Introduction to Robotics

Presented by

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Forward Kinematics

End Effector Position??

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) + \ell_3 \cos(\theta_1+\theta_2+\theta_3)$$
$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1+\theta_2) + \ell_3 \sin(\theta_1+\theta_2+\theta_3)$$
$$\phi_e = \theta_1 + \theta_2 + \theta_3$$

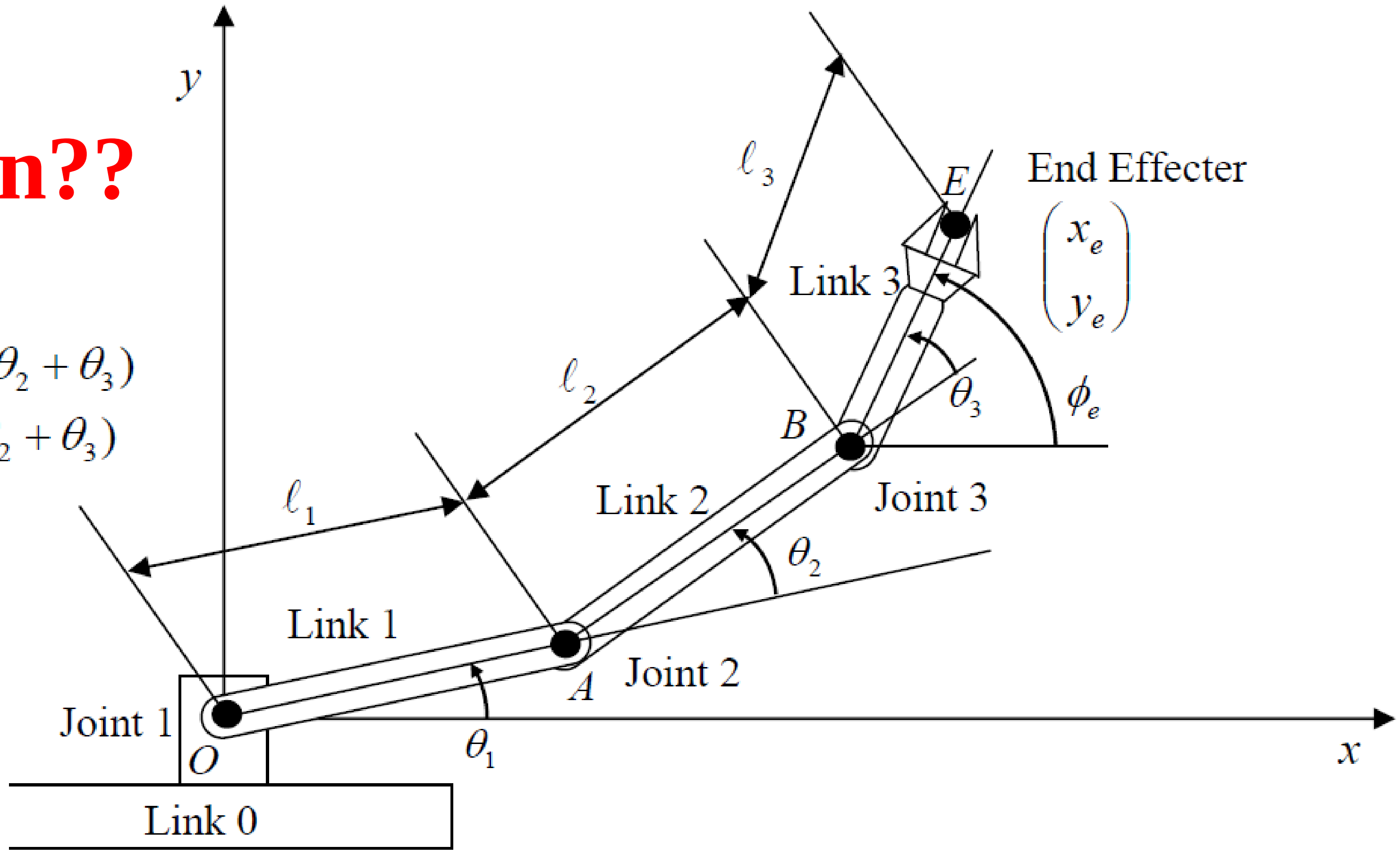


Figure 4.1.1 Three dof planar robot with three revolute joints

Representing Positions

Point P w.r.t frame 0

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix},$$

Point P w.r.t frame 1

$$p^1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

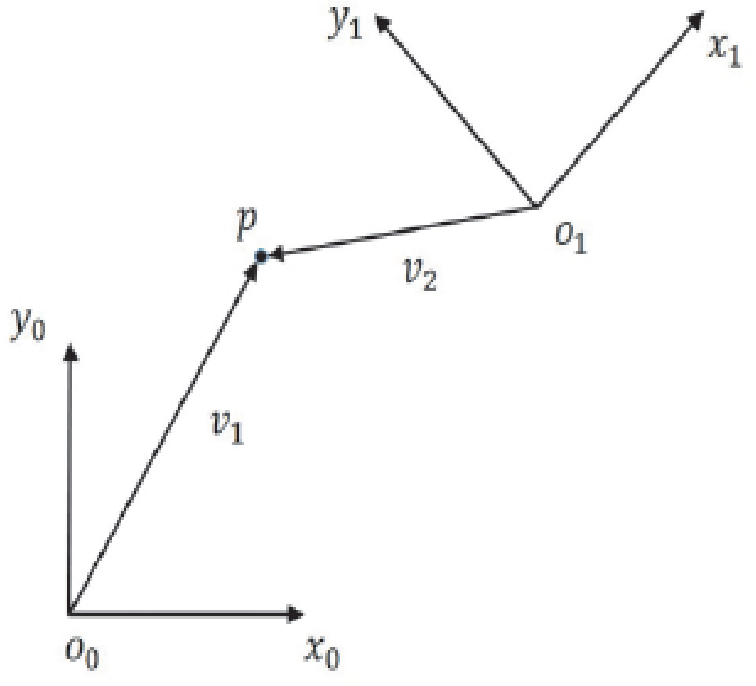
Point O1 w.r.t frame 0

$$o_1^0 = \begin{bmatrix} 12 \\ 8 \end{bmatrix},$$

Point O0 w.r.t frame 1

$$o_0^1 = \begin{bmatrix} -16 \\ 3 \end{bmatrix}$$

The position varies according to the reference frame.



: Two coordinate frames, a point p, and two vectors v_1 and v_2.

