

Discrete Mathematics

Yuanfu Li

Contents

Chapter 9

9.1 Relations and Their Properties

9.1.1 Introduction

Definition 9.1.1

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

We use the notation aRb to denote that $(a, b) \in R$.

9.1.2 Function as Relations

The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. The graph of f is a subset of $A \times B$, it is a relation from A to B .

Note:-

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R , then a function can be defined with R as its graph.

9.1.3 Relations on a Set

Definition 9.1.2

A *relation on a set A* is a relation from A to A .

Example 9.1.1

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$.

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b , we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

9.1.4 Properties of Relations

Definition 9.1.3

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element

Note:-

We see that a relation on A is reflexive if every element of A is related to itself

Definition 9.1.4

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

In other words, a relation is symmetric if and only if a is related to b always implies that b is related to a . A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a . That is, the only way to have a related to b and b related to a is for a and b to be the same element.

Definition 9.1.5

A relation R on a set A is called *transitive* is whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

9.1.5 Combining Relations

Definition 9.1.6

Let R be a relation from a set A to a set B and S a relation from B to a set C . The *composite* of R and S is the relation consisting of ordered pairs (a, c) , where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Note:-

Computing the composite of two relations requires that we find elements that are the *second* element of ordered pairs in the *first* relation and the *first* element of ordered pairs in the *second* relation.

Example 9.1.2

What is the composite of the relations R and S , where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

Definition 9.1.7

Let R be a relation on the set A . The powers $R^n, n = 1, 2, 3, \dots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Theorem 9.1.1

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: We first prove the "if" part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R . To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. ☺

So fucking elusive and tedious.

9.2 n -ary Relations and Their Applications

9.2.1 Introduction

Relations among elements from more than two sets are called n -ary relations.

9.2.2 n -ary Relations

Definition 9.2.1

Let A_1, A_2, \dots, A_n be sets. An n -ary relation on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$. The sets A_1, A_2, \dots, A_n are called the *domains* of the relation, and n is called its *degree*.

Example 9.2.1

Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of triples (a, b, c) , where a, b and c are integers with $a < b < c$

9.2.3 Databases and Relations

Understand the concept of **relational data model**, that's it.

9.2.4 Operations on n -ary Relations

Definition 9.2.2

Let R be an n -ary relation and C a condition that elements in R may satisfy. Then the *selection operator* s_C maps the n -ary relation R to the n -ary relation of all n -tuples from R that satisfy the condition C .

Definition 9.2.3

The *projection* P_{i_1, i_2, \dots, i_m} , where $i_1 < i_2 < \dots < i_m$, maps the n -tuple (a_1, a_2, \dots, a_n) to the m -tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$, where $m \leq n$.

In other words, the projection P_{i_1, i_2, \dots, i_m} deletes $n - m$ of the components of an n -tuple, leaving the i_1 th, i_2 th, ..., and i_m th components.

Definition 9.2.4

Let R be a relation of degree m and S a relation of degree n . The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$ -tuples $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$, where the m -tuple $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p)$ belongs to R and the n -tuple $(c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ belongs to S .

Definition 9.2.5

The *count* of an itemset I , denoted by $\sigma(I)$, in a set of transactions $T = \{t_1, t_2, \dots, t_k\}$, where k is a positive integer, is the number of transactions that contain this itemset. That is, $\sigma(I) = |\{t_i \in T \mid I \subseteq t_i\}|$.

The *support* of an itemset I is the probability that I is included in a randomly selected transaction from T . That is,

$$\text{support}(I) = \frac{\sigma(I)}{|T|}$$

The **support threshold** s is specified for a particular application. **Frequent itemset mining** is the process of finding itemset I with support greater than or equal to s . Such itemsets are said to be **frequent**.

Definition 9.2.6

If I and J are subsets of a set T of transactions, then

$$\text{support}(I \rightarrow J) = \frac{\sigma(I \cup J)}{|T|}$$

and

$$\text{support}(I \rightarrow J) = \frac{\sigma(I \cup J)}{\sigma(I)}.$$

9.4 Closure of Relations

9.4.1 Introduction

We can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R . Here, S is the smallest transitive relation that contains R . This relation is called the **transitive closure** of R .

9.4.2 Different Types of Closures

Definition 9.4.1

If R is a relation on a set A , then the **closure** of R is with respect to P , if it exists, is the relation S on A with property **P** that contains R and is a subset of every subset of $A \times A$ containing R with property **P**.

Example 9.4.1

The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive. By adding $(2, 2)$ and $(3, 3)$ to R which are the only pairs of the form (a, a) that are not in R , we can get a new relation which contains R .

The above example creates the **reflexive closure** of R .

We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a, a) | a \in A\}$ is the **diagonal relation** on A .

Example 9.4.2

The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric. By adding $(2, 1)$ and $(1, 3)$ to R which are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R , we can get a new symmetric relation contains R .

The new relation is called the **symmetric closure** of R .

9.4.3 Path in Directed Graphs

A path in a directed graph is obtained by traversing along edges (in the same direction as indicated by the arrow on the edge).

Definition 9.4.2

A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ in G , where n is a non-negative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ and has *length* n . We view the empty set of edges as a path of length zero from a to a . A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

The term *path* also applies to relations. Carrying over the definition from directed graphs to relations, there is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \dots$, and $(x_{n-1}, b) \in R$. Theorem 1 can be obtained from the definition of a path in a relation.

