# Introduction to Robotics

Presented by


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# Composition of Rotations

- Rotation with Respect to the Current Frame:

We make a rotation with respect to Frame 0 to get Frame 1, then another rotation w.r.t the current frame **(Frame 1)** to get Frame 2.

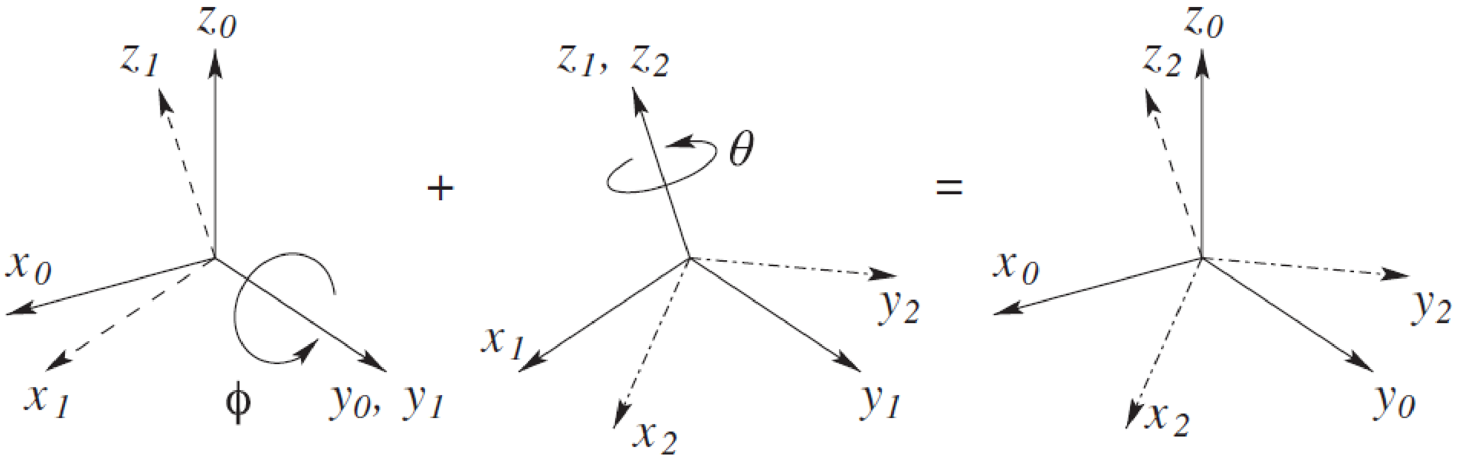



Figure 2.8: Composition of rotations about current axes.

- Rotation with Respect to the Fixed Frame:

We make a rotation with respect to Frame 0 to get Frame 1, then another rotation w.r.t the Fixed frame **(Frame 0)** to get Frame 2.

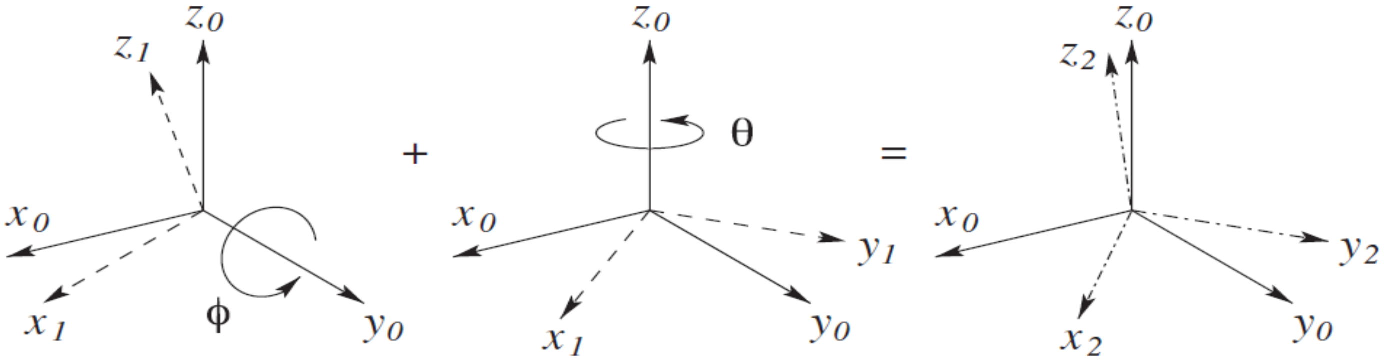



Figure 2.9: Composition of rotations about fixed axes.

# Composition of Rotations

- Rotation with Respect to the Current Frame:

$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2 \\
p^0 &= R_1^0 R_2^1 p^2
\end{aligned}
$$

$$R_2^0 = R_1^0 R_2^1$$

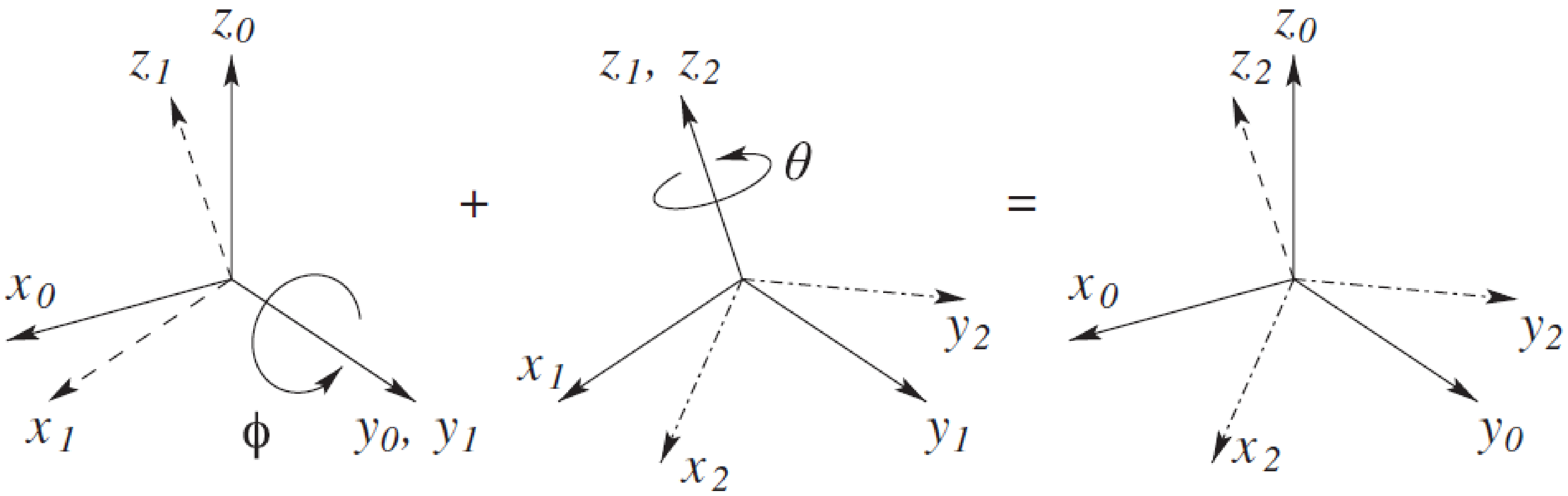



Figure 2.8: Composition of rotations about current axes.

If we make a rotation with respect to Frame 0 and get Frame 1 $R_1^0$), then another rotation w.r.t the current frame (Frame 1) to get Frame 2 ($R_2^1$), the rotation matrix of Frame 2 w.r.t Frame 0 = $R_1^0$ $R_2^1$ →Rule: The last rotation multiplied to the right of the previous multiplication (post-multiplication) if the rotation is referred to the **current frame**.

# Composition of Rotations

- Rotation with Respect to the Fixed Frame:

It is often desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames.

For example, we may wish to perform a rotation about x0 followed by a rotation about y0 (not y1!). We will refer to o0x0y0z0 as the fixed frame. In this case, the composition law in the previous slides is invalid.

