Solving Equation of Motion with Explicit FEM

The equation of Motion for non-linear damped systems is:

$$M\ddot{u} + C\dot{u} + f^{\text{int}} = F^{\text{ext}}, \tag{1}$$

where M is the mass, $\ddot{u}$ is the acceleration, C is the damping constant, f^{int} is the non-linear internal force, and F^{ext} is the external force.
Using an explicit scheme, where the subscript i denotes the time step, the equation becomes:

$$M\ddot{u}_i + C\dot{u}_i + f_i^{\text{int}} = F_i^{\text{ext}}. \tag{2}$$

The acceleration can be solved simply as:

$$\ddot{u}_i = M^{-1} \cdot \left(F_i^{\text{ext}} - f_i^{\text{int}} - C\dot{u}_i \right) \tag{3}$$

The update of the velocity and the displacement requires a time integration scheme. Second order convergence scheme is explained.

Second Order Central Differences Scheme

Assuming central difference scheme for the acceleration and the velocity,

$$\ddot{u}_i = \frac{\dot{u}_{i+\frac{1}{2}} - \dot{u}_{i-\frac{1}{2}}}{\Delta t}, \quad \ddot{u}_i = \frac{\dot{u}_i - \dot{u}_{i-\frac{1}{2}}}{\Delta t/2}, \quad \dot{u}_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{\Delta t}, \tag{4}$$

The update at every time step is as follows:

$$\ddot{u}_i = M^{-1} \cdot \left(F_i^{\text{ext}} - f_i^{\text{int}} - C\dot{u}_i \right), \tag{5}$$

$$\dot{u}_i = \dot{u}_{i-\frac{1}{2}} + \frac{\Delta t}{2} \cdot \ddot{u}_i, \tag{6}$$

$$\dot{u}_{i+\frac{1}{2}} = \dot{u}_{i-\frac{1}{2}} + \Delta t \cdot \ddot{u}_i, \tag{7}$$

$$u_{i+1} = u_i + \Delta t \cdot \dot{u}_{i+\frac{1}{2}}. \tag{8}$$

The initial conditions can be applied as follows:

$$u_0 = U, \quad \dot{u}_0 = V, \tag{9}$$

$$\ddot{u}_0 = M^{-1} \cdot \left(F_0^{\text{ext}} - f_0^{\text{int}} - C\dot{u}_0 \right), \tag{10}$$

$$\dot{u}_{-\frac{1}{2}} = \dot{u}_0 - \frac{\Delta t}{2} \cdot \ddot{u}_0. \tag{11}$$