Kinematic Modelling

I. Forward Kinematics

The forward kinematics shows the transformation from one frame into another one, starting at the base and ending at the end-effector. A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg or DH convention, as shown in Figure 2. In this method, to set one reference frame relative to another, only four parameters are needs instead of six, which are normally required for 3D motion. These parameters are (di, ai, θi, and αi), which tell the location of a link-frame of the robot from a previous link-frame. The transformation matrix between two neighboring frames is expressed, as shown in Eq

$$T_i = R_{z,\theta_i}\, T_{z,d_i}\, T_{x,a_i}\, R_{x,\alpha_i}$$

$$T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_i = \begin{bmatrix} c\theta_i & -s\theta_i\, c\alpha_i & s\theta_i\, s\alpha_i & a_i\, c\theta_i \\ s\theta_i & c\theta_i\, c\alpha_i & -c\theta_i\, s\alpha_i & a_i\, s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

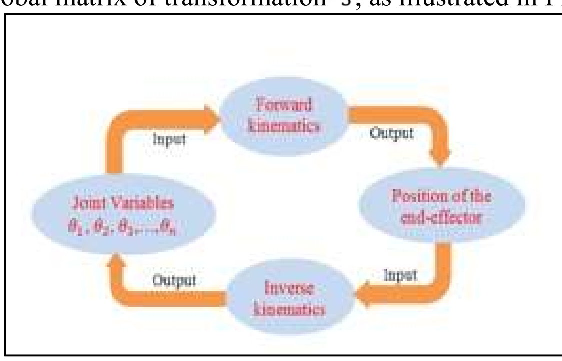
Where**:**

P_x : Position of the end-effector in x-direction = $a_i\, c\theta_i$

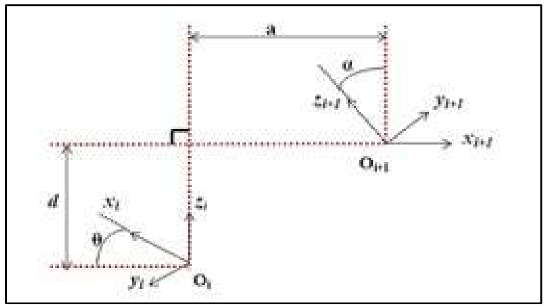
P_y : Position of the end-effector in y-direction = $a_i\, s\theta_i$

P_z : Position of the end-effector in z-direction = d_i a_i : Link length α_i : Link twist d_i : Link offset θ_i : Joint angle

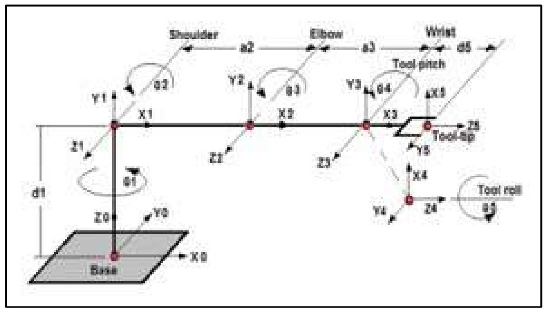
The parameters for the 5 DOF robotic arm used are listed in Table 1, where shows rotation about the z-axis, rotation about the x-axis, transition along the z-axis, and transition along the x-axis. By substituting the D-H parameters in Table 1 into Eq (1), we can obtain the individual transformation matricesT_1^0 to T_5^4, and a global matrix of transformationT_5^0, as illustrated in Figure 3.



Kinematic block diagram



D H frame



Link coordinate diagram of the robotic arm [9]

Denavit-Hartenberg Parameter Table

Link	a_i (mm)	α_i (degree)	d_i (mm)	θ _i(degree)
1	0	90	105	θ1
2	105	0	0	θ2
3	100	0	0	θ3
4	0	90	0	θ4
5	0	0	150	θ5

$$ {}^{0}_{1} = R_{z,\theta_1}\, T_{z,d_1}\, T_{x,a_1}\, R_{x,\alpha_1} \quad T $$

$$ {}^{0}_{1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & T\,1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\,(90\ ) & -s(90) & 0 \\ 0 & s(90\ ) & c(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ T_1^0 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

In a similar way will find the T_2^1, T_3^2, T_4^3, and T_5^4.

