Eigendecomposition (a.k.a., Spectrial Decomposition)

A brief introduction to eigendecomposition and some R syntax for carrying out eigendecomposition.

AUTHOR

Andrew Zieffler

PUBLISHED

Sept. 8, 2020

TABLE OF CONTENTS

Eigendecomposition is a method of decomposing or factoring a matrix into a set of product matrices made up of *eigenvalues* and *eigenvectors*. Only square matrices that can be expressed as

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

where **D** is a diagonal matrix and **P** is an invertible matrix, can be eigendecomposed. If **A** can be expressed this way it is referred to as a *diagonalizable* matrix.

Eigenvalues

An eigenvalue of matrix **A** is any non-zero scalar λ such that, $\mathbf{A} - \lambda\mathbf{I}$ is singular (e.g., determinant is zero). Consider an example,

$$\underset{n \times n}{\mathbf{A}} = \begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix}$$

Our goal is to find the values of λ that satisfy

$$\det\left(\begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Using matrix algebra,

$$\det\left(\begin{bmatrix}-3 & 5\\ 4 & -2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}-3 & 5\\ 4 & -2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix}-3-\lambda & 5\\ 4 & -2-\lambda\end{bmatrix} = 0$$

$$(-3-\lambda)(-2-\lambda) - 20 = 0$$

Computing the determinant, distributing the first set of terms (i.e., FOIL), subtracting 20 we obtain

$$(-3-\lambda)(-2-\lambda) - 20 = 0$$

$$6 + 3\lambda + 2\lambda + \lambda^2 - 20 = 0$$

$$\lambda^2 + 5\lambda - 14 = 0$$

Factoring the left-hand side,

$$(\lambda + 7)(\lambda - 2) = 0$$

Solving for λ, we find that $\lambda = -7$ and $\lambda = 2$. We could also have applied the quadratic formula to solve this for λ.

Remember the quadratic formula? The roots of the polynomial $Ax^2 + Bx + C$ are computed using:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Quick recap at Khan Academy if you need it.

Now that we have the eigenvalues we can double-check that $\mathbf{A} - \lambda\mathbf{I}$ is singular. (I will skip this here, but plug in the values for λ, one at a time, and ensure that the determinant is zero.) The equation you solved,

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

is referred to as the *characteristic equation*. Solving the characteristic equation gives the eigenvalues. Sometimes the eigenvalues are referred to as the characteristic roots of matrix **A**.

Eigenvectors

Eigenvectors

If λ is an eigenvalue of matrix **A**, then it is possible to find a vector **v** (an eigenvector) that satisfies

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Here since **A** is a 2×2 matrix, **v** will be a 2×1 vector to make the matrix multiplication work.

$$\underset{2\times 2}{\mathbf{A}}\;\underset{2\times 1}{\mathbf{v}} = \lambda\;\underset{2\times 1}{\mathbf{v}}$$

We can use matrix algebra to solve for the elements of vector **v** using each eigenvalue.

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

$$\begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -7\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

This produces a system of two equations with two unknowns:

$$\begin{aligned} -3v_1 + 5v_2 &= -7v_1 \\ 4v_1 - 2v_2 &= -7v_2 \end{aligned}$$

Simplifying this, we get

$$\begin{aligned} 4v_1 + 5v_2 &= 0 \\ 4v_1 + 5v_2 &= 0 \end{aligned}$$

The homogeneous set of equations means that there are an infinite number of solutions. The general solution here is to express one variable (say v_2) as a function of the other.

$$v_1 = \theta$$

$$v_2 = -\frac{4}{5}\theta$$

Any set of v_1 and v_2 in which $v_2 = -\frac{4}{5}v_1$ will satisfy this set of equations. Although there are an infinite number of solutions, one that is particularly nice is that whose a sum of squared values is equal to 1.

$$v_1^2 + v_2^2 = 1$$

$$\theta^2 + (-\frac{4}{5}\theta)^2 = 1$$

$$\frac{41}{25}\theta^2 = 1$$

$$\theta^2 = \frac{25}{41}$$

$$\theta = \sqrt{\frac{25}{41}}$$

$$= \frac{5}{\sqrt{41}}$$

Which implies that,

$$v_1 = \frac{5}{\sqrt{41}}$$

$$v_2 = -\frac{4}{\sqrt{41}}$$

And the eigenvector corresponding to the eigenvalue of -7 is

$$\mathbf{v} = \begin{bmatrix} \frac{5}{\sqrt{41}} \\ -\frac{4}{\sqrt{41}} \end{bmatrix}$$

We could verify this by ensuring that the the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ holds:

$$\begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{41}} \\ -\frac{4}{\sqrt{41}} \end{bmatrix} = -7 \begin{bmatrix} \frac{5}{\sqrt{41}} \\ -\frac{4}{\sqrt{41}} \end{bmatrix}$$

We can follow the same process for the second eigenvector which corresponds to the eigenvalue of 2. This produces an eigenvector of

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Which can again be verified.

Diagonalization

Let's place the two eigenvectors we found in a matrix **P**.

$$\mathbf{P} = \begin{bmatrix} \frac{5}{\sqrt{41}} & \frac{1}{\sqrt{2}} \\ -\frac{4}{} & \frac{1}{} \end{bmatrix}$$

$$\left\lfloor -\overline{\sqrt{41}} \qquad \overline{\sqrt{2}} \right\rfloor$$

We can obtain a diagonal matrix **D** such that,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

```
# Create A and P
A = matrix(c(-3, 4, 5, -2), nrow = 2)
P = matrix(c(5/sqrt(41), -4/sqrt(41), 1/sqrt(2), 1/sqrt(2)), nrow = 2)

# Compute D
solve(P) %*% A %*% P
```

```
              [,1] [,2]
[1,] -7.000000e+00    0
[2,]  4.440892e-16    2
```

Note that **D** is a diagonal matrix with the two eigenvalues on the main diagonal. Namely,

$$\mathbf{D} = \begin{bmatrix} -7 & 0 \\ 0 & 2 \end{bmatrix}$$

Recall that we said matrix **A** was a diagonalizable matrix, which essentially boils down to the fact that we can find a matrix **P** (that is invertible) and a diagonal matrix **D** such that

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

We find that **P** is a matrix composed of eigenvectors, and **D** is a diagonal matrix of eigenvalues. This is sometimes called the *eigenstructure* of matrix **A**.

Using R to find the Eigenstructure of a Matrix

We can use the `eigen()` function to compute the eigenvalues and eigenvectors for a diagonalizable matrix.

```
# Compute eigenstructure
eigen_decomp = eigen(A)

# View output
eigen_decomp
```

```
eigen() decomposition
$values
[1] -7  2
```

```
$vectors
           [,1]       [,2]
[1,] -0.7808688 -0.7071068
[2,]  0.6246950 -0.7071068
```

We can compute on this output by coercing each list element into a matrix.

```r
# Create P
P = matrix(eigen_decomp$vectors, nrow = 2)

# View P
P
```

```
           [,1]       [,2]
[1,] -0.7808688 -0.7071068
[2,]  0.6246950 -0.7071068
```

```r
# Create D
D = diag(eigen_decomp$values)

# View D
D
```

```
     [,1] [,2]
[1,]   -7    0
[2,]    0    2
```

```r
# Compute A
P %*% D %*% solve(P)
```

```
     [,1] [,2]
[1,]   -3    5
[2,]    4   -2
```