June 2022

Let $A, B \subset R$ be bounded above. Show that

$$A + B = \{x + y \in R \mid x \in A,\ y \in B\}$$

is bounded above, and

$$\sup(A + B) = \sup A + \sup B$$

Solution: Recall, we say that $\sup A$ is a supremum for the set A if and only if

1. $\sup A$ is an upper bound of A
   - i.e. $\forall_{a \in A} \quad a \leq \sup A$
2. $\sup A$ is the least upper bound of A
   - i.e. if α is also an upper bound of A then $\alpha \leq \sup A$

($\rightarrow$) In which we show that

$$\sup(A + B) \leq \sup A + \sup B$$

As $\sup A, \sup B$ are supremums of their respective sets

$$\forall_{a \in A} \forall_{b \in B} \quad a \leq \sup a \ \wedge \ b \leq \sup B$$

Combining these inequalities we find,

$$\forall_{a \in A} \forall_{b \in B} \quad a + b \leq \sup A + \sup B$$

And so, $\sup A + \sup B$ is an upper bound for the set $A + B$, yet as $\sup(A + B)$ is the least upper bound of $A + B$ it must be that

$$\sup(A + B) \leq \sup A + \sup B$$

($\leftarrow$) In which we show that

$$\sup(A + B) \geq \sup A + \sup B$$

As $\sup A, \sup B$ are supremum of their respective sets

$$\forall_{\epsilon > 0} \exists_{a \in A} \exists_{b \in B} \quad \sup a \leq a + \frac{\epsilon}{2} \ \wedge \ \sup b \leq b + \frac{\epsilon}{2}$$

Combining these inequalities,

$$\forall_{\epsilon>0}\exists_{a\in A}\exists_{b\in B}\quad \sup a + \sup b \le a + b + \epsilon$$

And as ϵ is arbitrary, in the limit as $\epsilon \to 0$ the inequality becomes

$$\forall_{\epsilon>0}\exists_{a\in A}\exists_{b\in B}\quad \sup a + \sup b \le a + b$$

Finally, we make use of the fact that $\sup(A+B)$ is an upper bound on the set $A+B$, in particular, $\forall_{a\in A, b\in B} a + b \le \sup(A+B)$ such that

$$\sup a + \sup b \le \sup(A+B)$$

The statement follows from the two inequalities demonstrated above.