frac{\partial u^{\langle 1\rangle}}{\partial \xi^1}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + \left(\frac{1}{\xi^1}\frac{\partial u^{\langle 1\rangle}}{\partial \xi^2} - \frac{u^{\langle 2\rangle}}{\xi^1}\right)\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + \left(\xi^1\frac{\partial}{\partial \xi^1}\left(\frac{u^{\langle 2\rangle}}{\xi^1}\right) + \frac{u^{\langle 2\rangle}}{\xi^1}\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1 \\
&\quad + \left(\frac{\partial}{\partial \xi^2}\left(\frac{u^{\langle 2\rangle}}{\xi^1}\right) + \frac{1}{\xi^1}u^{\langle 1\rangle}\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 \\
&= \frac{\partial u^{\langle 1\rangle}}{\partial \xi^1}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + \left(\frac{1}{\xi^1}\frac{\partial u^{\langle 1\rangle}}{\partial \xi^2} - \frac{u^{\langle 2\rangle}}{\xi^1}\right)\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + \left(\frac{\partial u^{\langle 2\rangle}}{\partial \xi^1}\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1 + \left(\frac{1}{\xi^1}\frac{\partial u^{\langle 2\rangle}}{\partial \xi^2} + \frac{u^{\langle 1\rangle}}{\xi^1}\right)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2
\end{aligned}$$

which matches with the corresponding result from the previous Example 2.18.1.

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EXERCISES

2.1 For the following vector and matrix pairs, compute the expressions $a_i a_i$, $a_i a_j$, A_{ii}, $A_{ij} a_j$, and specify whether they are scalars, vectors, or matrices:

(a) $a_i = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(b) $a_i = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(c) $a_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

2.2 Decompose the matrices A_{ij} in Exercise 2.1 into the sum of symmetric $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$ and antisymmetric $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$ matrices. For each case, verify that $A_{(ij)}$ and $A_{[ij]}$ satisfy the conditions specified in the last paragraph of Section 2.3.

2.3 Verify in general that the product $A_{ij}B_{ij}$ will vanish if A_{ij} is symmetric and B_{ij} is antisymmetric.

2.4 Given the three matrices: A_{ij} (symmetric), B_{ij} (antisymmetric), and C_{ij} (general), show that

$$\begin{aligned} &A_{ij}C_{ij} = A_{ij}C_{ji} = A_{ij}C_{(ij)} \\ &B_{ij}C_{ij} = -B_{ij}C_{ji} = B_{ij}C_{[ij]} \\ &\det(B_{ij}) = 0 \end{aligned}$$

2.5 Explicitly verify the following properties of the Kronecker delta:

$$\delta_{ij}a_{jk} = a_{ik}, \quad \delta_{ij}a_{kj} = a_{ki}$$

2.6 Formally expand the determinant relation (2.5.1) and justify that either index notation form yields a result that matches the traditional expansion of $\det[A_{ij}]$.

2.7 Explicitly justify relation (2.5.2) for the 2×2 case, and then use this result to verify orthogonal properties (2.5.3).

2.8 Using property (2.5.4), verify relations (2.5.5)–(2.5.8).

2.9 Using the general definitions of the dot and cross product given in Section 2.10, explicitly justify properties (2.6.5).

2.10 Consider a new set of axes obtained by rotating the original system 90° about the x_2-axis. Positive rotation is the same as defined in Fig. 2.3. Determine the transformation matrix Q_{ij}, and find the components of A_{ij} and a_i given in Exercise 2.1 in the new system. Verify that the length of the vector $|\boldsymbol{a}| = (a_i a_i)^{1/2}$ and the quantity A_{ii} are the same in each coordinate system, that is, they are invariant with respect to the rotation.

2.11 Differentiate with respect to time the orthogonality condition $\boldsymbol{QQ}^T = \boldsymbol{I}$, and show that

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T = -(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)^T$$

2.12 Consider the two-dimensional coordinate transformation shown in Fig. 2.9. Through the counter-clockwise rotation θ, a new polar coordinate system is created. Show that the transformation matrix for this case is given by

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

If $b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, A_{ij} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ are the components of a first- and second-order tensor in the x_1,x_2-system, calculate their components in a rotated Cartesian system aligned with the polar coordinates.

2.13 The *axial* or *dual vector* a_i, of a second-order tensor A_{ij} is defined by $a_i = -\frac{1}{2}\varepsilon_{ijk}A_{jk}$. Show that the dual vector of a symmetric tensor is zero, and thus one can write $a_i = -\frac{1}{2}\varepsilon_{ijk}A_{[jk]}$ or in direct notation $\boldsymbol{a} = -A_{23}\boldsymbol{e}_1 + A_{13}\boldsymbol{e}_2 - A_{12}\boldsymbol{e}_3$. Also verify that one can invert the previous expression to get $A_{[jk]} = -\varepsilon_{ijk}a_i = -\varepsilon_{jki}a_i$.