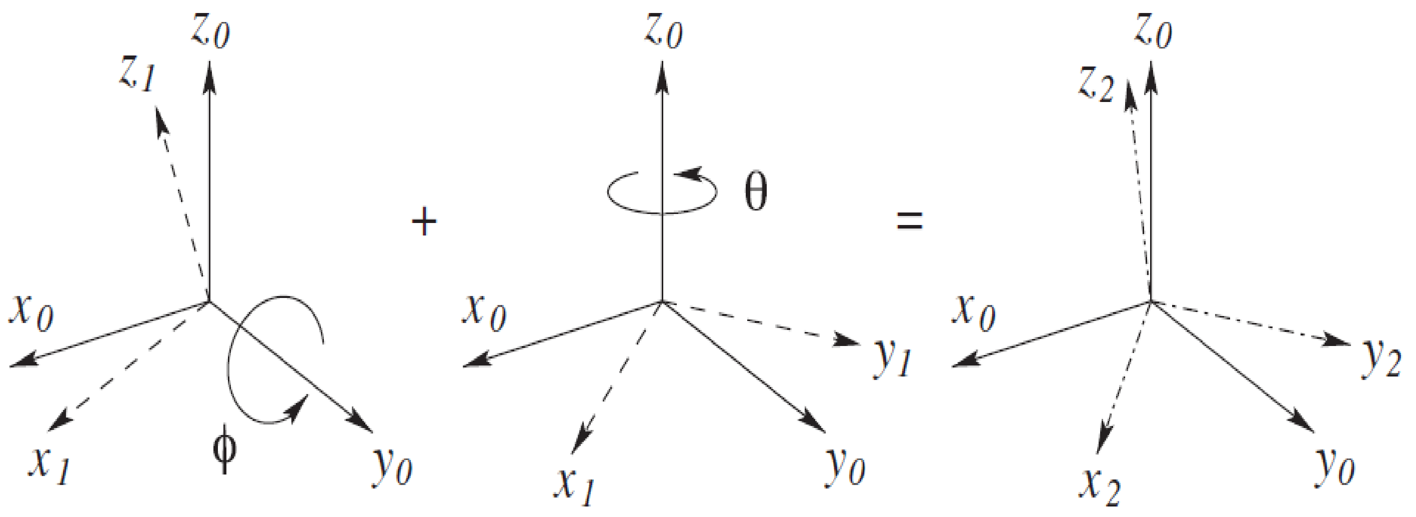



Figure 2.9: Composition of rotations about fixed axes.

$$R_2^0 = RR_1^0$$

If we make a rotation with respect to Frame 0 and get Frame 1 ($R_1^0$), then another rotation w.r.t the Fixed Frame (Frame 0) to get Frame 2 (R), the rotation matrix of Frame 2 w.r.t Frame 0 = $R\ R_1^0$ →Rule: The last rotation multiplied to the left of the previous multiplication (pre-multiplication) if the rotation is referred to the **Fixed frame**.

**Example 2.8.** *Suppose $R$ is defined by the following sequence of basic rotations in the order specified:*

1. *A rotation of $\theta$ about the current $x$-axis*
2. *A rotation of $\phi$ about the current $z$-axis*
3. *A rotation of $\alpha$ about the fixed $z$-axis*
4. *A rotation of $\beta$ about the current $y$-axis*
5. *A rotation of $\delta$ about the fixed $x$-axis*

*In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or postmultiply as the case may be to obtain*

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.24}$$

# Parameterizations of Rotations

Any rotation matrix R can be represented by three methods:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

First, rotate about the z-axis by the angle φ. Next, rotate about the **current** y-axis by the angle θ. Finally, rotate about the **current** z-axis by the angle ψ.

Euler Angles
(φ, θ, ψ)

First, a roll about $x_0$ through an angle ψ, then pitch about the $y_0$ by an angle θ. Finally, yaw about the $z_0$ by an angle φ. Since the successive rotations are relative to the **fixed frame**.

Roll, Pitch, Yaw Angles
(ψ, θ, φ)

In fact, any rotation matrix R can be represented by a single rotation about a suitable axis (k) in space by a suitable angle (θ).

Axis-angle
(θ)

End Effector

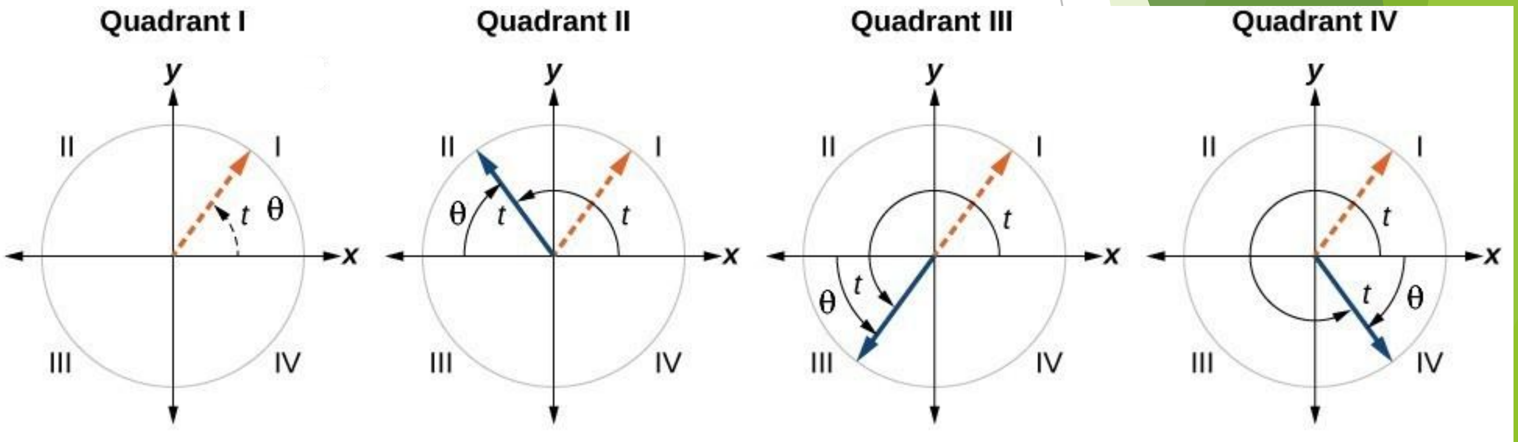
# Euler Angles

First, rotate about the z-axis by the angle φ. Next, rotate about the **current** y-axis by the angle θ. Finally, rotate about the **current** z-axis by the angle ψ.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Determine a set of Euler angles ø,θ,andψ so that

**First Case:**
Not both of $r_{13}$, $r_{23}$ are zero ($s_\theta \neq 0$, $r_{33} \neq \pm 1$)




$$\cos\theta = r_{33}$$

*If* $\sin\theta > 0$

$$\tan\emptyset = \frac{r_{23}}{r_{13}} \quad ?\ Quad$$

$$\tan\psi = \frac{r_{32}}{-r_{31}} \quad ?\ Quad$$

or

$$\cos\theta = r_{33}$$

*If* $\sin\theta < 0$

$$\tan\emptyset = \frac{-r_{23}}{-r_{13}} \quad ?\ Quad$$

$$\tan\psi = \frac{-r_{32}}{r_{31}} \quad ?\ Quad$$

**Second Case**:

Both of $r_{13}$, $r_{23}$ are zero. This implies that:

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

**If $r_{33}$=1, then θ=0 and R becomes:**

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \phi + \psi &= \operatorname{atan2}(r_{11}, r_{21}) \\ &= \operatorname{atan2}(r_{11}, -r_{12}) \end{aligned}$$

There are infinitely many solutions. In this case, we may take ø=0.

