Some Vector Geometry

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Vectors

Vectors "live" in a space of some dimension. Dimension is a mathematical abstraction which can sometimes be represented and other times not.

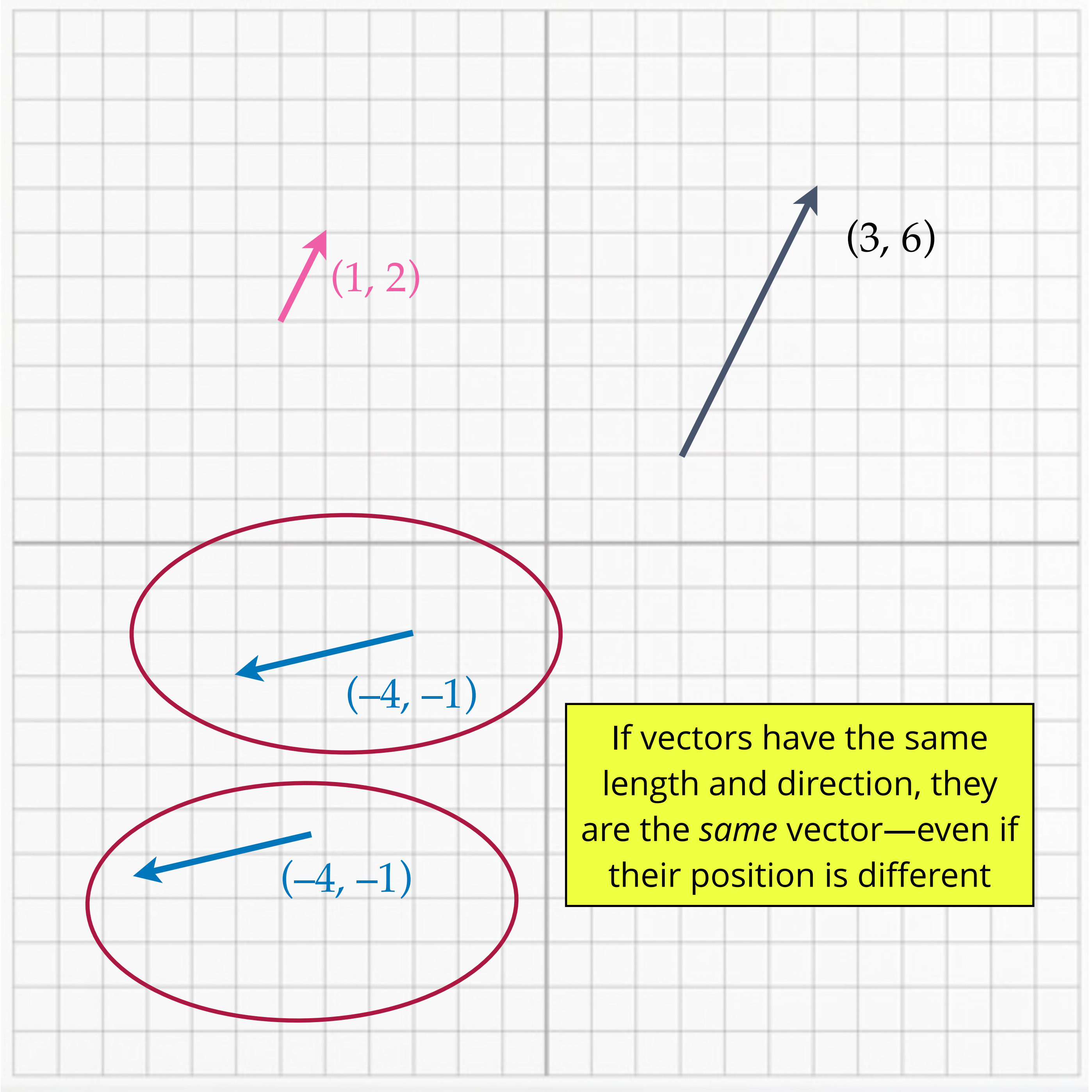
Vectors in two-dimensional space

$$\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

The number of elements in the vector enumerate the vector's dimension.

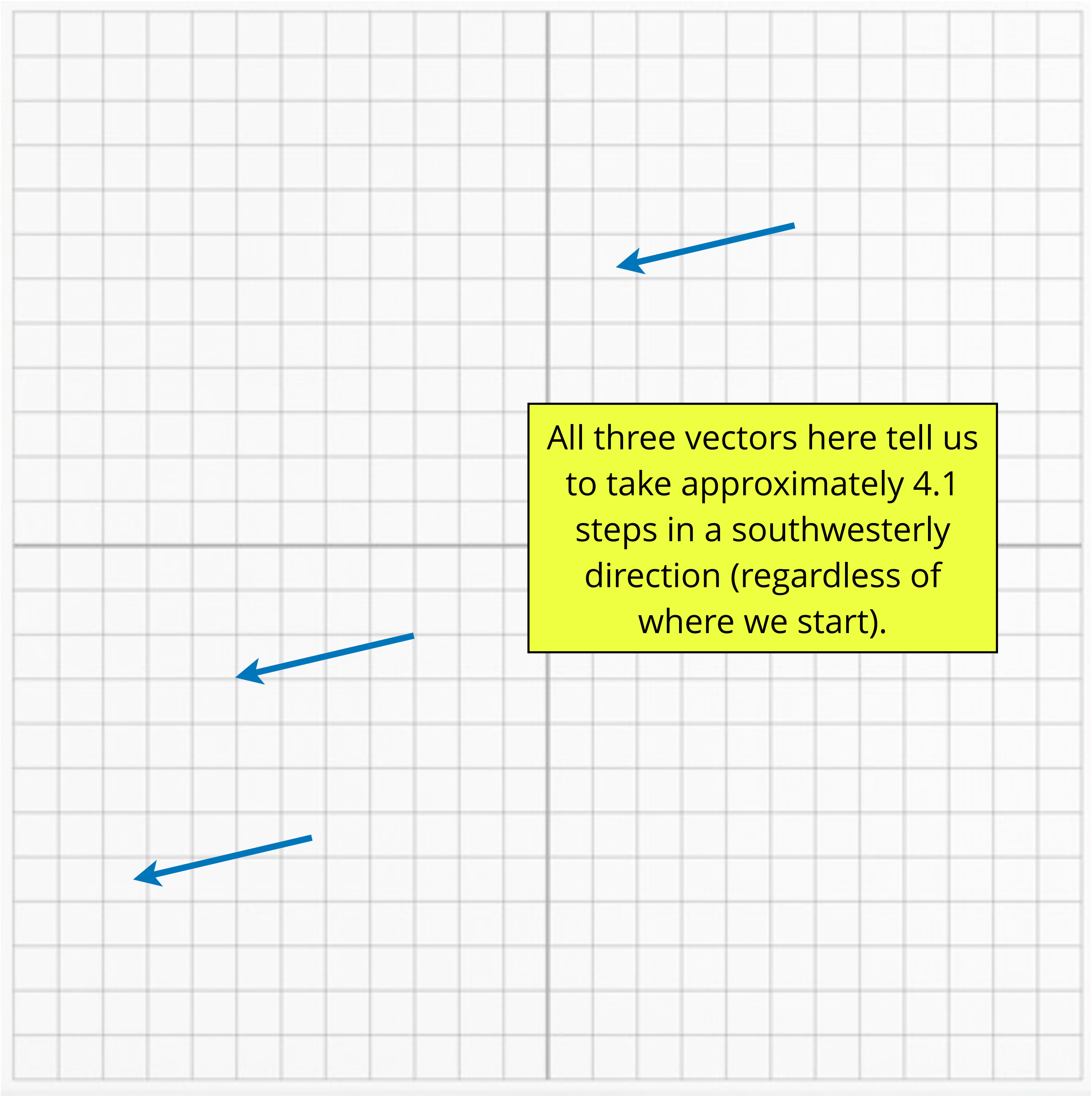
Aside from dimension, the only other two properties of a vector are its **length** and a **direction**

Position is a convenience, *not* a vector property.



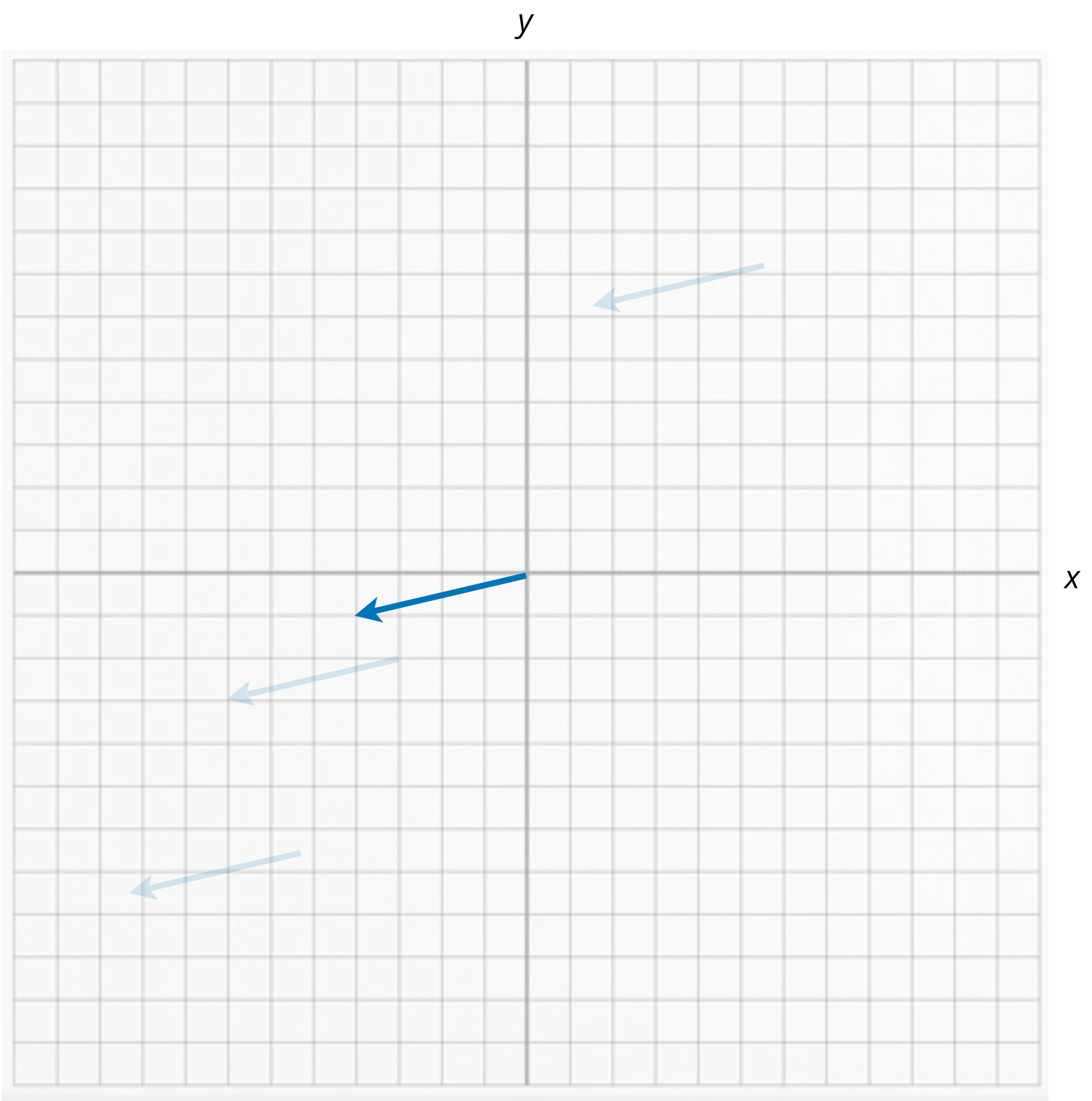
If vectors have the same length and direction, they are the *same* vector—even if their position is different

To help with this concept, think about vectors is in terms of movement. Essentially a vector is an instruction to take **L steps** (length) in a **particular direction** (direction).



