Gaussian minimax lowerbounds

Eric Ziebell

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Chapter 1

Minimax lowerbounds

Definition 1.0.1. Let $(\mathcal{X}_n, \mathcal{F}_n, (\mathbb{P}_{\vartheta,n})_{\vartheta\in\Theta})$ be a statistical model and (Θ, d) a metric space[1]. Suppose that $(v_n)_{n\in\mathbb{N}}$ is a null sequence. Then, $(v_n)_{n\in\mathbb{N}}$ is called optimal (minimax) convergence rate over Θ if

1. There exists an estimator $\hat{\vartheta}_n^*$ such that
$$\limsup_{n\to\infty} v_n^{-2} \sup_{\vartheta\in\Theta} \mathbb{E}_{\vartheta,n}[d(\hat{\vartheta}_n^*, \vartheta)^2] < \infty.$$

2. We have the uniform lowerbound
$$\liminf_{n\to\infty} v_n^{-2} \inf_{\hat{\vartheta}_n} \sup_{\vartheta\in\Theta} \mathbb{E}_{\vartheta,n}[d(\hat{\vartheta}_n, \vartheta)^2] > 0,$$
where the infimum is taken over all measurable functions (estimators) in model n.

Proposition 1.0.2 (Reduction scheme to a testing problem)**.** *Let $(\mathcal{X}_n, \mathcal{F}_n, (\mathbb{P}_{\vartheta,n})_{\vartheta\in\Theta})$ be a statistical model and (Θ, d) a metric space. Suppose that $(v_n)_{n\in\mathbb{N}}$ is a null sequence satisfying an upper bound (1) and*
$$\liminf_{n\to\infty} \inf_{\psi_n} \max_{j=1,\dots,M} \mathbb{P}_{\vartheta_j,n}(\psi_n \neq j) > 0.$$
Then, v_n is the optimal minimax convergence rate over Θ.

Proof of Proposition 1.0.2.

1. With Markov's inequality, we observe that forall $\alpha > 0$, we have
$$v_n^{-2}\mathbb{E}_{\vartheta,n}[d(\hat{\vartheta}_n, \vartheta)^2] \geq \alpha^2 \mathbb{P}_{\vartheta,n}(d(\hat{\vartheta}_n, \vartheta) \geq \alpha v_n),$$
such that
$$\liminf_{n\to\infty} v_n^{-2} \inf_{\hat{\vartheta}_n} \sup_{\vartheta\in\Theta} \mathbb{E}_{\vartheta,n}[d(\hat{\vartheta}_n, \vartheta)^2] \geq \liminf_{n\to\infty} \inf_{\hat{\vartheta}_n} \sup_{\vartheta\in\Theta} \alpha^2 \mathbb{P}_{\vartheta,n}(d(\hat{\vartheta}_n, \vartheta) \geq \alpha v_n).$$

2. For any subset $\{\vartheta_1, \dots, \vartheta_M\} \subset \Theta$, we obtain the lowerbound
$$\liminf_{n\to\infty} \inf_{\hat{\vartheta}_n} \sup_{\vartheta\in\Theta} \alpha^2 \mathbb{P}_{\vartheta,n}(d(\hat{\vartheta}_n, \vartheta) \geq \alpha v_n) \geq \liminf_{n\to\infty} \inf_{\hat{\vartheta}_n} \max_{j=1,\dots,M} \alpha^2 \mathbb{P}_{\vartheta_j,n}(d(\hat{\vartheta}_n, \vartheta_j) \geq \alpha v_n).$$

3. Suppose that $\{\vartheta_1, \dots, \vartheta_M\} \subset \Theta$ are seperated according to
$$d(\vartheta_i, \vartheta_j) > 2\alpha v_n = \gamma, \quad i \neq j, \quad i, j = 1, \dots, M.$$

[1] In general $d : \Theta \times \Theta \to [0, \infty)$ is also allowed to be a semi-distance.

Let us now consider the minimum distance test $\psi^* : \mathcal{X}_n \to \{1, \dots, M\}$:

$$\psi^* = \mathrm{argmin}_{k=1,\dots,M} d(\hat{\vartheta}_n, \vartheta_k).$$

Since the hypothesis are seperated with radius γ, we observe that $\{\hat{\vartheta}_n \in B_\gamma(\vartheta_j)\} \subset \{\psi^* = j\}$. Thus, we clearly have the inclusion $\{\psi^* \neq j\} \subset \{\hat{\vartheta}_n \in B_\gamma(\vartheta_j)\}^c = \{\hat{\vartheta}_n \in B_\gamma(\vartheta_j)^c\}$. However, it is still possible that $\hat{\vartheta}_n$ is the closest to ϑ_j but not inside of a γ ball, so the inclusion might be strict. All in all, we obtain the bound

$$\mathbb{P}_{\vartheta_j,n}(d(\hat{\vartheta}_n, \vartheta_j) \geq \gamma) \geq \mathbb{P}_{\vartheta_j,n}(\psi^* \neq j), \quad j = 1, \dots, M.$$

4. Since for any $\hat{\vartheta}_n$, we can construct such a test, we may replace the infimum over all estimators with an infimum over all tests in model n, yielding the inequality

$$\liminf_{n\to\infty} \inf_{\hat{\vartheta}_n} \max_{j=1,\dots,M} \alpha^2 \mathbb{P}_{\vartheta_j,n}(d(\hat{\vartheta}_n, \vartheta_j) \geq \alpha v_n) \geq \alpha^2 \liminf_{n\to\infty} \inf_{\psi_n} \max_{j=1,\dots,M} \mathbb{P}_{\vartheta_j,n}(\psi_n \neq j).$$

Thus, if the latter term is positive we obtain the desired lowerbound. $\square$

Bibliography