Rejection Sampling

Overview

If $f(x)$ can be calculated, then we can use rejection sampling to obtain a random draw from exactly the target distribution. This strategy relies on sampling candidates from an easier distribution and then correcting the sampling probability through random rejection of some candidates.

Let g denote another density from which we know how to sample and for which we can easily calculate $g(x)$. Let $e(\cdot)$ denote an envelope, having the property $e(x) = g(x)/\alpha \geq f(x)$ for all x for which $f(x) > 0$ for a given constant $\alpha \leq 1$. Rejection sampling proceeds as follows:
1. Sample $Y \sim g$.
2. Sample $U \sim \text{Unif}(0, 1)$.
3. Reject Y if $U > f(Y)/e(Y)$. In this case, do not record the value of Y as an element in the target random sample. Instead, return to step 1 .
4. Otherwise, keep the value of Y. Set $X = Y$, and consider X to be an element of the target random sample. Return to step 1 until you have accumulated a sample of the desired size.

The draws kept using this algorithm constitute an i.i.d. sample from the target density f; there is no approximation involved. To see this, note that the probability that a kept draw falls at or below a value y is

$$\begin{aligned}
P[X \leq y] &= P\left[Y \leq y \mid U \leq \frac{f(Y)}{e(Y)}\right] \\
&= P\left[Y \leq y \text{ and } U \leq \frac{f(Y)}{e(Y)}\right] / P\left[U \leq \frac{f(Y)}{e(Y)}\right] \\
&= \int_{-\infty}^{y} \int_{0}^{f(z)/e(z)} du g(z) dz / \int_{-\infty}^{\infty} \int_{0}^{f(z)/e(z)} du g(z) dz \\
&= \int_{-\infty}^{y} f(z) dz
\end{aligned}$$

which is the desired probability.

Example

Suppose we want to estimate $S = E[X^2]$ when X has the density that is proportional to $q(x) = exp\{-|x|^3/3\}$. Use rejection sampling to estimate $E[X^2]$. Let g denote standard normal(Gaussian) distribution, which we know how to sample and $g(x)$ can be easily calculated. Let $e(\cdot)$ denote an envelope, having property that $e(x) = g(x)/\alpha \geq f(x)$.
First, we want to find α and $e(\cdot)$. To find α such that $g(x)/\alpha \geq f(x)$ for all x.

$$exp\{-x^2/2\}/\sqrt{2\pi}/\alpha \geq exp\{-|x|^3/3\}$$
$$1/\alpha \geq exp\{-|x|^3/3 + x^2/2\}\sqrt{2\pi}$$
$$1/\alpha \geq exp\{(3x^2 - 2|x|^3)/6\}\sqrt{2\pi}$$
$$1/\alpha = 2.961$$
$$e(x) = g(x) * 2.961$$

Code f(x) in R:

```
f_x <- function(x) {
  exp(-abs(x)^3/3)
}
```

Code e(x) in R:

```r
e_x <- function(x) {
  2.961*exp(-x^2/2)/sqrt(2*pi)
}
```

Step 1: Sample $Y \sim g$

```r
set.seed(1693)
N<- 10^6
Y<- rnorm(N)
```

Step 2: Sample $U \sim Unif(0, 1)$

```r
U<- runif(N)
```

Step 3 and 4: Reject Y if $U > f(Y)/e(Y)$, Record X=Y if $U \leq f(Y)/e(Y)$.

```r
laccept=U<=f_x(Y)/e_x(Y)
X <- Y[laccept]
#Count and report Acceptance Ratio
length(X)/N
```

```
## [1] 0.869932
```

The acceptance ratio is about 0.87.
Step 5: Estimate $E[X^2]$ by taking the mean of recorded X.

```r
mean(X^2)
```

```
## [1] 0.7764799
```

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2}) \sqrt{2\pi} exp(-\frac{x^2}{2}) x^2 dx$$

Where $h(x) = \sqrt{2\pi} exp(-\frac{x^2}{2}) x^2$ and $f(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$, $x \sim N(0, 1)$
Thus, $\int_{-\infty}^{\infty} x^2 exp(-x^2) dx = E(h(x))$

```r
set.seed(1693)
h_x_2<- function(x){
  (sqrt(2*pi)*exp(-x^2/2)*x^2)
}
n<-10^6
x<-rnorm(n)
mean(sapply(x,h_x_2))
```

```
## [1] 0.8865135
```

Example 3

The joint uniform density is constant over a given range. In this case, the $f_{x,y}(x, y) = 1/4$ in range $[-1, 1] \times [-1, 1]$ and $f_{x,y}(x, y) = 0$ outside the range.

$$\int_{-1}^{1} \int_{-1}^{1} |x - y| dxdy$$

$$\int_{-1}^{1} \int_{-1}^{1} 4|x - y| \frac{1}{4} dxdy$$

Where $h(x, y) = |x - y|$ and $f_{x,y}(x, y) = \frac{1}{4}$ Thus, $\int_{-1}^{1} \int_{-1}^{1} |x - y| dxdy = E(h(x, y))$

```r
set.seed(1693)
h_x_3<- function(x,y){
  4*(abs(x-y))
}
n<-10^6
x<-runif(n,-1,1)
y<-runif(n,-1,1)
mean(h_x_3(x,y))
```

```
## [1] 2.665548
```