Econ 325 Section 003/004

Notes on Variance, Covariance, and Summation Operator

By Hiro Kasahara

Properties of Summation Operator

For a sequence of the values $\{x_1, x_2, ..., x_n\}$, we write the sum of x_1, x_2, ..., x_{n-1}, and x_n using the summation operator as

$$x_1 + x_2 + ... + x_n = \sum_{i=1}^{n} x_i. \tag{1}$$

Given a constant c,

$$\sum_{i=1}^{n} cx_i = cx_1 + cx_2 + ... + cx_n = c \times (x_1 + x_2 + ... + x_n) = c \sum_{i=1}^{n} x_i. \tag{2}$$

- For example, consider the case that $n = 2$ with the values of $\{x_1, x_2\}$ given by $x_1 = 0$ and $x_2 = 1$. Suppose that $c = 4$. Then, $\sum_{i=1}^{2} 4 \times x_i = 4 \times 0 + 4 \times 1 = 4 \times (0 + 1) = 4 \sum_{i=1}^{2} x_i$.
- In the special case of $x_1 = x_2 = ... = x_n = 1$, we have $\sum_{i=1}^{n} cx_i = \sum_{i=1}^{n} c \times 1 = c \times \sum_{i=1}^{n} 1 = c \times (1 + 1 + ... + 1) = nc$.

Consider another sequence $\{y_1, y_2, ..., y_m\}$ in addition to $\{x_1, x_2, ..., x_n\}$. Then, we may consider double summations over possible values of x's and y's. For example, consider the case of $n = m = 2$. Then, $\sum_{i=1}^{2} \sum_{j=1}^{2} x_i y_j$ is equal to $x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$ because

$$\begin{aligned}
&x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \\
&= x_1(y_1 + y_2) + x_2(y_1 + y_2) \quad \text{(by factorization)} \\
&= \sum_{i=1}^{2} x_i(y_1 + y_2) \quad \text{(by def. of the summation operator by setting } c = (y_1 + y_2) \text{ in (2))} \\
&= \sum_{i=1}^{2} x_i \left(\sum_{j=1}^{2} y_j \right) \quad \text{(because } y_1 + y_2 = \textstyle\sum_{j=1}^{2} y_j) \\
&= \sum_{i=1}^{2} \left(\sum_{j=1}^{2} x_i y_j \right) \quad \text{(because } x_i \textstyle\sum_{j=1}^{2} y_j = x_i(y_1 + y_2) = (x_iy_1 + x_iy_2) = \sum_{j=1}^{2} x_iy_j) \\
&= \sum_{i=1}^{2} \sum_{j=1}^{2} x_i y_j.
\end{aligned}$$

- Note that $\sum_{i=1}^{2} \sum_{j=1}^{2} x_i y_j = \sum_{j=1}^{2} \sum_{i=1}^{2} x_i y_j$. In general case of $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_m\}$, we have $\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j = \sum_{j=1}^{m} \sum_{i=1}^{n} x_i y_j$.
- Note that $\sum_{j=1}^{2} x_i y_j = x_i \sum_{j=1}^{2} y_j$ using (2) because x_i is treated as a constant in the summation operator over j's. Hence, we can write

$$\sum_{i=1}^{2} \sum_{j=1}^{2} x_i y_j = \sum_{i=1}^{2} x_i \sum_{j=1}^{2} y_j = \sum_{j=1}^{2} y_j \sum_{i=1}^{2} x_i.$$

In general, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} y_j = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} x_i. \tag{3}$$

That is, when we have double summations, we can take x_i's out of the summation over j's. Similarly, we can take y_j's out of the summation over i's.

