

1 The Pennsylvania State University
2 The Graduate School
3 Eberly College of Science

4 **DEVELOPMENT OF SCALABLE APPROACHES TO NEUTRINO MASS**
5 **MEASUREMENT WITH THE PROJECT 8 EXPERIMENT**

6 A Thesis in
7 The Physics Department
8 by
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10 © 2023 Andrew Douglas Ziegler

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²⁰ **Abstract**

²¹ Some shit goes here.

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¹⁴⁶ **Acknowledgments**

¹⁴⁷ Shout out to all the haters.

¹⁴⁸ **Dedication**

¹⁴⁹ Something heartfelt.

¹⁵⁰ **Chapter 1 |**
¹⁵¹ **Introduction and Summary**

¹⁵² **Chapter 2 |**

¹⁵³ **Neutrinos and Neutrino Masses**

¹⁵⁴ **2.1 Introduction**

¹⁵⁵ In this chapter I provide a cursory overview of background information relevant to
¹⁵⁶ neutrinos and neutrino mass measurements.

¹⁵⁷ In Section 2.2 I provide some background information on the history of neutrinos and
¹⁵⁸ beta-decay. In Section 2.3 I describe the discover of neutrino oscillations, which proved
¹⁵⁹ unambiguously that neutrinos have non-zero masses. In Section 2.4 I discuss the current
¹⁶⁰ state of the theoretical understanding of neutrino masses in the standard model. Lastly,
¹⁶¹ in Section 2.5 I discuss methods for measuring the absolute scale of the neutrino mass.

¹⁶² **2.2 Neutrinos and Beta-decay**

¹⁶³ Late in the 19th century the phenomena of radioactivity was first observed in experiments
¹⁶⁴ performed by Henri Becquerel with uranium, and further studied using thorium and
¹⁶⁵ radium by Marie and Pierre Curie. Early work in radioactivity classified different forms
¹⁶⁶ of radiation based on it's ability to penetrate different materials. Rutherford was the first
¹⁶⁷ to separate radioactive emissions into two types, alpha and beta radiation. Alpha rays
¹⁶⁸ can be easily stopped by a piece of paper or thin foil of metal, whereas beta radiation
¹⁶⁹ could penetrate metals several millimeters thick. Later a third form of radiation was
¹⁷⁰ identified by Villard, which was still more penetrating, and was eventually termed gamma
¹⁷¹ radiation by Rutherford.

¹⁷² When these forms of radioactivity were first discovered it was unclear what physically
¹⁷³ constituted an alpha, beta, or gamma particle. Experiments with radioactivity in
¹⁷⁴ magnetic fields was eventually able to identify the charge composition of different forms
¹⁷⁵ of radiation. In particular, experiments by Becquerel identified that beta radiation had

176 an identical charge-to-mass ratio to the electron discovered by Thompson in his work on
177 cathode rays. This was strongly suggestive that beta particles were indeed electrons.

178 Further studies of beta radiation lead to the discovery that radioactivity resulted
179 in the transmutation of elements caused by the decay of a heavier nucleus to a lighter
180 species. One feature of beta radiation, which we now properly call beta-decay, that
181 was different from alpha-decays and gamma radiation is that the electrons produced by
182 beta-decay have a continuous spectrum of kinetic energies, whereas, alpha and gamma
183 particles are emitted with discrete energies. This feature of beta-decay was first observed
184 by Chadwick in 1914, and was extremely puzzling at the time since the continuous
185 spectrum apparently violates energy conservation.

186 Famously, in 1930 Pauli proposed the existence of a new neutral particle, which he
187 termed the "neutron", that was also produced during beta-decay in order to resolve the
188 missing energy problem posed by the beta-decay spectrum. Because this particle carried
189 no charge, it was hypothesized at the time that it had simply not been observed in any
190 experiments up to that time. This "neutron", which was initially estimated to have a
191 mass no larger than that of an electron, was eventually renamed the "neutrino" by Fermi
192 after the discovery of the neutron by Chadwick in 1932. Later, in 1933, Fermi developed
193 a quantum mechanical theory for beta-decay in which both an electron and neutrino are
194 produced by the decay of a neutron to a proton inside the radioactive nucleus.

195 Little more than a speculation when first introduced, indirect evidence for the existence
196 of neutrinos was obtained in 1938 by the simultaneous observation of the electron and
197 recoiling nucleus in cloud chambers by Crane and Halpern. However, it wasn't until the
198 Cowan-Reines experiment in 1956 that direct evidence for the existence of neutrinos
199 was observed by detecting the inverse beta-decays caused by neutrinos from a nuclear
200 reactor interacting with protons contained in water molecules. The difficulty in detecting
201 neutrinos is caused by their weak interactions with other particles. Further, experiments
202 revealed that different types of neutrinos existed based on the nature of the leptons
203 produced in neutrino charged-current interactions, but the existence of a neutrino mass
204 remained an open question that would take more than 40 year to resolve.

205 **2.3 Neutrino Oscillations**

206 The first hint of neutrino flavor transitions or neutrino oscillations was indicated by
207 the solar neutrino problem, which referred to discrepancies between the predicted flux
208 of ν_e from the standard solar model and measurements of the solar neutrino flux such

as the famous experiment at the Homestake mine by Ray Davis Jr. and collaborators in the 1960's. Essentially, fewer electron-type neutrinos than expected were being observed from the sun. Finally, in the early 2000's the SNO experiment was able to resolve the solar neutrino problem by identifying neutrino oscillations as the cause of the observed deficit. Furthermore, measurements of the atmospheric flux of neutrinos by the Super-Kamiokande experiment and others revealed that fewer muon-type neutrinos survived passage through the earth than expected providing strong evidence for neutrino oscillations for both flavors.

The origin of neutrino oscillations is that the weak eigenstates are distinct from the mass eigenstates. The neutrino mass eigenstates represent physical particles in the sense that they are solutions to the free-particle Hamiltonian, whereas, the neutrino weak eigenstates correspond to the neutrino states that interact via the weak charged-current interaction. The neutrino weak eigenstates are a linear superposition of the neutrino mass eigenstates

$$\nu_\ell = \sum_i U_{\ell i} \nu_i, \quad (2.1)$$

where $\ell = e, \mu, \tau$ and $i = 1, 2, 3$. The matrix elements $U_{\ell i}$ are the elements of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix that describes the mixing between the neutrino flavor and mass states.

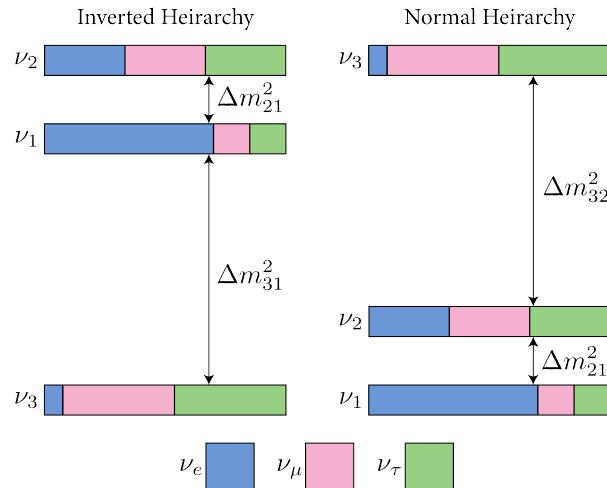


Figure 2.1: A diagram of two different neutrino mass ordering scenarios. In the inverted hierarchy (inverted mass ordering) the lightest neutrino mass is m_3 , whereas, in the normal hierarchy (normal mass ordering) m_1 is the lightest neutrino. What cannot be measured by neutrino oscillations is the neutrino absolute mass scale, which is essentially the mass of the lightest neutrino mass eigenstate.

226 One standard parameterization of the PMNS matrix is

$$\begin{aligned}
U_{PMNS} &= \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.2}
\end{aligned}$$

227 where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. The parameters α_1 and α_2 are only included in the
228 PNMS matrix if neutrinos are Majorana particles, something which represents a current
229 area of research in neutrino physics. The phase δ quantifies the degree of CP-violation
230 in the neutrino sector. Including the Majorana phases the PMNS matrix contains six
231 independent parameters. In addition, neutrino oscillation probabilities depend on the
232 squared mass differences between neutrino mass eigenstates

$$\Delta m_{ij}^2 = m_i^2 - m_j^2, \tag{2.3}$$

233 where $ij = 12, 32, 31$ respectively. Because $\Delta m_{32}^2 = \Delta m_{31}^2 - \Delta m_{21}^2$, this adds an additional
234 two parameters that must be constrained by neutrino oscillations.

235 A giant experimental effort over the past couple of decades has greatly contained the
236 majority of parameters in the PMNS matrix, many to relative uncertainties of only a
237 few percent. However, some parameters still remain relatively unconstrained, which is
238 the origin of the current uncertainty in the ordering of the neutrino masses (see Figure
239 2.1). The neutrino masses can be organized by their relative mass. The current neutrino
240 oscillation data can confirm that $m_2 > m_1$, however, the sign of Δm_{32}^2 is still unknown.
241 This leads to two scenarios where neutrino masses follow the ordering $m_3 > m_2 > m_1$,
242 which is called the normal mass ordering (NMO), or alternatively neutrino masses may
243 be ordered $m_2 > m_1 > m_3$, which is called the inverted mass ordering (IMO). Next-
244 generation neutrino oscillation experiments such as JUNO, Hyper-Kamiokande, and
245 DUNE are poised to resolve this ambiguity in the coming years.