Vectors can be positioned wherever it is convenient to put them.

Here we place the three vectors so each tail is at the origin (0, 0).



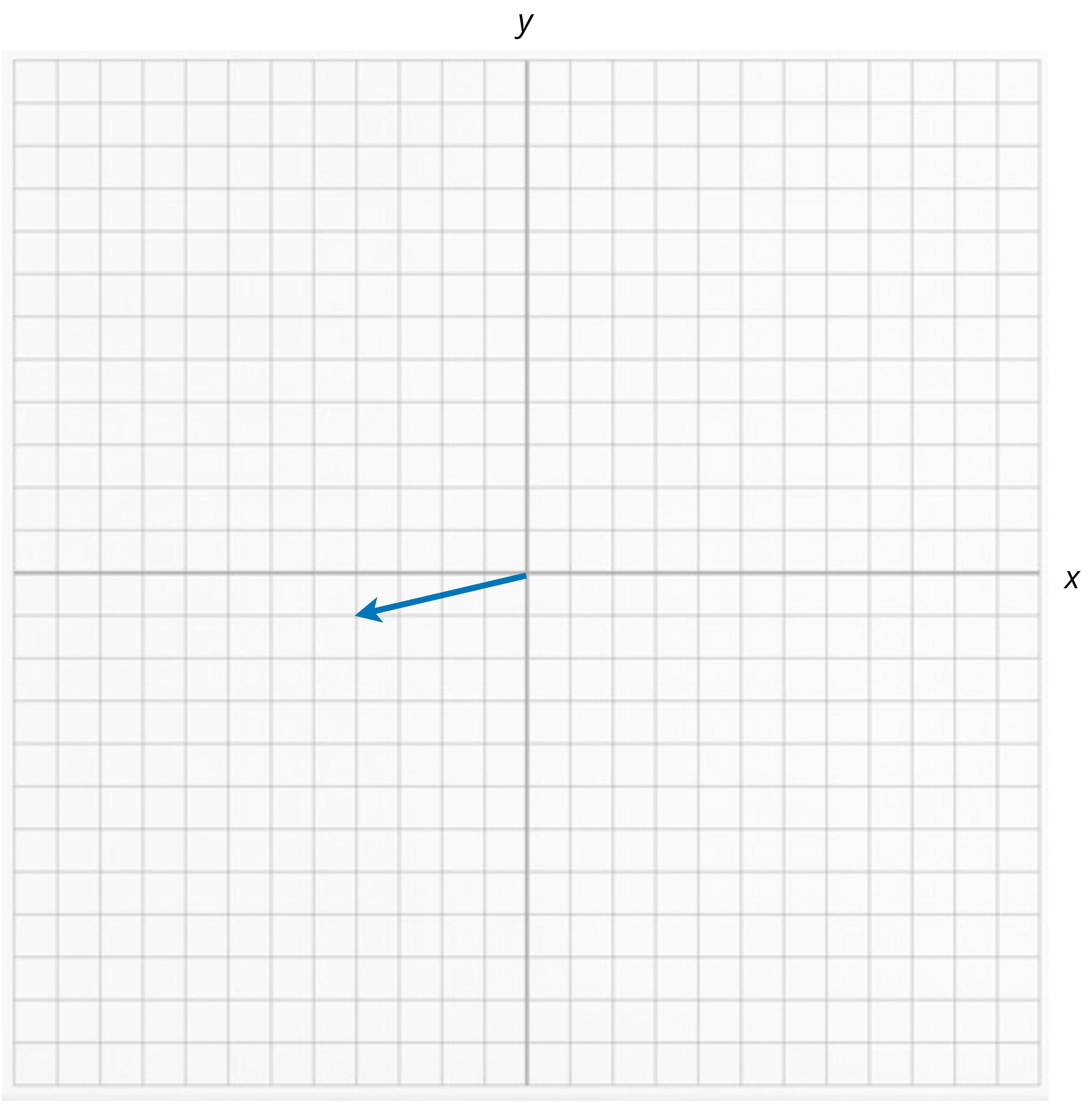
The actual elements of a vector give us another way to consider movement, so long as we have defined a **coordinate system**.

$$\mathbf{C} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

In the Cartesian plane, this vector tells us to take 4 steps in the negative *x* direction and 1 step in the negative *y* direction.

The key to this is that we need to know the directions of the two axes and also the scale of the units (e.g., how far is a step).

Both of these are also arbitrary and a matter of mathematical convenience.



Length of a Vector

More importantly, we can use the elements to compute the length of the vector.

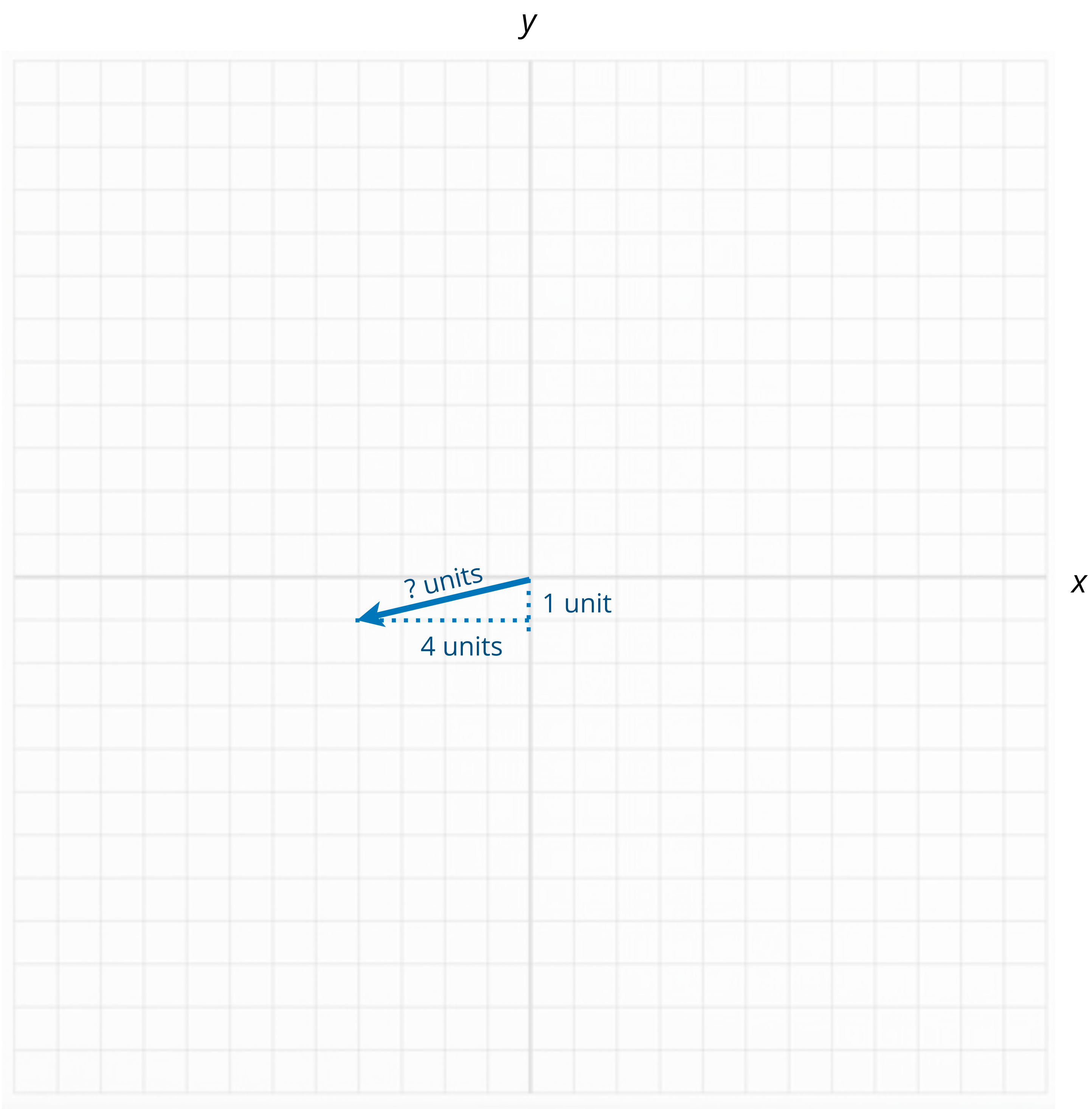
$$\mathbf{C} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

Find the length of vector **C**.

Hint: Remember the Pythagorean Theorem.

