Solving Equation of Motion with Explicit FEM

The equation of Motion for non-linear damped systems is:

$$M\ddot{u} + C\dot{u} + f^{\text{int}} = F^{\text{ext}}, \tag{1}$$

where M is the mass, $\ddot{u}$ is the acceleration, C is the damping constant, f^{int} is the non-linear internal force, and F^{ext} is the external force.
Using an explicit scheme, where the subscript i denotes the time step, the equation becomes:

$$M\ddot{u}_i + C\dot{u}_i + f_i^{\text{int}} = F_i^{\text{ext}}. \tag{2}$$

The acceleration can be solved simply as:

$$\ddot{u}_i = M^{-1} \cdot \left(F_i^{\text{ext}} - f_i^{\text{int}} - C\dot{u}_i \right) \tag{3}$$

The update of the velocity and the displacement requires a time integration scheme. The different schemes are derived using Taylor series expansion, and explained in the following section.

Time Integration Schemes

In order to have a second order convergence, a second order scheme is needed to approximate the acceleration and velocity. However, the current velocity, $\dot{u}_i$, which goes into the damping term, $C\dot{u}_i$, in eq. (3), uses a first order forward difference scheme. This leads to second order convergence in case of undamped systems and first order convergence in case of damped systems. The derivation of the order of convergence is shown below.

Second Order Central Differences

Taylor series expansions around $\dot{u}_{i+\frac{1}{2}}$ and $\dot{u}_{i-\frac{1}{2}}$:

$$\begin{aligned}
T_1 \quad &\Rightarrow \quad \dot{u}_{i+\frac{1}{2}} = \dot{u}_i + \frac{(\Delta t/2)}{1!}\ddot{u}_i + \frac{(\Delta t/2)^2}{2!}\dddot{u}_i + \frac{(\Delta t/2)^3}{3!}\ddddot{u}_i + \cdots \\
T_2 \quad &\Rightarrow \quad \dot{u}_{i-\frac{1}{2}} = \dot{u}_i - \frac{(\Delta t/2)}{1!}\ddot{u}_i + \frac{(\Delta t/2)^2}{2!}\dddot{u}_i - \frac{(\Delta t/2)^3}{3!}\ddddot{u}_i + \cdots \\
T_1 - T_2 \quad &\Rightarrow \quad \dot{u}_{i+\frac{1}{2}} - \dot{u}_{i-\frac{1}{2}} = \Delta t \cdot \ddot{u}_i + \frac{(\Delta t)^3}{24}\ddddot{u}_i + \cdots \\
&\Rightarrow \quad \ddot{u}_i = \frac{\dot{u}_{i+\frac{1}{2}} - \dot{u}_{i-\frac{1}{2}}}{\Delta t} - \frac{(\Delta t)^2}{24}\ddddot{u}_i + \cdots \\
&\Rightarrow \quad \ddot{u}_i = \frac{\dot{u}_{i+\frac{1}{2}} - \dot{u}_{i-\frac{1}{2}}}{\Delta t} + \mathcal{O}(\Delta t^2)
\end{aligned}$$

This is reorganized to approximate the next middle velocity at each step as in,

$$\dot{u}_{i+\frac{1}{2}} \approx \dot{u}_{i-\frac{1}{2}} + \Delta t \cdot \ddot{u}_i. \tag{4}$$

The displacement for next time step, u_{i+1}, can then be updated using the middle velocity, $\dot{u}_{i+\frac{1}{2}}$. Taylor series expansions around u_{i+1} and u_i would read:

$$
\begin{aligned}
T_3 \quad &\Rightarrow \quad u_{i+1} = u_{i+\frac{1}{2}} + \frac{(\Delta t/2)}{1!}\dot{u}_{i+\frac{1}{2}} + \frac{(\Delta t/2)^2}{2!}\ddot{u}_{i+\frac{1}{2}} + \frac{(\Delta t/2)^3}{3!}\dddot{u}_{i+\frac{1}{2}} + \cdots \\
T_4 \quad &\Rightarrow \quad u_i = u_{i+\frac{1}{2}} - \frac{(\Delta t/2)}{1!}\dot{u}_{i+\frac{1}{2}} + \frac{(\Delta t/2)^2}{2!}\ddot{u}_{i+\frac{1}{2}} - \frac{(\Delta t/2)^3}{3!}\dddot{u}_{i+\frac{1}{2}} + \cdots \\
T_3 - T_4 \quad &\Rightarrow \quad u_{i+1} - u_i = \Delta t \cdot \dot{u}_{i+\frac{1}{2}} + \frac{(\Delta t)^3}{24}\dddot{u}_{i+\frac{1}{2}} + \cdots \\
&\Rightarrow \quad \dot{u}_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta t} - \frac{(\Delta t)^2}{24}\dddot{u}_{i+\frac{1}{2}} + \cdots \\
&\Rightarrow \quad \dot{u}_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta t} + \mathcal{O}(\Delta t^2)
\end{aligned}
$$

