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Mathematical Institute

Exercises for Optimization I
Sheet 8*

You can submit in your solutions for the Numerical Exercise until Sunday, July 3 at 9pm, Discussion of this Exercise Sheet will be on June 27-28.

Exercise 1: Let $f(x) = \frac{1}{x^\alpha}$. For a fixed $p, 1 \leq p < \infty$, find the range for α such that $f \in L^p([1,\infty))$.

Exercise 2: For $\Omega = (0,1)$, find an element f such that $f \in L_1^{\text{loc}}(\Omega)$ but $f \notin L^1(\Omega)$.

Exercise 3: Let $f \in C(\mathbb{R})$ by $f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$. Prove that the weak derivative of f is is the step function

$$\chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Exercise 4: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \begin{cases} x_1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

Prove that this function has directional derivatives in all directions at $(0,0)$ but is not Gâteaux differentiable at $(0,0)$.

Exercise 5: Let A be an $n \times n$ matrix and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the functional $f(\mathrm{x}) = \mathrm{x}^T A \mathrm{x}$. Prove that $(\delta f)(\mathrm{x}_0) = \mathrm{x}_0^T \left(A + A^T\right)$.

Exercise 6: Let $f : R^2 \to R^2$ defined by

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^4 y}{x^6 + y^3} & x^2 + y^2 > 0 \end{cases}.$$

Prove that f has 0 as its Gâteaux derivative at the origin but fails to be continuous there.

*Group 8 will present the solutions.

Numerical Exercise 4: Use Newton's method to find:

a) the root of

$$f(x) = \exp(-x) - 10^{-9}$$

with $x^0 = 0$, and compare the results $(x^k, f(x^k))$ for the stopping criteria

(i) $|f(x^k)| < \epsilon_a$,

(ii) $|f(x^k)| < \epsilon_r |f(x^0)| + \epsilon_a$,

(iii) $|x^{k+1} - x^k| < \epsilon_a$,

(iv) $|x^{k+1} - x^k| < \epsilon_r |f(x^0)| + \epsilon_a$,

with all possible combinations of $\epsilon_a, \epsilon_r = 10^{-3}$ and 10^{-10} .

b) the extremum of

$$F(x) = (x_1 - 1)^4 + 2(x_1 - 1)^2(x_2 - 1)^2 + (x_2 + 1)^4 - 2(x_2 - 1)^2 - (2x_2 + 1)^2 + 1$$

with $x^0 = (1.21, -1.15)^\top$ and the stopping criteria

(i) $\|x^{k+1} - x^k\|_\infty < 10^{-4}$,

(ii) $\|x^{k+1} - x^k\|_\infty < \epsilon_r \|\nabla F(x^0)\| + 10^{-4}$,

where $\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$. In (ii) test different values for ϵ_r and compare the results.