**If $r_{33}$=1, then θ=π and R becomes:**

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\phi - \psi = \operatorname{atan2}(-r_{11}, -r_{12})$$

There are infinitely many solutions. In this case, we may take ø=0.

***Example***

$$R = \begin{bmatrix} 0 & -0.8660 & 0.5 \\ 0.5 & -0.433 & -0.75 \\ 0.8660 & 0.25 & 0.433 \end{bmatrix}$$

***Solution:***

$$\cos\theta = 0.433$$
$$\theta = 64.43$$

*If* $\sin\theta > 0$

$$\tan\phi = \frac{-0.75}{0.5} \quad 4^{th}\ Quad,\ \phi = -56.3$$

$$\tan\psi = \frac{0.25}{-0.866} \quad 2^{nd}\ Quad,\ \psi = 163.9$$

***or***

$$\cos\theta = 0.433$$
$$\theta = 295$$

*If* $\sin\theta < 0$

$$\tan\phi = \frac{0.75}{-0.5} \quad 2^{nd}\ Quad,\ \phi = 123.7$$

$$\tan\psi = \frac{-0.25}{0.866} \quad 4^{th}\ Quad,\ \psi = 343.9$$

**Matlab code**

```
R=[0 -0.866 0.5, 0.5 -0.433 -0.75, 0.866 0.25 0.433 ]
euler_angles=tr2eul(R)
```

# Roll, Pitch, Yaw Angles

First, a roll about $x_0$ through an angle ψ, then pitch about the $y_0$ by an angle θ. Finally, yaw about the $z_0$ by an angle φ. Since the successive rotations are relative to the **fixed frame**.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{aligned} R &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \theta_1 &= \sin^{-1}(-R_{31}) \\ \theta_2 &= \pi - \theta_1 = \pi - \sin^{-1}(-R_{31}) \\ \psi_1 &= \operatorname{atan2}\left(\frac{R_{32}}{\cos\theta_1}, \frac{R_{33}}{\cos\theta_1}\right) \\ \psi_2 &= \operatorname{atan2}\left(\frac{R_{32}}{\cos\theta_2}, \frac{R_{33}}{\cos\theta_2}\right) \end{aligned}$$

$$\begin{aligned} \phi_1 &= \operatorname{atan2}\left(\frac{R_{21}}{\cos\theta_1}, \frac{R_{11}}{\cos\theta_1}\right) \\ \phi_2 &= \operatorname{atan2}\left(\frac{R_{21}}{\cos\theta_2}, \frac{R_{11}}{\cos\theta_2}\right) \end{aligned}$$

$$(\psi_1, \theta_1, \phi_1)$$
$$(\psi_2, \theta_2, \phi_2)$$

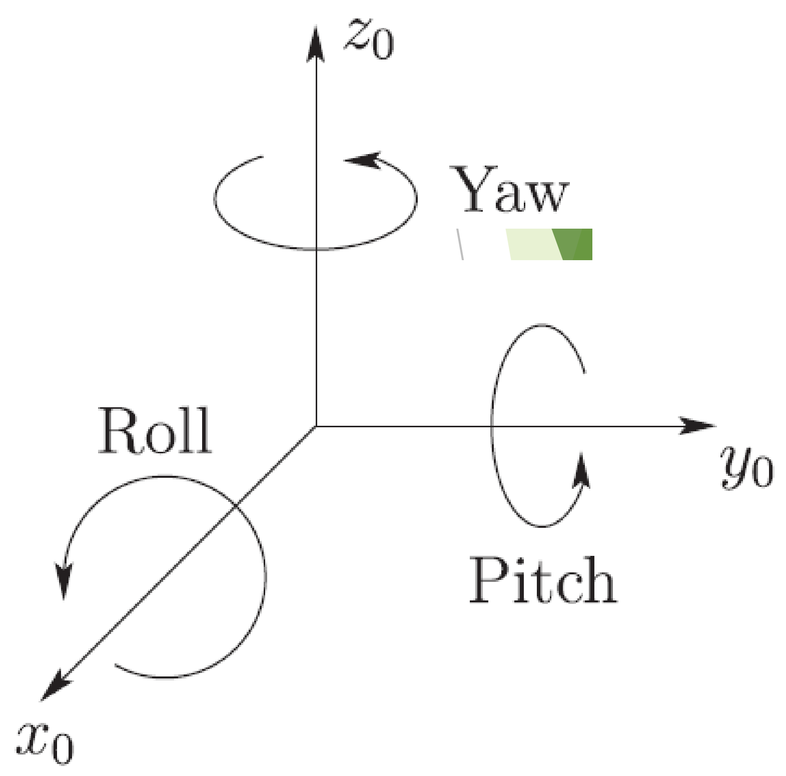



Figure 2.11: Roll, pitch, and yaw angles.

***Example***

$$R = \begin{bmatrix} 0 & -0.8660 & 0.5 \\ 0.5 & -0.433 & -0.75 \\ 0.8660 & 0.25 & 0.433 \end{bmatrix}$$

***Solution:***

$$\sin\theta = -r_{31} \quad , \quad \tan\psi = \frac{r_{32}}{r_{33}} \text{ or } \frac{-r_{32}}{-r_{33}} \;\; 3^{rd} \text{ Quad} \quad , \quad \tan\emptyset = \frac{r_{21}}{r_{11}} \text{ or } \frac{-r_{21}}{-r_{11}} \; 3^{rd} \text{ Quad}$$

$$\sin\theta = -60 \text{ or } 240 \qquad \tan\psi = 30 \; or \; 210 \qquad \tan\emptyset = 90 \; or \; -90$$

**Matlab code**

```
R=[0 -0.866 0.5, 0.5 -0.433 -0.75, 0.866 0.25 0.433 ]
rpyangles=tr2rpy(R)   % tranformation to roll-pitch-yaw
```

***Example***

$$R = \begin{bmatrix} 0 & 0 & -1 \\ 0.8660 & 0.5 & 0 \\ 0 & -0.8660 & 0 \end{bmatrix}$$

***Solution:***

$\sin\theta = -r_{31}$ $\quad\quad$ $\tan\psi = \frac{r_{32}}{r_{33}}$ $\quad\quad$ $\tan\emptyset = \frac{r_{21}}{r_{11}}$

$\sin\theta = 0$ $\quad\quad$ $\tan\psi = \frac{-0.866}{0} = -\mathrm{inf}$ $\quad\quad$ $\tan\emptyset = \frac{0.866}{0} = +\mathrm{inf}$

**Ψ= -90 or 90** $\quad\quad$ **θ= 0 or 180** $\quad\quad$ **Ø =90 or -90**

**Matlab code**

```
R=[0 0 -1; 0.8660 0.5 0 ; 0 -0.8660 0]
rpyangles=tr2rpy(R)   % tranformation to roll-pitch-yaw
```

# Axis-angle representation

In fact, any rotation matrix R can be represented by a single rotation about a suitable axis (k) in space by a suitable angle (θ).