The length of a vector is called its *norm* and it is denoted:

$$\|\mathbf{C}\|$$

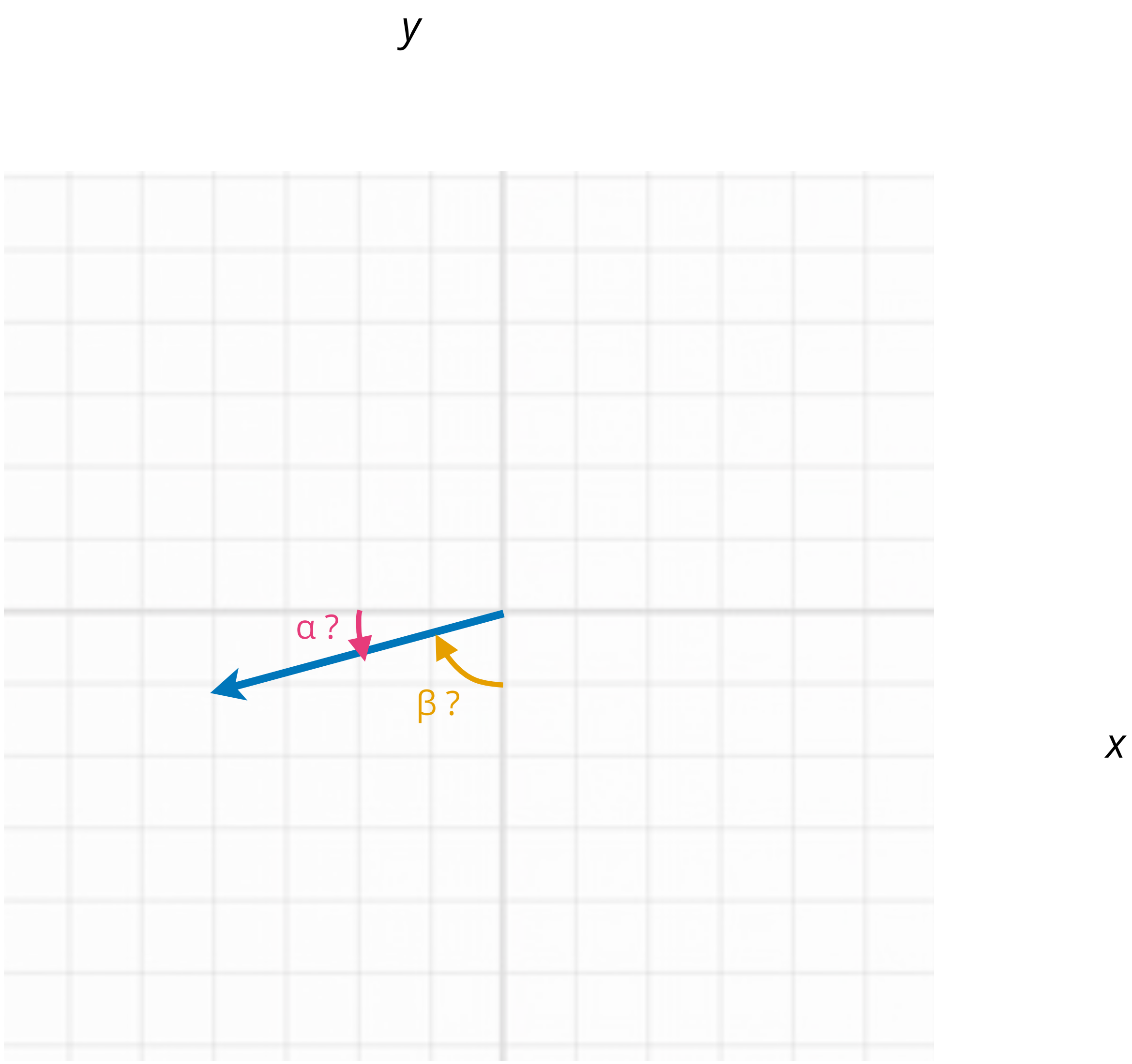


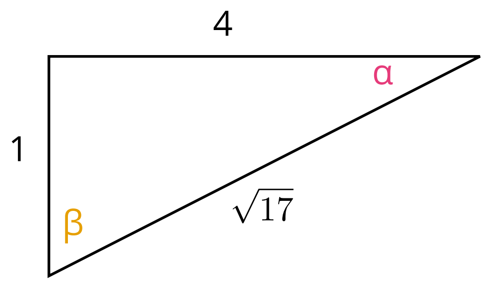
Direction of a Vector

We can also compute the direction of the vector by determining the angle of rotation from the *x*- and *y*-axis.

$$\mathbf{C} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

α measures the angle from the *x*-axis and β measures the angle from the *y*-axis





$$\cos(\alpha) = \frac{4}{\sqrt{17}}$$

$$\cos(\beta) = \frac{1}{\sqrt{17}}$$

In general, these formulae are

$$\cos(\alpha) = \frac{c_1}{\|\mathbf{C}\|}$$

$$\cos(\beta) = \frac{c_2}{\|\mathbf{C}\|}$$

Scaling Vectors

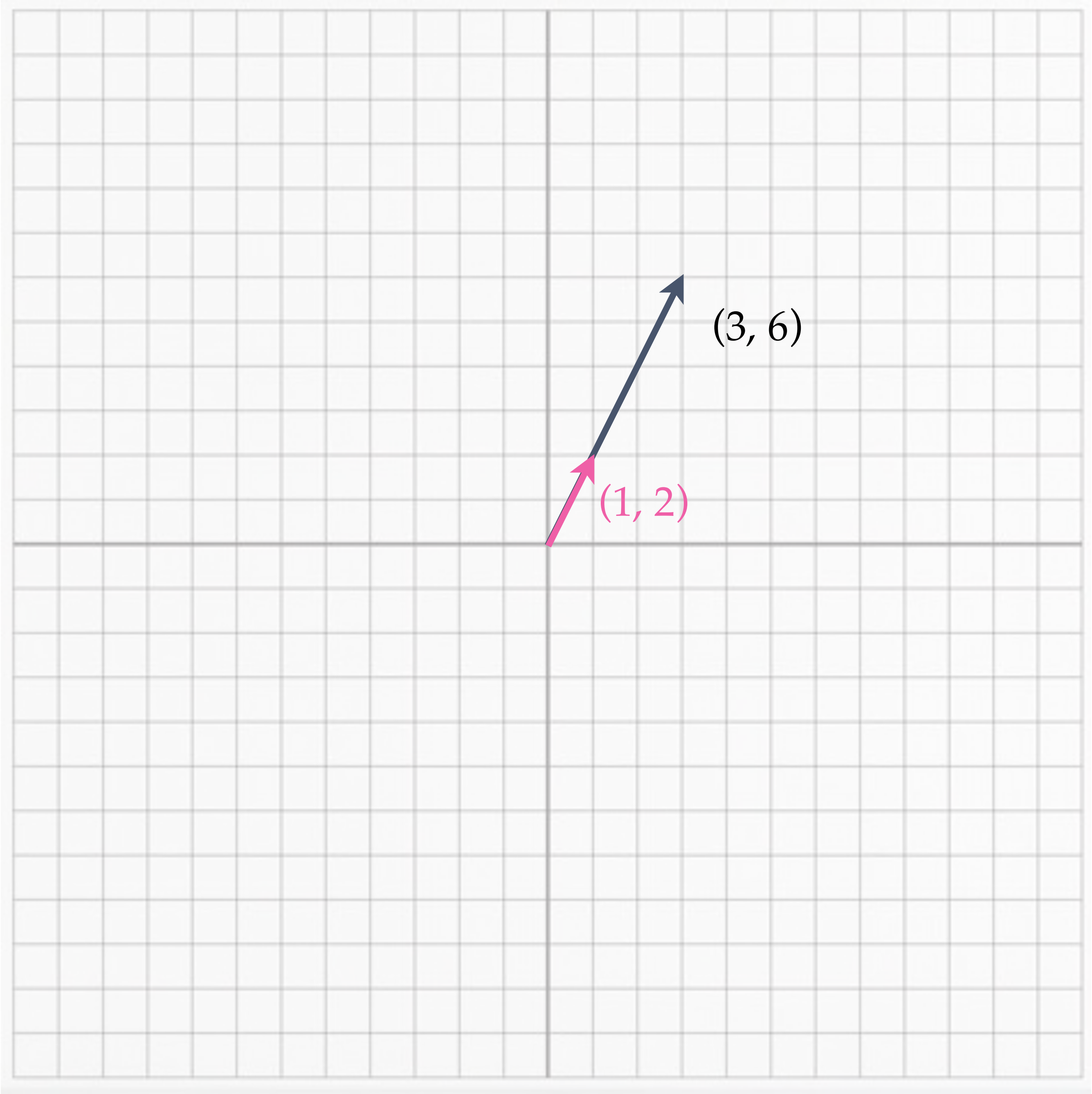
Scaling a vector by a positive factor changes the length (makes it shorter or longer), but not the direction.

Scaling a vector by 3

$$3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}$$

In this example, the vector [3, 6] is three times as long as the vector [1, 2].

Using a scalar value that is positive and below 1 reduces the length of the vector.

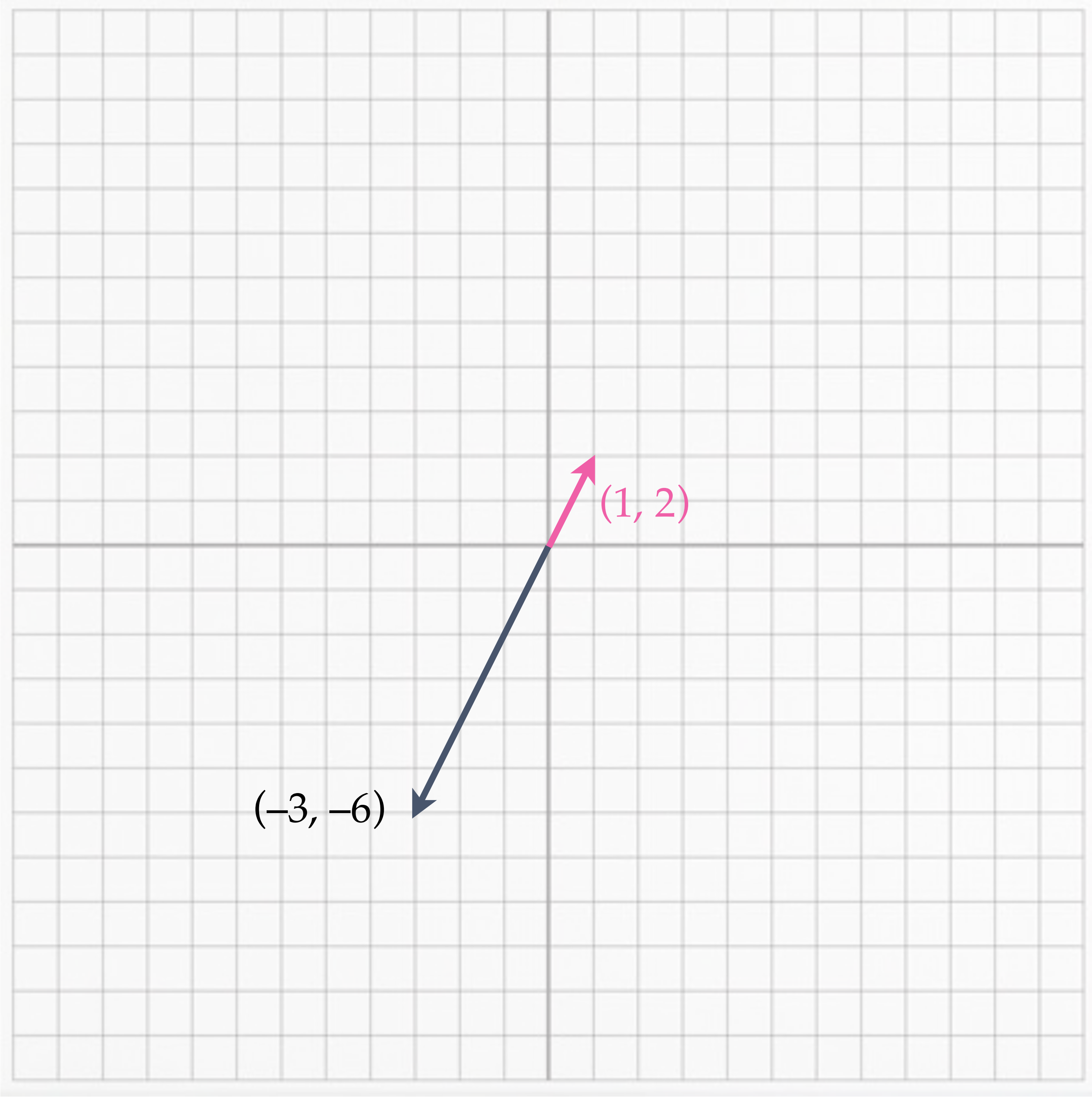


Scaling vectors by a negative factor changes the length (makes it shorter or longer), and the direction (step in the opposite direction)

Scaling a vector by –3

$$-3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-3\\-6\end{bmatrix}$$

In this example, the vector [–3, –6] is three times as long as the vector [1, 2] and it lies in completely the opposite direction.

