t-SNE

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1 Finding the derivative of t-SNE

Define: $q_{ij} = q_{ji} = \dfrac{(1 + \left\|y_i - y_j\right\|^2)^{-1}}{\sum_{k,l\neq k}(1 + \left\|y_k - y_l\right\|^2)^{-1}} = \dfrac{E_{ij}^{-1}}{\sum_{k,l\neq k} E_{kl}^{-1}} = \dfrac{E_{ij}^{-1}}{Z}$

Notice that $E_{ij} = E_{ji}$, thus we have the loss function

$$
\begin{aligned}
C = \mathbf{KL}(p_i||q_i) &= \sum_{k,l\neq k} p_{lk} \log \frac{p_{lk}}{q_{ik}} \\
&= \sum_{k,l\neq k} p_{lk} \left(\log p_{lk} - \log q_{lk}\right) \\
&= \sum_{k,l\neq k} p_{lk} \log p_{lk} - p_{lk} \log E_{kl}^{-1} + p_{lk} \log Z
\end{aligned}
$$

We derive with respect to y_i.

$$
\begin{aligned}
\Rightarrow -\frac{\partial L}{\partial y_i} &= \sum_{k,l\neq k} -p_{ik} \log \partial E_{kl}^{-1} + p_{lk} \partial \log Z \\
&= -2 \sum_{j\neq i} p_{ji} \frac{E_{ij}^{-2}}{E_{ij}^{-1}} (-2(y_i - y_j)) + \frac{1}{Z} \sum_{k',l'\neq k'} \partial E_{kl}^{-1} \\
&= 4 \sum_{j\neq i} p_{ji} E_{ij}^{-1} (y_i - y_j) - 4 \sum_{j\neq i} q_{ij} E_{ji}^{-1} (y_i - y_j) * \\
&= 4 \sum_{j\neq i} (p_{ji} - q_{ij})(1 + \left\|y_i - y_j\right\|^2)(y_i - y_j)
\end{aligned}
$$

* We derived this using the fact that $\sum_{k,l\neq k} p_{kl} = 1$ and that Z does not depend on k or l

2 Finding the derivative of SNE

Define: $q_{j|i} = \dfrac{exp^{-\left\|y_i - y_j\right\|^2}}{\sum_{k \neq i} exp^{-\left\|y_i - y_k\right\|^2}} = \dfrac{E_{ij}}{\sum_{k \neq i} E_{ik}} = \dfrac{Eij}{Z_i}$

Notice that $E_{ij} = E_{ji}$, thus we have the loss function

$$
\begin{aligned}
C = \mathbf{KL}(p_i || q_i) &= \sum_{k, l \neq k} p_{l|k} \log \frac{p_{l|k}}{q_{i|k}} \\
&= \sum_{k, l \neq k} p_{l|k} \left(\log p_{l|k} - \log q_{l|k} \right) \\
&= \sum_{k, l \neq k} p_{l|k} \log p_{l|k} - p_{l|k} \log E_{kl} + p_{l|k} \log Z
\end{aligned}
$$

We derive with respect to y_i.