CHAPTER

General Conservation or Balance Laws

5

We now wish to explore a number of fundamental physical laws that apply to continuum material behavior. These laws have been developed from many years of past research and are appropriate for deformations, loadings, and rate effects found in typical engineering applications where relativistic and nuclear behaviors can be neglected. These relations commonly represent some type of *conservation principle* and are also often referred to as *balance laws*. They are applicable to all continuum materials regardless of whether they are solids, fluids, elastic, plastic, etc. Our presentation will include *conservation of mass, linear and angular momentum*, as well as *energy* (first law of thermodynamics). In addition, we will also present the *Clausius–Duhem inequality* which is a form of the *second law of thermodynamics*. The balance laws will be initially formulated in *integral form* and then later reduced to *differential field equations* that apply to all continuum points within a body under study. While these relations will include many previously defined tensor fields, they will also introduce a few new variables in our study.

5.1 GENERAL CONSERVATION PRINCIPLES AND THE REYNOLDS TRANSPORT THEOREM

Fundamentally balance principles start as axioms involving integral relations over material body configurations. Consider first the time rate of change of certain integrals. Clearly for a *fixed region of space R*, and with $\boldsymbol{G}(\boldsymbol{x}, t)$ being an arbitrary tensor field

$$\frac{\partial}{\partial t}\int_R \boldsymbol{G}\, dv = \int_R \frac{\partial}{\partial t}\boldsymbol{G}\, dv \tag{5.1.1}$$

The time derivative passes through the volume integral sign since the limits of integration are time independent.

However, referring to our previous discussion in Section 3.1, we next reconsider this time rate of change over a *fixed group of continuum particles* that occupy the region of space R_m at some particular point in time. For this case, not only does the integrand change with time, but so does the spatial volume over which the integral is taken. We thus wish to define a material time derivative of a volume integral in such a way that it measures the rate of change of the total amount of some quantity carried by the given mass system in R_m. Hence, we want to consider

$$\frac{D}{Dt}\int_{R_m} \boldsymbol{G}\, dV$$

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Note that $R = R_m$ at the instant of time under consideration, and the differential volume elements dv and dV are related through the Jacobian determinant by relation (3.7.4), $dv = J\,dV$. Some authors refer to R as the *control volume* and R_m as the *material volume*.

Thus, we can write

$$\begin{aligned}\frac{D}{Dt}\int_{R_m} \boldsymbol{G}\,dv &= \int_R \frac{D}{Dt}\boldsymbol{G}\,dv + \int_R \boldsymbol{G}\frac{D}{Dt}dv = \int_R \dot{\boldsymbol{G}}\,dv + \int_{R_m} \boldsymbol{G}\dot{J}\,dV \\ &= \int_R \dot{\boldsymbol{G}}\,dv + \int_{R_m} \boldsymbol{G}v_{k,k}J\,dV = \int_R \dot{\boldsymbol{G}}\,dv + \int_R \boldsymbol{G}v_{k,k}\,dv \\ &= \int_R \frac{\partial}{\partial t}\boldsymbol{G}\,dv + \int_R (\boldsymbol{G}_{,k}v_k + \boldsymbol{G}v_{k,k})\,dv = \int_R \frac{\partial}{\partial t}\boldsymbol{G}\,dv + \int_R (\boldsymbol{G}v_k)_{,k}\,dv\end{aligned} \tag{5.1.2}$$

where we have used (3.13.9) and (3.3.3), with v_k being the velocity field. Next, using the Divergence Theorem on the second integral produces

$$\frac{D}{Dt}\int_R \boldsymbol{G}\,dV = \int_R \frac{\partial}{\partial t}\boldsymbol{G}\,dv + \int_{\partial R} \boldsymbol{G}v_k n_k\,ds \tag{5.1.3}$$

where n_k is the unit outward normal vector to the surface ∂R which encloses the region R, as shown in Fig. 5.1. Result (5.1.3) is often called the *Reynolds Transport Theorem*, and we see from this result that the material time rate of change is given by two terms: one related to the simple time rate of change within the region R, whereas another term related to a particular amount entering (flux) through the boundary ∂R.

Most of our physical laws will be first stated in a general global equation of balance or conservation in the form

$$\frac{D}{Dt}\int_R \rho\Psi\,dV = -\int_{\partial R} \boldsymbol{I}\cdot\boldsymbol{n}\,ds + \int_R \rho\boldsymbol{S}\,dv \tag{5.1.4}$$

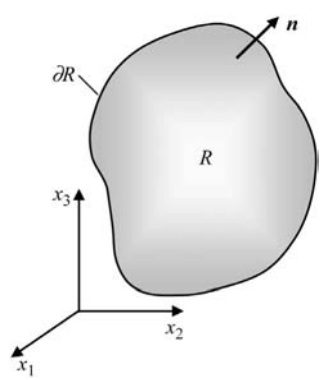

FIGURE 5.1

Typical control volume.

where $\boldsymbol{\Psi}$ is some tensor field, $\boldsymbol{I}$ is an influx term due to transfer through the boundary ∂R, $\boldsymbol{S}$ is an internal source term, and the minus sign is needed since $\boldsymbol{n}$ is the outward normal. We now proceed to several specific conservation or balance relations.