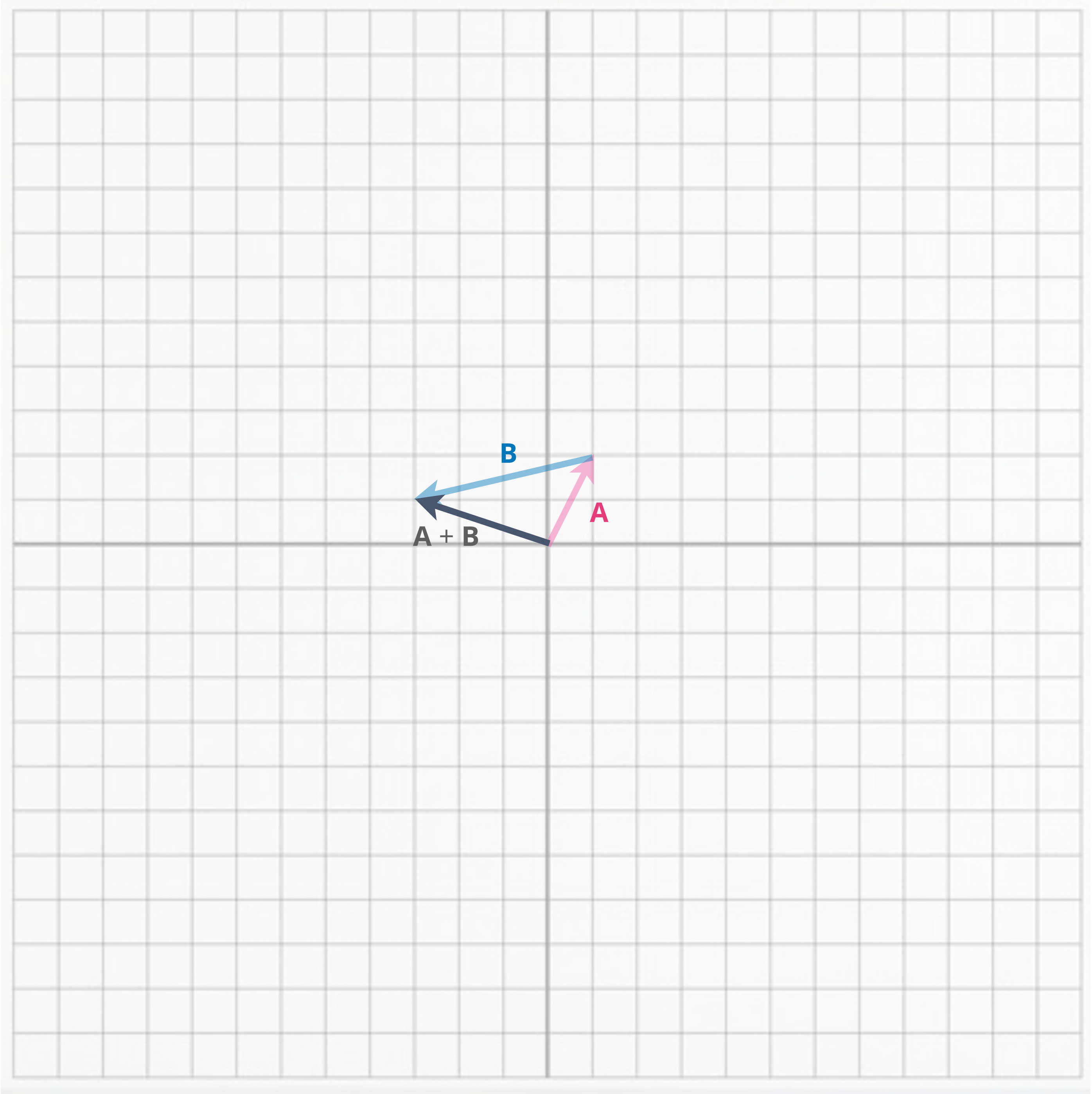


Adding Vectors

Adding (subtracting) vectors amounts to moving according to length and direction of first vector and then, from the point you ended, moving according to length and direction of second vector.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The overall distance and direction **from the place you started** is the result of adding the two vectors.

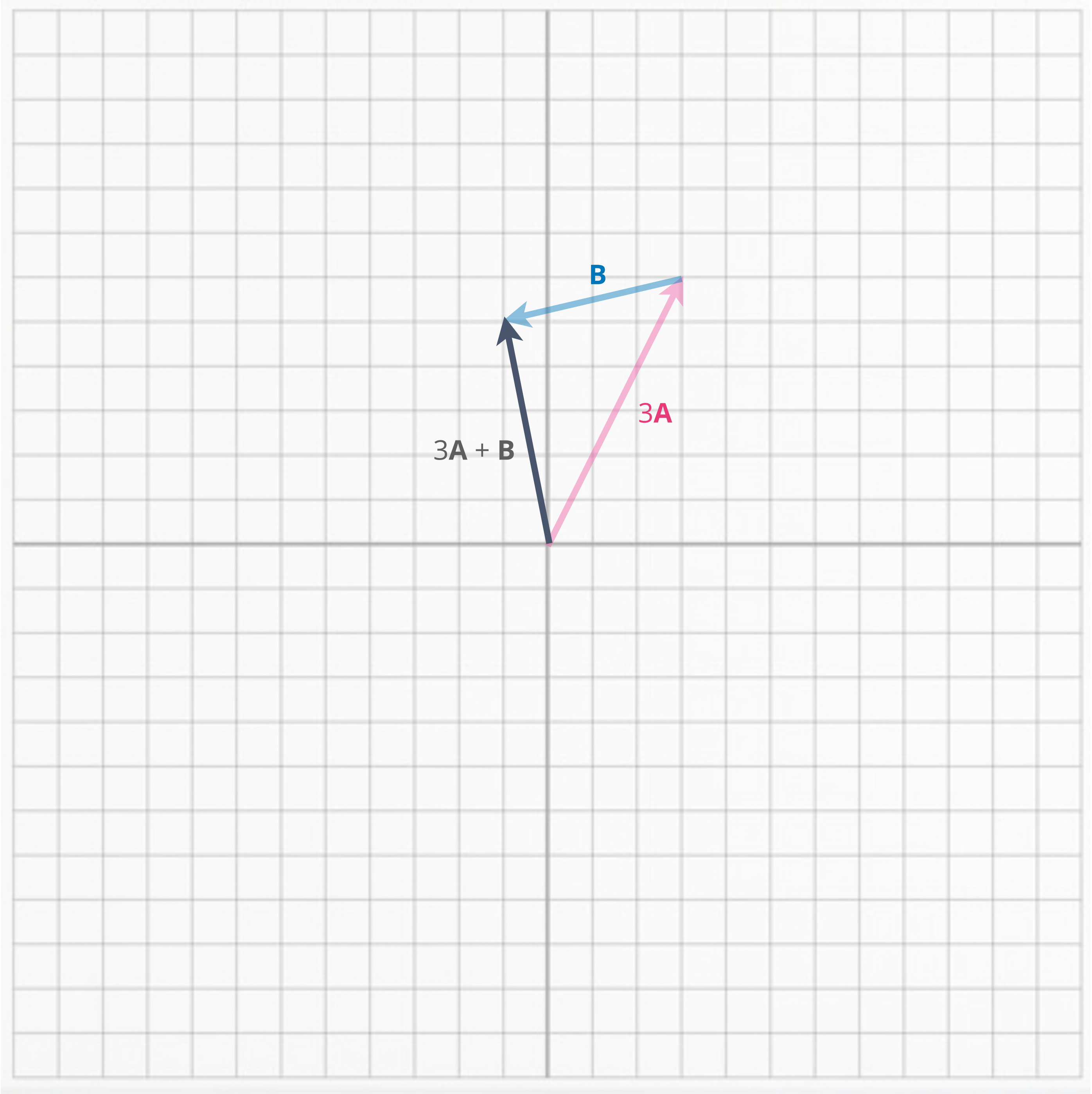


Linear Combinations

Linear combinations of vectors essentially combines scaling and adding.

$$3\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-4\\-1\end{bmatrix} = \begin{bmatrix}-1\\5\end{bmatrix}$$

Linear combinations are at the heart of regression.



Dot Product

What if we computed **A•A**?

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

The dot product is a mathematical operation for vectors that multiplies the corresponding elements of two vectors together (element-wise multiplication) and then sums the results.

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = 12 - 3 + 0 - 9$$

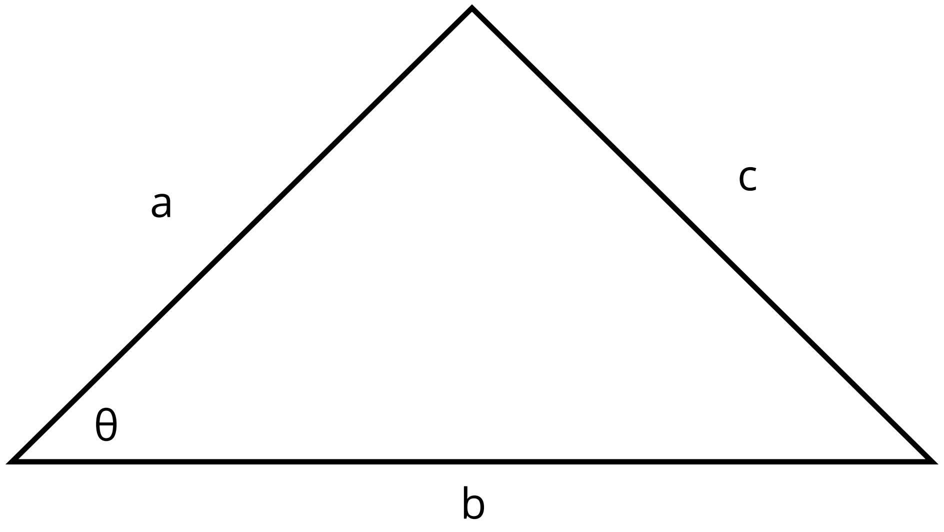
How would we express the length of **A** using a dot product?