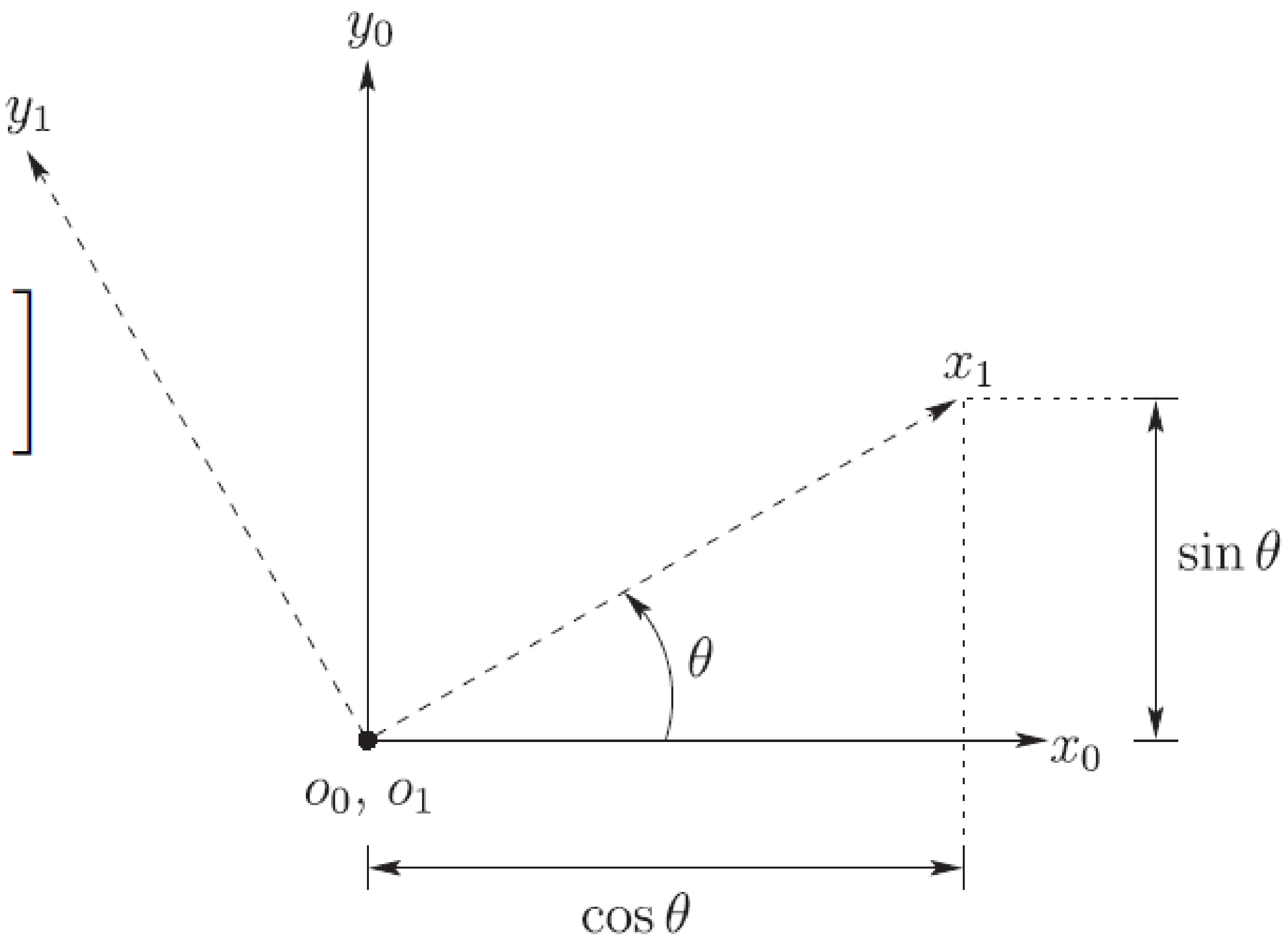
Representing Rotations in plane

Frame 1 w.r.t Frame 0

$$R_1^0 = [x_1^0 \mid y_1^0]$$

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Representing Rotations in 3D

Rotation by θ about the Z axis, Frame 1 w.r.t Frame 0