5.2 CONSERVATION OF MASS

Considering an arbitrary portion of a material R_m, the global principle of conservation of mass can be simply stated that total mass in R_m should remain constant at all times. Thus, we can write

$$\frac{D}{Dt}\int_{R_m} \rho\, dV = 0 \tag{5.2.1}$$

Incorporating the Reynolds Transport Theorem (5.1.2) or (5.1.3) gives the results

$$\frac{D}{Dt}\int_{R_m} \rho\, dV = \int_R \frac{\partial}{\partial t}\rho\, dv + \int_{\partial R} \rho v_k n_k\, ds = \int_R \left(\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k}\right) dv = 0 \tag{5.2.2}$$

Note that the first form of (5.2.2) indicates that the rate of change of mass in R is equal to the density change plus the total rate of mass entering the region through the boundary ∂R. Employing the Localization Theorem (2.17.16) implies that the common integrand in (5.2.2) must be zero:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \\ &\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = 0 \end{aligned} \tag{5.2.3}$$

Relation (5.2.3) is known as the *differential statement of the conservation of mass* and is a point-wise relation that applies at all continuum points within R. It is also often referred to as the *continuity equation* in fluid mechanics. This relation can be written in the alternative form as

$$\begin{aligned} &\frac{D\rho}{Dt} + \rho v_{k,k} = 0 \\ &\frac{D\rho}{Dt} + \rho \nabla\cdot \boldsymbol{v} = 0 \end{aligned} \tag{5.2.4}$$

Note that if the material is *incompressible*, then $D\rho / Dt = 0$, and thus

$$v_{k,k} = 0, \quad \nabla\cdot \boldsymbol{v} = 0 \tag{5.2.5}$$

For a fixed group of continuum particles, this conservation principle requires the mass to be the same in all configurations. Thus, we may use the reference and current configurations and write

$$\int_{R_o} \rho_o(\boldsymbol{X}, t)\, dV = \int_R \rho(\boldsymbol{x}, t)\, dv \tag{5.2.6}$$

where R_o and ρ_o are the reference volume and density, respectively, and R and ρ are the respective current values. Using (3.7.4) in (5.2.6) gives

$$\int_{R_o} (\rho_o - \rho J)\, dV = 0 \tag{5.2.7}$$

which is again true for all regions R_o, and so

$$\rho_o - \rho J = 0 \Rightarrow \frac{D}{Dt}(\rho J) = 0 \tag{5.2.8}$$

where we have used the fact that in the reference configuration $J_o = 1$. Relations (5.2.7) and (5.2.8) are sometimes referred to as the *Lagrangian or reference form of the conservation of mass*. Notice that expanding (5.2.8) gives the usual form (5.2.4).

$$\frac{D}{Dt}(\rho J) = \dot{\rho} J + \rho \dot{J} = J(\dot{\rho} + \rho v_{k,k}) = 0 \Rightarrow \dot{\rho} + \rho v_{k,k} = 0$$

Note that with the conservation of mass, the Reynolds Transport Theorem then gives the general result that for any tensor field $\boldsymbol{G}$:

$$\frac{D}{Dt}\int_R \rho \boldsymbol{G}\, dV = \int_R \rho \frac{D}{Dt}\boldsymbol{G}\, dv \tag{5.2.9}$$

EXAMPLE 5.2.1 CONSERVATION OF MASS CHECK

A continuum has a mass density given by $\rho = \dfrac{\rho_o}{a + bt}$, and a motion specified by $x_1 = (a + bt)X_1, x_2 = X_2, x_3 = X_3$, where ρ_o, a, and b are constants.

(a) Verify that the differential form of the conservation of mass is satisfied.

(b) Verify the global mass balance on a particular Cartesian unit cubical element defined by R: $\{1 \le x_1 \le 2; 0 \le x_2 \le 1; 0 \le x_3 \le 1\}$.

Solution: First compute the velocity field: $v_1 = \dfrac{Dx_1}{Dt} = bX_1 = \dfrac{bx_1}{a + bt}, v_2 = v_3 = 0$

(a) Differential mass balance check:

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \rho v_{1,1} = 0 \Rightarrow -\frac{\rho_o b}{(a + bt)^2} + \frac{\rho_o}{a + bt}\frac{b}{a + bt} = 0$$

(b) For global mass balance check on particular element, use (5.2.1) and (5.2.2). With $v_2 = v_3 = 0$, only need to check x_1-faces for surface integrals $\Rightarrow$

$$\begin{aligned}
\frac{D}{Dt}\int_{R_m} \rho\, dV &= \int_R \frac{\partial}{\partial t}\rho\, dv + \int_{\partial R} \rho v_1 n_1\, ds \\
&= \int_0^1\int_0^1\int_1^2 \frac{\partial}{\partial t}\frac{\rho_o}{a + bt} dx_1\, dx_2\, dx_3 + \frac{\rho_o}{a + bt}\frac{b(1)}{a + bt}(-1)\int_0^1\int_0^1 dx_2\, dx_3 \\
&\quad + \frac{\rho_o}{a + bt}\frac{b(2)}{a + bt}(1)\int_0^1\int_0^1 dx_2\, dx_3 \\
&= -\frac{\rho_o b}{(a + bt)^2} + \frac{\rho_o b}{(a + bt)^2} = 0
\end{aligned}$$

5.3 CONSERVATION OF LINEAR MOMENTUM

The principle of conservation of linear momentum is basically a statement of Newton's second law for a collection of particles. It can be stated as *the time rate of change of the total linear momentum of a given group of continuum particles equals the sum of all the external forces acting on the group*. This concept is valid provided that Newton's third law of action–reaction governs the internal forces between the particles and thus all internal forces will cancel each other when summed over the entire system. Thus, consider a fixed group of continuum particles instantaneously occupying a region of space R and acted upon by external surfaces forces $\boldsymbol{t}$ and body forces $\boldsymbol{b}$ as shown in Fig. 5.2. Using previous force relations (4.1.1) and (4.1.2), we may express this concept as

$$\frac{D}{Dt}\int_R \rho \boldsymbol{v}\, dv = \int_{\partial R} \boldsymbol{t}\, ds + \int_R \rho \boldsymbol{b}\, dv \tag{5.3.1}$$