The dot product is also sometimes referred to as an **inner product**.

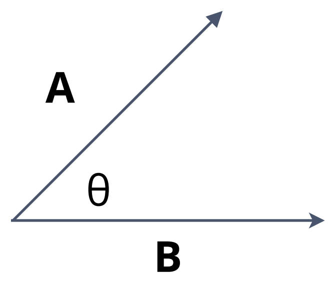
Law of Cosines

The Law of Cosines expresses the mathematical relationship between the side lengths of a triangle and one of its interior angles.

$$c^2 = a^2 + b^2 - 2(a)(b)\cos(\theta)$$

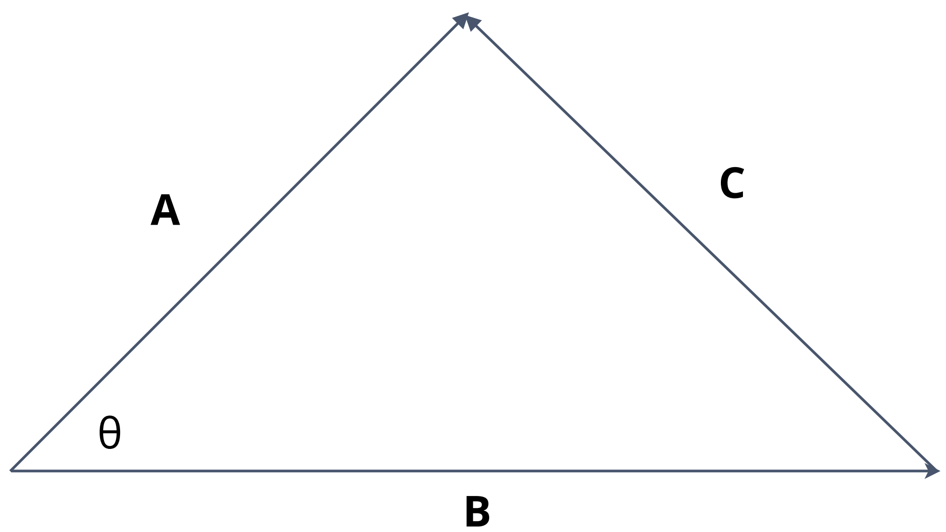


The Law of Cosines can also be used to express the mathematical relationship between any two vectors and the angle between them.



$$\mathbf{A} \bullet \mathbf{B} = \|\mathbf{A}\| \, \|\mathbf{B}\| \cos(\theta)$$

Law of Cosines (Proof)



Notice that: **C** = **A – B**

$$\mathbf{C} \bullet \mathbf{C} = (\mathbf{A} - \mathbf{B}) \bullet (\mathbf{A} - \mathbf{B})$$

$$= \mathbf{A} \bullet (\mathbf{A} - \mathbf{B}) - \mathbf{B} \bullet (\mathbf{A} - \mathbf{B})$$

$$= \mathbf{A} \bullet \mathbf{A} - \mathbf{A} \bullet \mathbf{B} - \mathbf{B} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2(\mathbf{A} \bullet \mathbf{B})$$

$$\|\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2(\mathbf{A} \bullet \mathbf{B})$$

According to the Law of Cosines

$$\|\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\,\|\mathbf{A}\|\,\|\mathbf{B}\|\cos(\theta)$$

Substituting

$$\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2(\mathbf{A} \bullet \mathbf{B}) = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\,\|\mathbf{A}\|\,\|\mathbf{B}\|\cos(\theta)$$

$$\mathbf{A} \bullet \mathbf{B} = \|\mathbf{A}\|\,\|\mathbf{B}\|\cos(\theta)$$

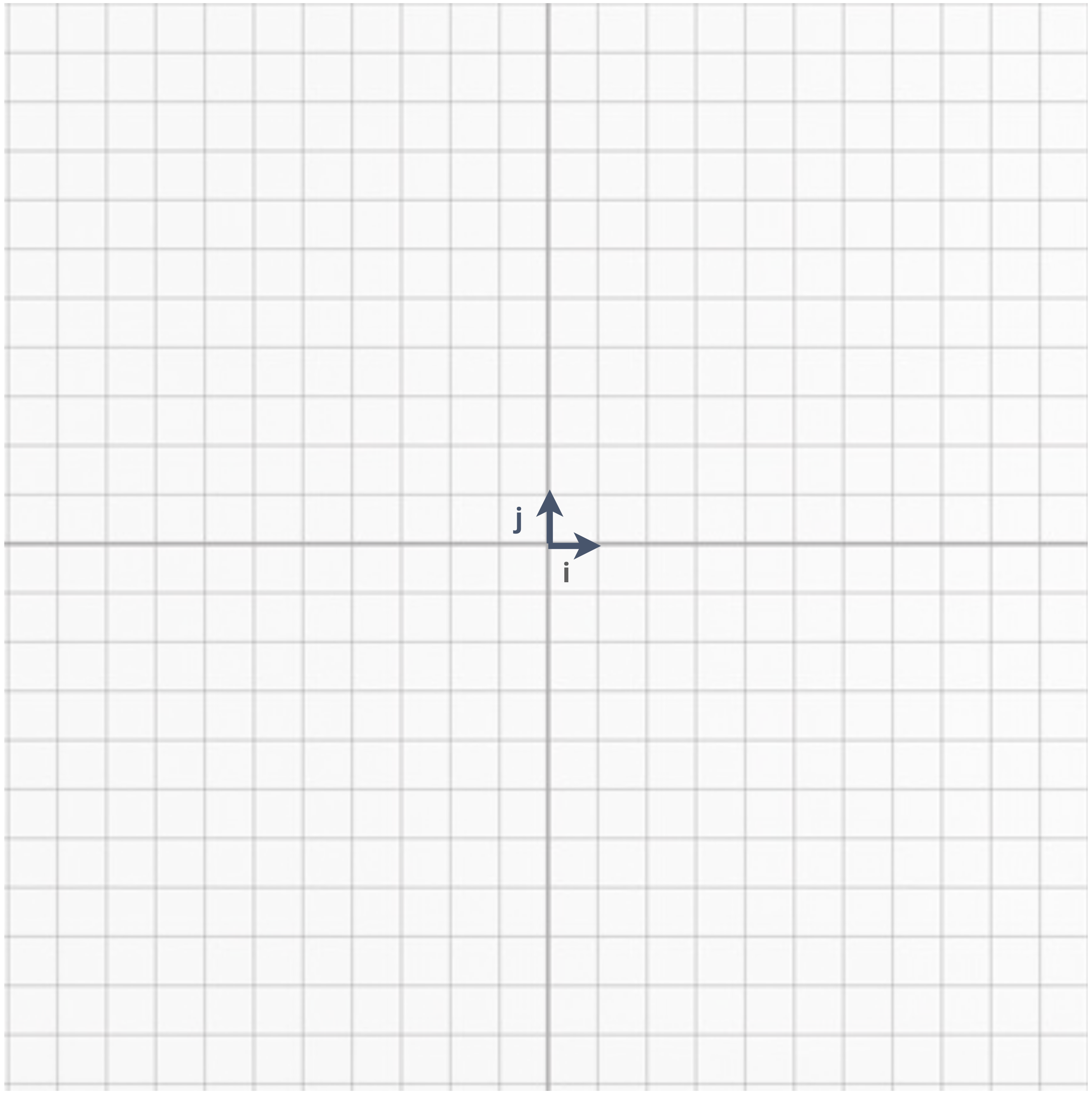
Basis Vectors

Basis Vectors

To define a coordinate system we need to define the “axes” of the system. We can do this with a set of **basis vectors**.

The basis vectors for the Cartesian plane we have been using are:

$$\vec{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



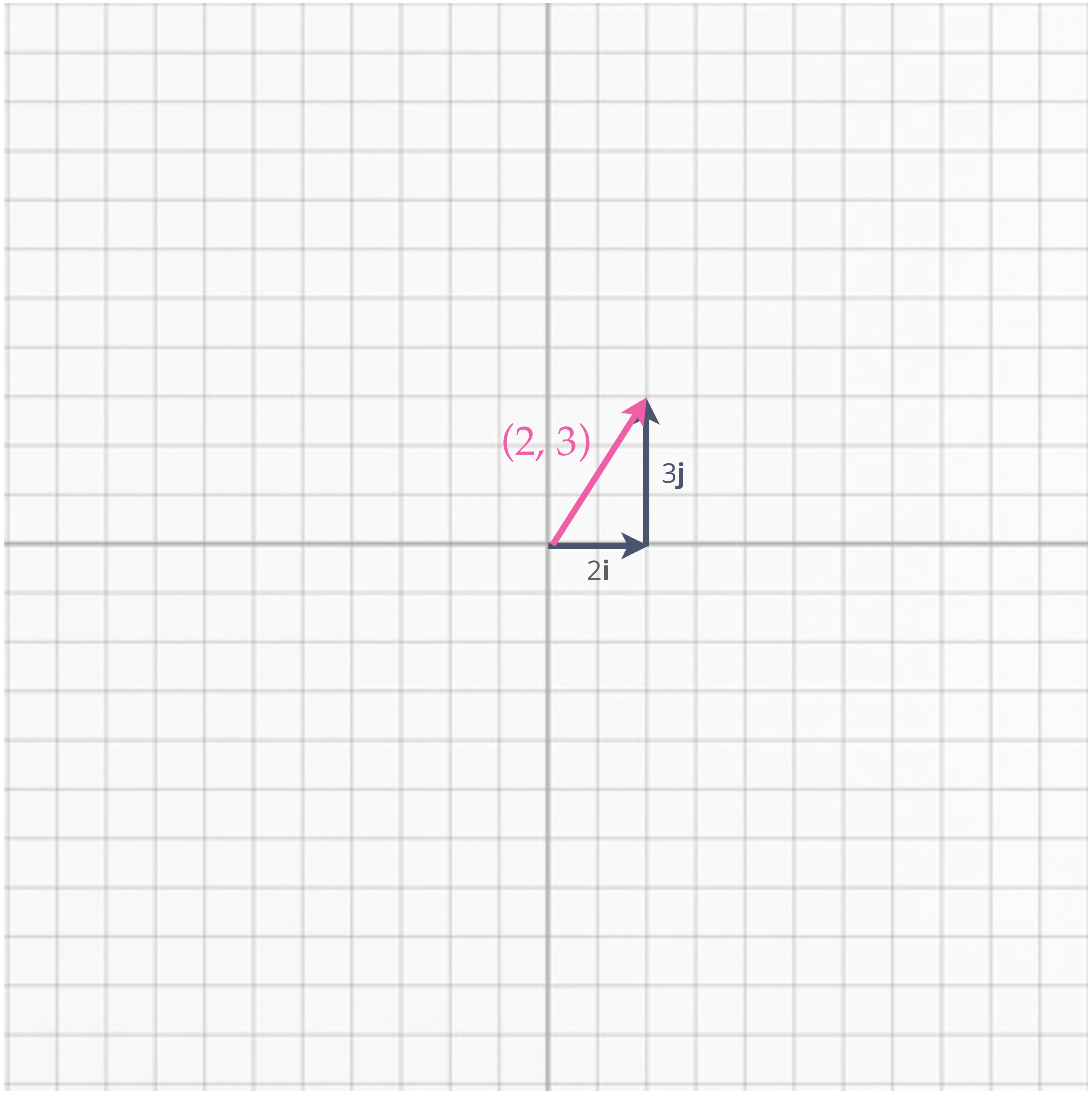
Any 2-dimensional vector can be expressed as a linear combination of the basis vectors/