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation by θ about the X axis, Frame 1 w.r.t Frame 0

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation by θ about the Y axis, Frame 1 w.r.t Frame 0

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

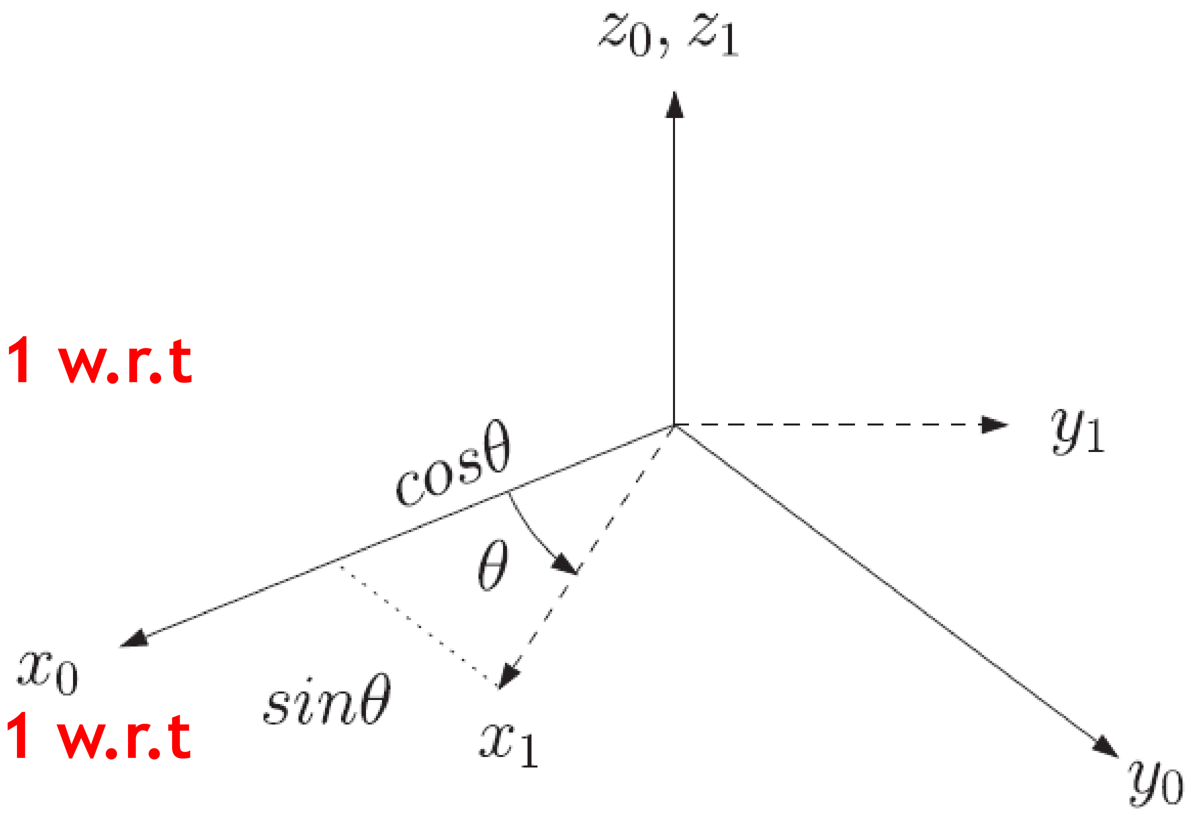


Figure 2.3: Rotation about z_0 by an angle θ.