Expectation, Variance, and Covariance

Let X and Y be two discrete random variables. The set of possible values for X is $\{x_1, \ldots, x_n\}$; and the set of possible values for Y is $\{y_1, \ldots, y_m\}$. The joint probability function is given by

$$p_{ij}^{X,Y} = P\left(X = x_i, Y = y_j\right), \qquad i = 1, \ldots n; j = 1, \ldots, m.$$

The marginal probability function of X is

$$p_i^X = P\left(X = x_i\right) = \sum_{j=1}^{m} p_{ij}^{X,Y}, \qquad i = 1, \ldots n,$$

and the marginal probability function of Y is

$$p_j^Y = P\left(Y = y_j\right) = \sum_{i=1}^{n} p_{ij}^{X,Y}, \qquad j = 1, \ldots m.$$

1. If c is a constant, then

$$E[cX] = cE[X]. \tag{4}$$

Proof: By definition of the expected value of cX,

$$\begin{aligned}
E[cX] &= \sum_{i=1}^{n}(cx_i)p_i^X \quad \text{(by def. of the expected value)}\\
&= cx_1p_1^X + cx_2p_2^X + cx_3p_3^X + ... + cx_{n-1}p_{n-1}^X + cx_np_n^X \quad \text{(by def. of the summation operator)}\\
&= c \times (x_1p_1^X + x_2p_2^X + x_3p_3^X + ... + x_{n-1}p_{n-1}^X + x_np_n^X) \quad \text{(because } c \text{ is a common factor)}\\
&= c \times \left(\sum_{i=1}^{n} x_i p_i^X\right) \quad \text{(by def. of the summation operator)}\\
&= c \times E[X] \quad \text{(by def. of the expected value of } X\text{)}\\
&= cE[X].
\end{aligned}$$

2.

$$E[X + Y] = E[X] + E[Y]. \tag{5}$$

Proof:

$$
\begin{aligned}
E(X+Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i+y_j)p_{ij}^{X,Y} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i p_{ij}^{X,Y} + y_j p_{ij}^{X,Y}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i p_{ij}^{X,Y} + \sum_{i=1}^{n}\sum_{j=1}^{m} y_j p_{ij}^{X,Y} \qquad (6) \\
&= \sum_{i=1}^{n} x_i \cdot \left(\sum_{j=1}^{m} p_{ij}^{X,Y}\right) + \sum_{j=1}^{m} y_j \cdot \left(\sum_{i=1}^{n} p_{ij}^{X,Y}\right) \qquad (7) \\
&\qquad \text{because we can take } x_i \text{ out of } \textstyle\sum_{j=1}^{m} \text{ because } x_i \text{ does not depend on } j\text{'s} \\
&= \sum_{i=1}^{n} x_i \cdot p_i^X + \sum_{j=1}^{m} y_j \cdot p_j^Y \\
&\qquad \text{because } p_i^X = \textstyle\sum_{j=1}^{m} p_{ij}^{X,Y} \text{ and } p_j^Y = \textstyle\sum_{i=1}^{n} p_{ij}^{X,Y} \\
&= E(X) + E(Y)
\end{aligned}
$$

Equation (6): To understand $\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i p_{ij}^{X,Y} + y_j p_{ij}^{X,Y}) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i p_{ij}^{X,Y} + \sum_{i=1}^{n}\sum_{j=1}^{m} y_j p_{ij}^{X,Y}$, consider the case of $n = m = 2$. Then,

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{j=1}^{2}(x_i p_{ij}^{X,Y} + y_j p_{ij}^{X,Y}) \\
&= (x_1 p_{11}^{X,Y} + y_1 p_{11}^{X,Y}) + (x_1 p_{12}^{X,Y} + y_2 p_{12}^{X,Y}) + (x_2 p_{21}^{X,Y} + y_1 p_{21}^{X,Y}) + (x_2 p_{22}^{X,Y} + y_2 p_{22}^{X,Y}) \\
&= (x_1 p_{11}^{X,Y} + x_1 p_{12}^{X,Y} + x_2 p_{21}^{X,Y} + x_2 p_{22}^{X,Y}) + (y_1 p_{11}^{X,Y} + y_2 p_{12}^{X,Y} + y_1 p_{21}^{X,Y} + y_2 p_{22}^{X,Y}) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} x_i p_{ij}^{X,Y} + \sum_{i=1}^{2}\sum_{j=1}^{2} y_j p_{ij}^{X,Y}.
\end{aligned}
$$