246 Neutrino oscillation probabilities are only sensitive to the neutrino masses via the
247 squared mass differences. Therefore oscillation probabilities are unaffected by the absolute

scale of the neutrino mass. However, oscillations can be used to obtain a lower bound on the neutrino masses by setting the mass of the lightest neutrino mass state to zero. This results in different lower limits depending on the ordering of the neutrino mass states. Current best-fit values with 1σ -uncertainties for the squared mass differences are

$$\Delta m_{21}^2 = (7.42^{+0.21}_{-0.20}) \times 10^{-5} \text{ eV}^2, \quad (2.4)$$

$$\Delta m_{31}^2 = (2.5176^{+0.026}_{-0.028}) \times 10^{-3} \text{ eV}^2 \text{ (NMO)}, \quad (2.5)$$

for the normal mass ordering, and in the case of the inverted ordering we have

$$\Delta m_{32}^2 = (-2.498^{+0.028}_{-0.028}) \times 10^{-3} \text{ eV}^2 \text{ (IMO).} \quad (2.6)$$

By letting the lightest neutrino mass in each ordering scenario (m_{least}) take on a range of values one can visualize the relative masses of the neutrinos as a function of m_{least} (see Figure 2.2).

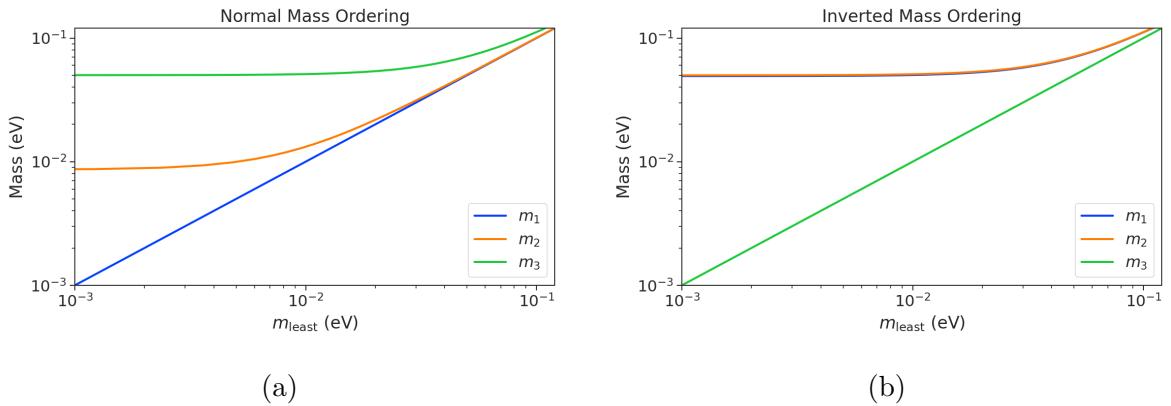


Figure 2.2: The masses of the neutrinos as a function of the lightest neutrino mass in both the normal (a) and inverted (b) mass ordering regimes.

2.4 Neutrino Masses in the Standard Model

Neutrinos are spin 1/2 particles and in modern quantum field theory spin-1/2 particles, or fermions, are described using the Dirac equation.

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0, \quad (2.7)$$

where the field that describes the particle is denoted as $\psi(x)$. In the standard fermions acquire mass through the Yukawa interaction, which add to the standard model Lagrangian terms of the form

$$\mathcal{L}_{\text{Yukawa}} = -Y_{ij}^\ell \bar{L}_{Li} \phi E_{Rj} + \text{h.c.}, \quad (2.8)$$

where Y_{ij}^ℓ is an element of the 3×3 Yukawa coupling matrix for leptons, L_{Li} is the left-handed lepton doublet for generation i , ϕ is the Higgs doublet, and E_{Rj} is the right-handed lepton field for generation j . In the standard model neutrinos are only represented as left-handed neutrinos and right-handed antineutrinos, consistent with experimental observations. Since there are no right-handed neutrino singlet fields and no Yukawa interaction terms for neutrinos are strictly massless, and non-zero neutrino masses is evidence for physics beyond the standard model. For the charged leptons, the Yukawa interaction leads to masses of the form

$$m_{ij}^\ell = Y_{ij}^\ell \frac{v}{\sqrt{2}}, \quad (2.9)$$

where v is the Higgs vacuum expectation value.

The observation of massive neutrinos motivates the extension of the standard model to explain the origin of neutrino masses, which can be approached in different way, but all methods add additional degrees of freedom to the standard model. One approach is to introduce to the standard model a right-handed neutrino field that allows one to introduce Yukawa terms of the form

$$\mathcal{L}_{\nu \text{Yukawa}} = -Y_{ij}^\ell \bar{L}_{Li} \phi \nu_{Rj} + \text{h.c.} \quad (2.10)$$

where ν_{Rj} is the right-handed neutrino singlet. Because experimental evidence strongly predicts only three active neutrinos these additional neutrinos are sterile and do not interact via the strong, weak, or electromagnetic interactions. After spontaneous symmetry breaking, the Yukawa interaction leads to mass terms given by

$$\mathcal{L}_D = -M_{Dij} \bar{\nu}_{Ri} \nu_{Lj} + \text{h.c.}, \quad (2.11)$$

which is called a Dirac mass term. One of the issues with constructing neutrino masses in this way is that the required Yukawa couplings are at least a factor of 10^6 smaller than that of an electron, which begs the question: why are the Yukawa couplings so small for the neutrinos?

An alternative approach is to allow the neutrinos to have a Majorana mass, which is

²⁸⁵ possible because neutrinos are electrically neutral particles. The Majorana mass terms
²⁸⁶ for the neutrino have the form

$$\mathcal{L}_M = -\frac{1}{2}(M_{Rij}\bar{\nu}_{Ri}\nu_{Rj}^c M_{Lij}\bar{\nu}_{Li}\nu_{Lj}^c) + \text{h.c.}, \quad (2.12)$$

²⁸⁷ where M_{Rij} and M_{Lij} are right-handed and left-handed Majorana mass matrices. A
²⁸⁸ consequence of neutrinos being Majorana particles is lepton number violation, which
²⁸⁹ predicts the occurrence of neutrino-less double beta-decay at a rate proportional to the
²⁹⁰ neutrino mass.

²⁹¹ In the most general case neutrinos have both Dirac and Majorana mass terms, which
²⁹² allows one to generate neutrino masses with Yukawa couplings similar to the rest of
²⁹³ the standard model. Considering just one generation of neutrinos for illustration, the
²⁹⁴ combined Lagrangian can be written as

$$\mathcal{L}_{D+M} = -m_D\bar{\nu}_R\nu_L - \frac{1}{2}(m_L\bar{\nu}_L\nu_L^c + m_R\bar{\nu}_R\nu_R^c) + \text{h.c.}, \quad (2.13)$$

²⁹⁵ or equivalently,

$$\mathcal{L}_{D+M} = -\frac{1}{2} \begin{bmatrix} \bar{\nu}_L & \bar{\nu}_R^c \end{bmatrix} \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} \nu_L^c \\ \nu_R \end{bmatrix} + \text{h.c..} \quad (2.14)$$

²⁹⁶ An example mass generation mechanism with this approach is the Type-I see-saw
²⁹⁷ mechanism, in which we take $m_L = 0$ and $m_R \gg m_D$. By diagonalizing Equation 2.14
²⁹⁸ one obtains the mass eigenvalues that represent the physical masses of the neutrinos.
²⁹⁹ The light neutrino mass eigenstate, which represents the observed neutrino mass, has a
³⁰⁰ mass given by

$$m_1 \approx \frac{m_D^2}{m_R}, \quad (2.15)$$

³⁰¹ and the heavy neutrino mass eigenstate, which represents the unobserved sterile neutrino,
³⁰² has a mass

$$m_2 \approx m_R. \quad (2.16)$$

³⁰³ For m_D similar to the other quark or lepton masses, one obtains physical neutrino masses
³⁰⁴ consistent with observations from sterile neutrino masses of $m_R \approx O(10^{15})$ GeV. This
³⁰⁵ mass scale is well beyond the capabilities of modern particle accelerators.

³⁰⁶ 2.5 Neutrino Absolute Mass Scale

³⁰⁷ The neutrino absolute mass scale or simply "neutrino mass" cannot be probed with
³⁰⁸ neutrino oscillations, since oscillation probabilities are determined by the squared mass
³⁰⁹ differences between neutrino mass eigenstates, therefore, alternative techniques are needed
³¹⁰ to perform an effective measurement of the neutrino mass.

