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Mathematical Institute

Exercises for Optimization I
Sheet 10*

You can submit in your solutions for the Numerical Exercise until Sunday, July 17 at 9pm, Discussion of this Exercise Sheet will be on July 11-12.

Exercise 1: Prove iii) from Corollary 3 in lecture 19.

Exercise 2: Prove "i) $\Rightarrow$ ii)" and "i) $\Rightarrow$ iii)" from Theorem 5 in lecture 19. Explain why this completes the proof from the lecture.

Exercise 3: Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Consider a Newton-like method with symmetric matrices M_k to find the minimum of f. Assume that the generated sequence $\left(x^k\right)$ converges to the solution $x^* \in \mathbb{R}^n$ and the sequence (M_k) converges to $\text{Hess}(f(x^*))$.

Show that in this setting the Dennis-Moré-condition

$$\|(M_k - \text{Hess}(f(x^*)))(x^{k+1} - x^k)\| = o(\|x^{k+1} - x^k\|) \quad (k \to \infty)$$

is fulfilled. Which additional assumption is needed to get q-superlinear convergence of (x^k) to x^*?

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*) **Group 10** will present the solutions.

Numerical Exercise 6: Consider the 1D-membrane-problem: Find a function $\mathbf{u} : [0,1] \to \mathbb{R}$ which solves the minimization problem

$$\min_{\mathbf{u}} \frac{1}{2}\int_0^1 |\mathbf{u}'(x)|^2\,dx - \int_0^1 \mathbf{u}(x)\,dx,$$

and satisfies the boundary condition $\mathbf{u}(0) = \mathbf{u}(1) = 0$. On the grid $0 = x_0 < x_1 < \ldots < x_{N+1}$ with $x_i := \frac{i}{N+1}$ we define the hat functions b_j by the requirements

i) $b_j(x_i) = \delta_{ij}$ for $i, j = 0, \ldots, N+1$,

ii) $b_j \in \mathscr{C}([0,1])$,

iii) $b_j|_{[x_k, x_{k+1}]}$ linear for $j = 0, \ldots, N+1$ and $k = 0, \ldots, N$.

For the sought function $\mathbf{u}$ we make the Ansatz $\mathbf{u}(x) = \sum_{i=1}^N u_i b_i(x)$, which allows to rewrite our minimization problem in the form

$$\min_{u\in\mathbb{R}^N} F(u) := \frac{1}{2}u^\top A_N u - \mathbf{1}^\top M_N u, \tag{1}$$

with $u = (u_i)_{i=1}^N$ and $\mathbf{1} = (1)_{i=1}^N$ and matrices

$$A_N = (N+1)\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ \vdots & & & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \quad \text{and } M_N = \frac{1}{6(N+1)}\begin{pmatrix} 4 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & 1 & & & \vdots \\ 0 & 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ \vdots & & & 1 & 4 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 4 \end{pmatrix}.$$

a) For $N \in \{10, 20, 40, 80\}$ and $u^0 = (0, \ldots, 0)^\top$ solve problem (1) with the descent method with Armijo stepsize rule. Use as stopping criterion

$$\|\nabla F(u)\| \leq 10^{-4}(1 + \|\nabla F(u^0)\|).$$

b) Solve the same problem using the Conjugate Gradient (CG) method, which is available in Matlab as the function *pcg*, invoking the same stopping criterion.

Compare the number of iterations, the function value $F(u^*)$ and the norm of the gradient $\|\nabla F(u^*)\|$. Provide and iteration monitor which compares both methods in every iteration. The monitor should include $\|\nabla F(u^k)\|, F(u^k)$ and $\frac{\|x^{k+1}-x^k\|}{\|x^k - x^{k-1}\|}$.