Theorem 9.4.1

Let R be a relation on a set A . There is a path of length n , where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$.

Assume that the theorem is true for the positive integer n . This is the inductive hypothesis. There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c , so $(a, c) \in R$, and a path of length n from c to b , that is, $(c, b) \in R^n$. Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$. Therefore, there is a path of length $n + 1$ from a to b if and only if $(a, b) \in R^{n+1}$. This completes the proof. \odot

9.4.4 Transitive Closures

Definition 9.4.3

Let R be a relation on a set A . The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R .

Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b , it follows that R^* is the union of all the sets R^n . In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

The connectivity relation is useful in many models.

Example 9.4.3

Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^* ?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most $n - 1$ changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary.

Theorem 9.4.2

The transitive closure of a relation R equals the connectivity relation R^* .

Lemma 9.4.1

Let A be a set with n elements, and let R be a relation on A . If there is a path of length at least one in R from a to b , then there is such a path with length not exceeding n . Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b , then there is such a path with length not exceeding $n - 1$.

From Lemma 1, we see that the transitive closure of R is the union of R, R^2, R^3, \dots , and R^n . This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i , for some positive integer i with $i \leq n$. Because

$$R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n$$

and the zero-one matrix representing a union of relations is the join of the zero-one matrices of these relations, the zero-one matrix for the transitive closure is the join of the zero-one matrices of the first n powers of the zero-one matrix of R .

Theorem 9.4.3

Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \dots \vee M_R^{[n]}$$

Example 9.4.4

Find the zero-one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: By Theorem 3, it follows that the zero-one matrix of R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

Because

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

it follows that

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithms 1

```

1  int** transitive_closure(M = zero-one n*n matrix)
2  {
3      int** A = M;
4      int** B = A;
5      for(int i = 2; i <= n; i++)
6      {
7          A = A xnor M;
8          B = B or A;
9      }
10     return B;
11 }
```


9.4.5 Warshall's Algorithm

Definition 9.4.4

Suppose that R is a relation on a set with n elements. Let v_1, v_2, \dots, v_n be an arbitrary listing of these n elements. The concept of the **interior vertices** of a path is used in Warshall's algorithm. If $a, x_1, x_2, \dots, x_{m-1}, b$ is a path, its interior vertices are x_1, x_2, \dots, x_{m-1} , that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.

Warshall's algorithm is based on the construction of a sequence of zero-one matrices. These matrices are W_0, W_1, \dots, W_n , where $W_0 = W_R$ is the zero-one matrix of this relation, and $W_k = [w_{ij}^{(k)}]$, where $W_k = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \dots, v_k\}$ (the first k vertices in the list) and is 0 otherwise. (The first and last vertices in the path may be outside the set of the first k vertices in the list). Note that $W_n = W_{R^*}$, because the (i, j) th entry of W_{R^*} is 1 if and only if there is a path from v_i to v_j , with all interior vertices in the set $\{v_1, v_2, \dots, v_k\}$ (but these are the only vertices in the directed graph).

Example 9.4.5

Let R be the relation $\{(a, d), (b, a), (b, c), (c, a), (c, d), (d, c)\}$. Let a, b, c, d be a listing of the elements of the set. Find the matrices W_0, W_1, W_2, W_3, W_4 . The matrix W_4 is the transitive closure R .

Solution: Let $v_1 = a, v_2 = b, v_3 = c, v_4 = d$. W_0 is the matrix of the relation. Hence,

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad W_2 = W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 9.4.2

Let $W_k = [w_{ij}^{[k-1]}]$ be the zero-one matrix that has a 1 in its (i, j) th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \dots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$$

Algorithm 2

```

1  int** Warshall(M = n*n zero-one matrix)
2  {
3      W = M;
4      /* Suppose the first index of an array is 1. */
5      for (int k = 1; k <= n; k++)
6      {
7          for (int i = 1; i <= n; i++)
8          {
9              for (int j = 1; j <= n; j++)
10                 W[i][j] = W[i][j] | (W[i][k] & W[k][j])
11             }
12         }
13     return W;
14 }
```

9.4.6 Equivalence Relations

Definition 9.4.5

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 9.4.6

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Definition 9.4.7

Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a . The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

9.4.7 Equivalence Classes and Partitions

R is an equivalence relation. We can see that R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major. Furthermore, these subsets are equivalence classes of R . This example illustrates how the equivalence classes of an equivalence relation partition a set into disjoint, non-empty subsets.

Theorem 9.4.4

Let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

$$aRb \qquad [a] = [b] \qquad [a] \cap [b] \neq \emptyset$$

Now we are going to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A . The union of the equivalence classes of R is all of A , because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 9.4.4, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset$$

when $[a]_R \neq [b]_R$

These two observations show that the equivalence classes form a partition of A , because they split A into disjoint subsets. More precisely, a **partition** of a set S is a collection of disjoint non-empty subsets of S that have S as their union. In other words, the collection of subsets $A_i, i \in I$ forms a partition of S if and only if

$$\begin{aligned} A_i &\neq \emptyset \text{ for } i \in I, \\ A_i \cap A_j &= \emptyset \text{ when } i \neq j, \end{aligned}$$

and

$$\bigcup_{i \in I} A_i = S$$

Theorem 9.4.5

Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i | i \in I\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.