Equation (7): This is a generalization of (3). To understand $\sum_{i=1}^{n}\sum_{j=1}^{m} x_i p_{ij}^{X,Y} = \sum_{i=1}^{n} x_i \cdot (\sum_{j=1}^{m} p_{ij}^{X,Y})$, consider the case of $n = m = 2$. Then,

$$
\begin{aligned}
\sum_{i=1}^{2}\sum_{j=1}^{2} x_i p_{ij}^{X,Y} &= x_1 p_{11}^{X,Y} + x_1 p_{12}^{X,Y} + x_2 p_{21}^{X,Y} + x_2 p_{22}^{X,Y} \\
&= x_1(p_{11}^{X,Y} + p_{12}^{X,Y}) + x_2(p_{21}^{X,Y} + p_{22}^{X,Y}) \\
&= \sum_{i=1}^{2} x_i(p_{i1}^{X,Y} + p_{i2}^{X,Y}) \\
&= \sum_{i=1}^{2} x_i(\sum_{j=1}^{2} p_{ij}^{X,Y}).
\end{aligned}
$$

Similarly, we may show that $\sum_{i=1}^{2}\sum_{j=1}^{2} y_j p_{ij}^{X,Y} = \sum_{j=1}^{2} y_j \cdot (\sum_{i=1}^{2} p_{ij}^{X,Y})$.

3. If a and b are constants, then $E[a + bX] = a + bE[X]$.

Proof:

$$
\begin{aligned}
E(a+bX) &= \sum_{i=1}^{n}(a+bx_i)p_i^X \\
&= \sum_{i=1}^{n}(ap_i^X + bx_ip_i^X) \\
&= \sum_{i=1}^{n} ap_i^X + \sum_{i=1}^{n} bx_ip_i^X \\
&= a\sum_{i=1}^{n} p_i^X + b\sum_{i=1}^{n} x_ip_i^X, \quad \text{(by using (2))} \\
&= a\cdot 1 + bE(X), \quad \text{where } \textstyle\sum_{i=1}^{n} p_i^X = \sum_{i=1}^{n} P(X=x_i) = 1 \text{ and } \sum_{i=1}^{n} x_ip_i^X = E(X) \\
&= a + bE(X).
\end{aligned} \tag{8}
$$

Equation (8): This is similar to (6). To understand $\sum_{i=1}^{n}(ap_i^X + bx_ip_i^X) = \sum_{i=1}^{n} ap_i^X + \sum_{i=1}^{n} bx_ip_i^X$, consider the case of $n=2$. Then, $\sum_{i=1}^{2}(ap_i^X + bx_ip_i^X) = (ap_1^x + bx_1p_1^X) + (ap_2^x + bx_2p_2^X) = (ap_1^x + ap_2^x) + (bx_1p_1^X + bx_2p_2^X) = \sum_{i=1}^{2} ap_i^X + \sum_{i=1}^{2} bx_ip_i^X$.

4. If c is a constant, then $Cov\,(X,c) = 0$.

   Proof: According to the definition of covariance,

   $$Cov(X,c) = E[(X-E(X))(c-E(c))].$$

   Since the expectation of a constant is itself, i.e., $E(c)=c$,

   $$
   \begin{aligned}
   Cov(X,c) &= E[(X-E(X))(c-c)] \\
   &= E[(X-E(X))\cdot 0] \\
   &= E[0] \\
   &= \sum_{i=1}^{n} 0\times p_i^X \\
   &= \sum_{i=1}^{n} 0 \\
   &= 0+0+\ldots+0 \\
   &= 0
   \end{aligned}
   $$

5. $Cov\,(X,X) = Var\,(X)$.

Proof: According to the definition of covariance, we can expand $Cov(X, X)$ as follows:

$$
\begin{aligned}
Cov(X, X) &= E[(X - E(X))(X - E(X))] \\
&= \sum_{i=1}^{n} [x_i - E(X)][x_i - E(X)] \cdot P(X = x_i), \quad \text{where } E(X) = \sum_{i=1}^{n} x_i p_i^X \\
&= \sum_{i=1}^{n} [x_i - E(X)][x_i - E(X)] \cdot p_i^X \\
&= \sum_{i=1}^{n} [x_i - E(X)]^2 \cdot p_i^X \\
&= E[(X - E(X))^2] \quad \text{(by def. of the expected value)} \\
&= Var(X).
\end{aligned}
$$