³¹¹ 2.5.1 Limits from Cosmology

³¹² In the Λ CDM model, which summarizes our current cosmological understanding of our
³¹³ universe, the mass-energy content of the universe is composed of approximately 27%
³¹⁴ dark matter and only 5% normal matter including neutrinos. From this observation, a
³¹⁵ rough limit on the neutrino mass can be obtained from the condition that neutrinos are
³¹⁶ not responsible for the entirety of the matter content of the universe. Using only this
³¹⁷ condition one can constrain the neutrino mass to be ...

³¹⁸ A prediction of the Λ CDM model is that the universe originated from a single
³¹⁹ expansion event colloquially called the "Big Bang". In the Big Bang scenario, our
³²⁰ universe originated as a hot spacetime singularity, which abruptly experience rapid
³²¹ expansion in a process called inflation. After the inflationary epoch the universe entered
³²² the reheating phase where the potential energy responsible for inflation decays into
³²³ standard model particles such as electrons, quarks, and gluons. The universe continued to
³²⁴ expand in size resulting in a decrease in energy density and lower temperature. Eventually
³²⁵ the temperature of the universe decreased enough to allow the formation of protons,
³²⁶ neutrons, and other baryons from quarks and gluons produced from the decays of the
³²⁷ inflationary fields.

³²⁸ Also produced during the Big Bang are electrons, neutrinos and other leptons as
³²⁹ well as a population of photons. These particles are kept in thermal equilibrium with
³³⁰ the rest of the quark-gluon plasma through interactions that take place at the high
³³¹ temperatures and densities of the early universe. However, as the universe continues
³³² to expand it's density and temperatures decreases leading to the eventual decoupling
³³³ of photons and leptons from the quarks and gluons. A prediction of inflation is that
³³⁴ this population of photons produced during the Big Bang should still be present, but
³³⁵ with a significantly reduced temperature due to the expansion of the universe. This is
³³⁶ consistent with the observation of the CMB (cosmic microwave background), which is a
³³⁷ population of microwave radiation with a blackbody temperature of 2.7 K. The CMB
³³⁸ is extremely uniform in all directions with slight anisotropies that can be analyzed to

study the evolution of the early universe. A series of experiments have measured the CMB with increasing levels of precision, which has lead to a significant increase in our current understanding of cosmology.

In addition to the CMB, inflation predicts the existence of a $C\nu B$ (cosmic neutrino background), which are the remnant neutrinos produced during the Big Bang. Since neutrinos only interact via the weak force, they decouple from the hot Big Bang plasma at an earlier time than the CMB radiation. The temperature at which the $C\nu B$ decouples depends on the neutrino rest mass. Neutrinos play a somewhat unique role in the Λ CDM model due to the fact that neutrinos act as radiation early in the universe but as matter in the late universe. This leads to unique signatures that impact anisotropies of the CMB as well as the distribution of matter in the universe. By combining measurements of the CMB with measurements of the large-scale structure (LSS) of the universe one can constrain the neutrino mass scale by fitting these datasets with the Λ CDM model. This analysis results in some of the most stringent constraints on the neutrino mass. A recent analysis was able to constrain the neutrino mass scale to

$$\Sigma_{m_\nu} \equiv \sum_i m_i < 0.12 \text{ eV}, \quad (2.17)$$

where m_i are the neutrino mass eigenstates.

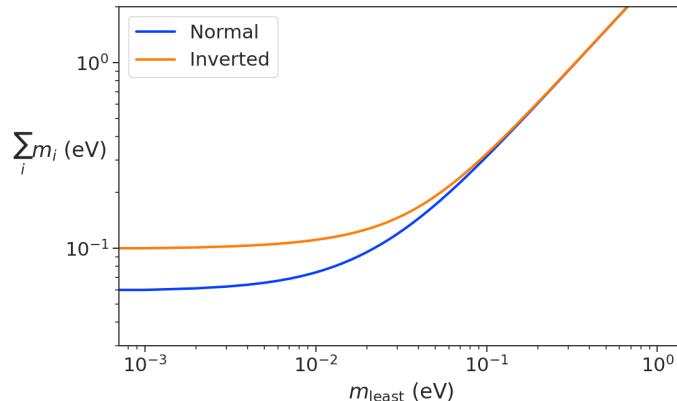


Figure 2.3: The neutrino mass observable measured by cosmology as a function of the lightest neutrino mass eigenstate.

The observable Σ_{m_ν} constrains the neutrino mass by setting the mass of the lightest neutrino mass eigenstate (m_{least}). In the normal mass ordering Σ_{m_ν} can be rewritten in

357 the form

$$\Sigma_{m_\nu} = m_{\text{least}} + \sqrt{\Delta m_{21}^2 + m_{\text{least}}^2} + \sqrt{\Delta m_{32}^2 + m_{\text{least}}^2}, \quad (2.18)$$

358 where it is clear that a measurement of Σ_{m_ν} effectively sets the neutrino mass scale
 359 through m_{least} . The analogous formula for the inverted mass ordering is

$$\Sigma_{m_\nu} = m_{\text{least}} + \sqrt{-\Delta m_{32}^2 + m_{\text{least}}^2} + \sqrt{-\Delta m_{31}^2 + m_{\text{least}}^2}. \quad (2.19)$$

360 In figure 2.3 we plot the observable Σ_{m_ν} as a function of m_{least} .

361 Upcoming experiments are planned to refine measurements of the CMB, LSS, and
 362 other cosmological observables. With this additional data it is possible that in the
 363 near future cosmological measurements will be able to positively constrain the neutrino
 364 absolute mass scale. However, the strength of these limits strictly depend on the accuracy
 365 of the Λ CDM model, which highlights the need for direct experimental measurements of
 366 the neutrino mass to confirm the predictions of cosmology and to fix the neutrino mass
 367 parameter in future cosmological analyses.

368 2.5.2 Limits from Neutrinoless Double Beta-decay Searches

369 If neutrinos are Majorana fermions then the neutrino is equivalent to its own antiparticle
 370 and lepton conservation is not an exact law of nature. Searches for lepton number
 371 violation, specifically the neutrinoless double beta-decay ($0\nu\beta\beta$) process, are some of the
 372 most powerful tests of lepton number conservation, which depend on the neutrinos being
 373 Majorana fermions. In double beta-decay two neutrons contained in the decay species
 374 nucleus spontaneously decay into two protons resulting in the production of two electrons
 and two neutrinos (see Figure 2.4). However, for $0\nu\beta\beta$ the two neutrinos self-annihilate

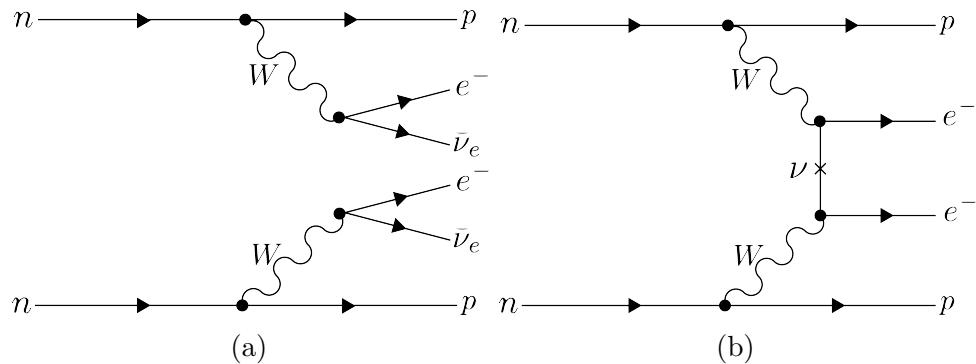


Figure 2.4: Feynman diagrams for double beta-decay (a) and $0\nu\beta\beta$ (b).

375

376 during the decay resulting only in the production of two electrons and a violation of
 377 lepton number by two.

378 Assuming that the exchange of two Majorana neutrinos is the dominant channel for
 379 $0\nu\beta\beta$, then a measurement of the $0\nu\beta\beta$ half-life for a particular isotope can be used to
 380 set the neutrino absolute mass scale. The half-life is written in terms of the effective
 381 neutrino mass for $0\nu\beta\beta$ ($m_{\beta\beta}$) using the equation

$$T_{1/2}^{0\nu} = \frac{1}{G|\mathcal{M}|^2 m_{\beta\beta}^2}, \quad (2.20)$$

382 where G is the phase-space factor for the decay and \mathcal{M} is the relevant nuclear matrix
 383 element. $m_{\beta\beta}$ is given by an incoherent sum of the neutrino mass eigenstates weighted
 384 by the PMNS mixing matrix parameters,

$$m_{\beta\beta} = \left| \sum_i U_{ei}^2 m_i \right|. \quad (2.21)$$

385 The information provided from $0\nu\beta\beta$ on the neutrino mass scale can be visualized
 386 by expressing the value of $m_{\beta\beta}$ in terms of m_{least} and two relative Majorana phases.
 387 The allowed regions for $m_{\beta\beta}$ as a function of m_{least} are shown in Figure 2.5 as the
 388 regions bounded by the black curves overlayed with the discovery probabilities of future
 0 $\nu\beta\beta$ decay experiments based on current neutrino data. Because of the possibility