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\,\overrightarrow{\mathbf{i}} + 3\,\overrightarrow{\mathbf{j}}$$

This is just scaling (stretching; shrinking) in the direction of each basis vector and then vector addition.

This can also be expressed as matrix multiplication, where the columns of the first matrix are the basis vectors and the column of the second matrix is the scalars:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

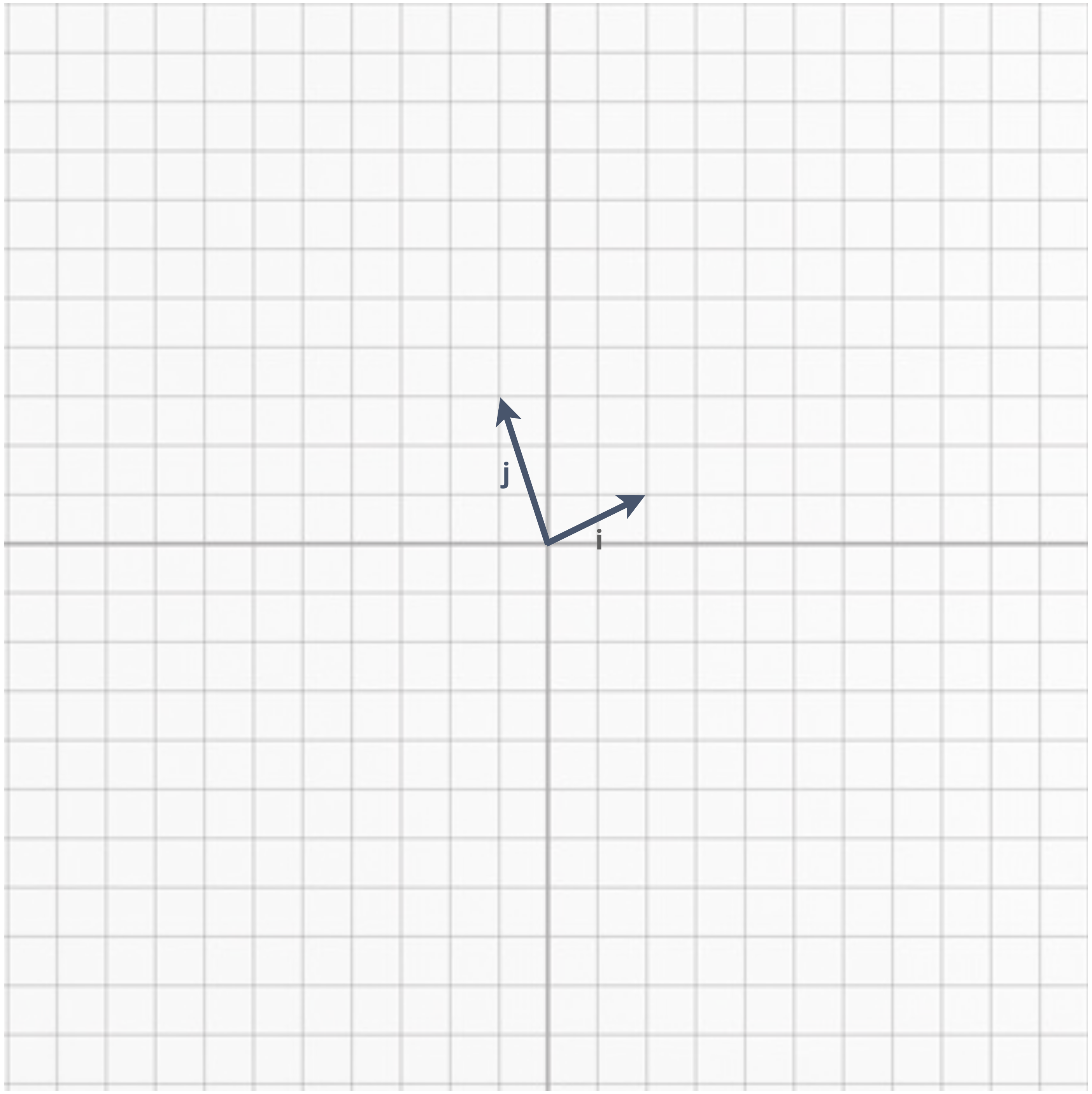


Changing Basis Vectors

The choice of basis vectors is a mathematical convenience. We could have chosen a different set of basis vectors which would result in a different coordinate system.

Here the new basis vectors (given in the cartesian coordinates) are:

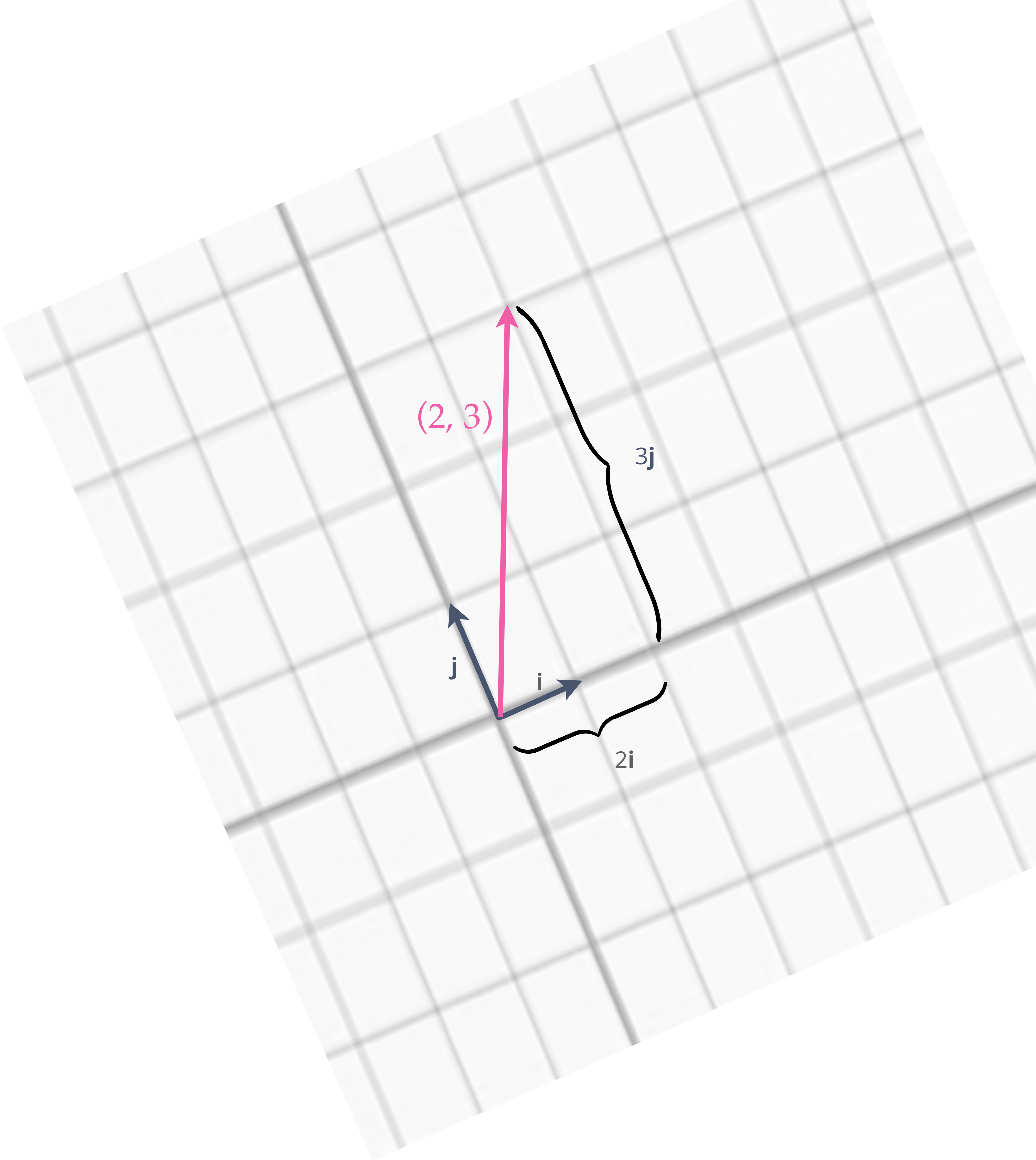
$$\vec{\mathbf{i}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{\mathbf{j}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



