Prof. Dr. Michael Hinze
Dr. Sema Coskun

Summer term 2022
Mathematical Institute

Exercises for Optimization I
Sheet 9*

You can submit in your solutions for the Numerical Exercise until Sunday, July 10 at 9pm, Discussion of this Exercise Sheet will be on July 4-5.

Exercise 1: Let $\hat{f}(x) := f(x) + \epsilon_f(x)$ be the perturbed version of the function f. We denote by ϵ_f, $\epsilon_{\nabla f}$ and $\epsilon_{\nabla^2 f}$ the errors in the function f, the gradient ∇f and the hessian $\nabla^2 f$.

a) Approximate the gradient and the hessian of f with central differences:

$$\nabla_h f(x) := \frac{\hat{f}(x+h) - \hat{f}(x-h)}{2h}.$$

Show that in this case the following relation of the errors holds:

$$\epsilon_{\nabla^2 f} = \mathcal{O}((\epsilon_{\nabla f})^{\frac{2}{3}}) = \mathcal{O}((\epsilon_f)^{\frac{4}{9}}).$$

b) Approximate the gradient of f with central differences and the hessian of f with forward differences:

$$\nabla_h^2 f(x) := \frac{\nabla\hat{f}(x+h) - \nabla\hat{f}(x)}{h}$$

Show that in this case the following relation of the errors holds:

$$\epsilon_{\nabla^2 f} = \mathcal{O}((\epsilon_{\nabla f})^{\frac{1}{2}}) = \mathcal{O}((\epsilon_f)^{\frac{1}{3}}).$$

Exercise 2: Show that the local Newton's method for optimization problems is invariant to affine-linear transformations $x = Ay + b$, where $A \in \mathbb{R}^{n\times n}$ invertible and $b \in \mathbb{R}^n$, i.e. the Newton's method applied to $f(x)$ generates a sequence $\left(x^k\right)$ and the Newton's method applied to $g(y) := f(Ay + b)$ generates a sequence $\left(y^k\right)$, such that

$$x^0 = Ay^0 + b \quad \Rightarrow \quad x^k = Ay^k + b \quad \forall\, k \in \mathbb{N}.$$

Exercise 3: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function, i.e. $f(x) = \frac{1}{2}x^\top Ax - b^\top x + c$, with $A \in \mathbb{R}^{n\times n}$ spd, $b \in \mathbb{R}^n, c \in \mathbb{R}$ and $t_{\min}$ the exact steplength $t_{\min} := \text{argmin}_{t\geq 0} f(x + td)$, with $x, d \in \mathbb{R}^n$, such that $\nabla f(x)^\top d < 0$. Then show

$$t_{\min} \in T_{\text{WP}}(x, d)$$

*Group 9 will present the solutions.

where the Wolfe-Powell steplength rule is given for $\sigma \in (0, \frac{1}{2})$ and $\rho \in [\sigma, 1)$ by

$$T_{\mathrm{WP}}(x,d) = \left\{ t > 0 \mid f(x+td) \leq f(x) + \sigma t f'(x) d \text{ and } f'(x+td) \geq \rho f'(x) d \right\}$$

with d being the descent direction of f at x, i.e. $f'(x)d < 0$.

Exercise 4: Show the left inequality from Lemma 2 iii) in lecture 11.

Numerical Exercise 5: Like Numerical Exercise 4 a) with $\epsilon_r = \epsilon_a = 10^{-3}$, but use the simplified Newton's method. What do you observe?