Adaptive Squeezed Rejection Sampling

Overview

Ordinary rejection sampling requires one evaluation of f for every candidate draw Y. In cases where evaluating f is computationally expensive but rejection sampling is otherwise appealing, we can achieve improved simulation speed by **squeezed rejection sampling.**

This strategy prevents the evaluation of f in some instances by using a nonnegative squeezing function, s. For s to be a suitable squeezing function, $s(x)$ must not exceed $f(x)$ anywhere on the support of f. An envelope, e, is also used; as with ordinary rejection sampling, $e(x) = g(x)/\alpha \geq f(x)$ on the support of f. The algorithm proceeds as follows:
1. Sample $Y \sim g$.
2. Sample $U \sim \text{Unif}(0,1)$.
3. If $U \leq s(Y)/e(Y)$, keep the value of Y. Set $X = Y$ and consider X to be an element in the target random sample. Then go to step 6.
4. Otherwise, determine whether $U \leq f(Y)/e(Y)$. If this inequality holds, keep the value of Y, setting $X = Y$. Consider X to be an element in the target random sample; then go to step 6 .
5. If Y has not yet been kept, reject it as an element in the target random sample.
6. Return to step 1 until you have accumulated a sample of the desired size.

The most challenging aspect is the construction of a suitable envelope. Gilks and Wild proposed an automatic envelope generation strategy for squeezed rejection sampling for a continuous, differentiable, log-concave density on a connected region of support. Define the squeezing function on T_k to be the exponential of the piecewise linear lower hull of ℓ formed by the chords between adjacent points in T_k. This lower hull is given by

$$s_k^*(x) = \frac{(x_{i+1} - x)\,\ell(x_i) + (x - x_i)\,\ell(x_{i+1})}{x_{i+1} - x_i} \quad \text{for } x \in [x_i, x_{i+1}]$$

and $i = 1, \ldots, k-1$. When $x < x_1$ or $x > x_k$, let $s_k^*(x) = -\infty$. Thus the squeezing function is $s_k(x) = \exp\{s_k^*(x)\}$.

Example

Suppose we want to estimate $\mathrm{S} = \mathrm{E}\left[\mathrm{X}^2\right]$ where X has the density that is proportional to q(x) = $\exp\{-|x|^3/3\}$. Using adaptive squeezed rejection sampling by setting $k = 3$ at $x = -1, 0, 1$ for construction the upper and inner piece hull (envelope and squeezing functions).

Envelope function:

$$e^*(x) = \begin{cases} \frac{2}{3} - |x|, \text{ if } x < -\frac{2}{3} \text{ or } x > \frac{2}{3} \\ 0, \text{ if } -\frac{2}{3} < x < \frac{2}{3} \end{cases}$$

or equivalently

$$e^*(x) = \begin{cases} \frac{2}{3} + x, \text{ if } x < -\frac{2}{3} \\ 0, \text{ if } -\frac{2}{3} < x < \frac{2}{3} \\ \frac{2}{3} - x, \text{ if } x > \frac{2}{3} \end{cases}$$

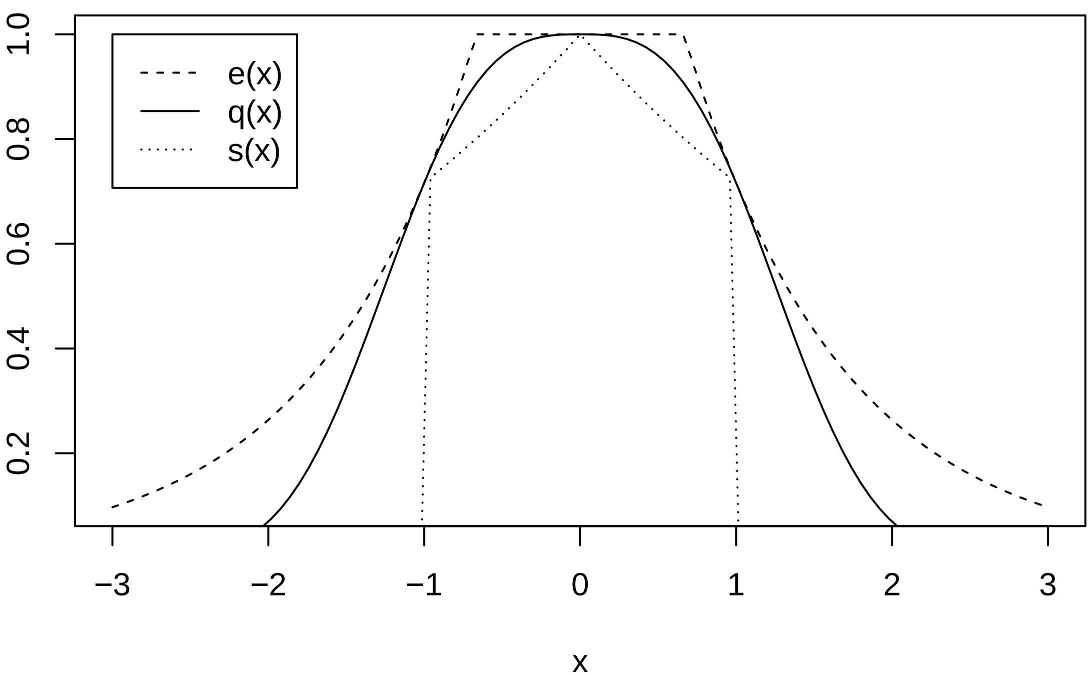
$$e(x) = \exp\left(e^*(x)\right) = \begin{cases} \exp\left\{\frac{2}{3} + x\right\}, \text{ if } x < -\frac{2}{3} \\ 1, \text{ if } -\frac{2}{3} < x < \frac{2}{3} \\ \exp\left\{\frac{2}{3} - x\right\}, \text{ if } x > \frac{2}{3} \end{cases}$$