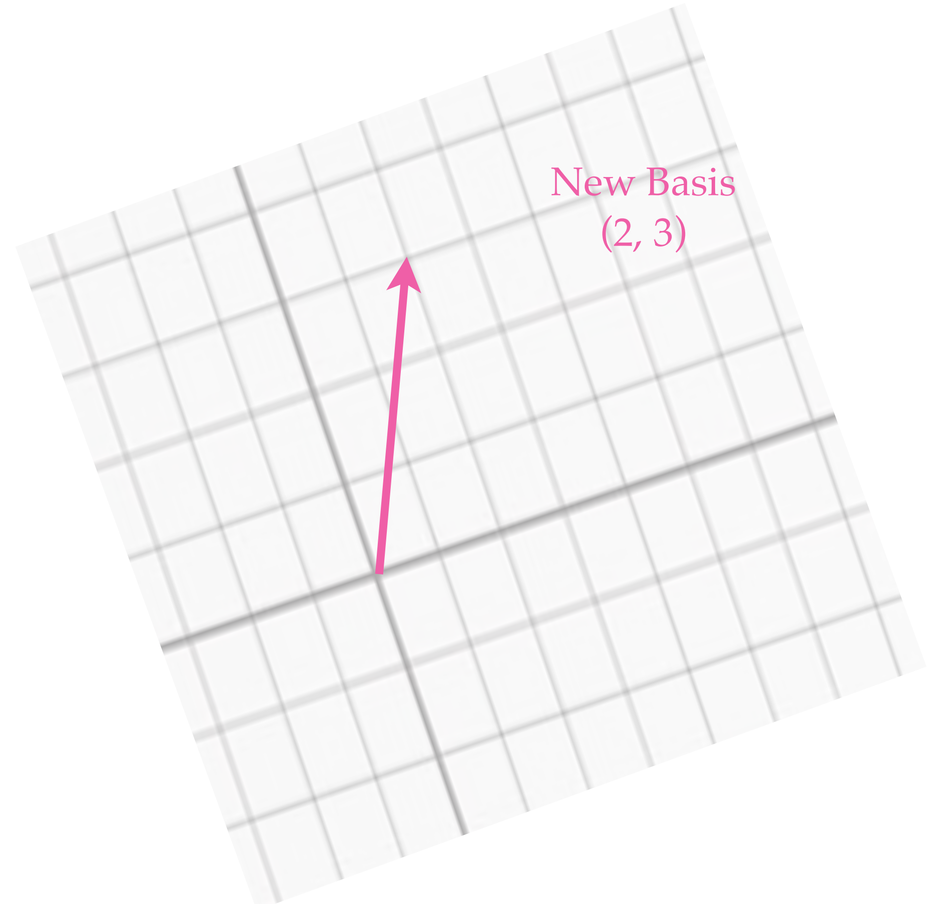
With this set of basis vectors, we have essentially changed (1) the direction of the axes, and (2) the unit length in each direction.

The “same” vector, (2, 3), now is completely different.

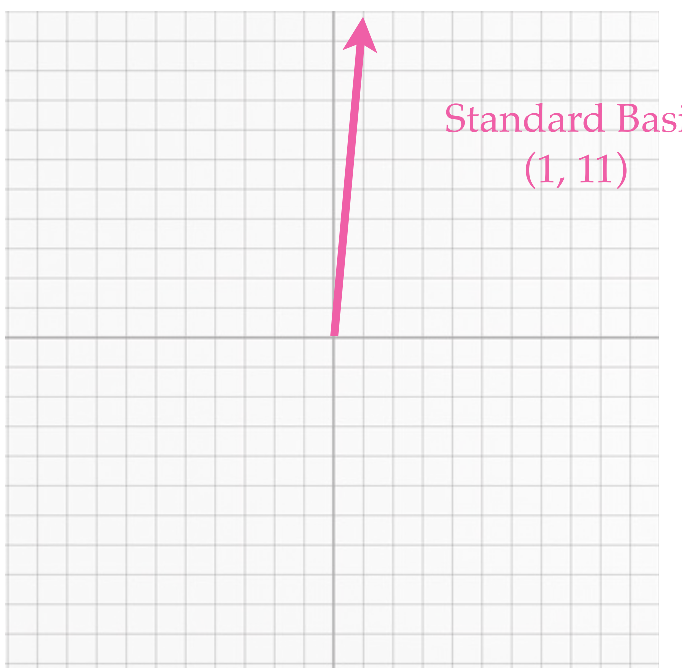
To represent this vector in the (1, 0)–(0, 1) basis we can use matrix multiplication:



Transforming to Standard Basis



What vector in the standard basis corresponds to (2, 3) in the new basis?

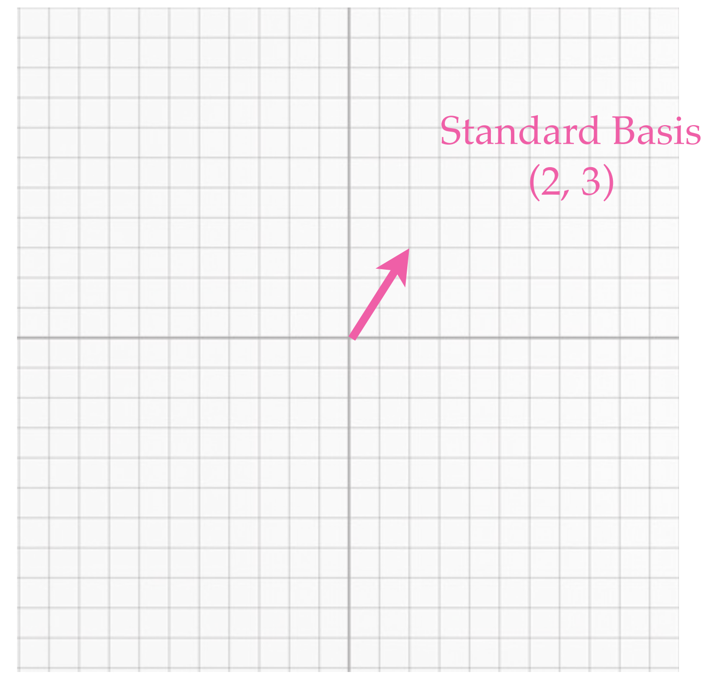


To represent this vector in the standard basis we can use matrix multiplication:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

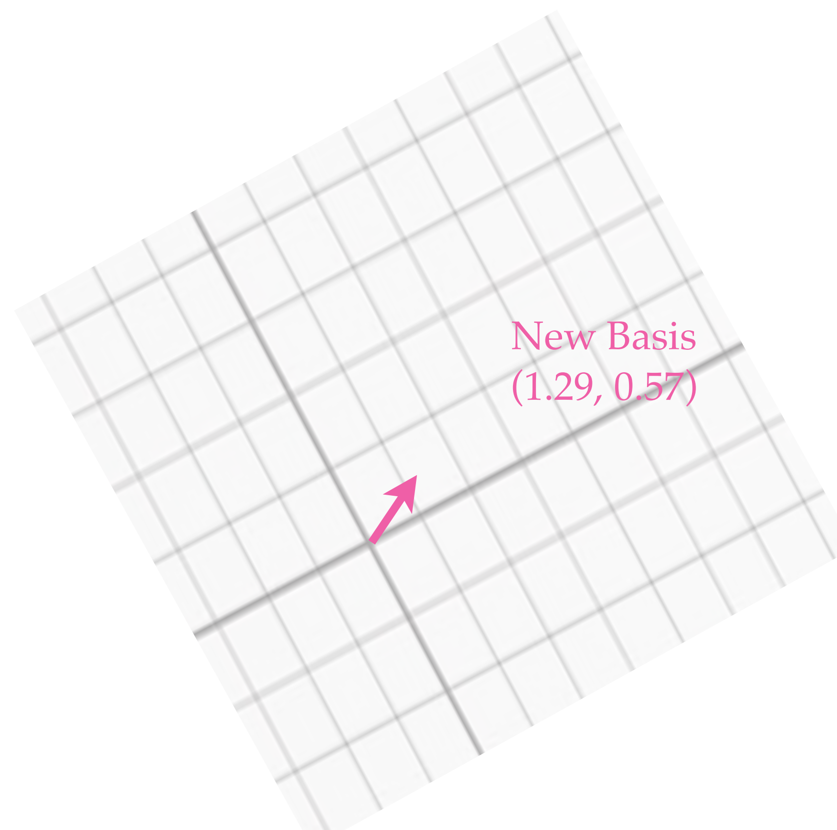
Transforming to New Basis

What vector in the new basis corresponds to (2, 3) in the standard basis?



To transform a vector in the standard basis to another basis we pre-multiply the vector of. scalars by the inverse of the basis matrix:

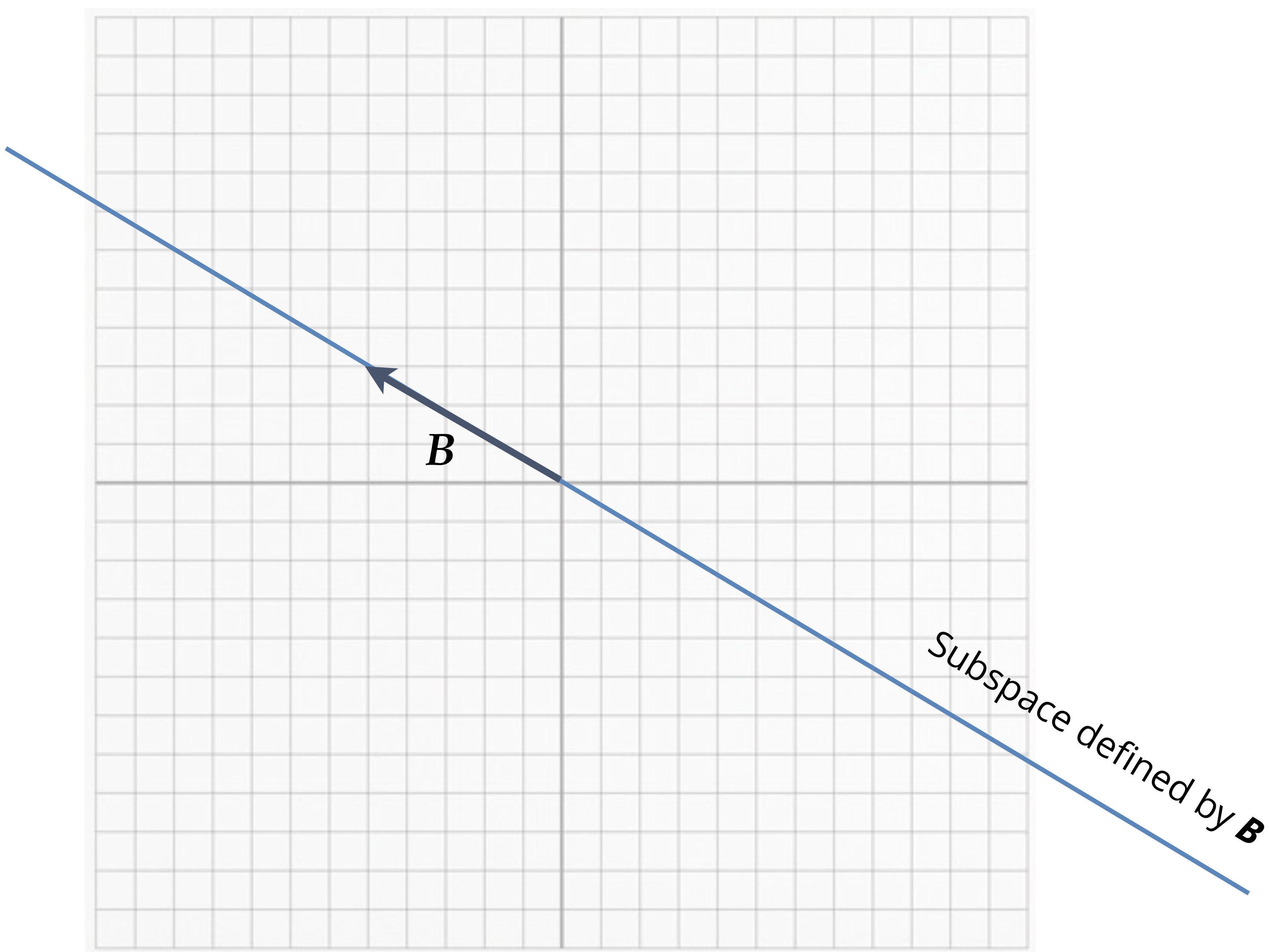
$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.29 \\ 0.57 \end{bmatrix}$$



