Introduction to Robotics

Presented by

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General Inverse Kinematics

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

find a solution, or possibly multiple solutions, of the equation

$$T_n^0(q_1, \ldots, q_n) = H$$

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n)$$

Example 5.1 (The Stanford Manipulator). *Recall the Stanford Manipulator of Section 3.3.5. Suppose that the desired position and orientation of the final frame are given by*

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, *and* θ_6 *we must solve the following simultaneous set of nonlinear trigonometric equations:*

Table 3.4: DH parameters for the Stanford Manipulator. $\theta_i, i = 1, 2, 4, 5, 6$ and d_3 are variable.

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	θ_1
2	d_2	0	$+90$	θ_2
3	d_3	0	0	0
4	0	0	-90	θ_4
5	0	0	$+90$	θ_5
6	d_6	0	0	θ_6

$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}
$$

Inverse Kinematics – Decoupling Method

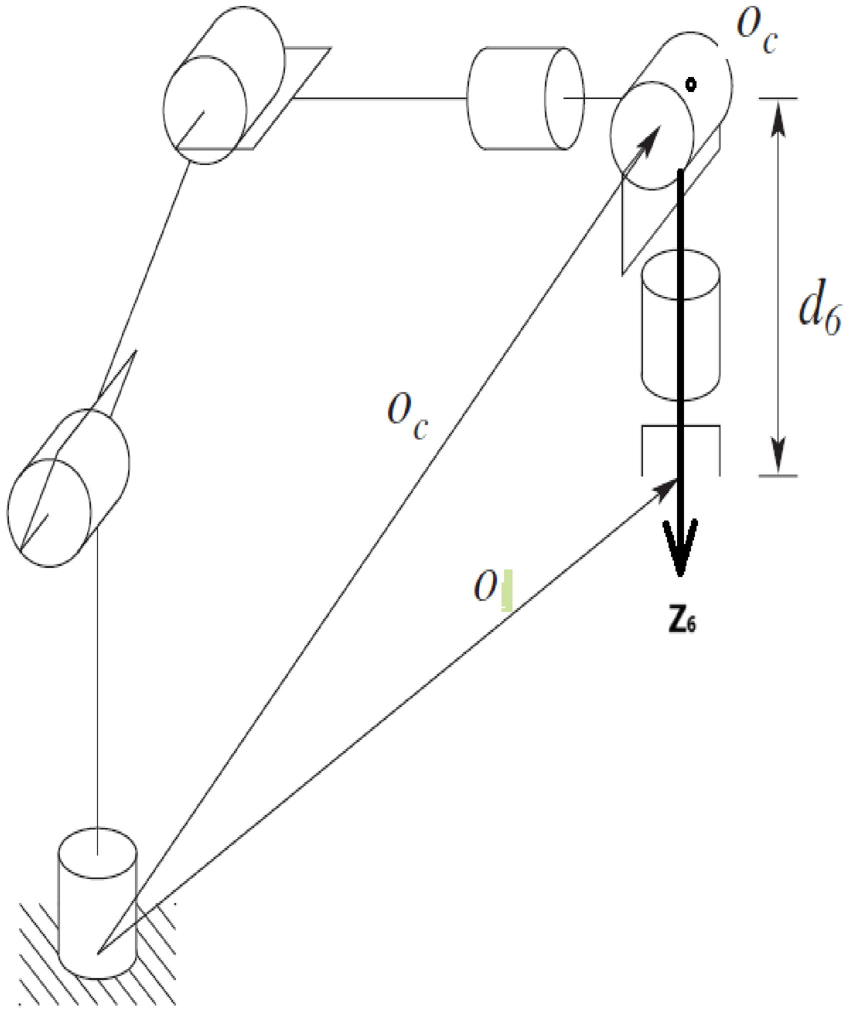
Inverse Position: Solve the IK for the position of Oc.

The assumption of a spherical wrist means that the axes z_3, z_4, and z_5 intersect at o_c and hence the origins o_4 and o_5 assigned by the DH convention will always be at the wrist center o_c. Often o_3 will also be at o_c, but this is not necessary for our subsequent development. The important point of this assumption for the inverse kinematics is that motion of the final three joints about these axes will not change the position of o_c, and thus the position of the wrist center is a function of only the first three joint variables.