Squeezing function: Let the log of the squeezing function to be $\log(s(x)) = -\frac{|x|}{3}$, if $-1 < x < 1$, which is equivalent to

$$\log(s(x)) = \begin{cases} -\frac{x}{3}, \text{ if } 0 < x < 1 \\ \frac{x}{3}, \text{ if } -1 < x < 0 \end{cases}$$

$$s(x) = \exp(\log(s(x))) = \begin{cases} \exp\left\{-\frac{x}{3}\right\}, \text{ if } 0 < x < 1 \\ \exp\left\{\frac{x}{3}\right\}, \text{ if } -1 < x < 0 \end{cases}$$

```r
rm(list = ls())
set.seed(1)
q <- function(x) {
  exp(-(abs(x))^3/3)
  }
e <- function(x) {
  ifelse((x > 2 / 3) | (x < -2 / 3),
         exp(2 / 3 - abs(x)),
         1
  )
  }
s <- function(x) {
  ifelse((x > -1) & (x < 1),
         exp(-abs(x) / 3),
         0
  )
  }
plot(e, -3, 3, lty = 2, xlab = "x", ylab = "")
plot(q, -3, 3, lty = 1, add = T)
plot(s, -3, 3, lty = 3, add = T)
legend(-3, 1, legend = c("e(x)", "q(x)", "s(x)"), lty = c(2, 1, 3))
```



Inverse CDFG^{-1} $\int_{-\infty}^{\infty} e(x)dx = \int_{-\infty}^{-\frac{2}{3}} \exp\left\{\frac{2}{3}+x\right\}dx + \int_{-\frac{2}{3}}^{\frac{2}{3}} \exp\{0\}dx + \int_{\frac{2}{3}}^{\infty} \exp\left\{\frac{2}{3}-x\right\}dx = \frac{3}{10}$. The normalizing constant is $\frac{3}{10}$.

$$G(x) = \begin{cases} \frac{3}{10}\exp\left\{\frac{2}{3}+x\right\}, \text{ if } x < -\frac{2}{3} \\ \frac{3}{10}x + \frac{1}{2}, \text{ if } -\frac{2}{3} < x < \frac{2}{3} \\ 1 - \frac{3}{10}\exp\left\{\frac{2}{3}+x\right\}, \text{ if } x > \frac{2}{3} \end{cases}$$

$$G^{-1}(y)=\begin{cases}\log\left(\frac{3}{10}y\right)-\frac{2}{3}, \text{ if } 0<y<\frac{3}{10}\\ \frac{3}{10}\left(y-\frac{1}{2}\right), \text{ if } -\frac{3}{10}\le y<\frac{7}{10}\\ \frac{2}{3}-\log\left(\frac{3}{10}(1-y)\right), \text{ if } \frac{7}{10}\le y<1\end{cases}$$

```
Ginv <- function(y) {
  ifelse(y < 3 / 10,
         log(10 / 3 * y) - 2 / 3,
         ifelse(y > 7 / 10,
                2 / 3 - log(10 / 3 * (1 - y)),
                10 / 3 * (y - 1 / 2)
                )
  )
  }
ars <- function(n) {
  i <- 0
  i1 <- 0
  j <- 0
  x <- rep(NA, n)
  while (i < n) {
    y <- Ginv(runif(1))
    u <- runif(1)
    if (u < s(y) / e(y)) {
      i <- i + 1
      i1 <- i1 + 1
      x[i] <- y
      } else if (u < q(y) / e(y)) {
        i <- i + 1
        x[i] <- y
        }
    j <- j + 1
    }
  return(list(x = x, acratio.sx = i1 / j, acratio = i / j))
}
res <- ars(1e6)
(ratio.sx <- res$acratio.sx) # Acceptance Ratio under s(x)
```

```
## [1] 0.5102933
```

```
(ratio <- res$acratio) # Acceptance Ratio
```

```
## [1] 0.7729116
```

```
(ans <- mean(res$x^2)) # Estimated E(X^2)
```

```
## [1] 0.7782454
```

The acceptance ratio under $s(x)$ is 0.5102933. The acceptance ratio is 0.7729116.E$\left(X^2\right)$ is approached by 0.7782454.