Shifting to index notation and introducing the Cauchy stress tensor

$$\begin{aligned}\frac{D}{Dt}\int_R \rho v_i\, dv &= \int_{\partial R} t_i\, ds + \int_R \rho b_i\, dv \\ &= \int_{\partial R} T_{ji} n_j\, ds + \int_R \rho b_i\, dv \\ &= \int_R T_{ji,j}\, dv + \int_R \rho b_i\, dv\end{aligned} \tag{5.3.2}$$

where we have used the Divergence Theorem. Employing the Reynolds Transport Theorem (5.1.2) and the conservation of mass (5.2.4) allows the evaluation

$$\begin{aligned}\frac{D}{Dt}\int_R \rho v_i\, dv &= \int_R \frac{D}{Dt}(\rho v_i)\, dv + \int_R \rho v_i v_{k,k}\, dv \\ &= \int_R \left(\frac{D\rho}{Dt} v_i + \rho \frac{Dv_i}{Dt}\right) dv + \int_R \rho v_i v_{k,k}\, dv \\ &= \int_R \rho \frac{Dv_i}{Dt}\, dv + \int_R v_i \left(\frac{D\rho}{Dt} + \rho v_{k,k}\right) dv \\ &= \int_R \rho \frac{Dv_i}{Dt}\, dv = \int_R \rho a_i\, dv\end{aligned} \tag{5.3.3}$$

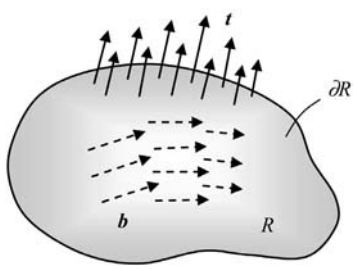

FIGURE 5.2

Body and surface forces acting on arbitrary portion of a continuum.

where a_i is the acceleration field. This result would also come more directly from (5.2.9).

Combining (5.3.2) and (5.3.3) yields

$$\int_R (T_{ji,j} + \rho b_i - \rho a_i)\, dv = 0 \tag{5.3.4}$$

Again using the Localization Theorem implies that the integrand must vanish

$$T_{ji,j} + \rho b_i = \rho a_i \tag{5.3.5}$$

or in a vector format

$$\nabla \boldsymbol{T}^T + \rho \boldsymbol{b} = \rho \boldsymbol{a} = \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) \tag{5.3.6}$$

or in a scalar format

$$\begin{aligned}
&\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} + \rho b_1 = \rho a_1\\
&\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} + \rho b_2 = \rho a_2\\
&\frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + \rho b_3 = \rho a_3
\end{aligned} \tag{5.3.7}$$

Relations (5.3.5)–(5.3.7) are known as *Cauchy's equations of motion*—a differential form of the conservation of linear momentum. For the case of small deformations, the acceleration expression (3.8.3) $\boldsymbol{a}(\boldsymbol{x},t) = \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2}$ is then used in the equations of motion. When the problem is in static equilibrium where there is no or negligible acceleration, Eqs. (5.3.5) reduce to the *equations of equilibrium*:

$$T_{ji,j} + \rho b_i = 0 \tag{5.3.8}$$

The previous equations of motion or equilibrium were expressed in terms of the Cauchy stress in the current configuration. We now wish to develop the conservation of linear momentum in the reference configuration by employing the first Piola–Kirchhoff (PK1) stress tensor $\boldsymbol{T}^o$ previously defined in relation (4.7.5), $\boldsymbol{T}^o = J\boldsymbol{T}(\boldsymbol{F}^{-1})^T$ or by the inverted relation (4.7.6) $\boldsymbol{T} = J^{-1}\boldsymbol{T}^o\boldsymbol{F}^T$. From our previous conservation of mass work

$$\rho_o\, dV = \rho\, dv \tag{5.3.9}$$

and from (4.7.1) and (4.7.2), we can write

$$t_i\, ds = T_i^R\, dS = T_{ij}^o N_j\, dS \tag{5.3.10}$$

Using these results, relations (5.3.2) and (5.3.3) can be transformed to the reference configuration as

$$\int_{R_o} \rho_o \frac{D}{Dt} v_i \, dV = \int_{\partial R_o} T_i^R \, dS + \int_{R_o} \rho_o B_i \, dV$$
$$= \int_{\partial R_o} T_{ij}^o N_j \, dS + \int_{R_o} \rho_o B_i \, dV \tag{5.3.11}$$

where B_i is the body force density in the reference configuration and all variables are functions of the material coordinates $\boldsymbol{X}$. Using the Divergence Theorem again on the surface integral term gives

$$\int_{R_o} \left(T_{ij,j}^o + \rho_o B_i - \rho_o \frac{Dv_i}{Dt} \right) dV = 0 \tag{5.3.12}$$

Finally invoking the Localization Theorem gives the differential form

$$\frac{\partial T_{ij}^o}{\partial X_j} + \rho_o B_i = \rho_o a_i \tag{5.3.13}$$

Result (5.3.13) then represents the desired form, the *conservation of linear momentum in the reference configuration*. This result can also be derived from the current configuration form (5.3.5) by eliminating the Cauchy stress using (4.7.6) and employing some detailed tensor analysis steps (see Exercise 5.11).

EXAMPLE 5.3.1 BODY FORCE: PRISMATIC BAR UNDER SELF-WEIGHT

Consider a simple static example of a uniform prismatic bar being loaded under its own self-weight as shown in Fig. 5.3. The bar is fixed at the top end, and assume a one-dimensional stress field with T_{11} being the only nonzero stress. Develop the equation of equilibrium for this case and integrate the single equation to determine the nonzero stress component.

Solution: The body force density (per unit mass) for this problem with the given coordinate system would be given by $b_1 = -g, b_2 = b_3 = 0$, where g is the local acceleration of gravity. For this case, the only nonzero equilibrium equation becomes

$$\frac{\partial T_{11}}{\partial x_1} + \rho b_1 = 0 \Rightarrow \frac{\partial T_{11}}{\partial x_1} - \rho g = 0$$

Note that since the bar is uniform, the mass density ρ is constant. This simple differential equation is easily integrated to give the stress result $T_{11} = \rho g x_1$, where the constant of integration has been set to zero to satisfy the boundary condition $T_{11}(0) = 0$. This is a rare case where we can solve for the stress without incorporating a material constitutive equation.

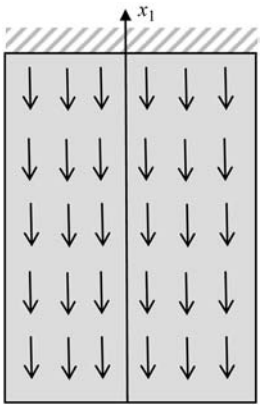

FIGURE 5.3

Bar under self-weight.