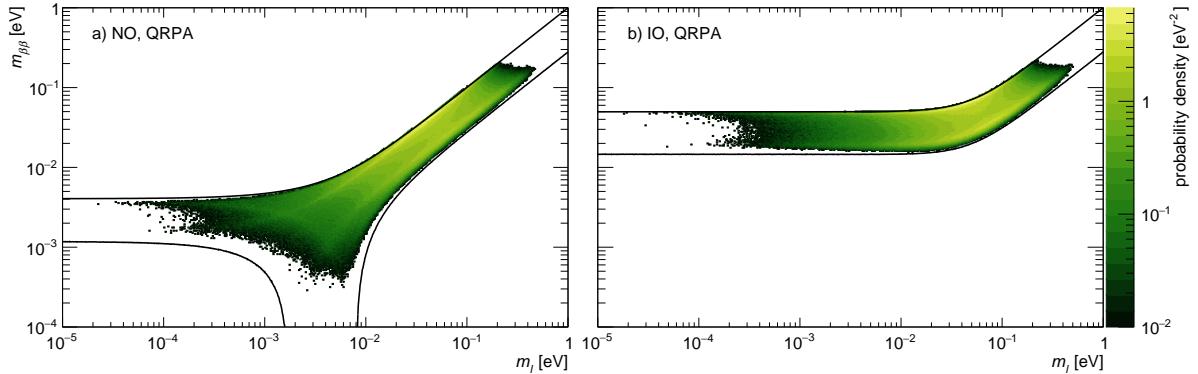


Figure 2.5: The discovery probabilities for the future generation of $0\nu\beta\beta$ experiments as a function of $m_{\beta\beta}$ and m_{least} .

389
 390 of cancellation due to the unknown Majorana phases included in the sum specified by
 391 Equation 2.21, the information gained is necessarily imperfect. Additionally, theoretical
 392 uncertainty in the calculation of the nuclear matrix elements complicates the calculation
 393 of $m_{\beta\beta}$ from a measurement of $0\nu\beta\beta$ half-life. Similar to cosmology there is a high degree

³⁹⁴ of complementarity between direct measurements of the neutrino mass and $0\nu\beta\beta$. In
³⁹⁵ particular, a measurement of m_{least} to less than than 0.1 eV sensitivity provides significant
³⁹⁶ information for $0\nu\beta\beta$ searches based of the discovery probabilities of Figure 2.5.

³⁹⁷ 2.5.3 Limits from Beta-decay

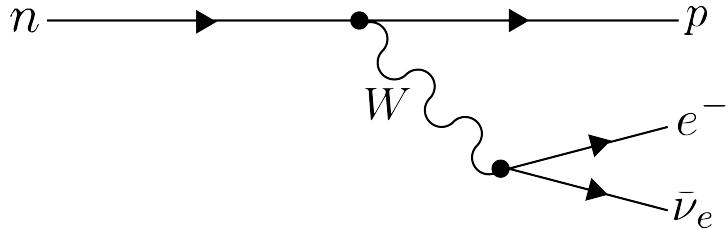


Figure 2.6: A Feynman diagram of beta decay

³⁹⁸ Certain processes involving neutrinos, in particular beta-decay (see Figure 2.6), have
³⁹⁹ initial states with well-defined total energies and final states where the kinetic energies
⁴⁰⁰ of each of the particles can be measured with high accuracy and precision. Beta-decay
⁴⁰¹ involves the decay of an unstable isotope where a neutron spontaneously converts to
⁴⁰² a proton and emits and electron and anti-neutrino ("neutrino" for brevity) to conserve
⁴⁰³ charge and lepton number. Therefore, by applying the principles of energy and momentum
⁴⁰⁴ conservation a measurement of the kinematics of the final state can be used to constrain
⁴⁰⁵ the neutrino mass as proposed by Fermi in his 1934 description of nuclear beta-decay
 (see Figure 2.7). Because the constraint on the neutrino mass from beta-decay depends

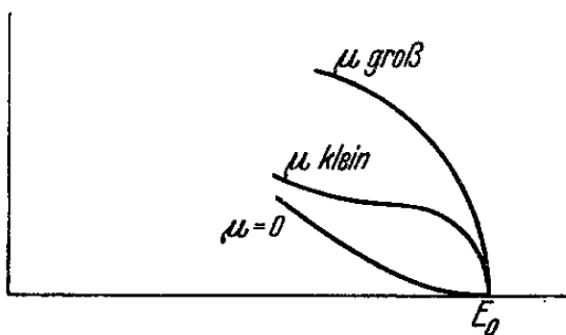


Figure 2.7: A figure from Fermi's 1934 paper on a theory of beta-decay depicting the kinetic energy spectrum of the emitted electron. The effect of the neutrino mass, written as μ , is to distort the shape of the spectrum near the endpoint from the zero-mass spectrum.

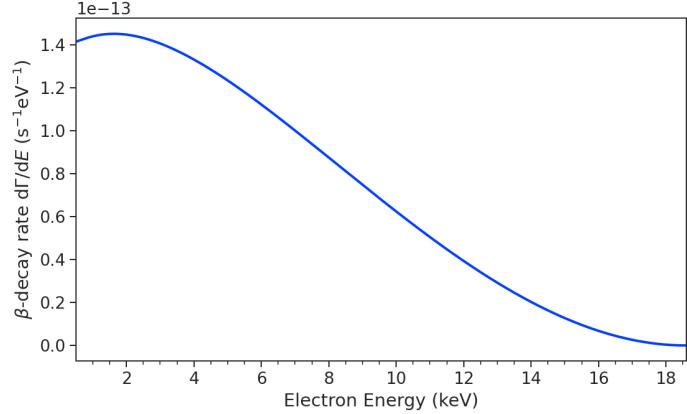
⁴⁰⁷ only on the final state measurement capabilities, such measurements of the neutrino mass
⁴⁰⁸ are often referred to as model-independent or direct in contrast to constraints on the
⁴⁰⁹ neutrino mass from cosmology and $0\nu\beta\beta$.

⁴¹⁰ The beta-decay isotope of choice for direct neutrino mass measurements with beta-
⁴¹¹ decay has been tritium (3H_2) for many decades, because it conveniently fulfills many
⁴¹² experimental requirements. Of upmost importance is a decay with a low Q-value, which
⁴¹³ is the available kinetic energy based on the mass difference between the initial and final
⁴¹⁴ states. The effect of a massive neutrino on the shape of the spectrum is magnified for
⁴¹⁵ low Q-values and tritium decays have an unusually low Q-value of 18.6 keV. Additionally,
⁴¹⁶ tritium beta-decay is a super-allowed decay, which means that it has a relatively short
⁴¹⁷ half-life of 12.3 years making it easy to obtain a high-activity source with a relatively
⁴¹⁸ small source mass. High-activity is desireable due to the low-activity near the tritium
⁴¹⁹ spectrum endpoint. For tritium beta-decays only a factor of 3×10^{-13} of the decays
⁴²⁰ occur in the last 1 eV of the spectrum. Isotopes with Q-values lower than tritium are
⁴²¹ known, but this is outweighed by exceedingly long half-lives leading to unobtainable
⁴²² source masses.

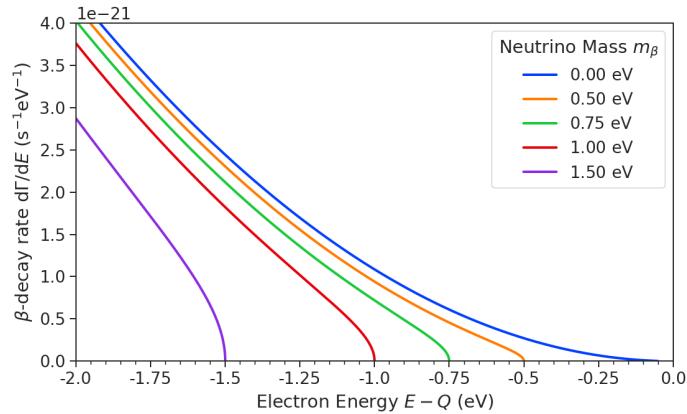
⁴²³ Neutrino mass measurements using beta-decay measure the effect of the neutrino's
⁴²⁴ mass on shape of the electron's kinetic energy spectrum near the endpoint. The kinetic
⁴²⁵ energy spectrum (see Figure ??) is given by

$$\frac{d\Gamma}{dE} = \frac{G_F^2 |V_{ud}|^2}{2\pi^3} (G_V^2 + 3G_A^2) F(Z, \beta) \beta (E + m_e)^2 (E_0 - E) \times \sum_{i=1,2,3} |U_{ei}|^2 [(E_0 - E)^2 - m_i^2]^{1/2} \Theta(E_0 - E - m_i), \quad (2.22)$$

⁴²⁶ where G_F is the Fermi coupling constant, V_{ud} is an element of the CKM matrix, E is
⁴²⁷ the kinetic energy of the electron, β is the velocity of the electron divided by the speed
⁴²⁸ of light, E_0 is the endpoint energy assuming zero neutrino mass, $F(Z, \beta)$ is the Fermi
⁴²⁹ function, and $\Theta(E_0 - E - m_i)$ is the Heaviside function, which enforces energy conservation.
⁴³⁰ One can see that the decay spectrum is actually a combination of three spectra with
⁴³¹ different endpoints based on the actual values of the neutrino mass eigenstates, m_i . This
⁴³² results in "kinks" in the spectrum shape due to the overlapping spectra but such an
⁴³³ effect would be nearly impossible to resolve given the finite energy resolution of a real
⁴³⁴ experiment.