Notes

the basic rotation matrix $R_{z,\theta}$ has the properties

$$
\begin{aligned}
R_{z,0} &= I \\
R_{z,\theta}R_{z,\phi} &= R_{z,\theta+\phi}
\end{aligned}
$$

The inverse of the rotation matrix corresponding to a rotation by angle θ can also be easily computed simply by constructing the rotation matrix for a rotation by the angle -θ:

$$
(R_{z,\theta})^{-1} = R_{z,-\theta}
$$

$$
(R_1^0)^T = (R_1^0)^{-1}
$$

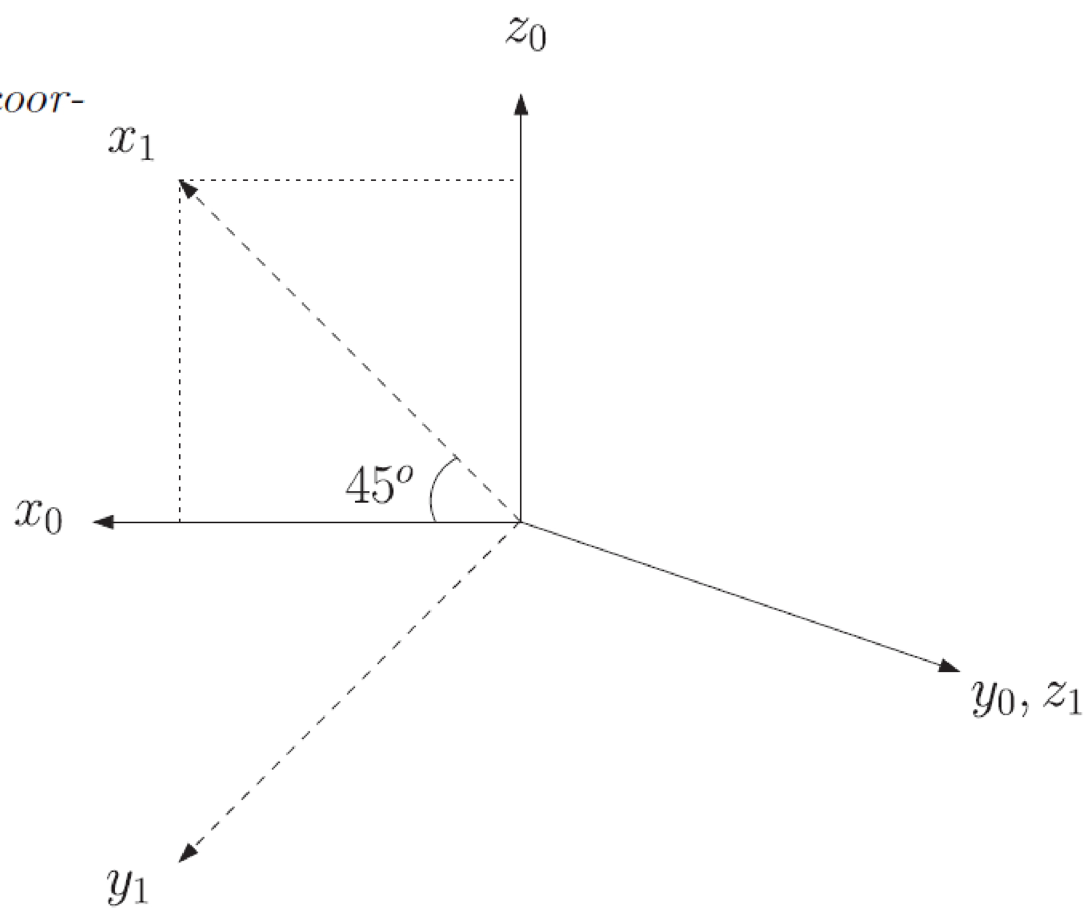
$$
\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T
$$