5.4 CONSERVATION OF MOMENT OF MOMENTUM

The angular or moment of momentum principle is another form of Newton's second law. For our application, it states that *the time rate of change of the total angular momentum of a group of particles must equal the sum of all external moments acting on the system with respect to some arbitrary point in space*. This concept is again valid provided that Newton's third law of action–reaction governs the internal forces between the particles and thus the moments of all internal forces will cancel each other.

So we again consider a fixed group of continuum particles instantaneously occupying a region of space R and acted upon by external surfaces forces $\boldsymbol{t}$ and body forces $\boldsymbol{b}$ as was shown in Fig. 5.2. Neglecting any distributed body of surface couples, we then can write

$$\frac{D}{Dt}\int_R \boldsymbol{r}\times\rho\boldsymbol{v}\,dv = \int_{\partial R}\boldsymbol{r}\times\boldsymbol{t}\,ds + \int_R \boldsymbol{r}\times\rho\boldsymbol{b}\,dv \tag{5.4.1}$$

where $\boldsymbol{r}$ is the position vector with respect to some arbitrary point. Working in index notation

$$\begin{aligned}
&\frac{D}{Dt}\int_R \varepsilon_{ijk}x_j\rho v_k\,dv = \int_{\partial R}\varepsilon_{ijk}x_j t_k\,ds + \int_R \varepsilon_{ijk}x_j\rho b_k\,dv\\
&\int_R \varepsilon_{ijk}\frac{D}{Dt}(x_j v_k)\rho\,dv = \int_R \varepsilon_{ijk}(x_j T_{lk})_{,l}\,dv + \int_R \varepsilon_{ijk}x_j\rho b_k\,dv\\
&\int_R \varepsilon_{ijk}[v_j v_k\rho + x_j a_k\rho - T_{jk} - x_j T_{lk,l} + x_j\rho b_k]dv = 0\\
&\int_R \varepsilon_{ijk}(v_j v_k\rho - T_{jk} - [T_{lk,l} + \rho b_k - \rho a_k]x_j)\,dv = 0
\end{aligned} \tag{5.4.2}$$

Now since $\varepsilon_{ijk}v_j v_k = 0$ (symmetric times antisymmetric) and $T_{lk,l} + \rho b_k - \rho a_k = 0$ from the equations of motion (5.3.5), we get

$$\int_R \varepsilon_{ijk}T_{jk}\,dv = 0 \Rightarrow \varepsilon_{ijk}T_{jk} = 0 \tag{5.4.3}$$

Since ε_{ijk} is antisymmetric, then T_{jk} must be symmetric, and this proves the *symmetry of the Cauchy stress tensor.* Thus,

$$\begin{aligned} &T_{ij} = T_{ji}, \boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}} \Rightarrow \\ &T_{12} = T_{21} \\ &T_{23} = T_{32} \\ &T_{31} = T_{13} \end{aligned} \tag{5.4.4}$$

Relation (5.4.4), sometimes called *Cauchy's Second Law of Motion*, is the result from the balance of the moment of momentum. Recall that this result hinges on the assumption of no distributed body or surface couples. We will later explore some specialized continuum mechanics theories for *polar materials* where we wish to include such internal loadings, and for such cases the Cauchy stress will no longer be symmetric.

5.5 CONSERVATION OF LINEAR MOMENTUM EQUATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

Similar to Sections 3.17 and 4.10, we now wish to list the Cauchy equations of motion (5.3.7) in curvilinear cylindrical and spherical coordinate systems. In direct notation, these equations read

$$\nabla \boldsymbol{T} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \tag{5.5.1}$$

where we have used the symmetry of the stress tensor. The various curvilinear components of the traction vector and stress tensor have been previously defined in Section 4.10. The spatial acceleration term follows from relation (3.4.3):

$$\boldsymbol{a}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \tag{5.5.2}$$

Section 2.18 outlined the basic procedures to determine the vector differential operator and various gradient operations necessary to construct the specific equations. Here, we only provide a listing of the final results. Appendix A includes these results along with other field equation forms.

Equations of motion in cylindrical coordinates:

$$\begin{aligned} &\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \rho b_r \\ &= \rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) \\ &\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta} + \rho b_\theta \\ &= \rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) \\ &\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + \rho b_z \\ &= \rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) \end{aligned} \tag{5.5.3}$$

Equations of motion in spherical coordinates:

$$
\begin{aligned}
&\frac{\partial T_{RR}}{\partial R}+\frac{1}{R}\frac{\partial T_{R\phi}}{\partial \phi}+\frac{1}{R\sin\phi}\frac{\partial T_{R\theta}}{\partial \theta}+\frac{1}{R}(2T_{RR}-T_{\phi\phi}-T_{\theta\theta}+T_{R\phi}\cot\phi)+\rho b_R\\
&=\rho\left(\frac{\partial v_R}{\partial t}+v_R\frac{\partial v_R}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_R}{\partial \theta}-\frac{v_\phi^2+v_\theta^2}{R}\right)\\
&\frac{\partial T_{R\phi}}{\partial R}+\frac{1}{R}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{1}{R\sin\phi}\frac{\partial T_{\phi\theta}}{\partial \theta}+\frac{1}{R}[(T_{\phi\phi}-T_{\theta\theta})\cot\phi+3T_{R\phi}]+\rho b_\phi\\
&=\rho\left(\frac{\partial v_\phi}{\partial t}+v_R\frac{\partial v_\phi}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_\phi}{\partial \theta}+\frac{v_R v_\phi}{R}-\frac{v_\theta^2\cot\phi}{R}\right)\\
&\frac{\partial T_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial T_{\phi\theta}}{\partial \phi}+\frac{1}{R\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{1}{R}(2T_{\phi\theta}\cot\phi+3T_{R\theta})+\rho b_\theta\\
&=\rho\left(\frac{\partial v_\theta}{\partial t}+v_R\frac{\partial v_\theta}{\partial R}+\frac{v_\phi}{R}\frac{\partial v_\theta}{\partial \phi}+\frac{v_\theta}{R\sin\phi}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\theta v_R}{R}+\frac{v_\phi v_\theta\cot\phi}{R}\right)
\end{aligned}
\tag{5.5.4}
$$

It is again noticeable that these cylindrical and spherical forms are considerably more complicated in form than the corresponding Cartesian relations. However, for many problems, these curvilinear equations will allow solutions to be generated that would not be possible using a Cartesian formulation.