(a)



(b)

Figure 2.8: The tritium beta-decay spectrum. The affect of a massive neutrino on the spectrum is to change it's shape near the endpoint by an amount proportional to the size of the neutrino mass. This suggests that a sufficiently high-statistic and high-resolution measurement of the spectrum endpoint would be able to measure the neutrino mass.

435

The neutrino mass scale variable measured by beta-decay is given by

$$m_\beta^2 = \sum_i |U_{ei}|^2 m_i^2, \quad (2.23)$$

436 where m_{beta} is the effective mass of the neutrino in beta-decay or simply neutrino mass
 437 for brevity. By assuming unitarity, the neutrino mass can be expressed in terms of
 438 the PMNS matrix elements, squared mass differences, and the lightest neutrino mass

⁴³⁹ eigenstate. For the normal mass ordering the equation is

$$m_\beta^2 = m_{\text{least}}^2 + |U_{e2}|^2 \Delta m_{21}^2 + |U_{e3}|^2 \Delta m_{31}^2, \quad (2.24)$$

⁴⁴⁰ and for the inverted ordering the equation is

$$m_\beta^2 = m_{\text{least}}^2 + |U_{e1}|^2 (-\Delta m_{32}^2 - \Delta m_{21}^2) + |U_{e2}|^2 (-\Delta m_{32}^2). \quad (2.25)$$

⁴⁴¹ Therefore, a measurement of the neutrino mass in combination with neutrino mixing
⁴⁴² parameters is effectively a measurement of m_{least} .

⁴⁴³ Since the neutrino mass is small (< 1 eV), it's effect on the spectrum is limited to
⁴⁴⁴ the endpoint region. The affect of a non-zero neutrino mass on the endpoint spectrum is
⁴⁴⁵ plotted for the reader in Figure 2.8. Resolving the small changes in the spectrum shape
⁴⁴⁶ requires an experimental technique with high statistics, excellent energy resolution, and
⁴⁴⁷ low background activity.

⁴⁴⁸ The KATRIN collaboration, utilizing a large MAC-E (magnetic adiabatic collimation
⁴⁴⁹ with electrostactic) filter spectrometer recently obtained the best direct measurement of
⁴⁵⁰ the neutrino mass, with a 90% confidence upper limit of 0.8 eV. With more statistics the
⁴⁵¹ KATRIN collaboration estimates an ultimate sensitivity to neutrino masses of 0.2 eV.

452 Chapter 3

453 Direct Measurement of the Neutrino Mass 454 with Cyclotron Radiation Emission Spec- 455 troscopy

456 3.1 Introduction

457 A promising technique for direct measurements of the neutrino mass beyond the projected
458 limit of the ongoing KATRIN experiment is tritium beta-decay spectroscopy with an
459 atomic tritium source [1]. Atomic tritium, combined with a large-volume, high-resolution
460 energy measurement technique, is capable of measuring the neutrino mass with sensitivity
461 below the 50 meV limit allowed by neutrino oscillations.

462 Cyclotron Radiation Emission Spectroscopy or CRES is a high-resolution energy
463 measurement technique compatible with atomic tritium production and storage that can
464 enable the next-generation of neutrino mass direct measurement experiments [2]. The
465 Project 8 collaboration is currently engaged in a program of research and development
466 (R&D) aimed at developing the technology necessary for a 40 meV sensitivity measurement
467 of the neutrino mass using CRES and atomic tritium [3].

468 In Section 3.2 we provide an introduction to the basics of the CRES technique
469 as well as the goals of the Project 8 experiment. Additionally, we sketch out the
470 phased experiment development plan being implemented by Project 8 to build towards a
471 next-generation neutrino mass experiment.

472 In Section 3.3 we give a brief overview of Phase II of the Project 8 experiment [4, 5],
473 which completed early in 2023. Although the bulk of the work presented in this thesis is
474 relevant to designs of future Project 8 experiments, a description of the work in Phase II
475 provides useful context for the rest of the work.

476 In Section 3.4 we introduce a CRES measurement concept based on antenna arrays [6],

477 which could be the basis for the ultimate Project 8 neutrino mass experiment. A significant
478 portion of the R&D efforts of Project 8 in Phase III were directed towards simulating
479 and modeling this experimental concept in order to understand the achievable sensitivity
480 to the neutrino mass.

481 Lastly, in Section 3.5 we introduce conceptual designs of pilot-scale experiments that
482 combine atomic CRES with a large-volume CRES detection technique. This includes a
483 design concept for an antenna array based experiment, but also a design for a resonant
484 cavity based experiment. Resonant cavities are discussed in more depth in Chapter 6
485 and have become the preferred choice for future CRES experiments in Project 8 over
486 antenna arrays.

487 **3.2 CRES and Project 8**

488 **3.2.1 Cyclotron Radiation Emission Spectroscopy — CRES**

489 Of the standard physical quantities the one that can be measured with the highest
490 precision is time and the inversely related quantity frequency. In fact it is often advan-
491 tageous to convert measurements of other physical quantities like mass or length into
492 frequency measurements due to the digital nature of frequency measurements that make
493 them immune to many sources of noise. Atomic clocks, which operate by measuring the
494 frequencies of various atomic transitions, have been used to measure time with astounding
495 relative uncertainties of 10^{-18} seconds. The extreme precision possible with frequency
496 measurements is often summarized using the a quote from the Physicist Arthur Schawlow
497 who said advise his students to "Never measure anything but frequency!".

498 Neutrino mass measurements using tritium beta-decay require us to measure a
499 perturbation of the 18600 eV tritium endpoint to precisions as low as 0.1 eV, therefore, a
500 spectroscopic technique with extremely high resolution is required for this measurement.
501 Part of the reason that frequency measurements are capable of such high resolutions is
502 that they are essentially counting measurements, which average the number of oscillations
503 of a physical system over time. By observing a rapidly oscillating system over a sufficient
504 length of time one can obtain essentially arbitrary precision on a frequency limited only
505 by the time available for measurement and the SNR of the system.

506 In order to perform frequency-based high-resolution spectroscopy of the tritium beta-
507 decay spectrum one needs to translate the kinetic energy of the electron into a frequency.
508 The simplest way to accomplish this is to place a gaseous supply of tritium into a magnetic

509 field. When one of the atoms decays the resulting electron will immediately begin to
 510 orbit around a magnetic field line at the cyclotron frequency which is proportional to
 511 its kinetic energy (see Figure 3.1). The acceleration caused by the orbit leads to the
 512 emission of cyclotron radiation that can be detected using an array of antennas or a
 513 different RF sensor such as a resonant cavity. The frequency of the radiation gives the
 514 electron's kinetic energy, which is used to build the beta-decay spectrum and measure
 515 the neutrino mass. The name for this measurement technique is Cyclotron Radiation
 516 Emission Spectroscopy or CRES.

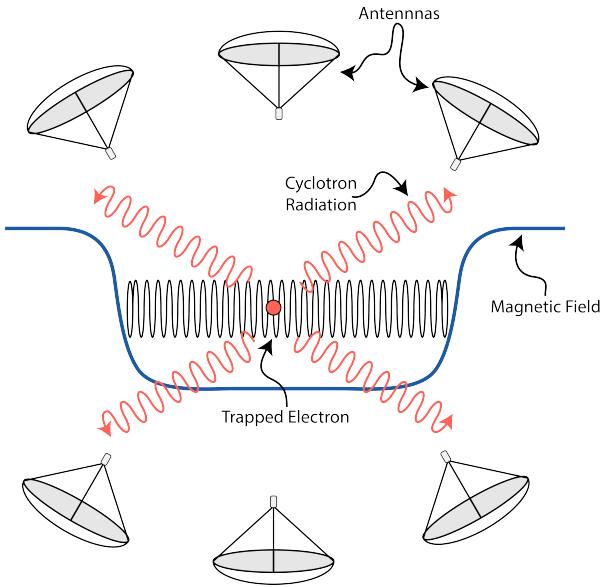


Figure 3.1: A cartoon illustration of the CRES technique. An electron is contained in a magnetic trap so that it's cyclotron radiation can be detected by an array of antennas. Detecting the cyclotron radiation allows us to measure its cyclotron frequency and determine its kinetic energy.

517 For non-relativistic particles the cyclotron frequency is only a function of the charge-
 518 to-mass ratio of the particle, however, from the relativistic form of the cyclotron frequency

$$f_c = \frac{qB}{2\pi m_e \gamma} = \frac{1}{2\pi} \frac{qB}{m_e + E_{\text{kin}}/c^2}, \quad (3.1)$$

519 one can see that the kinetic energy (E_{kin}) of the electron is directly proportional to the
 520 inverse of the cyclotron frequency (f_c). Electrons with kinetic energies of 18.6 keV are in
 521 the weakly relativistic regime with $\beta = \frac{v}{c} = 0.263$ and $\gamma = 1.036$.