$$ T_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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$$ T_3^2 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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$$ T_4^3 = \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & -c_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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$$ T_5^4 = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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The position and orientation for the wrist of the articulated robot can be obtained by multiply the matrices (T_1^0 , T_2^1 , and T_3^2):

$$T_3^0 = T_1^0\, T_2^1\, T_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1\ a_3 c_{23} + a_2 c_2 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_3 c_{23} + a_2 c_2) \\ s_{23} & c_{23} & 0 & a_3\, s_{23} + a_2 s_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_5^3 = T_4^3\, T_5^4$$

$$T_5^3 = \begin{bmatrix} c_4 c_5 & & & \\ s_4 c_5 & -s_5 c_4 & s_4 & d_5 s_4 \\ s_5 & -s_4 s_5 & -c_4 & -d_5 c_4 \\ 0 & c_5 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix}$$

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$$T_5^0 = T_3^0\, T_5^3$$

$$T_5^0 = \begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Inverse kinematics

$$[T_1^0]^{-1}\, T_5^0 = [T_1^0]^{-1}\, T_1^0\, T_2^1\, T_3^2 T_4^3\, T_5^4 \quad .$$

The equation above become:

$$\begin{bmatrix} c_1 n_{11} + s_1 n_{21} & c_1 n_{12} + s_1 n_{22} & c_1 n_{13} + s_1 n_{23} & c_1\, x + s_1\, y \\ -s_1 n_{11} + c_1 n_{21} & -s_1 n_{12} + c_1 n_{22} & -s_1 n_{13} + c_1 n_{23} & -s_1\, x + c_1\, y \\ n_{31} & n_{32} & n_{33} & z - d_1 \\ 0 & 0 \quad \dots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

$$\alpha_{11} = \big((\ {}_3 - s_2 s_3)(c_4 - 1) + (-c\ c_2 c\ \ {}_3)s\ \ {}_4\big)c\ \ {}_2 s_3 - s\ {}_2 c\ {}_5$$
$$\alpha_{12} = -\big({}_2 c_3 - s_2 s_3)(c_4 - 1) + (-\ (c\ \ {}_2 c_3\ \ )s_4\big)\ c_2 s_3 -\ s\ \ s_5$$
$$\alpha_{13} = -(c_3 - s_2 s_3)s_4 + (-c_2 s_3 -\ c_2 \qquad s_2 c_3)c\ {}_4$$
$$\alpha_{14} = (-\ {}_2 c_3 - s_2 s_3)s_4 + (-c_2 s_3\ \ (c\ \ ) \qquad - s_2 c_3)\ c_4$$
$$d_5\ \ (c_2 c_3 - s_2 s_3)a_4 + c_2 a_3\ \ +$$
$$\alpha_{21} = -s \qquad {}_5$$
$$\alpha_{22} = -c \qquad {}_5$$
$$\alpha_{23} = 0$$
$$\alpha_{24} = 0$$
$$\alpha_{31} = \big((\ {}_3 + c_2 s_3)(c_4 - 1) + (c_2 c\ s_2 c\ \ )s_4\ \ \big)c_5\ \ {}_3 - s_2 s\ {}_3$$
$$\alpha_{32} = -\ \ {}_2 c_3 + c_2 s_3)(c_4 - 1) + (c\big((s\ \ {}_3)s\ \ {}_4\big)s\ \ {}_2 c_3 - s\ {}_2 s\ {}_5$$
$$\alpha_{33} = -\ \ c_3 + c_2 s_3)s_4 + (c_2 c_3 - s\ (s_2 \qquad {}_2 s_3)c_4$$
$$\alpha_{34} = (-\ {}_2 c_3 + c_2 s_3)s_4 + (c_2 c_3 -\ \ (s\ \ {}_5 +\ \ (s\ s_2 s_3)c_4\ )d_2 c\ {}_3 + c_2 s_3)a_4 + s_2 a_3$$

$$= \begin{bmatrix} (c_{23}\ \ - 1) - s_{234})c_5 & -(c_{23}\ \ (c_4 - s\ \ {}_{234})\ (c_4 - 1)\ \ s_5 & s_{234} & -s_{234} d_5 + c_{23} a_4 + c_2 a_3 \\ -s_5 & -c_5 & 0 & 0 \\ (s_{23}(\ - 1) + c_{23} s_4)c_5 & -(s_{23}\ \ c_4\ \ c\ \ {}_{23} s_4\ (c_4 - 1)\ \ +)s\ {}_5 & c_{234} & c_{234} d_5 + s_{23} a_4 + s_2 a_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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From the equations we obtained the θ_1