This is reorganized to approximate the next displacement at each step as in,

$$
u_{i+1} \approx u_i + \Delta t \cdot \dot{u}_{i+\frac{1}{2}}. \tag{5}
$$

First Order Forward Differences

The previous section is sufficient in case of undamped systems, i.e. $\ddot{u}_i = M^{-1} \cdot \left(F_i^{\text{ext}} - f_i^{\text{int}}\right)$, because the current velocity, $\dot{u}_i$, is not needed. However, for damped systems, this term is needed and approximated by a first order scheme. The scheme is derived from a taylor series expansion around $\dot{u}_{i-\frac{1}{2}}$ (T_2 from previous section):

$$
\begin{aligned}
T_2 \quad &\Rightarrow \quad \dot{u}_{i-\frac{1}{2}} = \dot{u}_i - \frac{(\Delta t/2)}{1!}\ddot{u}_i + \frac{(\Delta t/2)^2}{2!}\dddot{u}_i - \frac{(\Delta t/2)^3}{3!}\ddddot{u}_i + \cdots \\
&\Rightarrow \quad \dot{u}_i - \dot{u}_{i-\frac{1}{2}} = \frac{(\Delta t/2)}{1!}\ddot{u}_i - \frac{(\Delta t/2)^2}{2!}\dddot{u}_i + \frac{(\Delta t/2)^3}{3!}\ddddot{u}_i + \cdots \\
&\Rightarrow \quad \ddot{u}_i = \frac{\dot{u}_i - \dot{u}_{i-\frac{1}{2}}}{\Delta t/2} + \frac{(\Delta t/2)}{2}\dddot{u}_i - \frac{(\Delta t/2)^2}{6}\ddddot{u}_i + \cdots \\
&\Rightarrow \quad \ddot{u}_i = \frac{\dot{u}_i - \dot{u}_{i-\frac{1}{2}}}{\Delta t/2} + \mathcal{O}(\Delta t)
\end{aligned}
$$

This is reorganized to approximate the current velocity at each step as in,

$$
\dot{u}_i \approx \dot{u}_{i-\frac{1}{2}} + \frac{\Delta t}{2} \cdot \ddot{u}_i. \tag{6}
$$

Time Iteration Loop

First, since the equation of motion is a second order ordinary differential equation, two initial values need to be defined, namely, $u(t=0)$ and $\dot{u}(t=0)$. Assume $u(t=0) = U$ and $\dot{u}(t=0) = V$. The initial values u_0 and $\dot{u}_0$ can be trivially applied as:

$$
u_0 = U, \quad \dot{u}_0 = V.
$$

Then, equation (6) is used to approximate $u_{-\frac{1}{2}}$ as:

$$
\dot{u}_{-\frac{1}{2}} = \dot{u}_0 - \frac{\Delta t}{2} \cdot \ddot{u}_0,
$$

where,

$$
\ddot{u}_0 = M^{-1} \cdot \left(F_0^{\text{ext}} - f_0^{\text{int}} - C\dot{u}_0\right).
$$

After the initial conditions are set, the time loop for **undamped systems** is performed by applying eq. (3) to compute the current acceleration, then eq. (4) to approximate the middle velocity, them eq. (5) to approximate the next displacement. For **damped systems**, eq. (6) is used to approximate the current velocity.

Example: SDOF Mass Spring Damper System

As an example to demonstrate the solution, a single degree of freedom (SDOF) mass spring damper is used. The analytical solution for a free vibration ($F^{\text{ext}} = 0$) undamped SDOF system is:

$$u(t) = \frac{\dot{u}(t=0)}{\omega}\sin(\omega t) + u(t=0)\cos(\omega t),$$

where ω is the eigen frequency of the system,

$$\omega = \sqrt{\frac{K}{M}}.$$

An example of a linear undamped free vibration system with a mass $M = 1.0$, a stiffness $K = 1.0$, and initial conditions $u(t=0) = 1.0$, $\dot{u}(t=0) = -1.0$ is shown in Figure 1. The analytical solution is:

$$u(t) = \cos(t) - \sin(t).$$

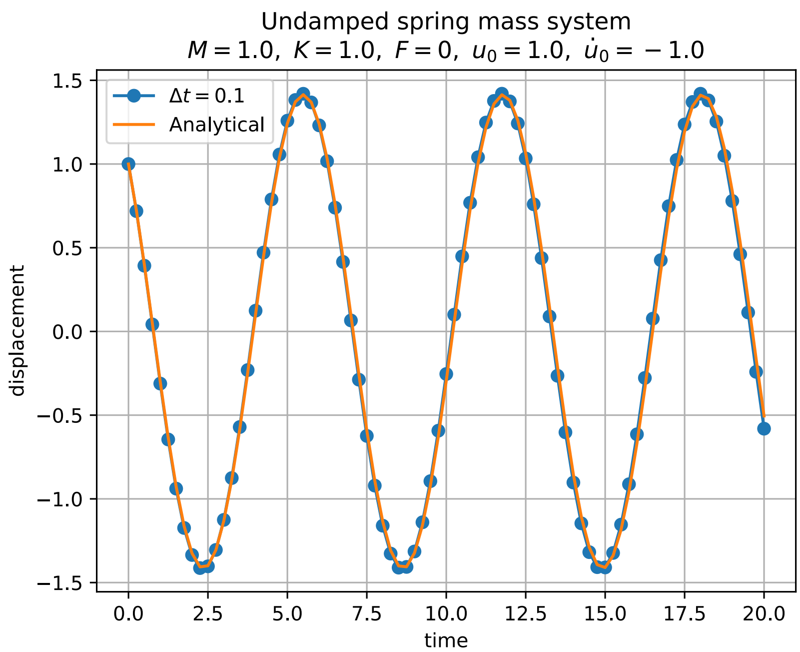


Figure 1: Solution of a linear undamped free vibration system using Explicit FEM.