2.14 Verify properties 7 and 8 in Table 2.1.

2.15 Show that the relation (2.10.9) for the inverse of a matrix satisfies the defining statement (2.10.8).

2.16 Verify the relationships (2.11.5) that express the invariants I_1, I_2, and I_3 in terms of the fundamental invariants I_A, II_A, and III_A. Next, invert these relations and express the fundamental invariants in terms of I_1, I_2, and I_3.

2.17 Verify that the fundamental invariants can be expressed in terms of the principal values as given by relations (2.11.8).

2.18 Determine the invariants, and principal values and directions of the following symmetric second-order tensors (matrices). Use the determined principal directions to establish a principal coordinate system, and following the procedures in Example 2.11.1, formally transform (rotate) the given matrix into the principal system to arrive at the appropriate diagonal form.

(a) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Answer: $\lambda_i = -3, -1, 0$)

(b) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (Answer: $\lambda_i = -2, 0, 2$)

(c) $\begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ (Answer: $\lambda_i = 3, 6, 9$)

You may check your work using the MATLAB code C-2.

2.19 Consider the principal value problem for matrix $\boldsymbol{A}$, $\boldsymbol{An} = \lambda_{(i)}\boldsymbol{n}$. First show that the principal values of $\boldsymbol{A}^{-1}$ are given by $1/\lambda_{(i)}$ and that the principal directions of $\boldsymbol{A}^{-1}$ are the same as $\boldsymbol{A}$. Then using principal coordinates, show that $I_{A^{-1}} = II_A / III_A$.

2.20 A symmetric tensor A_{ij} is said to be *positive definite*, if for all vectors b_i, $A_{ij}b_ib_j \geq 0$, with equality only if $b_i = 0$. Show that a symmetric tensor A_{ij} is positive definite if and only if (iff) all of its eigenvalues are positive.

2.21 Calculate the spherical and deviatoric parts of the following second-order tensors:

$$\text{(a) } A_{ij} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{(b) } A_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{(c) } A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

2.22 Verify relations (2.13.3) and (2.13.4) to express $\boldsymbol{A}^4$ and $\boldsymbol{A}^5$ in terms of powers ≤ 2.

2.23 Using the Polar Decomposition Theorem (2.16.1), establish relations (2.16.2).

2.24 Calculate the quantities $\nabla \cdot \boldsymbol{u}, \nabla \times \boldsymbol{u}, \nabla^2 \boldsymbol{u}, \nabla \boldsymbol{u}, tr(\nabla \boldsymbol{u})$ for the following Cartesian vector fields:

(a) $\boldsymbol{u} = x_1\boldsymbol{e}_1 + x_1x_2\boldsymbol{e}_2 + 2x_1x_2x_3\boldsymbol{e}_3$

(b) $\boldsymbol{u} = x_1^2\boldsymbol{e}_1 + 2x_1x_2\boldsymbol{e}_2 + x_3^3\boldsymbol{e}_3$

(c) $\boldsymbol{u} = x_2^2\boldsymbol{e}_1 + 2x_2x_3\boldsymbol{e}_2 + 4x_1^2\boldsymbol{e}_3$

2.25 Using index notation, verify the following vector identities:

(a) $(2.17.11)_{1,2,3}$

(b) $(2.17.11)_{4,5,6,7}$

(c) $(2.17.11)_{8,9,10}$

2.26 Extend the polar coordinate results in Example 2.18.1 and determine the forms of $\nabla f, \nabla \cdot \boldsymbol{u}$, $\nabla^2 f$ and $\nabla \times \boldsymbol{u}$ for a three-dimensional cylindrical coordinate system as shown in Fig. 2.6.

2.27 For the spherical coordinate system (R, ϕ, θ) in Fig. 2.7, show that

$$h_1 = 1, h_2 = R, h_3 = R\sin\phi$$

and the standard vector operations are given by

$$\nabla f = \hat{\boldsymbol{e}}_R \frac{\partial f}{\partial R} + \hat{\boldsymbol{e}}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{\boldsymbol{e}}_\theta \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}$$

$$\nabla^2 f = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla \times \boldsymbol{u} = \hat{\boldsymbol{e}}_R \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi\, u_\theta) - \frac{\partial u_\phi}{\partial \theta}\right)\right] + \hat{\boldsymbol{e}}_\phi \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} - \frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta)\right]$$
$$+ \hat{\boldsymbol{e}}_\theta \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(Ru_\phi) - \frac{\partial u_R}{\partial \phi}\right)\right]$$

2.28 For the case of *spherical coordinates*, determine: g_{ij}, g^{ij}, Γ^{i}_{jk} (calculate only those shown in answer).

Ans. Using $\xi^1 = R$, $\xi^2 = \phi$, $\xi^3 = \theta$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2 \phi \end{bmatrix}, \quad \Gamma^1_{11} = 0, \quad \Gamma^3_{13} = \frac{1}{R}, \quad \Gamma^2_{33} = -\sin\phi\cos\phi, \quad \Gamma^3_{23} = \cot\phi$$