5.6 CONSERVATION OF ENERGY

Our final conservation principle involves the energy balance, and this has the potential to bring into play many possible energy forms including mechanical, thermal, chemical, electrical, magnetic, and others. We will generally limit our study here to problems involving only mechanical and thermal phenomena and thus will develop various *thermomechanical material models*. However, continuum mechanics theories can easily be constructed in a similar manner to handle much broader physical behaviors.

The principle of conservation of energy may be stated as *the time rate of change of the kinetic and internal energy of a given group of continuum particles is equal to the sum of the rate of change of work done by the external forces, and the energy entering the system through the boundary*. The *internal energy* represents all energy associated with the microscopic modes of continuum motion which are not accounted for in the bulk mechanical energy terms. For most applications, the energy entering through the boundary will be in the form of *thermal energy*. Note that we can also include internal distributed sources as energy entering the system. This energy principle is usually referred to as the *First Law of Thermodynamics*. A more complete description of thermodynamics can be found in Tadmor et al. (2012).

Using our control volume shown in Fig. 5.1 to hold our fixed group of particles, the *kinetic energy* of the group is given by

$$K=\int_R \frac{1}{2}v^2\rho\,dv \tag{5.6.1}$$

where $v^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ is the square of the velocity field within R. The *internal energy* may be written as

$$E = \int_R \varepsilon \rho \, dv \tag{5.6.2}$$

where $\varepsilon = \varepsilon(\boldsymbol{x}, t)$ is the *internal energy density per unit mass* and is a thermodynamic state variable. The rate of working of the external surface and body forces, often referred to as the *external mechanical power*, can be expressed by

$$P_{ext} = \int_{\partial R} \boldsymbol{t} \cdot \boldsymbol{v} \, ds + \int_R \boldsymbol{b} \cdot \boldsymbol{v} \rho \, dv \tag{5.6.3}$$

Relation (5.6.3) can be rewritten in several alternative forms. Working on the surface integral term

$$\begin{aligned} \int_{\partial R} \boldsymbol{t} \cdot \boldsymbol{v} \, ds &= \int_{\partial R} t_i v_i \, ds = \int_{\partial R} T_{ij} n_j v_i \, ds \\ &= \int_R (T_{ij} v_i)_{,j} \, dv = \int_R (T_{ij,j} v_i + T_{ij} v_{i,j}) \, dv \end{aligned} \tag{5.6.4}$$

where we have used the Divergence Theorem to change the surface integral to volume integral. Next we note that using (3.13.3), $T_{ij} v_{i,j} = T_{ij}(D_{ij} + W_{ij}) = T_{ij} D_{ij}$, since W_{ij} is antisymmetric. Thus, the external mechanical power relation can be written as

$$\begin{aligned} P_{ext} &= \int_R [(T_{ij,j} + \rho b_i) v_i + T_{ij} D_{ij}] \, dv \\ &= \int_R [\rho a_i v_i + T_{ij} D_{ij}] \, dv \end{aligned} \tag{5.6.5}$$

The term $\int_R T_{ij} D_{ij} \, dv$ is commonly referred to as the *stress power*. The two variables T_{ij} and D_{ij} are often called *energetic conjugates*, since the integral of the double dot product gives the system energy due to the stress and strain fields. Introducing the first and second Piola–Kirchhoff stress tensors T_{ij}^o and S_{ij}, the stress power can be expressed in the following equivalent relations in the reference configuration:

$$\int_R T_{ij} D_{ij} \, dv = \int_{R_o} T_{ij}^o \dot{F}_{ij} \, dV = \int_{R_o} S_{ij} \dot{E}_{ij} \, dV \tag{5.6.6}$$

where $\dot{F}_{ij}$ is the material time rate of the deformation gradient tensor and $\dot{E}_{ij}$ is the material time rate of the Lagrangian strain (3.6.2). Notice that new conjugate stress-deformation rate pairs are developed in these alternative forms.

For our thermomechanical study, the energy entering system R both through the boundary ∂R and from internal sources may be written as

$$Q = -\int_{\partial R} \boldsymbol{q} \cdot \boldsymbol{n} \, ds + \int_R h \rho \, dv \tag{5.6.7}$$

with $\boldsymbol{q}$ being the *rate of heat flux per unit area* and h is the *specific energy source (supply) per unit mass*. A common example of a source term would be from a radioactive material generating heat in a distributed fashion.

The general energy balance statement then reads

$$\dot{K} + \dot{E} = P_{ext} + Q \tag{5.6.8}$$

and substituting in the previous specific results for the various energy pieces gives

$$\frac{D}{Dt}\int_R \frac{1}{2}v^2 \rho\, dv + \frac{D}{Dt}\int_R \varepsilon\rho\, dv = \int_R [\rho a_i v_i + T_{ij} D_{ij}]\, dv - \int_{\partial R} q_i n_i\, ds + \int_R h\rho\, dv \tag{5.6.9}$$

Using the Reynolds Transport and Divergence Theorems, this result can be expressed as

$$\int_R \rho \frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon\right) dv = \int_R [\rho a_i v_i + T_{ij} D_{ij}]\, dv - \int_R q_{i,i}\, dv + \int_R h\rho\, dv \tag{5.6.10}$$

