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Mathematical Institute

Exercises for Optimization I
Sheet 11*

You can submit in your solutions for the Numerical Exercise until Sunday, July 24 at 9pm, Discussion of this Exercise Sheet will be on July 18-19.

Exercise 1: Find the global minimum of the following problem:

$$\min_{x\in\mathbb{R}^2} f(x) := (1-x_1)(x_2-2)^2 - x_2$$
$$\text{s.t.} \qquad \begin{pmatrix} -x_1 \\ x_1-1 \\ (x_1+x_2-1)^2(x_1-x_2-1)^2 \\ x_2^2-0.5 \end{pmatrix} \leq 0.$$

Exercise 2: Let $\Delta > 0$ and $q(s) := \frac{1}{2}s^\top As - b^\top s$, with $A \geq 0$ symmetric. Consider the *Trust-Region problem*:

$$\min q(s) \qquad \text{s.t. } \|s\| \leq \Delta.$$

Show:

a) This problem has a solution.

b) If there exists $\lambda \geq 0$, such that

   i) $(A+\lambda Id)s^* = b$,
   ii) $A+\lambda Id \geq 0$ and
   iii) either $\|s^*\| = \Delta$ or $(\|s^*\| < \Delta$ and $\lambda = 0)$,

   are true, then s^* solves the Trust-region problem.

Exercise 3: Derive the necessary optimality condition of the problem

$$\min_{x\in\mathbb{R}^2} x_1^2 - 2x_1 + x_2^2 - 3 \qquad \text{s.t. } 2x_1 + 3x_2 - 15 = 0. \tag{1}$$

geometrically. Then calculate the stationary points and associated Lagrange multipliers to verify your geometric solution. Furthermore, check the sufficient second order conditions.

*Group 11 will present the solutions.

Exercise 4: Derive the necessary optimality condition of the problem

$$\min_{x\in C}(x_1-2)^2+(x_2-1)^2, \qquad C=[-1,1]\times[-1,1]. \tag{2}$$

geometrically. Then calculate the stationary points and associated Lagrange multipliers to verify your geometric solution. Furthermore, check the sufficient second order conditions.

Numerical Exercise 7: Use a projected gradient method with optimal stepsize $t_{\min}$ (see Exercise Sheet 4) and tolerance 10^{-6} to solve the quadratic restricted problem

$$\min_{x\in M}\|Ax-b\|^2, \qquad M=\{x\in\mathbb{R}^4 : x\geq 0\}. \tag{3}$$

with

$$A=\begin{pmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{pmatrix}, \quad b=\begin{pmatrix}2\\0\\-2\\-1\end{pmatrix}.$$

projected gradient method:

i) $x^0\in\mathbb{R}^n, \tau_r, \tau_a>0$ tolerances, $k=0$, $\beta\in(0,1)$ and $\sigma>0$

$$x^0(1):=P_\Omega(x^0-\nabla f(x^0))$$

ii) If $\|x^k-x^k(1)\|\leq\tau_a+\tau_r\|x^0-x^0(1)\|$: stop with $x^*:=x^k$

iii) $\lambda=\operatorname{argmax}\left\{\beta^m : f(x^k(\beta^m))-f(x^k)\leq-\frac{\sigma}{\beta^m}\|x^k-x^k(\beta^m)\|^2\right\}$ (Armijo step size)

$x^k:=x^k(\lambda):=P_\Omega(x^k-\lambda\nabla f(x^k))$,

$k=k+1$, go to ii)