6. $Cov\,(X, Y) = Cov\,(Y, X)\,.$

Proof: According to the definition of covariance, we can expand $Cov(X, Y)$ as follows:

$$
\begin{aligned}
Cov(X, Y) &= E[(X - E(X))(Y - E(Y))] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} [x_i - E(X)][y_j - E(Y)] \cdot p_{ij}^{X,Y}, \quad \text{where } E(X) = \sum_{i=1}^{n} x_i p_i^X \text{ and } E(Y) = \sum_{j=1}^{m} y_j p_j^Y \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} [y_j - E(Y)][x_i - E(X)] \cdot p_{ij}^{X,Y} \\
&= E[(Y - E(Y))(X - E(X))] \quad \text{(by def. of the expected value)} \\
&= Cov(Y, X). \quad \text{(by def. of the covariance)}
\end{aligned}
$$

7. $Cov\,(a_1 + b_1 X, a_2 + b_2 Y) = b_1 b_2 Cov\,(X, Y)\,,$ where $a_1, a_2, b_1,$ and b_2 are some constants.

Proof: Using $E(a_1 + b_1 X) = a_1 + b_1 E(X)$ and $E(a_2 + b_2 Y) = a_2 + b_2 E(Y)$, we can expand $Cov\,(a_1 + b_1 X, a_2 + b_2 Y)$ as follows:

$$
\begin{aligned}
Cov(X, Y) &= E[(a_1 + b_1 X - E(a_1 + b_1 X))(a_2 + b_2 Y - E(a_2 + b_2 Y))] \\
&= E[(a_1 + b_1 X - (a_1 + b_1 E(X)))(a_2 + b_2 Y - (a_2 + b_2 E(Y))] \\
&= E[(a_1 - a_1 + b_1 X - b_1 E(X))(a_2 - a_2 + b_2 Y - b_2 E(Y)] \\
&= E[(b_1 X - b_1 E(X))(b_2 Y - b_2 E(Y)] \\
&= E[b_1 (X - E(X)) \cdot b_2 (Y - E(Y))] \\
&= E[b_1 b_2 (X - E(X))(Y - E(Y))] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} b_1 b_2 (x_i - E(X))(y_j - E(Y)) \cdot p_{ij}^{X,Y} \\
&= b_1 b_2 \sum_{i=1}^{n} \sum_{j=1}^{m} [x_i - E(X)][y_j - E(Y)] \cdot p_{ij}^{X,Y} \quad \text{(by using (2))} \\
&= b_1 b_2 Cov(X, Y).
\end{aligned}
$$

8. If X and Y are independent, then $Cov\,(X,Y) = 0$.

   Proof: If X and Y are independent, by definition of stochastic independence, $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) = p_i^X p_j^Y$ for any $i = 1, ..., n$ and $j = 1, ..., m$. Then, we may expand $Cov\,(X,Y)$ as follows.

$$
\begin{aligned}
Cov(X,Y) &= E[(X - E(X))(Y - E(Y))] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}[x_i - E(X)][y_j - E(Y)] \cdot P(X = x_i, Y = y_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}[x_i - E(X)][y_j - E(Y)]p_i^X p_j^Y \\
&\qquad \text{because } X \text{ and } Y \text{ are independent} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\{[x_i - E(X)]p_i^X\}\{[y_j - E(Y)]p_j^Y\} \\
&= \sum_{i=1}^{n}[x_i - E(X)]p_i^X \left\{\sum_{j=1}^{m}[y_j - E(Y)]p_j^Y\right\} && (9) \\
&\qquad \text{because we can move } [x_i - E(X)]p_i^X \text{ outside of } \textstyle\sum_{j=1}^{m} \\
&\qquad \ \text{because } [x_i - E(X)]p_i^X \text{ does not depend on the index } j\text{'s} \\
&= \left\{\sum_{j=1}^{m}[y_j - E(Y)]p_j^Y\right\}\left\{\sum_{i=1}^{n}[x_i - E(X)]p_i^X\right\} && (10) \\
&\qquad \text{because we can move } \left\{\textstyle\sum_{j=1}^{m}[y_j - E(Y)]p_j^Y\right\} \text{ outside of } \textstyle\sum_{i=1}^{n} \\
&\qquad \ \text{because } \left\{\textstyle\sum_{j=1}^{m}[y_j - E(Y)]p_j^Y\right\} \text{ does not depend on the index } i\text{'s} \\
&= \left\{\sum_{i=1}^{n} x_i p_i^X - \sum_{i=1}^{n} E(X)p_i^X\right\} \cdot \left\{\sum_{j=1}^{m} y_j p_j^Y - \sum_{j=1}^{m} E(Y)p_j^Y\right\} \\
&= \left\{E(X) - \sum_{i=1}^{n} E(X)p_i^X\right\} \cdot \left\{E(Y) - \sum_{j=1}^{m} E(Y)p_j^Y\right\} \\
&\qquad \text{by definition of } E(X) \text{ and } E(Y) \\
&= \left\{E(X) - E(X)\sum_{i=1}^{n} p_i^X\right\} \cdot \left\{E(Y) - E(Y)\sum_{j=1}^{m} p_j^Y\right\} \\
&\qquad \text{because we can move } E(X) \text{ and } E(Y) \text{ outside of } \textstyle\sum_{i=1}^{n} \text{ and } \textstyle\sum_{j=1}^{m}\text{, respectively} \\
&= \{E(X) - E(X) \cdot 1\} \cdot \{E(Y) - E(Y) \cdot 1\} \\
&= 0 \cdot 0 = 0.
\end{aligned}
$$