522 The required frequency resolution needed for neutrino mass measurement can be

523 obtained by differentiating Equation 3.1,

$$\frac{df_c}{dE_{\text{kin}}} = \frac{1}{2\pi} \frac{-qBc^2}{(m_e c^2 + E_{\text{kin}})^2}, \quad (3.2)$$

524 from which we can obtain the relationship between fractional differences in energy and
525 frequency,

$$\frac{df_c}{f_c} = \frac{1 - \gamma}{\gamma} \frac{dE_{\text{kin}}}{E_{\text{kin}}}. \quad (3.3)$$

526 Therefore, an energy precision of 1 eV for an 18.6 keV electron requires a frequency
527 precision of approximately 2 ppm.

528 The minimum observation time required to achieve this resolution can be estimated
529 using the uncertainty principle as formulated by Gabor. Electron's from tritium beta-
530 decay experience random collisions with the background gas particles, which limits the
531 uninterrupted radiation lifetime. The time between collision events, referred to as track
532 length in the context of CRES measurements, is an exponentially distributed variable.
533 Differences in the track lengths of a population of mono-energetic electrons leads to
534 uncertainty or broadening in the distribution of measured frequencies proportional to
535 the mean track length, τ_λ . The resulting frequency distribution has a Lorentzian profile,
536 whose width is given by the Gabor limit,

$$\tau_\lambda \Delta f_c = \frac{1}{2\pi} \implies \Delta f_c = \frac{1}{2\pi\tau_\lambda}. \quad (3.4)$$

537 The cyclotron frequency for a 18.6-keV electron in a 1 T field is approximately
538 27 GHz, from which one can estimate the minimum observation time for 2 ppm frequency
539 resolution at approximately 3 μ sec. The Gabor limit is not the true lower bound on the
540 frequency resolution for a CRES signal, since it is based on the details of the Fourier
541 representation of a time-series with a fixed length. If one takes the approach of fitting the
542 CRES signal in the time-domain, then one finds that the limit on frequency precision is
543 given by the Cramér-Rao lower bound (CRLB), which depends on both the track length
544 as well as the SNR. In general, the CRLB allows for better precision on the cyclotron
545 frequency, however, the Gabor limit provides an illustrative limit with the correct order
546 of magnitude.

547 Ensuring that an electron remains under observation long enough so that it's frequency
548 can be properly measured requires a magnetic trap. A magnetic trap is a local minimum
549 in a background magnetic field generated an appropriate configuration of electromagnetic
550 coils. Since magnetic fields can do no work, there is no danger of the magnetic trap

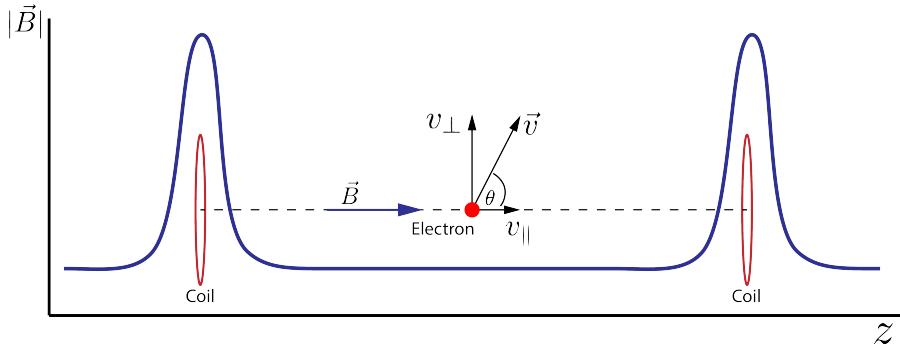


Figure 3.2: An illustration of an electron in a bathtub magnetic trap generated by two well-separated coils.

affecting the kinetic energy electron after it is emitted from the beta-decay. One common approach to creating a magnetic trap is the "bathtub" trap configuration, which in its simplest form consists of two high magnetic field pinch coils aligned on a central axis that are well separated (see Figure 3.2). This configuration produces a trap with a flat uniform bottom and relatively steep walls, which is ideal for CRES measurements.

Electrons produced in the trap oscillate back and forth between the trap walls at a frequency that depends upon the pitch angle, unless they are produced with pitch angles too small to be contained in the trap. Pitch angle is defined as the angle between the component of the electron's velocity perpendicular to the magnetic field and the component parallel to the magnetic field,

$$\tan \theta = \frac{v_{\perp}}{v_{\parallel}}. \quad (3.5)$$

The axial motion of the electron leads to variation in the cyclotron frequency due to the changing value of the magnetic fields. This leads to frequency modulation that generates sidebands in the cyclotron radiation spectrum. Resolving these sideband frequency components is necessary for a complete reconstruction of the CRES signal in the experiment.

Electrons trapped in a cylindrically symmetric trap have three primary components of motion (see Figure 3.3). The dominant component, typically with the highest frequency, is the electron's cyclotron orbit, which encodes information on the electron's kinetic energy. Axial motion from the electron's pitch angle leads to frequency modulation but also a shift in the average magnetic field experienced by an electron. This leads to a correlation between the kinetic energy of the electron and the pitch angle depending on the particular shape of the magnetic trap, which can negatively impact energy resolution.

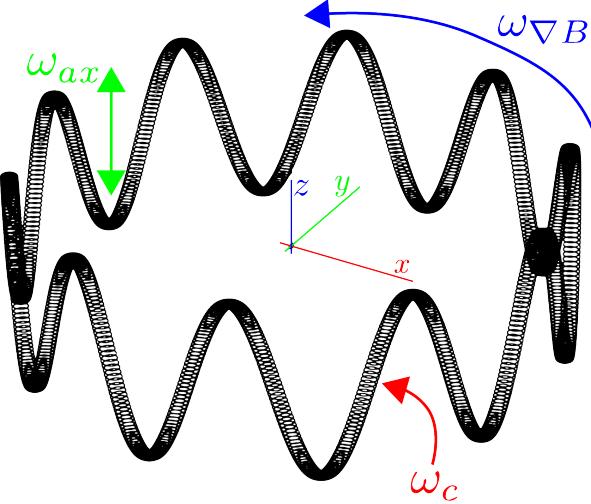


Figure 3.3: A plot of the main components of an electron's trajectory in a cylindrically symmetric trap.

To reduce this correlation one must engineer the trap to have a flat bottom with very steep wall both of which are more easily achieved with a small aspect ratio bathtub trap. Radial gradients in the trap oftentimes leads to a third component of motion called grad-B drift. The equation for the drift velocity is

$$\mathbf{v}_{\nabla B} = \frac{m_e v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla B}{B^2}. \quad (3.6)$$

These additional components of motion all influence the shape of the CRES signal so modeling their effects is critical to proper measurement of the kinetic energy.

The total power of the radiation emitted by an electron in a free-space environment is given by the Larmor equation

$$P(\gamma, \theta_p) = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 \omega_c^2}{c} (\gamma^2 - 1) \sin^2 \theta_p, \quad (3.7)$$

where ω_c is the cyclotron frequency multiplied by 2π and θ_p is the pitch angle to distinguish it from the spherical angle coordinate. A single electron with a 90° pitch angle and 18.6 keV of kinetic energy in a 1 T magnetic field emits a total radiation power of 1.2 fW, which is quite small compared with typical RF systems, furthermore, one is typically only able to receive a fraction of this total power with an antenna or other detection system. Therefore, RF systems in CRES experiments must be operated at cryogenic temperatures to limit the noise power such that adequate SNR can be achieved for signal detection and reconstruction. Alternatively, longer tracks enable detection of weaker

589 signals due to the increase in the total signal energy available for the detection algorithm.

590 **3.2.2 The Project 8 Collaboration**

591 The Project 8 collaboration is a group of institutions in the United States and Germany
592 aiming to measure the neutrino mass by developing a novel spectrometer technology
593 based on CRES. In the ultimate Project 8 experiment the CRES technique will be used
594 to measure the beta-decay spectrum using a large source of atomic tritium sufficient to
595 achieve the required statistics in the last $O(10)$ eV of the decay spectrum. Project 8 is
596 targeting a neutrino mass sensitivity below 50 meV, which exhausts the range of possible
597 neutrino masses under the inverted hierarchy and is a factor of four less than sensitivity
598 projections for the ongoing KATRIN experiment.

599 Project 8's proposed experiment requires the development of two novel technologies:
600 the production and trapping of a source of atomic tritium on cubic-meter scales and
601 technology to enable CRES measurements of individual electrons in the same volume.

602 **Atomic Tritium**

603 Previous measurements of the tritium beta-decay spectrum for neutrino mass measure-
604 ments have all relied on a sources of molecular tritium for their measurements due to the
605 numerous practical and technical challenges associated with the production and storage
606 of hydrogen isotopes.

607 To produce atomic hydrogen one must supply sufficient energy to the tritium molecule
608 to break the molecular bond between. Common approaches to this include the use of hot
609 coaxial filament atom crackers as well as plasma atom sources. Both approaches heat the
610 tritium atoms to temperatures > 2500 K, which must then be cooled to temperatures
611 on the order of a few mK so that the tritium atoms can be trapped. Cooling the atoms
612 requires the construction of a large tritium infrastructure and cooling system that can
613 supply a source of cold atoms to the trap.