Now the term $\rho a_i v_i = \rho \dfrac{Dv_i}{Dt} v_i = \dfrac{1}{2}\rho \dfrac{Dv^2}{Dt}$, and thus (5.6.8) reduces to

$$\int_R (\rho\dot{\varepsilon} - T_{ij} D_{ij} + q_{i,i} - \rho h)\, dv = 0 \tag{5.6.11}$$

Again employing the Localization Theorem (2.17.16), the integrand itself must vanish

$$\begin{aligned} &\rho\dot{\varepsilon} - T_{ij} D_{ij} + q_{i,i} - \rho h = 0 \\ &\rho\dot{\varepsilon} - tr(\boldsymbol{TD}) + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \rho h = 0 \end{aligned} \tag{5.6.12}$$

which is the *differential form of the energy balance equation* for thermomechanical continuum mechanics. Clearly, differential relation (5.6.12) is written in terms of local spatial variables, but could be restated in the reference configuration using (5.6.6) for the stress power and simply converting the other scalar variables to reference values. Since relation (5.6.12) involves additional unknowns of internal energy ε, and the heat flux vector $\boldsymbol{q}$ (usually the source term h is given), the energy equation requires additional relations among ε, $\boldsymbol{q}$, and the temperature θ. These addition relations are normally involved with *thermodynamical constitutive equations* sometimes called *equations of state*. These will be discussed later in Chapter 7.

EXAMPLE 5.6.1 STRESS POWER CALCULATION FOR SIMPLE SHEAR

Determine the stress power term $T_{ij}D_{ij}$, if the continuum undergoes simple shearing motion and has the particular Cauchy plane stress form

$$\begin{aligned} x_1 &= X_1 + \gamma(t)X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \qquad T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: For simple shearing deformation, the velocity field becomes $v_1 = \dot{\gamma}(t)x_2, v_2 = v_3 = 0$, and thus the velocity gradient and rate of deformation tensors are

$$\boldsymbol{L}=\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}=\begin{bmatrix} 0 & \dot{\gamma}(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{D}=\frac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^T\right)=\frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma}(t) & 0 \\ \dot{\gamma}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The stress power term is then

$$T_{ij}D_{ij}=tr(\boldsymbol{TD})=\frac{1}{2}tr\left\{\begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \dot{\gamma}(t) & 0 \\ \dot{\gamma}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$
$$=\frac{1}{2}tr\begin{bmatrix} T_{12}\dot{\gamma}(t) & T_{11}\dot{\gamma}(t) & 0 \\ T_{22}\dot{\gamma}(t) & T_{12}\dot{\gamma}(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}=T_{12}\dot{\gamma}(t)$$

5.7 SECOND LAW OF THERMODYNAMICS—ENTROPY INEQUALITY

The previous section developed the basic energy balance or conservation of energy principle commonly known as the first law of thermodynamics. The differential form (5.6.12) can be viewed as a measure of the interconvertibility of heat and work while maintaining a proper energy balance. However, the expression provides no restrictions on the direction of any such interconvertibility processes, and this is an important issue in thermomechanical behavior of materials. When considering thermal effects with dissipation phenomena, the direction of energy transfer must satisfy certain criteria, and this introduces *irreversible processes*. For example, a process in which friction changes mechanical energy into heat energy cannot be reversed. Another common observable restriction is that heat only flows from warmer regions to cooler regions and not the other way. Collectively, such restrictions are connected to the *second law of thermodynamics.* Of course various restrictions relate to the second law in different ways, and we wish to establish a mathematical relationship applicable to the thermomechanical behavior of continuum materials. One particular mathematical statement associated with the second law is the *Clausius–Duhem entropy inequality*, and this relation has broad applications for many materials we wish to study. As we shall see in later chapters, this inequality will place restrictions on the material response functions. More detailed background on this topic can be found in Haupt (2002), Holzapfel (2006), Asaro and Lubarda (2006), and Tadmor et al. (2012).

Basic to our development in this section is the definition of *entropy*. This rather abstract variable can be interpreted as a measure of the microscopic randomness or disorder of the continuum system. In classical thermodynamics, it is commonly

defined as a state function related to heat transfer. For a reversible process, the *entropy per unit mass*, $s(\boldsymbol{x}, t)$ is commonly defined by the relation

$$ds = \left(\frac{\delta q}{\theta}\right)_{rev} \tag{5.7.1}$$

where θ is the absolute temperature (Kelvin-scale, always positive) and δq is the heat input per unit mass over a *reversible process*. Since relation (5.7.1) is an exact differential, we may write it between two states 1 and 2, or for an entire cycle as

$$\Delta s = s_2 - s_1 = \int_1^2 \left(\frac{\delta q}{\theta}\right)_{rev} \quad \text{or} \quad \oint ds = \oint \left(\frac{\delta q}{\theta}\right)_{rev} = 0 \tag{5.7.2}$$

However, for irreversible processes (the real world), observations indicate that

$$\oint \left(\frac{\delta q}{\theta}\right)_{irrev} < 0 \tag{5.7.3}$$

Since we interpret $\delta q / \theta$ as the entropy input from the heat input δq, we conclude that over an irreversible cycle, the *net entropy input is negative.* However, as entropy is assumed to be a state variable, it must return to its initial value at the end of any cycle. Because of this, the negative entropy input shown in (5.7.3) implies that entropy has been created inside the system. In other words, *dissipative irreversible processes produce a positive internal entropy production.* Therefore, for an irreversible change of state $1 \rightarrow 2$, the entropy increase will be greater than the entropy input by heat transfer

$$\Delta s > \int_1^2 \left(\frac{\delta q}{\theta}\right)_{irrev} \tag{5.7.4}$$

The previous relations form the fundamental basis for the second law.