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

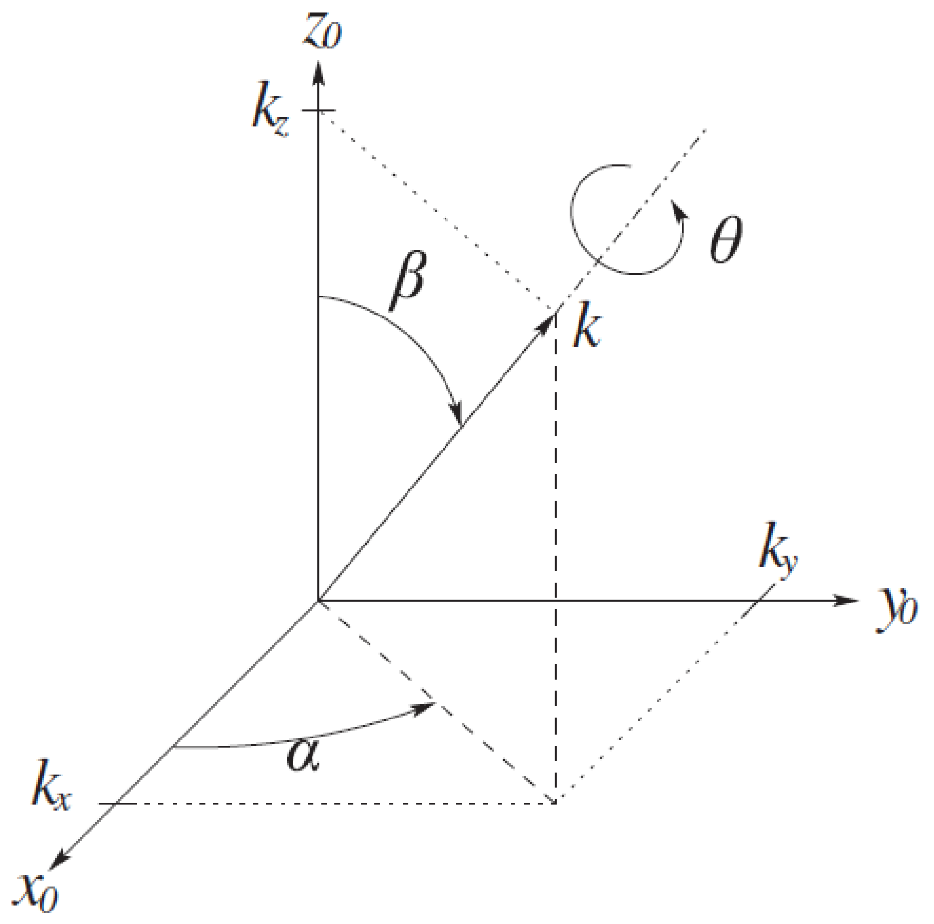



Figure 2.12: Rotation about an arbitrary axis.

**Example 2.9.** *Suppose $R$ is generated by a rotation of $90°$ about $z_0$ followed by a rotation of $30°$ about $y_0$ followed by a rotation of $60°$ about $x_0$. Then*

$$
\begin{aligned}
R &= R_{x,60}R_{y,30}R_{z,90} \\
&= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \\
\theta &= \cos^{-1}\left(-\frac{1}{2}\right) = 120°
\end{aligned}
$$

*The equivalent axis is given from Equation (2.46) as*

$$
k = \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right)
$$

**Matlab code**

```
R=rotx(60*pi/180)*roty(30*pi/180)*rotz(90*pi/180)
[theta v]= tr2angvec(R)
```

# Rigid Motions

$$p^0 = R_1^0 p^1$$

Pure Rotation

$$p^0 = R_1^0 p^1 + d_1^0$$

Frame 1 to Frame 0

$$p^1 = R_2^1 p^2 + d_2^1$$

Frame 2 to Frame 0

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$ (Origin Frame 1 wrt Frame 0)

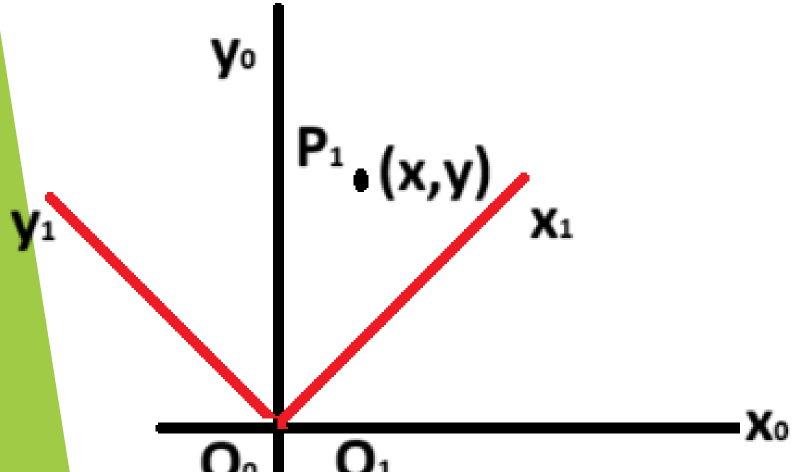
$$p^0 = R_2^0 p^2 + d_2^0$$ (Origin Frame 2 wrt Frame 0)

Where

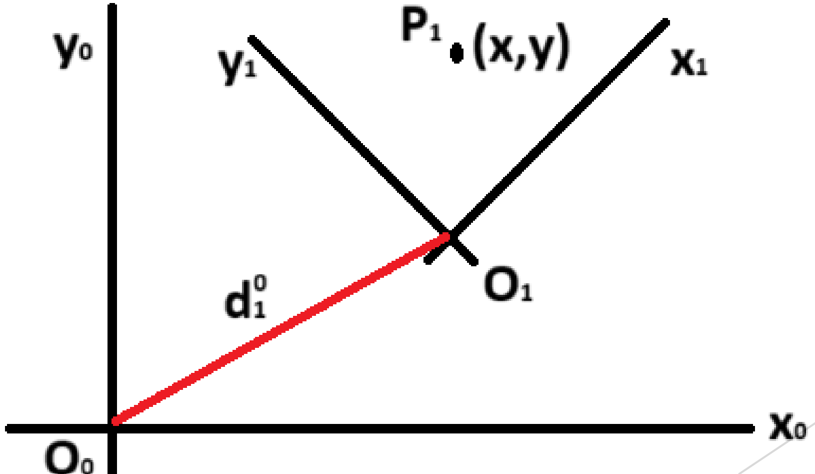
$$R_2^0 = R_1^0 R_2^1$$

$$d_2^0 = d_1^0 + R_1^0 d_2^1$$

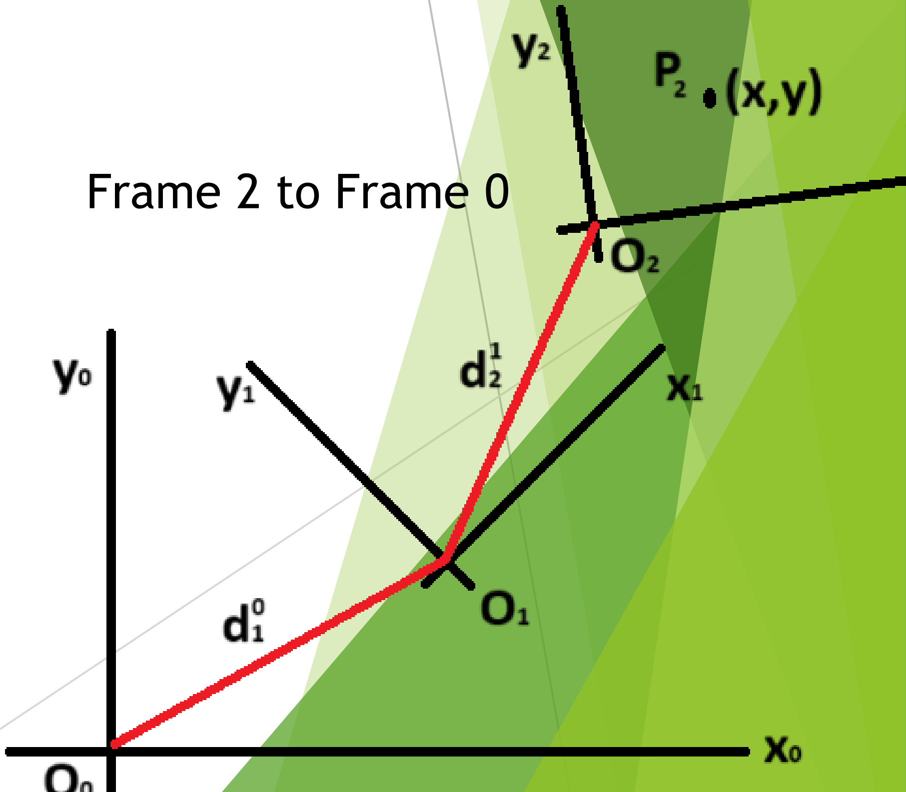
Pure Rotation




Frame 1 to Frame 0




Frame 2 to Frame 0



# Homogeneous Transformations

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

$$H_1^0 \qquad H_2^1 \qquad H_2^0$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$P^0 = H_1^0 P^1$$

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the above equation $n = (n_x, n_y, n_z)$ is a vector representing the direction of $x_1$ in the $o_0x_0y_0z_0$ frame, $s = (s_x, s_y, s_z)$ represents the direction of $y_1$, and $a = (a_x, a_y, a_z)$ represents the direction of $z_1$. The vector $d = (d_x, d_y, d_z)$ represents the vector from the origin $o_0$ to the origin $o_1$ expressed in the frame $o_0x_0y_0z_0$.

