Assignment 02

Matrix Decomposition

Answer Key

This assignment is worth 15 points. Use the matrix **A** throughout this assignment:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix}$$

Eigendecomposition

Questions 1-5 in this section should be solved by hand.

1. Find the eigenvalues of A.

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\
\det\left(\begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\
\det\left(\begin{bmatrix} 2-\lambda & 6 \\ -1 & 7-\lambda \end{bmatrix}\right) &= 0 \\
(2-\lambda)(7-\lambda) - (-6) &= 0 \\
\lambda^2 - 9\lambda + 20 &= 0 \\
(\lambda - 5)(\lambda - 4) &= 0 \\
\lambda = 4 \quad \text{and} \quad \lambda &= 5
\end{aligned}$$

2. Write the two specific characteristic equations.

$$\det(\mathbf{A} - 4\mathbf{I}) = 0$$

$$\det(\mathbf{A} - 5\mathbf{I}) = 0$$

3. Find the eigenvectors for A.

$$\begin{aligned}
\mathbf{Av} &= \lambda\mathbf{v} \\
\begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
\begin{bmatrix} 2v_1 + 6v_2 \\ -1v_1 + 7v_2 \end{bmatrix} &= \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix} \\
-v_1 + 3v_2 &= 0 \\
v_1 &= 3\theta \\
v_2 &= \theta \\
9\theta^2 + \theta^2 &= 1 \\
\theta^2 &= \frac{1}{10} \\
\theta &= \frac{1}{\sqrt{10}} \\
\mathbf{v} &= \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Av} &= \lambda\mathbf{v} \\
\begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= 5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
\begin{bmatrix} 2v_1 + 6v_2 \\ -1v_1 + 7v_2 \end{bmatrix} &= \begin{bmatrix} 5v_1 \\ 5v_2 \end{bmatrix} \\
-v_1 + 2v_2 &= 0 \\
v_1 &= 2\theta \\
v_2 &= \theta \\
4\theta^2 + \theta^2 &= 1 \\
\theta^2 &= \frac{1}{5} \\
\theta &= \frac{1}{\sqrt{5}} \\
\mathbf{v} &= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
\end{aligned}$$

Compute and verify the following:

4. $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$

$$\mathrm{tr}(\mathbf{A}) = 2 + 7 = 9$$

$$\sum_{i=1}^{n} \lambda_i = 4 + 5 = 9$$

5. $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$

$$\det(\mathbf{A}) = 2(7) - (-1)(6) = 20$$

$$\prod_{i=1}^{n} \lambda_i = 4(5) = 20$$

Use R for Questions 6–8. Show the syntax you used.

6. Verify that P is invertible.

```
# Create P
P = matrix(c(2/sqrt(5), 1/sqrt(5), 3/sqrt(10), 1/sqrt(10)), nrow = 2)

# Check that P has an inverse
solve(P)
```

```
##           [,1]      [,2]
## [1,] -2.236068  6.708204
## [2,]  3.162278 -6.324555
```

```
# Or check if determinant is 0
det(P) == 0
```

```
## [1] FALSE
```

7. Compute D using the eigenvectors you computed in Question 3.

```
# Create A
A = matrix(c(2, -1, 6, 7), nrow = 2)

# Compute D
solve(P) %*% A %*% P
```

```
##               [,1] [,2]
## [1,]  5.000000e+00    0
## [2,] -1.776357e-15    4
```

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$$

8. Verify that $\mathbf{A} = \mathbf{PDP}^{-1}$

```
# Create D
D = matrix(c(5, 0, 0, 4), nrow = 2)

# Compute A
P %*% D %*% solve(P)
```

```
##      [,1] [,2]
## [1,]    2    6
## [2,]   -1    7
```

9. Use the equation for A given in Question 8 and the properties of matrix inverses to express $\mathbf{A}^{-1}$ as a function of D and P.

$$\begin{aligned} \mathbf{A}^{-1} &= (\mathbf{PDP}^{-1})^{-1} \\ &= \mathbf{P}(\mathbf{PD})^{-1} \\ &= \mathbf{PD}^{-1}\mathbf{P}^{-1} \end{aligned}$$

10. Verify the equation you produced in Question 9 using R. Show your syntax.

```
# Inverse of A
solve(A)
```

```
##      [,1] [,2]
## [1,] 0.35 -0.3
## [2,] 0.05  0.1
```

```
# Verify equation
P %*% solve(D) %*% solve(P)
```

```
##      [,1] [,2]
## [1,] 0.35 -0.3
## [2,] 0.05  0.1
```

LU Decomposition

11. Explain why we can we carry out LU (or LUP) decomposition on A by referring to the properties of A.

Since **A** is a square matrix we can carry out LU/LUP decomposition.

12. Write out the system of equations, assuming L is a unit triangular matrix, that allow you to solve for the elements of L and U.

$$\begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

13. Give the L and U matrices.

$$\begin{aligned} 2 &= 1(u_{11}) + 0(0) \\ 6 &= 1(u_{12}) + 0(u_{22}) \\ -1 &= l_{21}(u_{11}) + 1(0) \\ 7 &= l_{21}(u_{12}) + 1(u_{22}) \end{aligned}$$

$$\begin{aligned} l_{21} &= -0.5 \\ u_{11} &= 2 \\ u_{12} &= 6 \\ u_{22} &= 10 \end{aligned}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 2 & 6 \\ 0 & 10 \end{bmatrix}$$

14. Verify that the matrices given in Question 13 produce A using R. Show your syntax.

```
# Create L
L = matrix(c(1, -0.5, 0, 1), nrow = 2)

# Create U
U = matrix(c(2, 0, 6, 10), nrow = 2)

# Verify
L %*% U
```

```
##      [,1] [,2]
## [1,]    2    6
## [2,]   -1    7
```

15. Use the LU decomposition to solve the following system of equations. Solve this by hand.

$$
\begin{aligned}
2x_1 + 6x_2 &= 24 \\
-x_1 + 7x_2 &= -38
\end{aligned}
$$

Step 1

$$
\begin{aligned}
\mathbf{LZ} &= \mathbf{Y} \\
\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} 24 \\ 38 \end{bmatrix} \\
z_1 &= 24 \\
z_2 &= 50
\end{aligned}
$$

Step 2

$$
\begin{aligned}
\mathbf{UX} &= \mathbf{Z} \\
\begin{bmatrix} 2 & 6 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 24 \\ 50 \end{bmatrix} \\
x_1 &= -3 \\
x_2 &= 5
\end{aligned}
$$