However, for use in continuum mechanics, the second law is normally rephrased in a different form. First, consider a fixed group (closed system) of continuum particles R_m that occupying spatial region R:

$$\textit{Entropy input rate} = \int_R \frac{\rho h}{\theta} dv - \int_{\partial R} \frac{\boldsymbol{q} \cdot \boldsymbol{n}}{\theta} ds \tag{5.7.5}$$

with again $\boldsymbol{q}$ being the *rate of heat flux per unit area* and h the *specific energy source per unit mass*. Note that the ds term in the surface integral is the differential surface area and not the differential entropy. Now according to relation (5.7.4), the rate of entropy increase in R must be greater than or equal to (for the reversible case) the entropy input rate, and thus

$$\frac{D}{Dt}\int_R s\rho \, dv \geq \int_R \frac{\rho h}{\theta} dv - \int_{\partial R} \frac{\boldsymbol{q} \cdot \boldsymbol{n}}{\theta} ds \tag{5.7.6}$$

Using our usual procedures on the integral formulations, we apply relation (5.2.9) and the Divergence Theorem to get

$$\int_R \left(\dot{s}\rho - \frac{\rho h}{\theta} + \left(\frac{q_i}{\theta}\right)_{,i} \right) dv \geq 0 \tag{5.7.7}$$

and this implies

$$\dot{s} \geq \frac{h}{\theta} - \frac{1}{\rho}\left(\frac{q_i}{\theta}\right)_{,i}$$
$$\dot{s} \geq \frac{h}{\theta} - \frac{1}{\rho}\nabla \cdot \left(\frac{\boldsymbol{q}}{\theta}\right) \tag{5.7.8}$$

Relations (5.7.7) and (5.7.8) are known as the *integral and differential forms of the Clausius–Duhem inequality*. They represent forms of the second law of thermodynamics for continuum mechanics applications.

It is easily shown that (5.7.8) can also be expressed as

$$\dot{s} \geq \frac{h}{\theta} - \frac{1}{\rho\theta}(\nabla \cdot \boldsymbol{q}) + \frac{1}{\rho\theta^2}(\boldsymbol{q} \cdot \nabla\theta) \tag{5.7.9}$$

By using the energy equation (5.6.12), the entropy inequality can be expressed as

$$\rho(\theta\dot{s} - \dot{\varepsilon}) + T_{ij}D_{ij} - \frac{1}{\theta}(\boldsymbol{q} \cdot \nabla\theta) \geq 0 \tag{5.7.10}$$

which is sometimes referred to as the *reduced Clausius–Duhem or Dissipation inequality*. In some applications, the *free energy* $\Psi = \varepsilon - s\theta$, is introduced, and relation (5.7.10) would become

$$-\rho(\dot{\Psi} + s\dot{\theta}) + T_{ij}D_{ij} - \frac{1}{\theta}(\boldsymbol{q} \cdot \nabla\theta) \geq 0 \tag{5.7.11}$$

It should also be pointed out that the observable and accepted concept that heat only flows from regions of higher temperature to lower temperature implies that

$$\boldsymbol{q} \cdot \nabla\theta \leq 0 \tag{5.7.12}$$

with equality only if $\nabla\theta = 0$. This relation is sometimes called the *classical heat conduction inequality*. Using this relation with (5.7.9), we can argue the stronger statement

$$\dot{s} \geq \frac{h}{\theta} - \frac{1}{\rho\theta}(\nabla \cdot \boldsymbol{q}) \tag{5.7.13}$$

which is known as the *Clausius–Planck inequality*. Another form of this relation can be found using (5.7.10):

$$\rho(\theta\dot{s} - \dot{\varepsilon}) + T_{ij}D_{ij} \geq 0 \tag{5.7.14}$$

5.8 SUMMARY OF CONSERVATION LAWS, GENERAL PRINCIPLES, AND UNKNOWNS

Since so many laws and principles have been developed in this chapter, we now provide a listing of these basic relations along with the associated unknowns appearing in the equations. Table 5.1 illustrates these relations and summarizes the

Table 5.1 Summary of general equations and unknowns

Equation	No. of Equations	Unknowns (No Repeats)	No. of Unknowns
Conservation of mass: $\frac{D\rho}{Dt} + \rho\nabla\cdot v = 0$	1	Density, velocity	4
Conservation of linear momentum: $\nabla T^T + \rho b = \rho a = \frac{\partial v}{\partial t} + v\cdot\nabla v$	3	Cauchy stress	6
Conservation of angular momentum: $T = T^T$	—	—	—
Conservation of energy: $\rho\dot{\varepsilon} - \mathrm{tr}(TD) + \nabla\cdot q - \rho h = 0$	1	Internal energy, heat flux vector	4
Clausius–Duhem inequality: $\dot{s} \geq \frac{h}{\theta} - \frac{1}{\rho}\nabla\cdot\left(\frac{q}{\theta}\right)$	—	Entropy, temperature	2
Strain–displacement: $E = \frac{1}{2}[(\nabla u) + (\nabla u)^T + (\nabla u)^T(\nabla u)]$ Strain Rate–velocity: $D = \frac{1}{2}(\nabla v + (\nabla v)^T)$	6	Strain or strain rate	6
Total	11		22

number of equations and unknowns for each general principle. We have also included a fundamental kinematic relation from Chapter 3 that relates either the strain and displacements or strain rates and velocities. The conservation of angular momentum simply gives symmetry of the Cauchy stress tensor (for nonpolar materials). This result is automatically incorporated into all other relations, and so this system of equations (5.4.4) is generally not included in the table total. Likewise, although the Clausius–Duhem inequality will place restrictions on material behavior, it is often not considered a governing equation and is used only on an irregular basis. Under these conditions, the thermomechanical system coming from conservation principles and kinematics totals to 11 governing equations with 22 unknowns including mass density, velocity (or equivalently motion or displacement), stress, internal energy, heat flux, and temperature. Entropy would be added to the unknown listing if the second law is included. For the reduced problem of a nonthermal mechanical system, the energy equation is dropped and thus the number of equations reduces to 10, while the unknowns reduce to 16 including mass density, velocity (or equivalently motion or displacement), and stress. Note that the body force and heat source variables are assumed to be given *a priori*.