Given a homogeneous transformation $H_1^0$ relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$

**Example 2.10.** *The homogeneous transformation matrix $H$ that represents a rotation by angle $\alpha$ about the current $x$-axis followed by a translation of $b$ units along the current $x$-axis, followed by a translation of $d$ units along the current $z$-axis, followed by a rotation by angle $\theta$ about the current $z$-axis, is given by*

$$
\begin{aligned}
H &= Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta} \\
&= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -d s_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & d c_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

```
H=trotx(alpha)*transl(b,0,0)*transl(0,0,d)*trotz(theta)
trplot(T)
tranimate(H)
```

The Toolbox has many functions to create homogeneous transformations.
ample we can demonstrate composition of transforms by

```
>> T = transl(1, 0, 0) * trotx(pi/2) * transl(0, 1, 0)
T =
    1.0000         0         0    1.0000
         0    0.0000   -1.0000    0.0000
         0    1.0000    0.0000    1.0000
         0         0         0    1.0000
>> trplot(T)
```

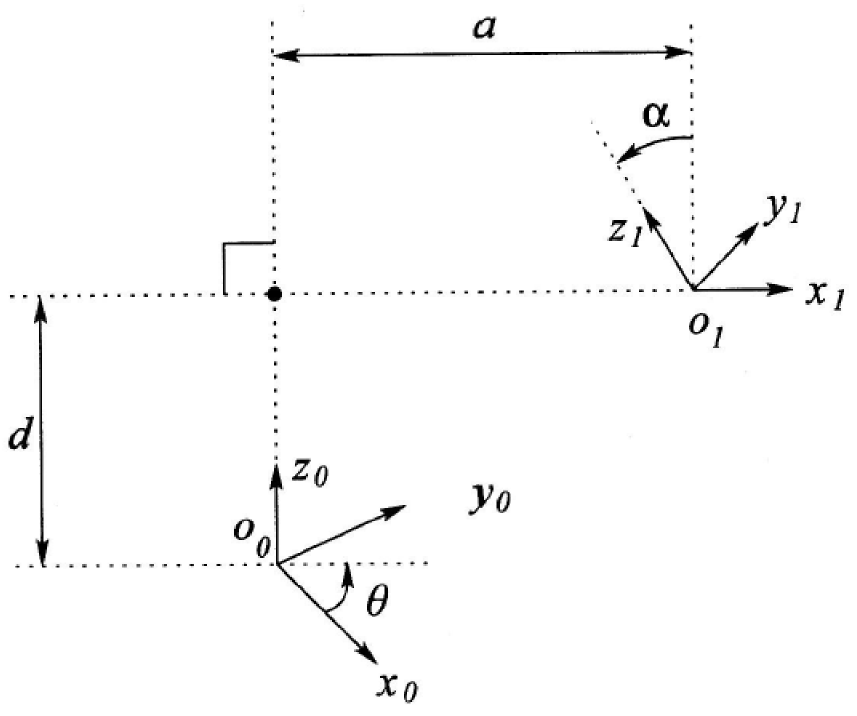
The rotation matrix component of T is

```
>> t2r(T)
ans =
    1.0000         0         0
         0    0.0000   -1.0000
         0    1.0000    0.0000
```

and the translation component is a vector

```
>> transl(T)'
ans =
    1.0000    0.0000    1.0000
```

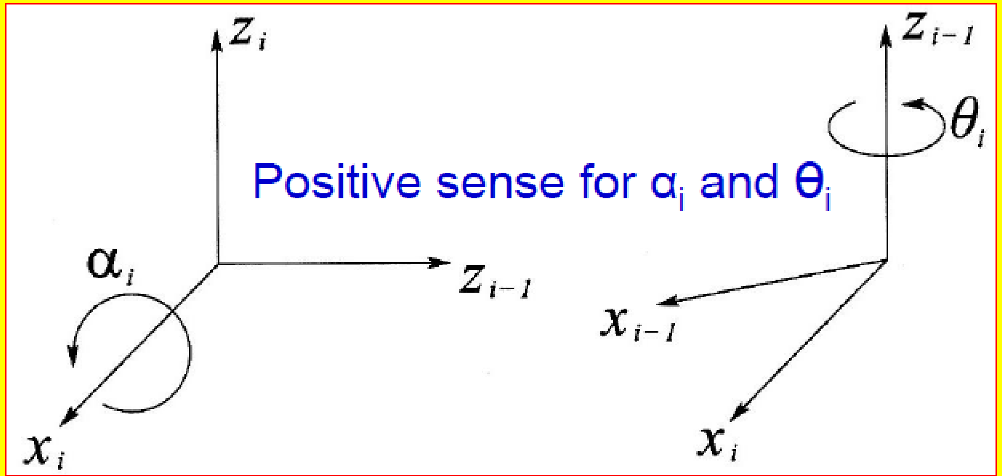
# The Denavit – Hartenberg Convention




**DH Coordinate Frame Assumptions**

**(DH1)** The axis $x_1$ is perpendicular to the axis $z_0$.

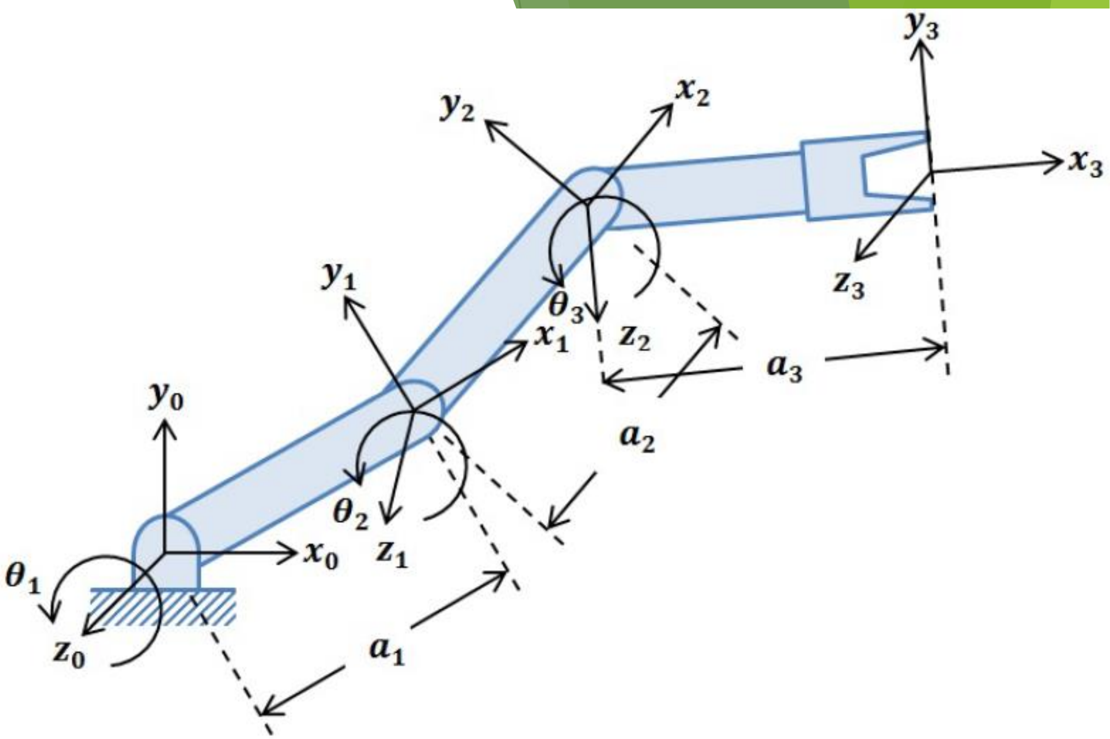
**(DH2)** The axis $x_1$ intersects the axis $z_0$.