   Equations (9) and (10): This is similar to equations (3) and (7). Please consider the case of $n = m = 2$ and convince yourself that (9) and (10) hold.

9. $Var\,(X + Y) = Var\,(X) + Var\,(Y) + 2Cov\,(X,Y)$.

Proof: By the definition of variance,

$$Var(X+Y) = E[(X+Y-E(X+Y))^2].$$

Then,

$$\begin{aligned}
Var(X+Y) &= E[(X+Y-E(X+Y))^2] \\
&= E[((X-E(X))+(Y-E(Y)))^2] \\
&= E[(X-E(X))^2+(Y-E(Y))^2+2(X-E(X))(Y-E(Y))] \\
&\quad \text{because for any } a \text{ and } b,\ (a+b)^2 = a^2+b^2+2ab \\
&= E[(X-E(X))^2]+E[(Y-E(Y))^2]+2E[(X-E(X))(Y-E(Y))] \quad \text{(by using (5))} \\
&= Var(X)+Var(Y)+2Cov(X,Y) \\
&\quad \text{by definition of variance and covariance}
\end{aligned}$$

10. $Var\,(X-Y) = Var\,(X) + Var\,(Y) - 2Cov\,(X,Y)$.

    Proof: The proof of $Var\,(X-Y) = Var\,(X)+Var\,(Y)-2Cov\,(X,Y)$ is similar to the proof of $Var\,(X+Y) = Var\,(X)+Var\,(Y)+2Cov\,(X,Y)$. First, we may show that $E(X-Y) = E(X)-E(Y)$. Then,

$$\begin{aligned}
Var(X-Y) &= E[(X-Y-E(X-Y))^2] \\
&= E[((X-E(X))-(Y-E(Y)))^2] \\
&= E[(X-E(X))^2+(Y-E(Y))^2-2(X-E(X))(Y-E(Y))] \\
&= E[(X-E(X))^2]+E[(Y-E(Y))^2]-2E[(X-E(X))(Y-E(Y))] \quad \text{(by using (5))} \\
&= Var(X)+Var(Y)-2Cov(X,Y)
\end{aligned}$$

11. Define $W = (X-E(X))/\sqrt{Var(X)}$ and $Z = (Y-E(Y))/\sqrt{Var(Y)}$. Show that $Cov(W,Z) = Corr(X,Z)$.

Proof: Expanding $Cov(W,Z)$, we have

$$\begin{aligned}
Cov(W,Z) &= E[(W-E(W))(Z-E(Z))] \\
&= E[WZ] \quad \text{(because } E[W]=E[Z]=0) \\
&= E\left\{\frac{X-E(X)}{\sqrt{Var(X)}}\cdot\frac{Y-E(Y)}{\sqrt{Var(Y)}}\right\} \\
&\quad \text{by definition of } W \text{ and } Z \\
&= E\left\{\frac{1}{\sqrt{Var(X)}}\cdot\frac{1}{\sqrt{Var(Y)}}\cdot[X-E(X)]E[Y-E(Y)]\right\} \\
&= \frac{1}{\sqrt{Var(X)}}\cdot\frac{1}{\sqrt{Var(Y)}}\cdot E\left\{[X-E(X)]E[Y-E(Y)]\right\} \quad \text{(by using (2) and (4))} \\
&\quad \text{because both } \tfrac{1}{\sqrt{Var(X)}} \text{ and } \tfrac{1}{\sqrt{Var(Y)}} \text{ are constant} \\
&= \frac{E\left\{[X-E(X)]E[Y-E(Y)]\right\}}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \qquad \text{(by definition of covariance)} \\
&= Corr(X,Y) \qquad \text{(by definition of correlation coefficient)}
\end{aligned}$$