Example

Consider the frames $o_0x_0y_0z_0$ *and* $o_1x_1y_1z_1$ *shown in Fig-*
Projecting the unit vectors x_1, y_1, z_1 *onto* x_0, y_0, z_0 *gives the coordinates of* x_1, y_1, z_1 *in the* $o_0x_0y_0z_0$ *frame as*

$$x_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad z_1^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$



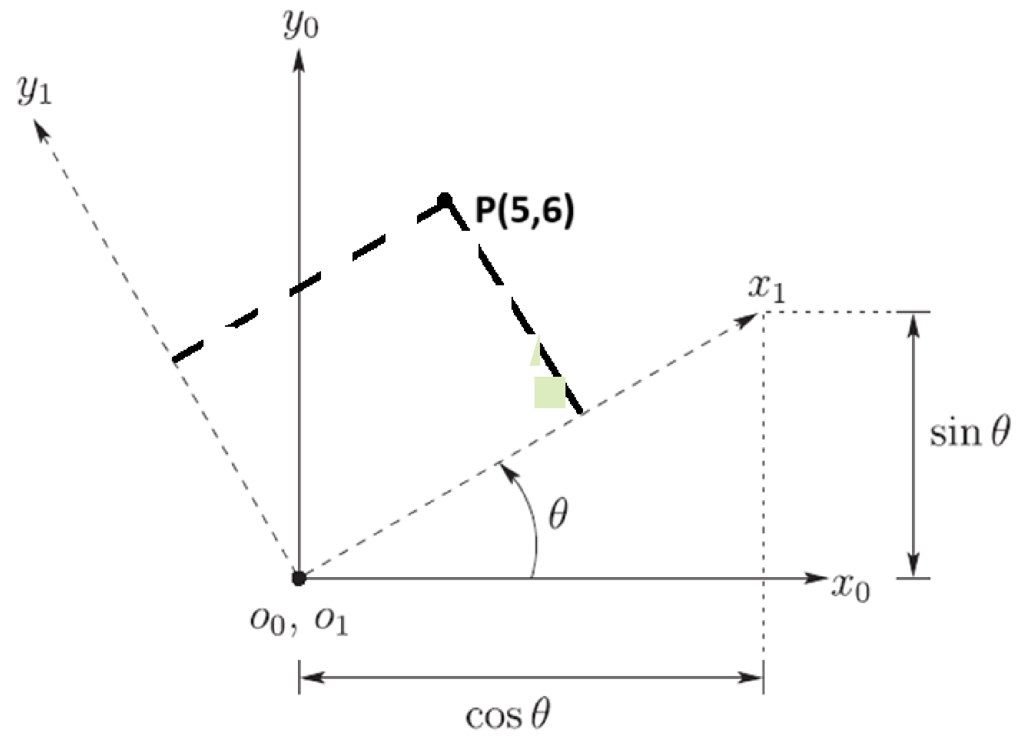
Rotational Transformations- Position

Thus, the rotation matrix R_1^0 can be used not only to represent the orientation of coordinate frame $o_1x_1y_1z_1$ with respect to frame $o_0x_0y_0z_0$, but also to transform the coordinates of a point from one frame to another. If a given point is expressed relative to $o_1x_1y_1z_1$ by coordinates p^1, then $R_1^0p^1$ represents the **same point** expressed relative to the frame $o_0x_0y_0z_0$.

$$p^0 = R_1^0 p^1 \qquad \boldsymbol{P^1} = \begin{bmatrix} \mathbf{5} \\ \mathbf{6} \end{bmatrix}, \ \boldsymbol{\theta=90}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{P^0} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{5} \\ \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{-6} \\ \mathbf{5} \end{bmatrix}$$



Rotational Transformations-Vector Rotation

Example 2.3. *The vector v with coordinates $v^0 = (0,1,1)$ is rotated about y_0 by $\frac{\pi}{2}$ as shown in Figure 2.7. The resulting vector v_1 is given by*

$$\begin{aligned} v_1^0 &= R_{y,\frac{\pi}{2}} v^0 \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