It can be proved that using the above two assumptions, there exist unique 4 numbers a, d, θ, α sufficient for the homogeneous transformation. The parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.

$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\,\mathrm{Trans}_{z,d_i}\,\mathrm{Trans}_{x,a_i}\,Rot_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

| link | $a_1$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1^*$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ |
| 3 | $a_3$ | 0 | 0 | $\theta_3^*$ |



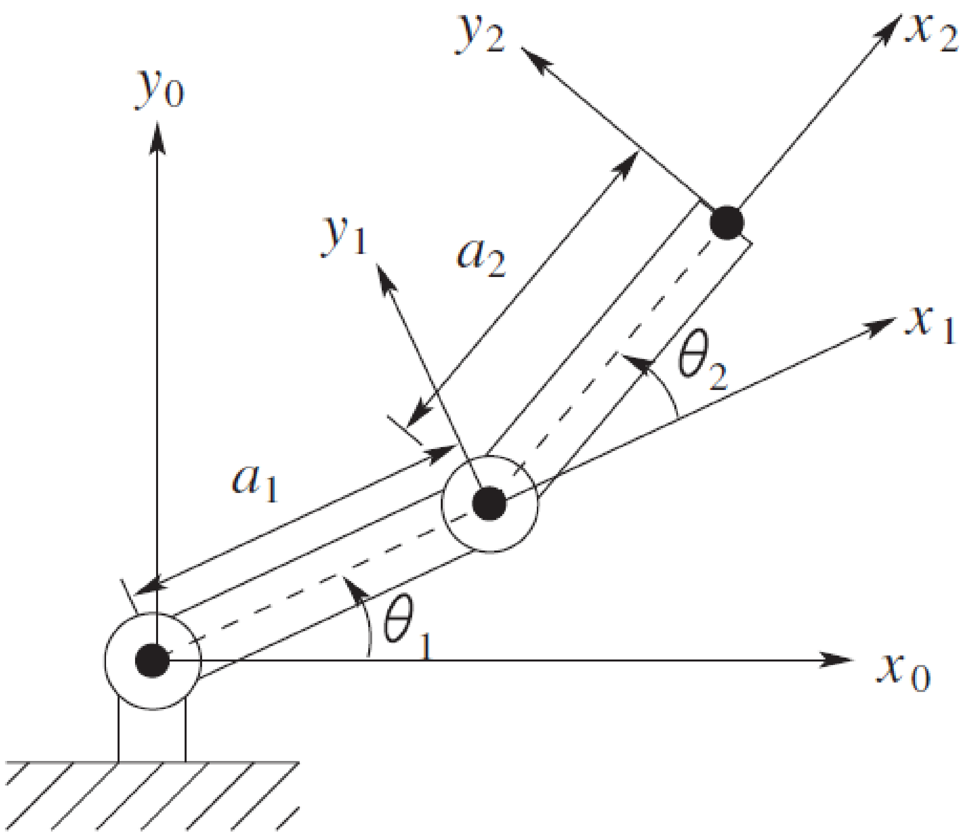

Using Equation (3-10) from SHV, we can generate the $A$ matrices.

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying together, we get $T_0^3$, the homogenous transformation that gives the position and orientation of the frame attached to the end effector in the base coordinate frame.

$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.




parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.

Figure 3.6: Two-link planar manipulator. The $z$-axes all point out of the page, and are not shown in the figure.

Table 3.1: DH parameters for 2-link planar manipulator. $\theta_1$ and $\theta_2$ are the joint variables.

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |

parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.

Table 3.2: DH parameters for 3-link cylindrical manipulator. $\theta_1$, $d_2$, and $d_3$ are the joint variables.

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $d_1$ | $\theta_1$ |
| 2 | 0 | $-90$ | $d_2$ | 0 |
| 3 | 0 | 0 | $d_3$ | 0 |

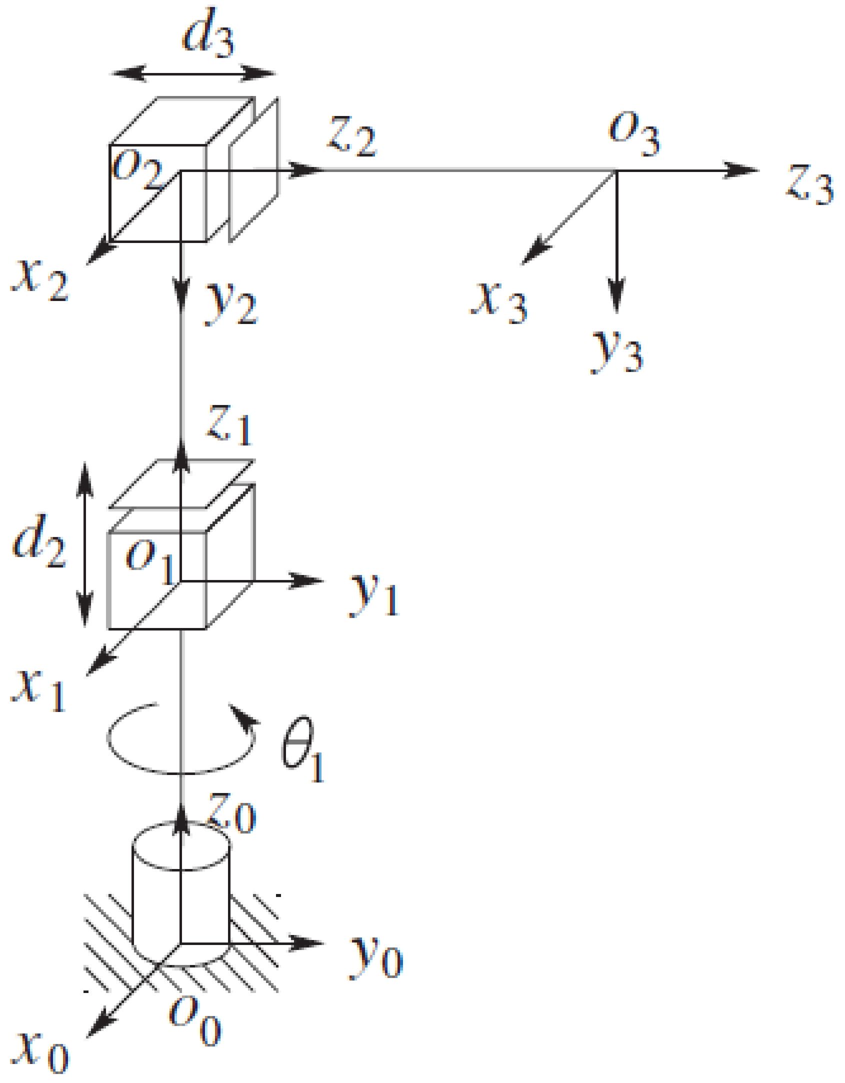



Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: DH parameters for 3-link cylindrical manipulator. $\theta_1$, $d_2$, and $d_3$ are the joint variables.