$$-s_1\,x + c_1\,y = 0$$

$$\therefore\ \theta_1 = atan2(y, x)$$

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In a similar way will find the other joint angles:

$$[T_1^0\,T_2^1]^{-1}\,T_5^0 = T_3^2\,T_4^3\,T_5^4$$

$$[T_1^0\,T_2^1\,T_3^2]^{-1}\,T_5^0 = T_4^3\,T_5^4$$

$$[T_1^0\,T_2^1\,T_3^2\,T_4^3]^{-1}\,T_5^0 = T_5^4$$

To find the θ_2

$$c_1x + s_1y = -s_{234}\ \ _5 + c_{23}a_4 + c_2a_3$$

$$c_1x + s_1y = -s_{234}d_5 + c_2(c_3a_4 + a_3)$$

$$c_2 = \frac{c_1x + s_1y + s_{234}d_5}{(c_3a_4 + a_3)}$$

$$z - d_1 = c_{234}d_5 + s_{23}a_4 + s_2a_3$$

$$z - d_1 = c_{234}d_5 + s_2(s_3a_4 + a_3)$$

$$s_2 = \frac{z - d_1 - c_{234}d_5}{(s_3a_4 + a_3)}$$

$$\therefore\ \theta_2 = atan2\ (s_2, c_2)$$

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For θ_3

$$c_1x + s_1y = -s_{234}d_5 + c_{23}a_4 + c_2a_3$$

$$z - d_1 = c_{234}d_5 + s_{23}a_4 + s_2a_3$$

$$[c_1x + s_1y + s_{234}d_5]^2 = [c_{23}a_4 + c_2a_3]^2$$

$$[z - d_1 - c_{234}d_5]^2 = [s_{23}a_4 + s_2a_3]^2$$

$$c_3 = \frac{[c_1x + s_1y + s_{234}d_5]^2 + [z - d_1 - c_{234}d_5]^2 - a_3a_4}{2\,a_3a_4}$$

$$s_3 = \pm\sqrt{1 - {c_3}^2}$$

$$\therefore\ \theta_3 = atan2\ (s_3, c_3)$$

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$$s_{234} = -\ c\ \ _1n_{13} + s_1n_{23}$$

$$c_{234} = n_{33}$$

$$\theta_{234} = atan\ \ 2\ (s_{234}, c_{234})$$

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3$$

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$$s_5 = s_1n_{11} - c_1n_{21}$$

$$c_5 = s_1n_{12} - c_1n_{22}$$

$$\therefore\ \theta_5 = atan2\ (s_5, c$$

3. Velocity kinematics

Jacobian matrix is used to calculate the velocity kinematics using a matrix, which depends on the changes of the joints velocities into Cartesian velocities. This matrix is important in control of the movement of the robotic arm, used to achieve smooth path planning, and used to determine the dynamic equation. The relationships between the joint velocity and the linear and angular velocity of the end effector are shown following:

$$\dot{p} = J_p\,(\theta)\dot{\theta} \quad .$$

$$w = J_w\,(\theta)\dot{\theta} \quad .$$

By combining, the equations (19, 20) will give the Jacobian matrix:

$$J_\theta = \begin{bmatrix} J_p \\ J_w \end{bmatrix} = [6\, * n] = \begin{bmatrix} J_{p1} & \cdots & J_{pn} \\ J_{w1} & \cdots & J_{wn} \end{bmatrix} \quad .$$

$$v = J(\theta)\dot{\theta} \quad .$$

Where:

θ : Joint angle. $\dot{\theta}$

: joint velocity.

$\dot{p}$: linear velocity of the end effector.

$\dot{w}$: angular velocity of the end effector.