The origin of the tool frame (whose desired coordinates are given by o) is simply obtained by a translation of distance d_6 along z_5 from o_c (see Table 3.3). In our case, z_5 and z_6 are the same axis, and the third column of R expresses the direction of z_6 with respect to the base frame. Therefore, we have

$$O = O_6$$

$$O = O_c^0 + d_6 * Z_6 \qquad o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Here, we will find the main joints except the wrist.

$$O_c^0 = O - d_6 * Z_6$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

$$\begin{array}{cccccc} & & \boldsymbol{x_a} & \boldsymbol{y_a} & \boldsymbol{z_a} & \boldsymbol{O_a} \\ T_a^0 & = & \left[\begin{matrix} \boxed{r_{11}} & \boxed{r_{12}} & \boxed{r_{13}} & \boxed{O_x} \\ \boxed{r_{21}} & \boxed{r_{22}} & \boxed{r_{23}} & \boxed{O_y} \\ \boxed{r_{31}} & \boxed{r_{32}} & \boxed{r_{33}} & \boxed{O_z} \\ 0 & 0 & 0 & 1 \end{matrix}\right. & & & & \left.\right] \end{array}$$

$$T_c^0 = A_1\ A_2 \ldots..\ A_c = \begin{bmatrix} r_{11} & r_{12} & r_{13} & O_x \\ r_{21} & r_{22} & r_{23} & O_y \\ r_{31} & r_{32} & r_{33} & O_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{O_c} = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

We may find the values of the first three joint variables. This determines the orientation transformation R_3^0 which depends only on these first three joint variables.

Inverse Rotation: Solve the IK for the orientation R_6^3

We can now determine the orientation of the end effector relative to the frame o3x3y3z3

$$R = R_3^0 R_6^3$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

The final three joint angles can then be found as a set of Euler angles corresponding to R_6^3. Note that the righthand side of the equation is completely known since R is given and R_3^0 can be calculated once the first three joint variables are known.

$$\theta_4 = \varphi$$
$$\theta_5 = \theta$$
$$\theta_6 = \psi$$

Euler Angles

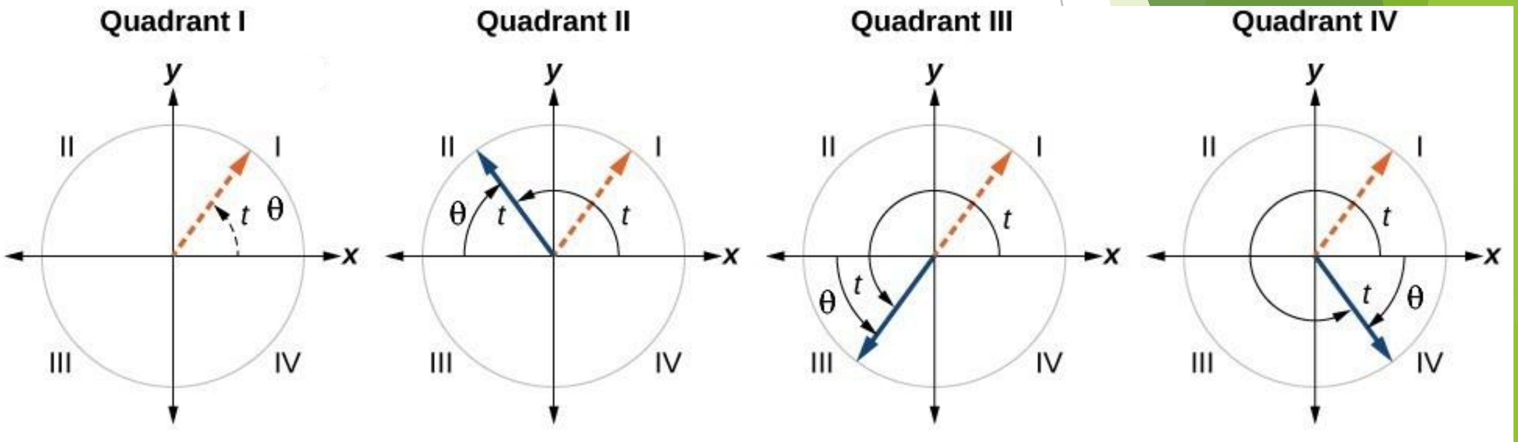
First, rotate about the z-axis by the angle φ. Next, rotate about the **current** y-axis by the angle θ. Finally, rotate about the **current** z-axis by the angle ψ.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Determine a set of Euler angles ø,θ,andψ so that