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $d_1$ | $\theta_1$ |
| 2 | 0 | $-90$ | $d_2$ | 0 |
| 3 | 0 | 0 | $d_3$ | 0 |

Table 3.3: DH parameters for the spherical wrist.

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 4 | 0 | 90 | 0 | $\theta_4$ |
| 5 | 0 | $-90$ | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |

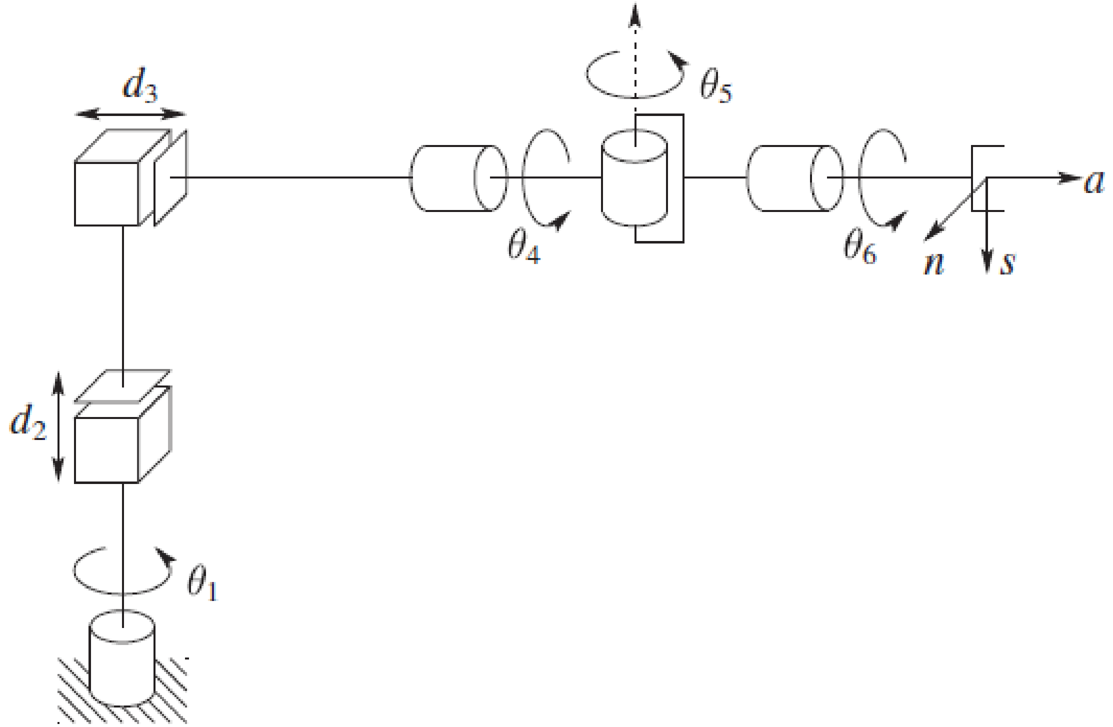
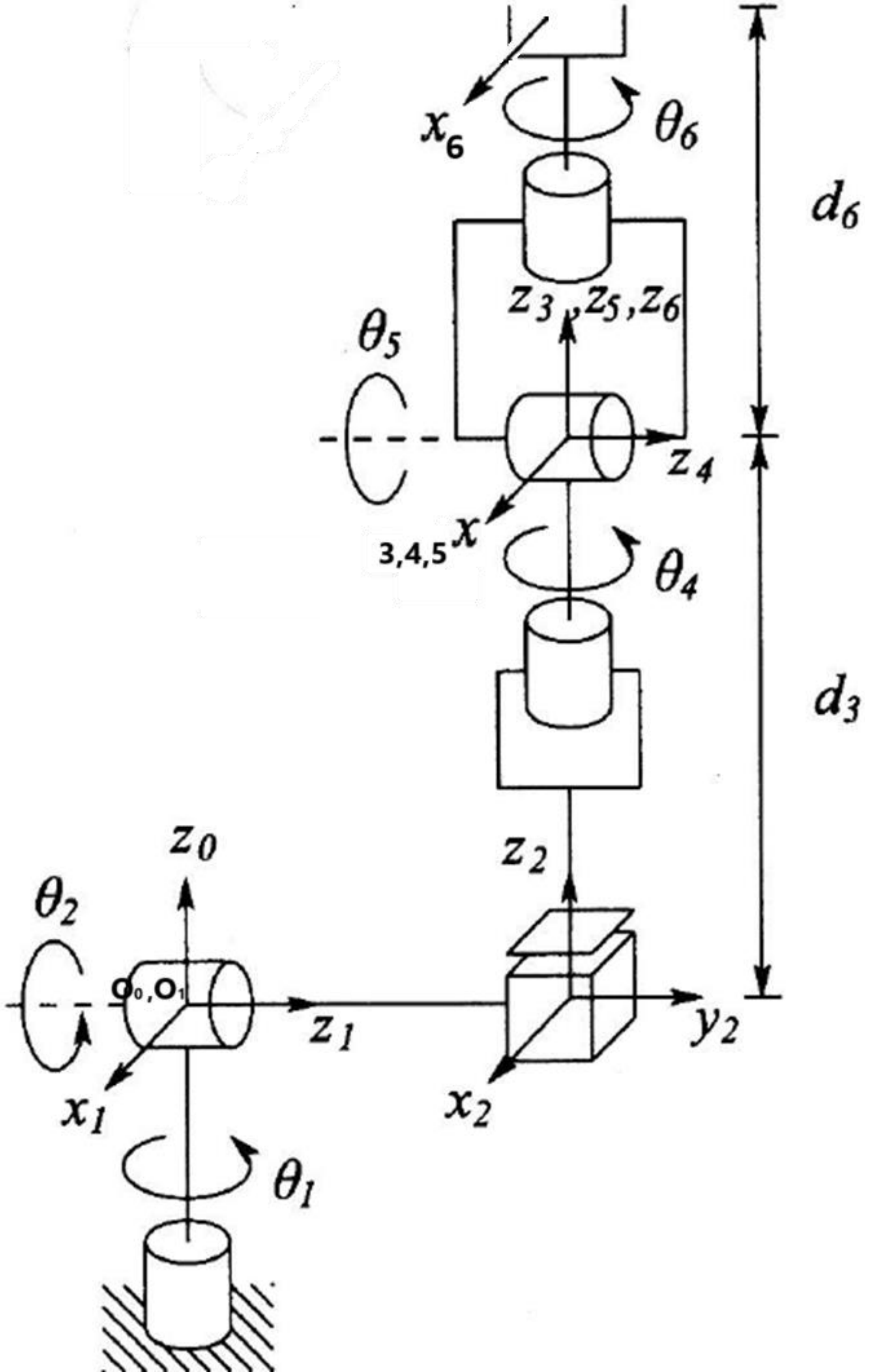



Figure 3.9: Cylindrical robot with spherical wrist.

parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.




parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.

| Link | $d_i$ | $a_i$ | $\alpha_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | −90 | $\theta_1^\star$ |
| 2 | $d_2$ | 0 | +90 | $\theta_2^\star$ |
| 3 | $d_3^\star$ | 0 | 0 | 0 |
| 4 | 0 | 0 | −90 | $\theta_4^\star$ |
| 5 | 0 | 0 | +90 | $\theta_5^\star$ |
| 6 | $d_6$ | 0 | 0 | $\theta_6^\star$ |