The number of rows in the Jacobian matrix equal to the number of DOF in the cartesian coordinate (three linear and three angular) while the number of columns equal to the number of DOF in the joint. The matrix of Jacobian can be obtained using the following equations:

$$J_{pi} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & for\ revolute\ joint\ (i) \\ z_{i-1} & for\ prismatic\ joint\ (i) \end{cases} \quad .$$

$$J_{wi} = \begin{cases} z_{i-1} & for\ revolute\ joint\ (i) \\ 0 & for\ prismatic\ joint\ (i) \end{cases} \quad .$$

Where:

J_{pi}: Linear Jacobian matrix.

J_{wi}: Angular Jacobian matrix.

For the 5 DOF robotic arm used, the Jacobian matrix will be in the form:

$$J(\theta) = \begin{bmatrix} z_0 \times (o_5 - o_0) & z_1 \times (o_5 - o_1) & \ldots & z_2 \times (o_5 - o_2) & z_3 \times (o_5 - o_3) & z_4 \times (o_5 - o_4) \\ z_0 & z_1 & & z_2 & z_3 & z_4 \end{bmatrix}$$

From the forward kinematics can be calculated the linear part of $J(\theta)$, the value of $(o_n - o_{i-1})$ for 5 DOF robotic arm used is:

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$o_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}$$

$$o_2 = \begin{bmatrix} a\ c_1c_2 \\ a\ s_1c_2 \\ a_2s\ + d_1 \end{bmatrix} \begin{matrix} 2 \\ 2 \\ 2 \end{matrix}$$

$$o_3 = \begin{bmatrix} a_3\ {}_1c_{23} + ac\ \ {}_2c_1c_2 \\ a_3\ {}_1c_{23} + as\ \ {}_2s_1c_2 \\ a_3s\ \ + a_2s\ {}_{23}\ {}_2 + d_1 \end{bmatrix}$$

$$o_4 = \begin{bmatrix} a_3\ {}_1c_{23} + ac\ \ {}_2c_1c_2 \\ a_3\ {}_1c_{23} + as\ \ {}_2s_1c_2 \\ a_3s\ \ + a_2s\ {}_{23}\ {}_2 + d_1 \end{bmatrix}$$

$$o_5 = \begin{bmatrix} d_5\ {}_1s_{234} + c\ \ a_3c_1c_{23} + a_2c_1c_2 \\ d_5\ {}_1s_{234} + s\ \ a_3s_1c_{23} + a_2s_1c_2 \\ -d\ c_{234} + \ {}_5\ a_3s_{23} + a_2s_2 + d_1 \end{bmatrix}$$

Also, the angular part of $J(\theta)$ can be obtained using the forward kinematics, from the equations (2-9) the values of z_i are:

$$z_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ Then

$$z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$z_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$z_3 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} c_1\, s_{234} \\ s_1\, s_{234} \\ -c_{234} \end{bmatrix}$$

$$z_0 \times (o_5 - o_0) = \begin{bmatrix} -d_5 s_1 s_{234} - a_3 s_1 c_{23} - a_2 s_1 c_2 \\ d_5 c_1 s_{234} + a_3 c_1 c_{23} + a_2 c_1 c_2 \\ 0 \end{bmatrix} = J_{11} .$$

$$z_1 \times (o_5 - o_1) = \begin{bmatrix} d_5 c_1 c_{234} - a_3 c_1 s_{23} - a_2 s_2 c_1 \\ d_5 s_1 c_{234} - a_3 s_1 s_{23} - a_2 s_2 s_1 \\ d_5 s_{234} + a_3 c_{23} + a_2 c_2 \end{bmatrix} = J_{12} .$$

$$z_2 \times (o_5 - o_2) = \begin{bmatrix} d_5 c_1 c_{234} - a_3 c_1 s_{23} \\ d_5 s_1 c_{234} - a_3 s_1 s_{23} \\ d_5 s_{234} + a_3 c_{23} \end{bmatrix} = J_{13} .$$

$$z_3 \times (o_5 - o_3) = \begin{bmatrix} d_5 c_1 c_{234} \\ d_5 s_1 c_{234} \\ d_5 s_{234} \end{bmatrix} = J_{14} .$$

$$z_4 \times (o_5 - o_4) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = J_{15} .$$

The Jacobian matrix will become:

$$J(\theta) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ 0 & s_1 & s_1 & s_1 & c_1\, s_{234} \\ 0 & -c_1 & -c_1 & -c_1 & s_1\, s_{234} \\ 1 & 0 & 0 & 0 & -c_{234} \end{bmatrix} .$$