Homework 3

Atlas42

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1 X^TX is invertible when X is full rank

For X to be full rank, either when its rows or its columns are linearly independent. For example: If our matrix is an m $\times$ n matrix with $m < n$, then it has full rank when its m rows are linearly independent or Rank $(X) = m$. If $m > n$, the matrix has full rank when its n columns are linearly independent or Rank $(X) =$ n. If $m = n$, the matrix has full rank either when its rows or its columns are linearly independent (when the rows are linearly independent, so are its columns in this case).
Assume that $m \leq n$ and Rank $(X) =$ m, and let $X^TX_u = 0$ for some $u \in R^m$
We need to show that $u = 0$, We also have that

$$0 = (X^TX_u, u) = (Xu, Xu),$$

and thus $X_u = 0$. But as Rank $(X) = m$. (Otherwise, the columns of X would be linearly dependent, and hence its rank less than m.) Assume that $X^TX \in R^{m\times m}$ is invertible. Then $m =$ Rank $(X^TX) \leq$ Rank $(X) \leq \min\{m, n\}$. Thus $\min\{m, n\} = m$, Rank $(X) = m$ and $m \leq n$

2 To be continued