* joint variable

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

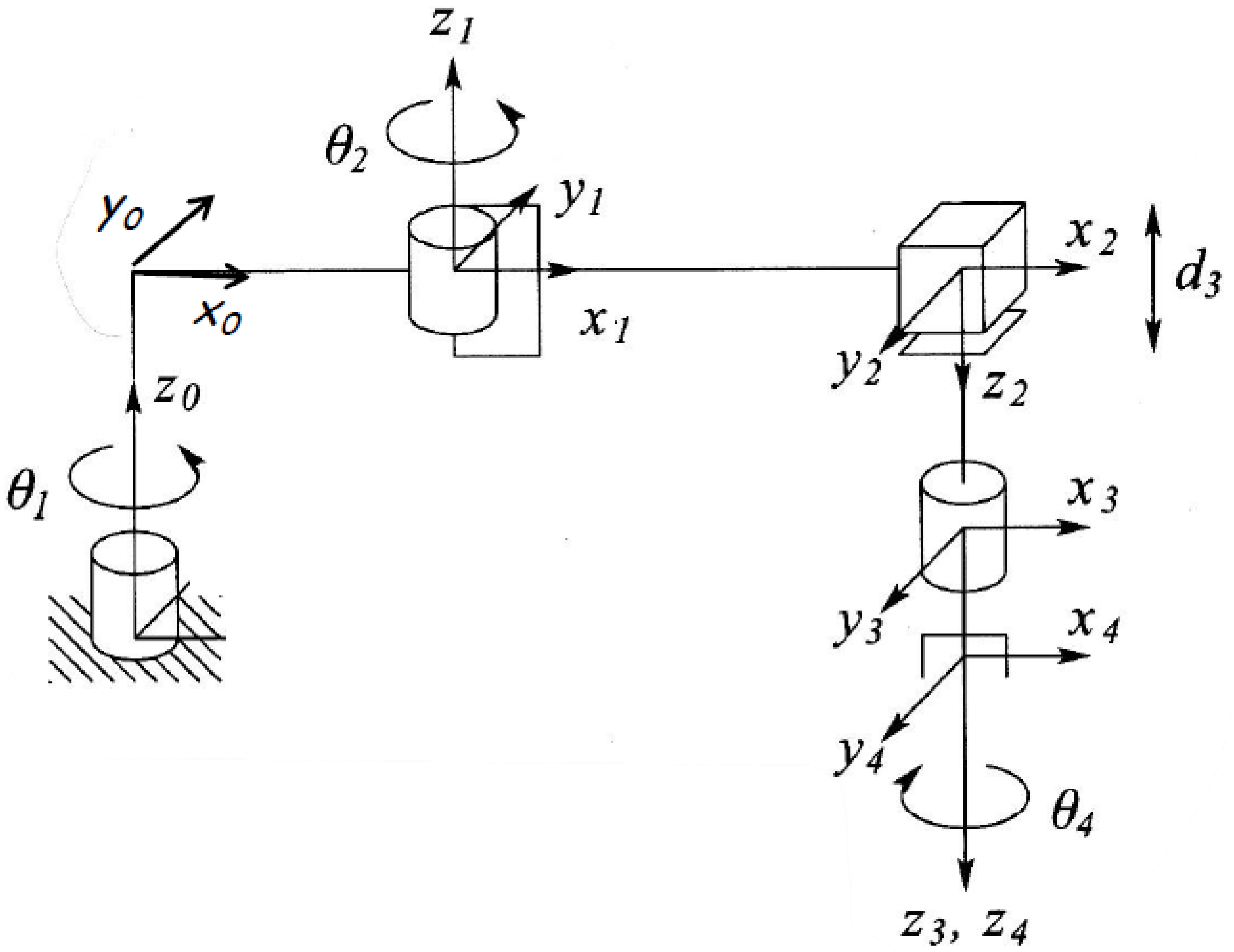
$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

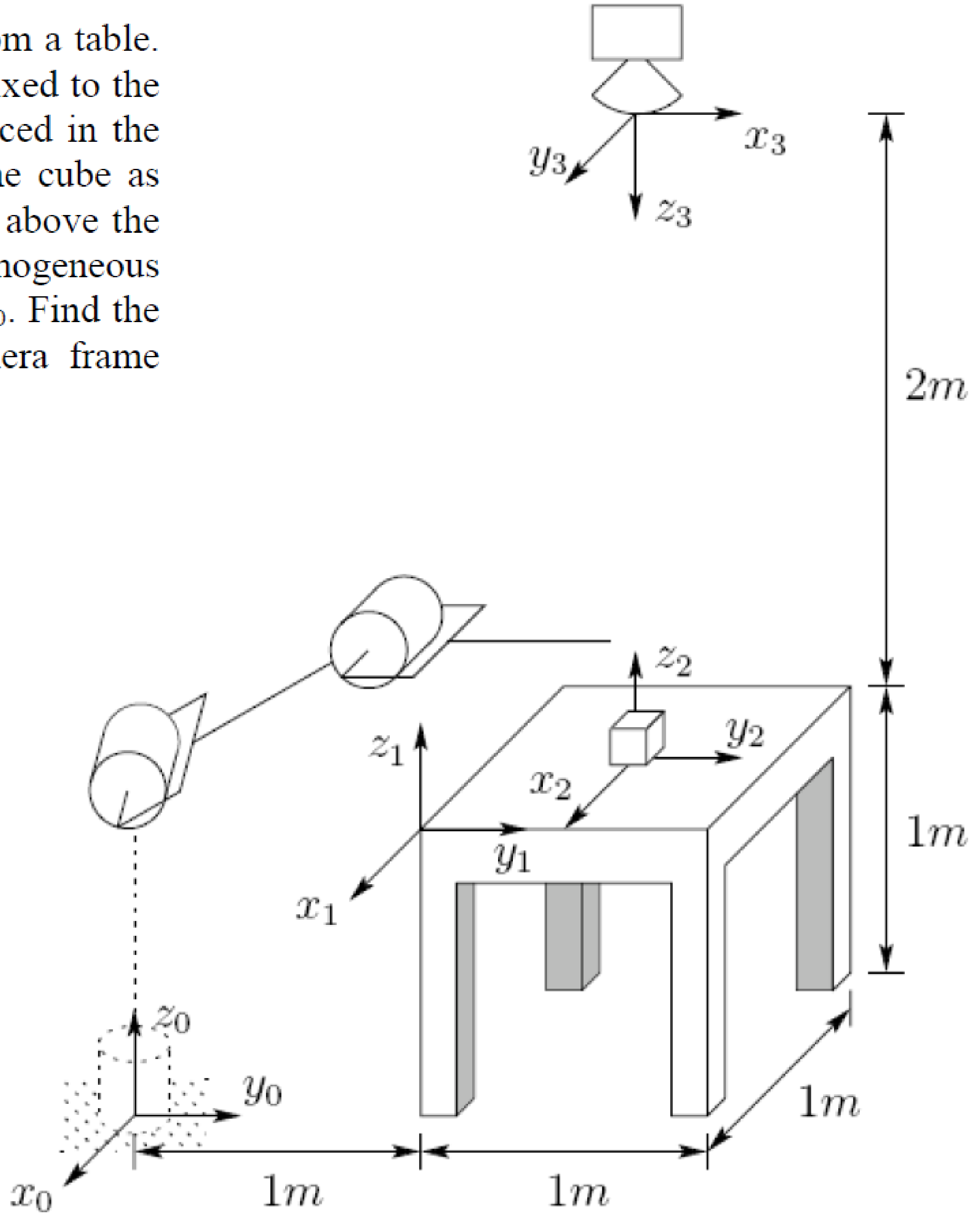



| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1^\star$ |
| 2 | $a_2$ | 180 | 0 | $\theta_2^\star$ |
| 3 | 0 | 0 | $d_3^\star$ | 0 |
| 4 | 0 | 0 | $d_4$ | $\theta_4^\star$ |

* joint variable

parameter a is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle α is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for α is determined from $z_0$ to $z_1$ by the right-handed rule. The parameter d is the perpendicular distance from the origin $o_0$ to the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, θ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$.

Consider the diagram of the figure below. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2m above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.




$$\gg T_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gg T_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\gg T_3^0 =$

$$T_2^3 = {T_2^0}^{-1} * T_3^0$$

$$T_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$