First Case:

Not both of r_{13}, r_{23} are zero ($s_\theta \neq 0$, $r_{33} \neq \pm 1$)



$$\cos\theta = r_{33}$$

$$\cos\theta = r_{33}$$

If $\sin\theta > 0$

$$\tan\emptyset = \frac{r_{23}}{r_{13}} \ ?\,Quad$$

$$\tan\psi = \frac{r_{32}}{-r_{31}} \ ?\,Quad$$

or

If $\sin\theta < 0$

$$\tan\emptyset = \frac{-r_{23}}{-r_{13}} \ ?\,Quad$$

$$\tan\psi = \frac{-r_{32}}{r_{31}} \ ?\,Quad$$

Example

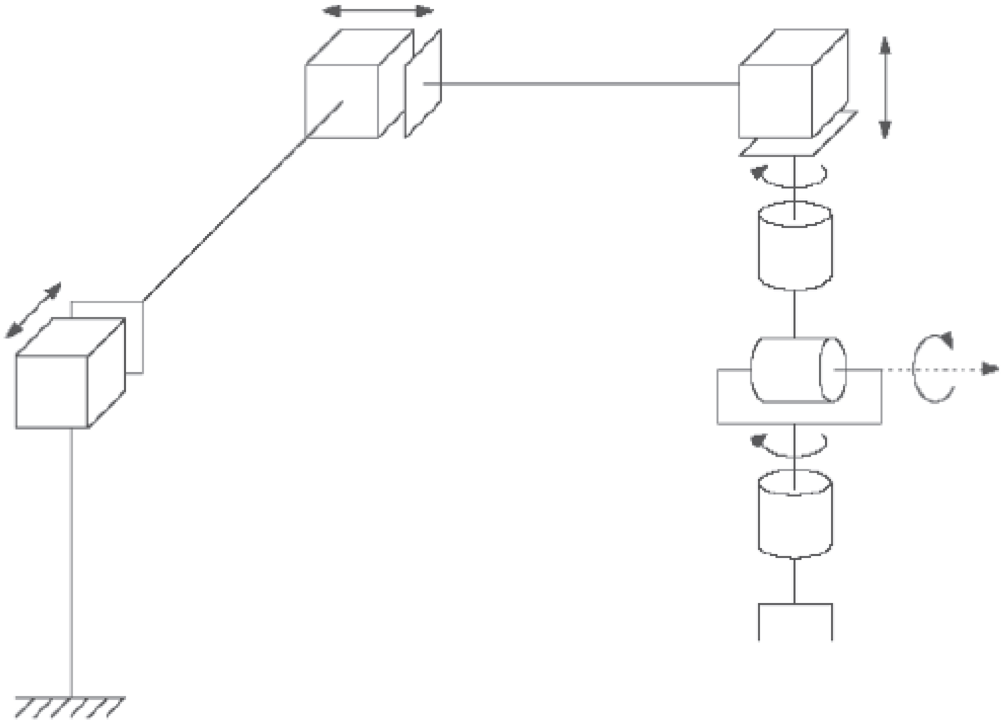
$$H=\begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & -2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ d=1$$

solution

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} 0.5-0 \\ -2-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -3 \\ 1 \end{bmatrix}$$

$$T_3^0 = A_1\ A_2\ A_3 = \begin{bmatrix} 0 & 0 & -1 & -d_3^* \\ 0 & -1 & 0 & d_2^* \\ -1 & 0 & 0 & d_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} -d_3^* \\ d_2^* \\ d_1^* \end{bmatrix} = \begin{bmatrix} 0.5 \\ -3 \\ 1 \end{bmatrix},\ d_3^* = -0.5\ ,\ d_2^* = -3\ ,\ d_1^* = 1$$



Cartesian manipulator with spherical wrist.

Link	a	α	d	θ
1	0	-90	d_1^*	0
2	0	-90	d_2^*	90
3	0	0	d_3^*	0

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$R_3^0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; R_6^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_6^3 = {R_3^0}^T R_6^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

Find Euler angles

$$\theta_4 = \varphi$$
$$\theta_5 = \theta$$
$$\theta_6 = \psi$$