For both mechanical and thermomechanical problems, our number counting finds insufficient numbers of equations to solve for all of the unknown variables. For the mechanical model, we need six more equations, while the thermomechanical case

needs 10 additional relations. These additional equations are needed to properly close the mathematical system. Up until this point, our field equations have been applicable to all continuum materials irrespective of their material properties. We have not brought into consideration any specific material behavior relations among either mechanical or thermodynamic variables. From common experience, we know that different materials will behave differently even under identical loading conditions. Therefore, the missing equations are to be found from *material response relations or constitutive relations*. These will typically involve stress–strain and/or stress–strain rate mechanical relations, and thermodynamic equations involving heat flux–temperature and internal energy specification. The next few chapters will develop these relations in detail, and all of this will eventually be included into overall system models in order to formulate complete continuum theories for a large variety of material types.

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EXERCISES

5.1 Explicitly justify relation (5.2.9).

5.2 Using the results in Section 2.18, develop the conservation of mass equation for cylindrical coordinates

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\left[\frac{\partial(r\rho v_r)}{\partial r}+\frac{\partial(\rho v_\theta)}{\partial \theta}+r\frac{\partial(\rho v_z)}{\partial z}\right]=0$$

5.3 Show that if the spatial velocity field is expressible by $\boldsymbol{v}=-\nabla\Phi$, then $\nabla\times\boldsymbol{v}=0$ and if the density is a constant (incompressible case), then conservation of mass implies that $\nabla^2\Phi=0$.

5.4 For the incompressible case, check if the given velocity fields satisfy conservation of mass

(a) $v_1=a(t)x_1, v_2=b(t)x_2, v_3=c(t)x_3$, with $a(t)+b(t)+c(t)=0$

(b) $v_1=kx_1x_2, v_2=-\frac{k}{2}x_2^2, v_3=0 \quad k=constant$

(c) $v_1=2k(x_1^2-x_2^2), v_2=-2kx_1x_2, v_3=kx_1x_2, \quad k=constant$

5.5 For two-dimensional flow of an incompressible fluid, the velocity field can be represented by a scalar *Lagrange stream function* $\psi(x_1, x_2)$ as $v_1 = \dfrac{\partial \psi}{\partial x_2}, v_2 = -\dfrac{\partial \psi}{\partial x_1}$. Show that this representation satisfies conservation of mass for any ψ with appropriate continuous derivatives.

5.6 In an *ideal nonviscous fluid*, there can be no shear stress, and so the Cauchy stress tensor must be hydrostatic (spherical), i.e. $T_{ij} = -p\delta_{ij}$. For this case, show that the equations of motion reduce to *Euler's equations of motion*

$$-\frac{1}{\rho}\nabla p + \boldsymbol{b} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}$$

5.7 Show that the following stress fields in the absence of body forces satisfy the equations of equilibrium (a, b, α all constants):

$$(a) \quad \boldsymbol{T} = \begin{bmatrix} \frac{a}{2}x_1^2 & -ax_1x_2 & 0 \\ -ax_1x_2 & \frac{a}{2}x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$(b) \quad \boldsymbol{T} = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix};$$

$$(c) \quad \boldsymbol{T} = \begin{bmatrix} ax_2 & bx_3 & 0 \\ bx_3 & ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5.8 For the plane stress case defined by relation (4.6.1), assume that the nonzero stresses can be defined in terms of the *Airy stress function*, $\phi(x_1, x_2)$:

$$T_{11} = \frac{\partial^2 \phi}{\partial x_2^2}, T_{22} = \frac{\partial^2 \phi}{\partial x_1^2}, T_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

Show that this representation identically satisfies the equilibrium equations (with no body forces) and hence is a *self-equilibrated form*.

5.9 Express the cylindrical equations of motion (5.5.3) for the simplified case of *axisymmetry* where all stresses and velocities are independent of the angle θ, and $v_\theta = 0$.

5.10 As a continuation of Exercise 5.9, express the cylindrical equations of motion (5.5.3) for the simplified case of *axisymmetry* where all stresses and velocities are independent of coordinates θ and z, with $v_\theta = v_z = 0$. Note for this case all partial derivatives become d/dr. Can any of the resulting ordinary differential equations be integrated using standard methods?

5.11 Using (4.7.6), eliminate the Cauchy stress from the equations of motion (5.3.5), and develop (5.3.13), the conservation of linear momentum equations in the reference configuration.

5.12 Recall the energetically conjugate form in the stress power relation $\int_R T_{ij} D_{ij}\, dv$.

Develop the corresponding forms using each of the two Piola–Kirchhoff stress tensors as given in relation (5.6.6).

5.13 Calculate the stress power term $T_{ij}D_{ij}$, if the continuum undergoes the following motion and has the particular Cauchy stress form:

$$(a)\quad \begin{aligned} x_1 &= 2tX_1 + 4t^2 X_2 \\ x_2 &= 4tX_2 \\ x_3 &= X_3 \end{aligned} \qquad T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (b)\quad \begin{aligned} x_1 &= 2tX_1 \\ x_2 &= tX_1 + 2tX_2 \\ x_3 &= X_3 \end{aligned} \qquad T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5.14 Consider the case of an ideal nonviscous fluid (see Exercise 5.6) with no internal heat sources and assume that the transfer of heat is governed by *Fourier's law* $\boldsymbol{q} = -k\nabla\theta$, where k is a constant known as the material *thermal conductivity*. Show that the energy equation reduces to

$$\rho\dot{\varepsilon} = k\nabla^2\theta - p(\nabla\cdot\boldsymbol{v})$$

5.15 Starting with the Clausius–Duhem inequality (5.7.8), develop the equivalent forms (5.7.9) and (5.7.10).

5.16 For a reversible process, the Clausius–Planck inequality (5.7.14) now becomes an equality, $\rho(\theta\dot{s} - \dot{\varepsilon}) + T_{ij}D_{ij} = 0$. Using the energy equation for this case with zero source term, show that

$$\rho\dot{s} = -\frac{q_{i,i}}{\theta}$$

5.17 For an isothermal process (constant temperature distribution), show that the Clausius–Duhem inequality (5.7.11) reduces to $\dot{\Psi} \le \frac{1}{\rho} T_{ij} D_{ij}$.