614 Once cold tritium atoms are produced they cannot make contact with any surfaces
615 to avoid recombination of the atoms to molecules. Therefore, a magnetic trap is required
616 to store the atoms for a sufficient length of time that they have a chance to decay before
617 escaping the trap. Trapping the atoms at this scale requires the construction of a large
618 and complex magnet system that must be cooled to cryogenic temperatures to avoid
619 heating of the atoms, which leads to their escape from the trap.

620 The significant experimental complexity caused by atomic tritium makes a molecular

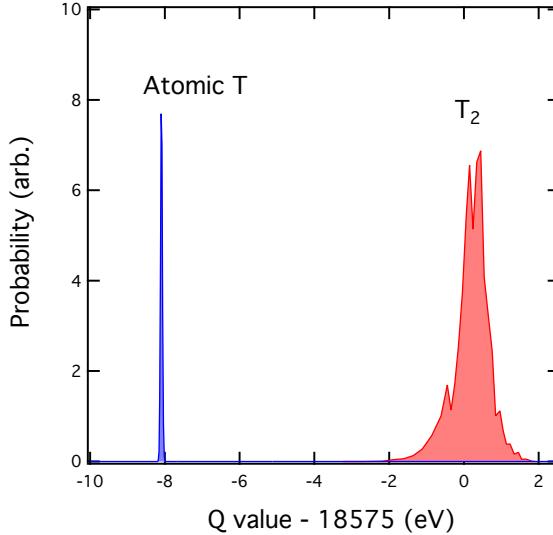


Figure 3.4: A plot of the final state distributions of atomic and molecular tritium. The final state distribution provides the primary contribution to the width of the molecular spectrum whereas thermal doppler broadening is responsible for the width of the atomic spectrum.

source the obvious choice from practical considerations. However, the drawback of molecular tritium for neutrino mass measurement is the irreducible broadening in the electron's kinetic energy due to the final state spectrum of molecular tritium (see Figure 3.4). The broadening of the final state spectra has a RMS amplitude of 436 meV caused by variation in the final vibrational state of the daughter molecule. For atomic tritium the primary sources of broadening in the final state spectrum are magnetic hyperfine splittings ($O(10^{-5})$ eV) and thermal Doppler broadening caused by the motion of the trapped atom. For atomic tritium at a temperature of 1 mK thermal broadening is the dominant contribution, providing about 1 meV RMS of broadening to the electron's kinetic energy.

The larger energy broadening with molecular tritium leads to an irreducible statistical uncertainty that limits the achievable sensitivity to approximately 100 meV at 90% confidence. For previous direct measurements of the neutrino mass this uncertainty is an insignificant contribution to the overall uncertainty budget, however, for experiments like Project 8 atomic tritium is a key component to the success of the experiment.

636 CRES for Neutrino Mass Measurement

637 Several promising features of the CRES technique make it a particularly attractive choice
638 for a next generation neutrino mass measurement experiment. For example, with a CRES

639 experiment the volume of the source gas can be the same as the volume of the CRES
640 spectrometer. This is due to the fact that CRES is a remote-sensing technique that can
641 observe the energy of the electron without altering its trajectory or directly interacting
642 with the electron. Given that tritium gas is transparent to cyclotron radiation the kinetic
643 energies of electrons can be measured with an appropriate sensing technology, such as a
644 cavity or antenna array, located directly outside the atom trapping volume.

645 The current state-of-the-art tritium beta-decay spectroscopy experiment, KATRIN,
646 utilizes the magnetic adiabatic collimation with an electrostatic filter (MAC-E filter)
647 technique to measure the beta-decay spectrum of molecular tritium. In this approach, a
648 source of molecular tritium is located outside of the spectrometer. When a beta-decay
649 occurs the electron must exit the tritium source and travel through the MAC-E filter
650 before it can be detected on the other side of the filter using a charge sensor. With this
651 approach the measurement statistics are limited by the transverse areas of the tritium
652 source and MAC-E filter due to the need to travel through the detector without scattering.
653 This scaling is less favorable than the volumetric scaling that one has with CRES due to
654 the ability to co-locate source and detector.

655 Another promising aspect of the CRES technique is the inherently high precision
656 of frequency based measurements. The endpoint of the molecular tritium beta-decay
657 spectrum is approximately 18.6 keV, which dwarfs the neutrino mass scale of $< 1 \text{ eV}/c^2$
658 by at least a factor of 10^5 . Measuring the effect of such a small mass on a high energy
659 electron requires excellent energy resolution. Since frequency measurements are essentially
660 counting measurements they are intrinsically quite accurate due to the ability to measure
661 the cyclotron frequency by effectively averaging over millions of cyclotron orbits. Using
662 off-the-shelf RF components its is possible to achieve part-per-million accuracy on the
663 kinetic energy with the CRES technique.

664 A final aspect of the CRES technique that is attractive for a next-generation experi-
665 ment is the relative immunity to backgrounds. Since CRES operates via non-destructive
666 measurements of the electron's cyclotron frequency potential sources of background elec-
667 trons are effectively filtered out by limiting the frequency bandwidth of the measurement.
668 The fiducial volume of the experiment is free from any surfaces that could introduce
669 stray electrons and electrons from sources outside the fiducial volume can be prevented
670 from entering the experiment.

671 **Neutrino Mass Sensitivity Goals**

672 Project 8's ultimate goal is to combine CRES with atomic tritium to measure the neutrino mass with 40 meV sensitivity at the 90% confidence level (see Figure 3.5). This sensitivity

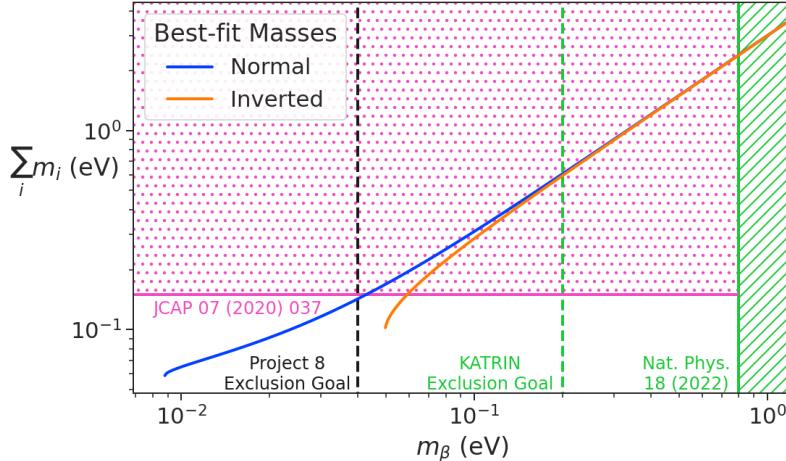


Figure 3.5: Neutrino mass exclusion plot including limits from cosmological measurements and the KATRIN experiment. Allowed ranges for neutrino masses under the normal and inverted hierarchies are shown as the blue and orange lines respectively. The black dashed line shows Project 8's goal neutrino mass sensitivity for the Phase IV experiment.

673
674 is sufficient to fully exhaust the range of allowable neutrino masses under the inverted
675 neutrino mass ordering regime and is approximately an order of magnitude less than the
676 projected final sensitivity of the KATRIN experiment. Excluding the full neutrino mass
677 parameter space would require a sensitivity an order of magnitude lower than what is
678 proposed by Project 8, which would require an experiment whose size and complexity
679 are currently well beyond proposals for the next-generation of neutrino mass direct
680 measurement experiments.

681 **3.2.3 Project 8 Phased Development Plan**

682 Reaching 40 meV sensitivity will require the simultaneous development and eventually
683 combination of two novel technologies. The first is the technology required to supply a
684 source of atomic tritium of the appropriate size, density, purity, and temperature along
685 so that the atoms can be trapped and their beta-decays measured in the spectrometer.
686 The second is a CRES measurement technology that is both compatible with the tritium
687 atom trap and is capable of reconstructing CRES events with sufficient energy resolution

688 to achieve the required sensitivity.

689 These technologies require a significant up-front research and development (R&D)
690 investment to build-out the required capabilities for a 40 meV CRES experiment. There-
691 fore, Project 8 is following a phased experiment plan in which incremental progress can
692 be made towards the ultimate goal of a 40 meV neutrino mass measurement with CRES.

693 **Phase I and II: Proof of Principle and First Tritium Measurements**

694 The earlier phases of the Project 8 experiment, Phase I and II, were focused on demon-
695 stration and development of the CRES technique itself as well as a proof-of-principle
696 measurement of the neutrino mass using the CRES technique.

697 In Phase I, Project 8 performed a proof-of-principle measurement of the ^{83m}Kr
698 spectrum using CRES, which marked the first ever energy spectrum measurement with
699 CRES. The experiment included all of the main components expected for the full-scale
700 version of the experiment. An electron source consisting of a gas of ^{83m}Kr was supplied
701 to a waveguide gas cell constructed out of a segment of WR-42 waveguide and sealed
702 with Kapton windows at the top and bottom. A magnetic trapping region was created
703 in the waveguide cell using a single electromagnetic coil wrapped around the waveguide
704 which provided a trapping volume on the order of a few cubic-millimeters. Detection of
705 the cyclotron radiation was performed by connecting the waveguide cell to an additional
706 segment of waveguide that transmitted the radiation to a cryogenic amplifier.