12. Let $\{x_i : i=1,\dots,n\}$ and $\{y_i : i=1,\dots,n\}$ be two sequences. Define the averages

$$\begin{aligned}
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i, \\
\bar{y} &= \frac{1}{n}\sum_{i=1}^{n} y_i.
\end{aligned}$$

(a) $\sum_{i=1}^{n}(x_i-\bar{x})=0.$

Proof:

$$\begin{aligned}
\sum_{i=1}^{n}(x_i-\bar{x}) &= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\bar{x} \\
&= \sum_{i=1}^{n} x_i - n\bar{x} \\
&\quad \text{because } \textstyle\sum_{i=1}^{n}\bar{x} = \bar{x}+\bar{x}+\dots+\bar{x} = n\bar{x} \\
&= n\frac{\sum_{i=1}^{n} x_i}{n} - n\bar{x} \\
&\quad \text{because } \textstyle\sum_{i=1}^{n} x_i = \frac{n}{n}\sum_{i=1}^{n} x_i = n\frac{\sum_{i=1}^{n} x_i}{n} \\
&= n\bar{x} - n\bar{x} \\
&\quad \text{because } \bar{x} = \tfrac{\sum_{i=1}^{n} x_i}{n} \\
&= 0.
\end{aligned}$$

(b) $\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n} x_i(x_i-\bar{x}).$

Proof: We use the result of 2.(a) above.

$$\begin{aligned}
\sum_{i=1}^{n}(x_i-\bar{x})^2 &= \sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x}) \\
&= \sum_{i=1}^{n}x_i(x_i-\bar{x}) - \sum_{i=1}^{n}\bar{x}(x_i-\bar{x}) \\
&= \sum_{i=1}^{n}x_i(x_i-\bar{x}) - \bar{x}\sum_{i=1}^{n}(x_i-\bar{x}) \\
&\text{because } \bar{x} \text{ is constant and does not depend on } i\text{'s} \quad = \sum_{i=1}^{n}x_i(x_i-\bar{x}) - \bar{x}\cdot 0 \\
&\text{because } \textstyle\sum_{i=1}^{n}(x_i-\bar{x}) = 0. \text{ as shown above} \\
&= \sum_{i=1}^{n}x_i(x_i-\bar{x}).
\end{aligned}$$

(c) $\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \sum_{i=1}^{n}y_i(x_i-\bar{x}) = \sum_{i=1}^{n}x_i(y_i-\bar{y})$.

Proof: The proof is similar to the proof of 2.(b) above.

$$\begin{aligned}
\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) &= \sum_{i=1}^{n}(x_i-\bar{x})y_i - \sum_{i=1}^{n}(x_i-\bar{x})\bar{y} \\
&= \sum_{i=1}^{n}(x_i-\bar{x})y_i - \bar{y}\sum_{i=1}^{n}(x_i-\bar{x}) \\
&= \sum_{i=1}^{n}(x_i-\bar{x})y_i - \bar{y}\cdot 0 \\
&= \sum_{i=1}^{n}y_i(x_i-\bar{x}).
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) &= \sum_{i=1}^{n}x_i(y_i-\bar{y}) - \sum_{i=1}^{n}\bar{x}(y_i-\bar{y}) \\
&= \sum_{i=1}^{n}x_i(y_i-\bar{y}) - \bar{x}\sum_{i=1}^{n}(y_i-\bar{y}) \\
&= \sum_{i=1}^{n}x_i(y_i-\bar{y}) - \bar{x}\cdot 0 \\
&= \sum_{i=1}^{n}x_i(y_i-\bar{y}).
\end{aligned}$$