Prof. Dr. Michael Hinze
Dr. Sema Coskun

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Mathematical Institute

Exercises for Optimization I
Sheet 6*

You can submit in your solutions for the Numerical Exercise until Sunday, June 12 at 9pm, Discussion of this Exercise Sheet will be on June 13-14.

Exercise 1: Show the left inequality from Lemma 2 iii) in lecture 11.

Exercise 2: Let $f : \mathbb{R} \to \mathbb{R}, f(x) := -\frac{1}{4}x^4$ with starting point $x^0 \in \mathbb{R}$ arbitrary. Show that the sequence (x^k) generated by Newton's method converges to $x^* = 0$. What kind of extremum is x^*?

Exercise 3: Show that the simplified Newton's method (for an optimization problem) with $M_k = \mathrm{Hess}(f(x^0))$, where x^0 close enough to x^*, admits linear convergence, i.e.

$$\|x^{k+1} - x^*\| \leq c\, \|x^k - x^*\| \quad \forall k, \qquad \text{with } c < 1.$$

Exercise 4:

a) Given the matrices $U, V \in \mathbb{R}^{n\times m}$ and the invertible matrix $A \in \mathbb{R}^{n\times n}$. Show that if $(Id_m - V^\top A^{-1}U)$ is invertible, then $M := A - UV^\top$ is invertible with

$$M^{-1} = A^{-1} + A^{-1}U(Id_m - V^\top A^{-1}U)^{-1}V^\top A^{-1}.$$

Hint: Calculate MM^{-1} and $M^{-1}M$.

(b) Show that $M = A - uv^\top$, with $u, v \in \mathbb{R}^n$ is invertible, if and only if $1 - v^\top A^{-1}u \neq 0$. In this case it holds

$$M^{-1} = A^{-1} + \frac{A^{-1}uv^\top A^{-1}}{1 - v^\top A^{-1}u}.$$

*[*] **Group 6** will present the solutions.

Numerical Exercise 3: (Minimal triangulated graphs)

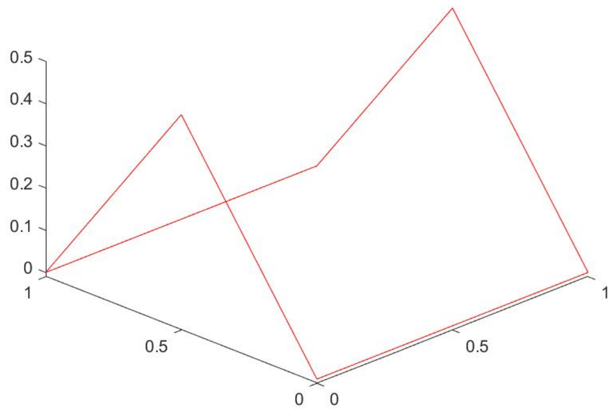

Figure 1: Boundary condition

Compute the minimal triangular graph over $\Omega := (0,1)^2$, i.e find a function $q : \bar{\Omega} \to \mathbb{R}$, which is piecewise linear and continuous on a given triangulation of Ω, and which satisfies the boundary conditions $q_{|\partial\Omega} = \frac{1}{2} - |x_2 - \frac{1}{2}|$, and whose graph has minimal surface area among all piecewise linear and continuous functions defined on the same triangulation and satisfying the same boundary conditions.

Output: Surface area and a figure showing the triangulated graph.

Proceed as follows: Subdivide Ω into triangles (everything consists of triangles (Plato)) with corners $Q_1(x_1^1, x_2^1), \dots, Q_m(x_1^m, x_2^m)$. Consider functions q, which are continuous on $\bar{\Omega}$ and linear on each triangle. Construct a function $A : \mathbb{R}^n \to \mathbb{R}$ which defines the surface area of the graph of q (what is n?).

Solution algorithm: Steepest descent with Armijo step size rule:

a) Choose $x^0 \in \mathbb{R}^n$, and $\beta, \gamma \in (0,1)$,

b) for $k = 0, 1, 2, \dots$ do

(i) $\nabla f(x^k) = 0 \to$ STOP with $x^* = x^k$.

(ii) $s^k := -\nabla f(x^k)$,

(iii) $\sigma_k := \max\{\sigma > 0; \sigma \in \{1, \beta, \beta^2, \dots\}\} : f(x^k) - f(x^k + \sigma s^k) \geq -\gamma\sigma\nabla f(x^k)^t s^k$

(iv) $x^{k+1} = x^k + \sigma_k s^k$

Choose $\beta = \frac{1}{2}$, $\gamma \in [10^{-3}, 10^{-2}]$ and stop if in b) (ii) $\|s^k\| \leq 10^{-8}$ is satisfied.