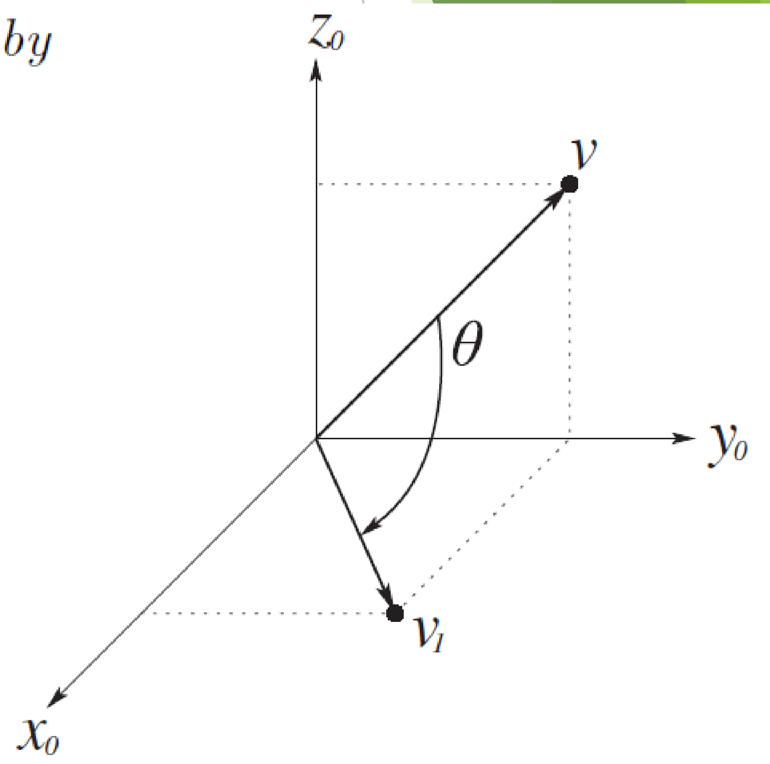


Figure 2.7: Rotating a vector about axis y_0.

Rotational Transformations

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.

2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.

3. It is an operator taking a vector and rotating it to give a new vector in the same coordinate frame.

Composition of Rotations

- Rotation with Respect to the Current Frame:

We make a rotation with respect to Frame 0 to get Frame 1, then another rotation w.r.t the current frame **(Frame 1)** to get Frame 2.

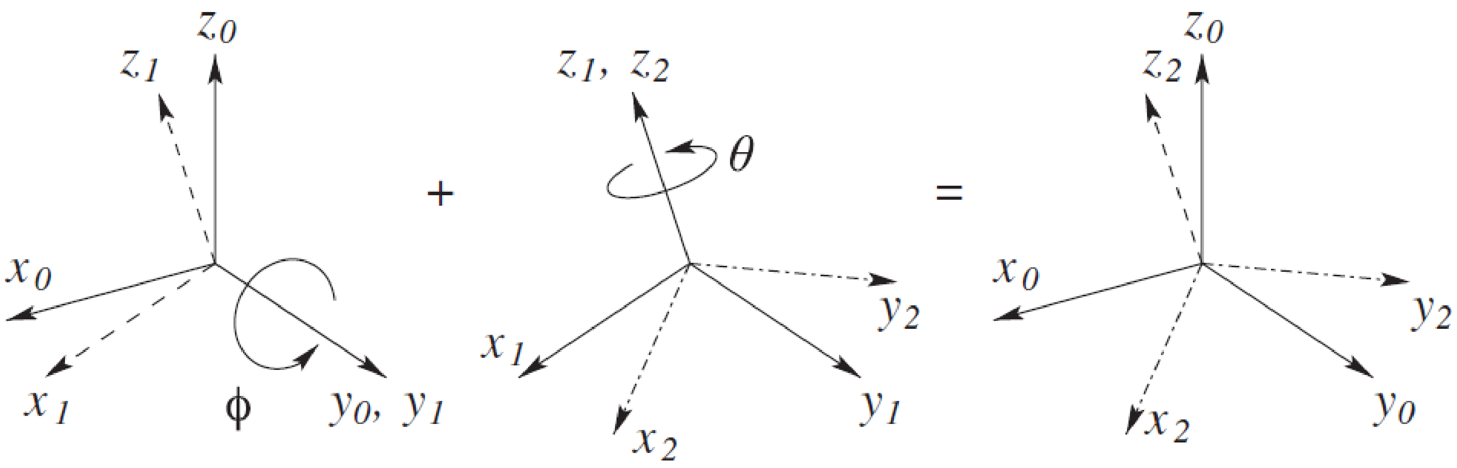


Figure 2.8: Composition of rotations about current axes.

- Rotation with Respect to the Fixed Frame:

We make a rotation with respect to Frame 0 to get Frame 1, then another rotation w.r.t the Fixed frame **(Frame 0)** to get Frame 2.

