Monte Carlo Simulation

Overview

When an i.i.d. random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is obtained from f, we can approximate μ by a sample average:

$$\hat{\mu}_{\mathrm{MC}} = \frac{1}{n} \sum_{i=1}^{n} h\left(\mathbf{X}_i\right) \rightarrow \int h(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \mu$$

Examples

Example 1

$$\int_1^3 \frac{x}{(1+x^2)^2} dx$$

$$= \int_1^3 2 \frac{x}{(1+x^2)^2} \frac{1}{2} dx$$

Where $h(x) = 2\frac{2x}{(1+x^2)^2}$ and $f(x) = \frac{1}{2}$, $x \sim Unif(1,3)$

Thus, $\int_1^3 \frac{x}{(1+x^2)^2} dx = E(h(x))$ Using Monte Carlo approximation:

```r
set.seed(1693)
#Define function h(x)
h_x_1<- function(x){
  (2*x/(1+x^2)^2)
}
#Monte Carlo approximation
n<-10^6
#sample n from Unif(1,3)
x<-runif(n,1,3)
#compute MC approximation
mean(sapply(x,h_x_1))
```

```
## [1] 0.2001542
```

Example 2

$$\int_{-\infty}^{\infty} x^2 exp(-x^2) dx$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2}) \sqrt{2\pi} exp(-\frac{x^2}{2}) x^2 dx$$

Where $h(x) = \sqrt{2\pi} exp(-\frac{x^2}{2}) x^2$ and $f(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$, $x \sim N(0,1)$

Thus, $\int_{-\infty}^{\infty} x^2 exp(-x^2) dx = E(h(x))$

```r
set.seed(1693)
h_x_2<- function(x){
  (sqrt(2*pi)*exp(-x^2/2)*x^2)
}
n<-10^6
x<-rnorm(n)
mean(sapply(x,h_x_2))
```

```
## [1] 0.8865135
```

Example 3

The joint uniform density is constant over a given range. In this case, the $f_{x,y}(x,y) = 1/4$ in range $[-1,1] \times [-1,1]$ and $f_{x,y}(x,y) = 0$ outside the range.

$$\int_{-1}^{1}\int_{-1}^{1} |x-y| dxdy$$

$$\int_{-1}^{1}\int_{-1}^{1} 4|x-y|\frac{1}{4}dxdy$$

Where $h(x,y) = |x-y|$ and $f_{x,y}(x,y) = \frac{1}{4}$ Thus, $\int_{-1}^{1}\int_{-1}^{1} |x-y|dxdy = E(h(x,y))$

```
set.seed(1693)
h_x_3<- function(x,y){
  4*(abs(x-y))
}
n<-10^6
x<-runif(n,-1,1)
y<-runif(n,-1,1)
mean(h_x_3(x,y))
```

```
## [1] 2.665548
```