Other Concepts Related to Vectors that are Important for Statistical Modeling

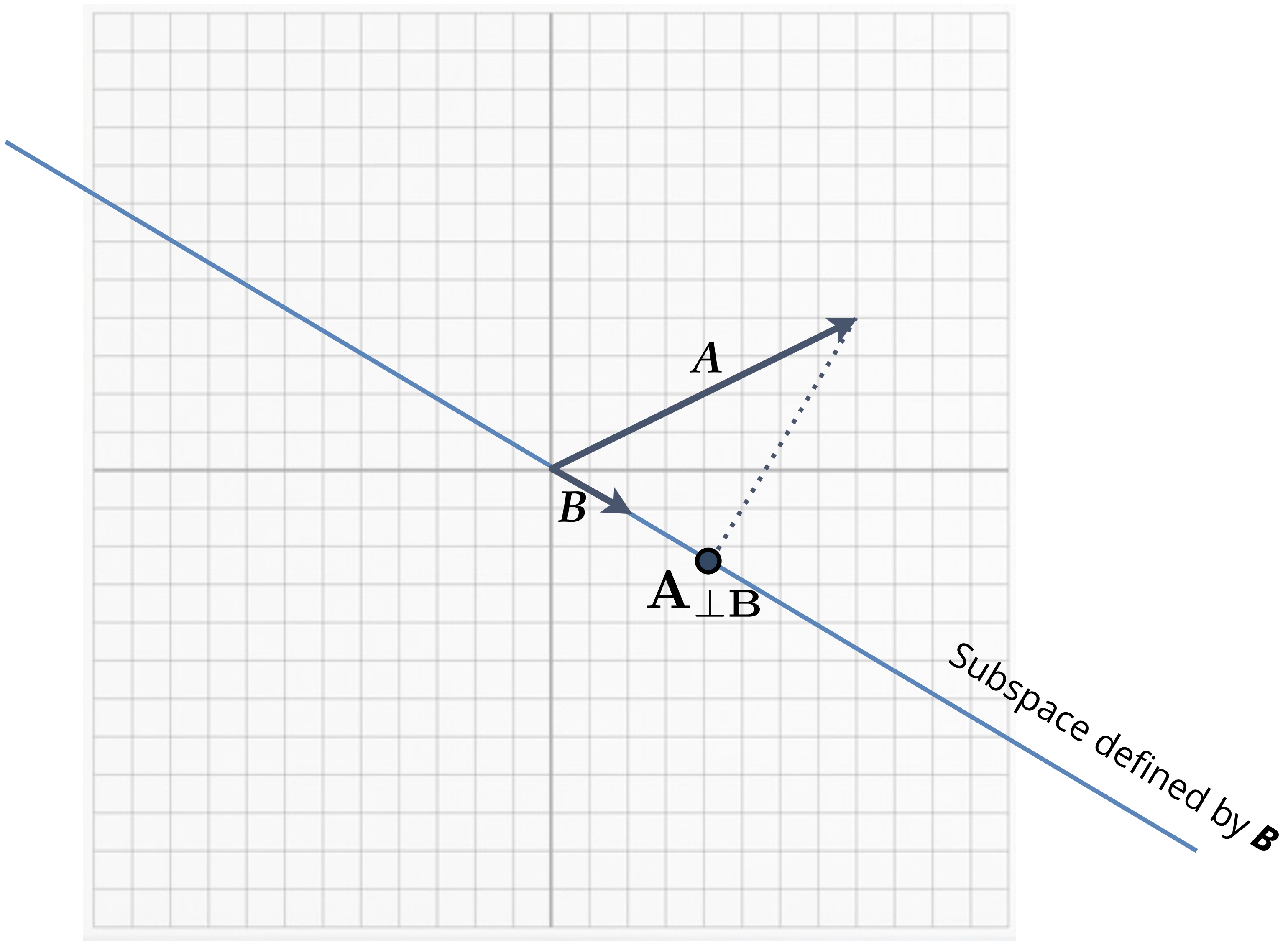
Subspace

A subspace is a part of the entire dimensional space defined by all of the points that can be reached by scaling a particular vector.



Projection

Projecting vector **A** onto a subspace defined by vector **B** finds the point in the subspace of **B** as close as possible to **A**



Unit Vector

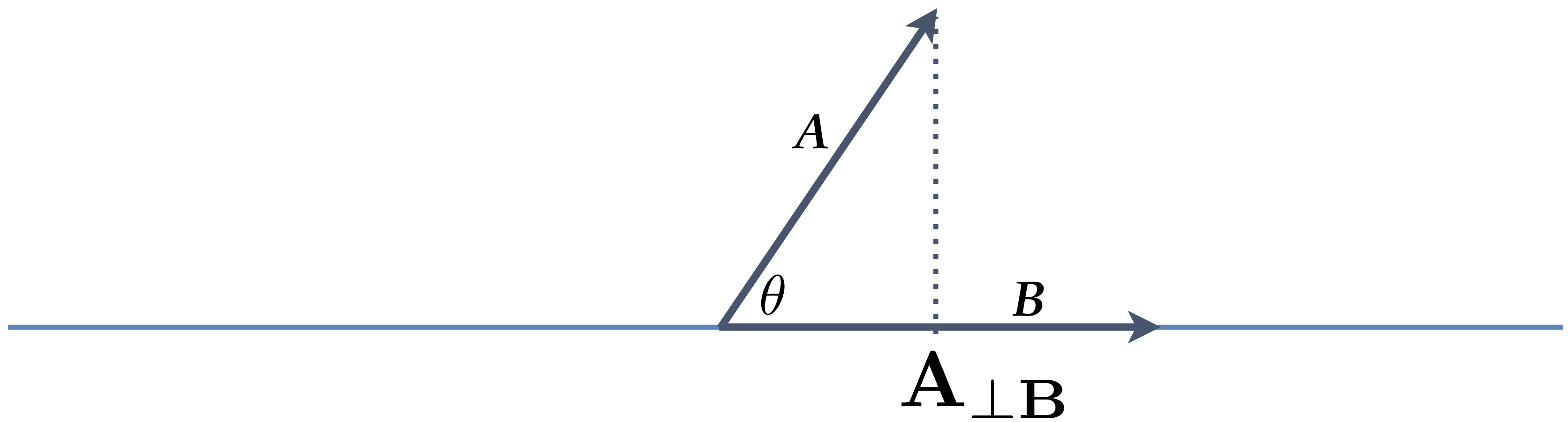
A unit vector is a vector that has length = 1.

Any vector can be scaled to have length 1 by dividing by its length.

$$\mathbf{1} = \frac{\mathbf{B}}{||\mathbf{B}||}$$

The resulting unit vector will be in the same direction as the original vector.

Length of a Projection



Definition of cosine,

$$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

from earlier,

$$\mathbf{A} \bullet \mathbf{B} = ||\mathbf{A}||\,||\mathbf{B}||\cos(\theta)$$

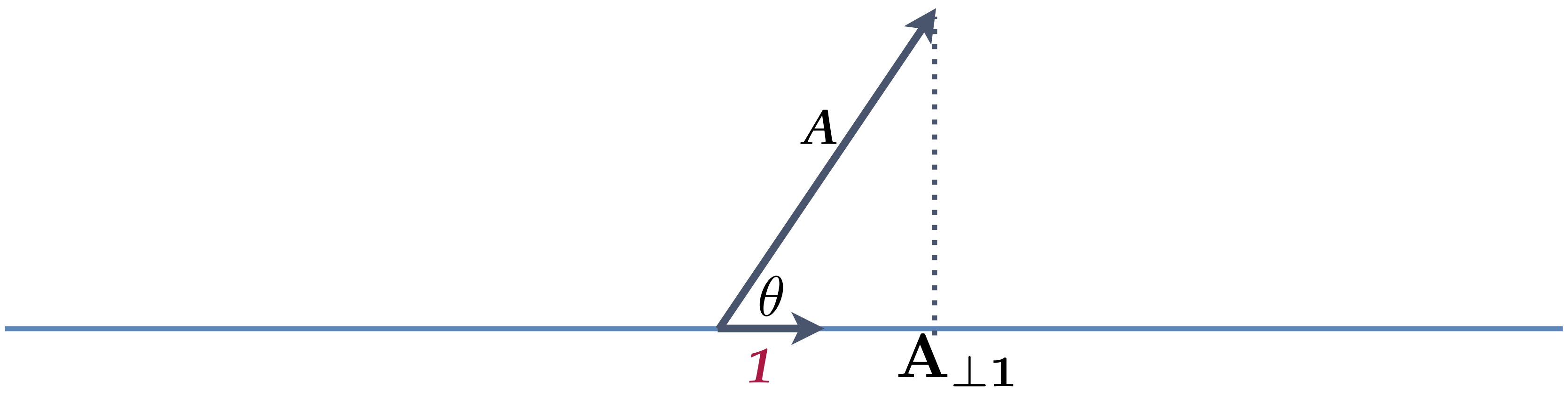
$$\text{adjacent side} = \cos(\theta) \times \text{hypotenuse}$$

$$||\mathbf{A}_{\perp\mathbf{B}}|| = ||\mathbf{A}||\cos(\theta)$$

$$\frac{\mathbf{A} \bullet \mathbf{B}}{||\mathbf{B}||} = ||\mathbf{A}||\cos(\theta)$$

$$||\mathbf{A}_{\perp\mathbf{B}}|| = \frac{\mathbf{A} \bullet \mathbf{B}}{||\mathbf{B}||}$$

Length of a Projection: Unit Vector



$$||\mathbf{A}_{\perp\mathbf{1}}|| = \frac{\mathbf{A} \bullet \mathbf{1}}{||\mathbf{1}||}$$

$$||\mathbf{A}_{\perp\mathbf{1}}|| = \mathbf{A} \bullet \mathbf{1}$$

The length of the projection of **A** onto the unit vector is equal to the dot product between **A** and the unit vector