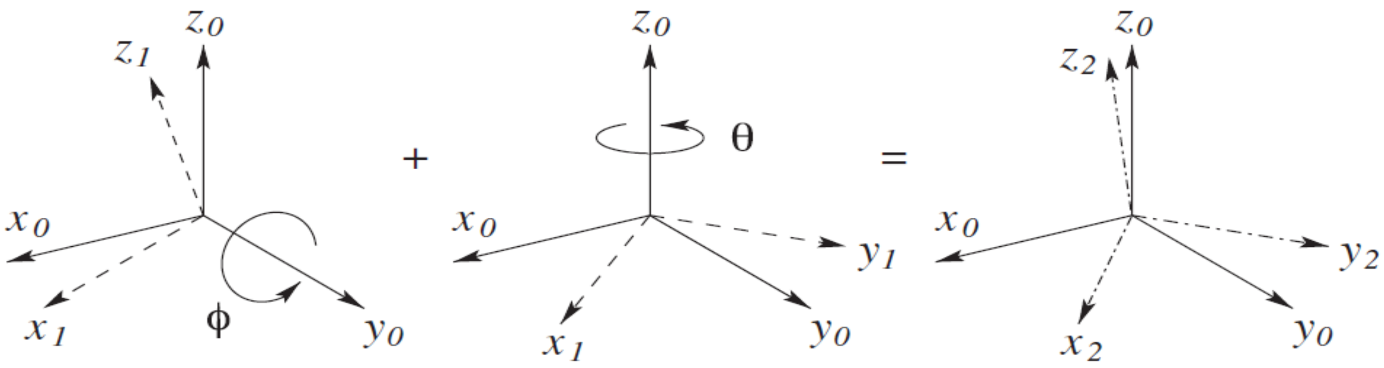


Figure 2.9: Composition of rotations about fixed axes.

Composition of Rotations

- Rotation with Respect to the Current Frame:

$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2 \\
p^0 &= R_1^0 R_2^1 p^2
\end{aligned}
$$

$$R_2^0 = R_1^0 R_2^1$$

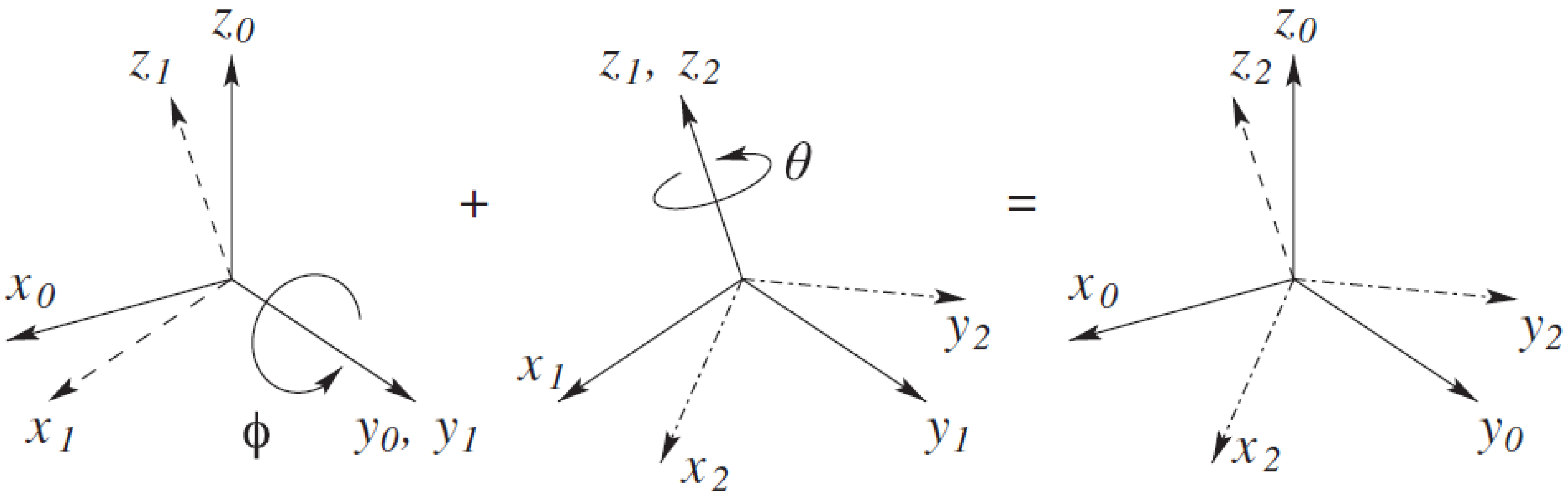


Figure 2.8: Composition of rotations about current axes.