Another example of a non-linear damped system with a mass $M = 1.0$, a non-linear stiffness $K = \sin(u)$, a damping constant $C = 0.03$, a transient external force $F^{\text{ext}} = 3\cos(t)$, and initial conditions $u(t=0) = 1.0$, $\dot{u}(t=0) = 0.0$ is shown in Figure 2.

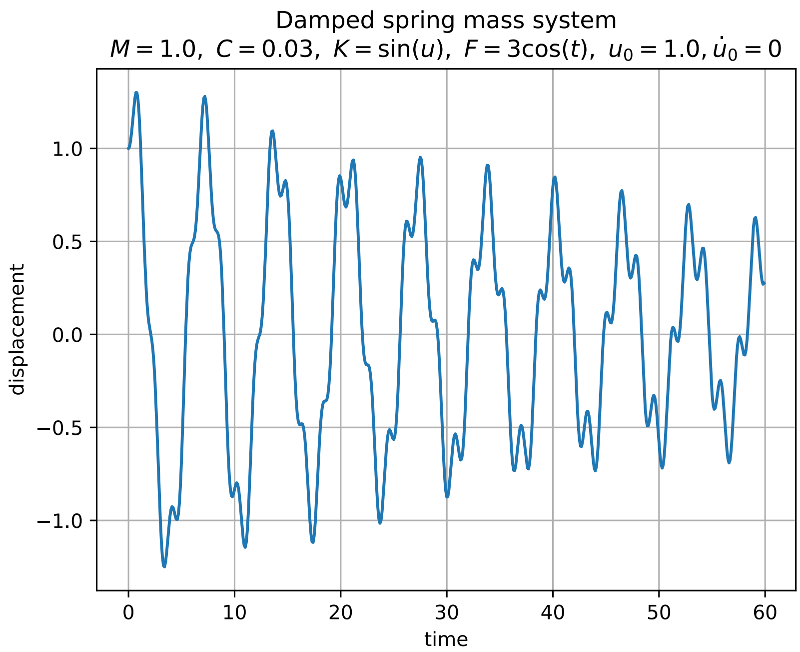
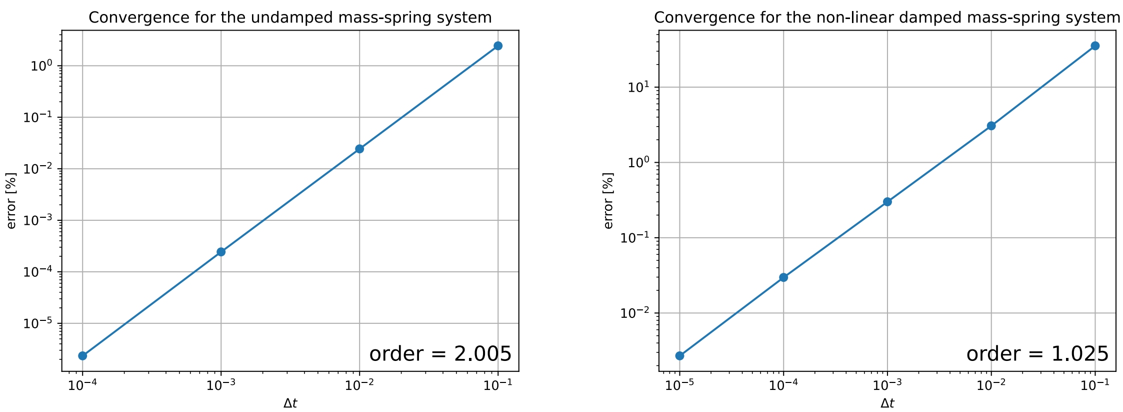


Figure 2: Solution of a non-linear damped system using Explicit FEM.

The order of convergence can be depicted from Figure (3), where it shows the order of convergence to be ≈ 2 for undamped and ≈ 1 for non-linear damped examples. The error, ε, is calculated using the L2 norm, such that

$$\varepsilon = \frac{\|\tilde{u} - u\|_2}{\|u\|_2},$$

where $\tilde{u}$ is the Explicit FEM approximate solution and u is the accurate solution. For the undamped case, the analytical solution is used as an accurate solution since it is known a priori, and for the damped case, a solution with $\Delta t = 1e-6$ is used.



(a) Undamped

(b) Damped

Figure 3: Error plots for the SDOF mass-spring systems.