If we make a rotation with respect to Frame 0 and get Frame 1 R_1^0), then another rotation w.r.t the current frame (Frame 1) to get Frame 2 (R_2^1), the rotation matrix of Frame 2 w.r.t Frame 0 = R_1^0 R_2^1 →Rule: The last rotation multiplied to the right of the previous multiplication (post-multiplication) if the rotation is referred to the **current frame**.

Example 2.5. *Suppose a rotation matrix R represents a rotation of angle ϕ about the current y-axis followed by a rotation of angle θ about the current z-axis as shown in Figure 2.8. Then the matrix R is given by*

$$\begin{aligned} R &= R_{y,\phi}R_{z,\theta} \qquad (2.18) \\ &= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix} \end{aligned}$$

Example 2.6. *Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis. Then the resulting rotation matrix is given by*

$$\begin{aligned}
R' &= R_{z,\theta}R_{y,\phi} \qquad (2.19)\\
&= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}\\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned}$$

Comparing Equations (2.18) and (2.19) we see that $R \neq R'$.

Composition of Rotations

- Rotation with Respect to the Fixed Frame:

It is often desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames.

For example, we may wish to perform a rotation about x0 followed by a rotation about y0 (not y1!). We will refer to o0x0y0z0 as the fixed frame. In this case, the composition law in the previous slides is invalid.

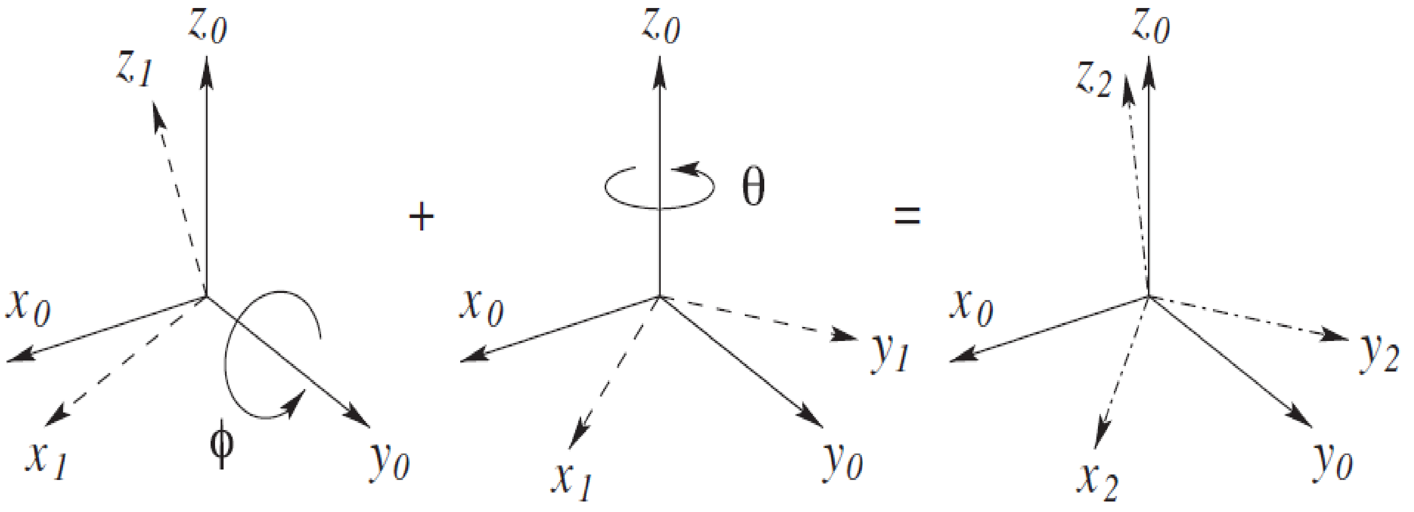


Figure 2.9: Composition of rotations about fixed axes.

$$R_2^0 = RR_1^0$$

If we make a rotation with respect to Frame 0 and get Frame 1 (R_1^0), then another rotation w.r.t the Fixed Frame (Frame 0) to get Frame 2 (R), the rotation matrix of Frame 2 w.r.t Frame 0 = $R\ R_1^0$ →Rule: The last rotation multiplied to the left of the previous multiplication (pre-multiplication) if the rotation is referred to the **Fixed frame**.

Example 2.8. *Suppose R is defined by the following sequence of basic rotations in the order specified:*

1. *A rotation of θ about the current x-axis*
2. *A rotation of ϕ about the current z-axis*
3. *A rotation of α about the fixed z-axis*
4. *A rotation of β about the current y-axis*
5. *A rotation of δ about the fixed x-axis*

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or postmultiply as the case may be to obtain

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.24}$$