707 Success in Phase I was achieved with the 2014 publication of the measured ^{83m}Kr
708 conversion spectrum, which contains a mono-energetic 17.8-keV as well as several other
709 conversion lines at higher energies. Publication of this result marked the official end of
710 Phase I and the start of Phase II in which Project 8 shifted its focus to the demonstration
711 of the first tritium beta-decay spectrum using CRES. Phase II successfully concluded
712 in 2023 with the submission of the papers demonstrating the first tritium beta-decay
713 spectrum endpoint and neutrino mass measurement using CRES. For more information
714 on Phase II please see Section 3.3.

715 **Phase III: Research and Development and a Pilot-scale Experiment**

716 With the completion of Phase II Project 8 has shifted into a phase focused on the
717 construction of an experiment that demonstrates all the technologies of the final experi-
718 ment in Phase IV. The goal for this pilot-scale experiment is to successfully retire all
719 technological and engineering risks associated with the Phase IV experiment, while being

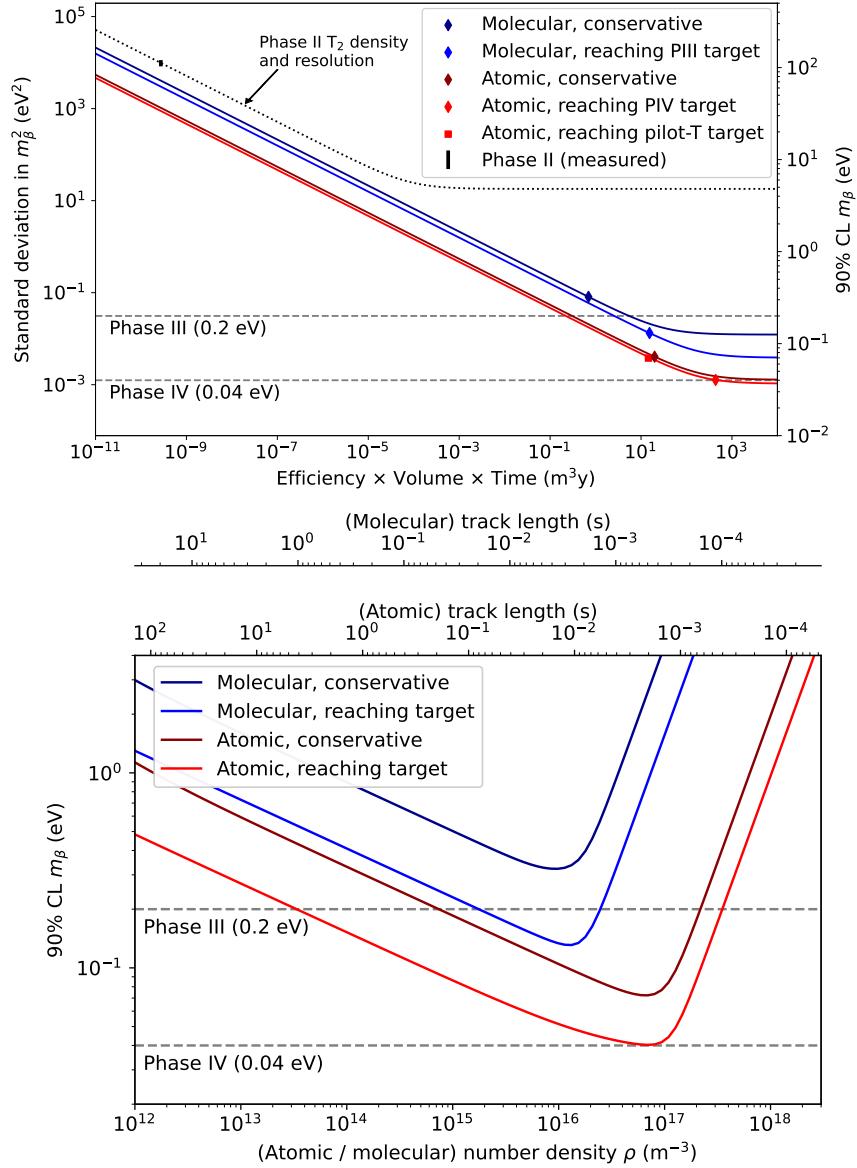


Figure 3.6: Sensitivity calculations for a cavity based CRES experiment that demonstrate the neutrino mass measurement goals of the Project 8 collaboration throughout the phased development plan. The blue tinged curves indicate molecular tritium sources and the red tinged curves indicate atomic tritium sources. In the current plan Phase III contains two tritium experiments. The first is the Low-frequency Apparatus (LFA) which is a molecular tritium experiment and the second is the atomic tritium pilot-scale experiment that ends Phase III. The sensitivity of these experiments is primarily a function of statistics, however, there is a critical density beyond which CRES electrons do not have enough time to radiate between collisions for a high-resolution frequency measurement leading to worse sensitivity.

720 a scientifically interesting experiment in it's own right that has sensitivity to neutrino
721 masses on par with KATRIN's final projected sensitivity.

722 Phase III R&D is divided into two equally important efforts — atomic tritium and
723 CRES detection techniques. Atomic tritium development in Phase III includes the
724 development of all aspects of the tritium system required for the pilot-scale experiment.
725 This includes the production of tritium atoms, atomic cooling and recirculation systems,
726 purity and isotope concentration monitoring, and trapping. Currently, Project 8 is
727 operating small scale demonstrator systems developing atom crackers to show that atom
728 production at the estimated rates needed for Phase IV is achievable. Future efforts
729 will continue the current developments on atom production and expand to include
730 demonstrations of atomic cooling with an evaporative beam line as well as atom trapping
731 using Halbach magnet arrays.

732 The need for new CRES detection techniques is driven by the drastic increase in scale
733 from Phase II to the Phase IV and the pilot-scale experiments. The physical volume
734 used for CRES in Phase II was on the order of a few cubic-centimeters, and achieving
735 Project 8's sensitivity target of 40 meV requires an experiment volume on the multi-cubic
736 meter scale. Therefore, the waveguide gas cell CRES detection technique used in Phase
737 II is not a feasible option for the future of Project 8 due to it's inability to scale to the
738 required size.

739 Two alternative CRES detection techniques have been proposed for the pilot-scale
740 experiment — antenna arrays and resonant cavities (see Section 3.4 and Chapter 6
741 respectively). Both approaches have relative advantages and disadvantages, however,
742 the improved understanding of the antenna array and cavity approaches to CRES in
743 the recent years has led to cavities being the preferred technology for the pilot-scale
744 experiment due to the estimated reduced cost and complexity of this approach. Since a
745 large degree of the work presented in this thesis is focused on the development of the
746 antenna array CRES technique as well as the design of demonstrator experiments, we
747 described the proposed R&D plan for antenna array CRES in Phase III in Section 3.4.

748 Cavity CRES R&D in Phase III consists of a series of demonstrator experiments
749 intended to demonstrate cavity CRES at a variety of scales and magnetic fields using
750 electrons from ^{83m}Kr , an electron gun, and potentially molecular tritium sources. The
751 near-term cavity effort in Project 8 is the cavity CRES apparatus (CCA), which is a
752 small-scale cavity experiment operating near 26 GHz, that will perform the first CRES
753 measurements using a small cavity. This experiment will pave the way towards larger
754 scale cavity experiments in preparation for the eventual pilot-scale tritium experiment.

755 The pilot-scale experiment is the first experiment, which will combine atomic tritium
756 and large-volume CRES detection in the same experiment. It will directly demonstrate
757 all the technologies required for Phase IV such that no technical risks remain for scaling
758 the experiment to required scale. A robust approach to scaling the pilot-scale experiment
759 is to simply build multiple copies of it for the Phase IV experiment.

760 **Phase IV: Project 8's Ultimate Neutrino Mass Experiment**

761 The design of Phase IV should be a direct extension of the pilot-scale CRES experiment
762 that marks the official end of Phase III (see Section 3.5). The Phase IV experiment
763 represents the final experiment in the Project 8 neutrino mass measurement experiment
764 plan and will have sensitivity to neutrino masses of 40 meV.

765 **3.3 Phase II: First Tritium Beta Decay Spectrum and**
766 **Neutrino Mass Measurement with CRES**

767 In Phase II Project 8 demonstrate the first ever measurement of the tritium beta-decay
768 spectrum endpoint using the CRES technique, which lead to the first neutrino mass
769 measurement by the Project 8 collaboration. This milestone was made possible by
770 many improvements in the CRES technique and more developed understanding of CRES
771 systematics, which takes an important first step towards larger scale measurements of
772 the tritium beta-decay spectrum with CRES. In this section, I shall briefly describe some
773 the important elements of the Phase II experiment, with the goal of contextualizing the
774 research and development efforts for Phases III and IV of Project 8. For more complete
775 descriptions of the work that lead to Project 8's Phase II results please refer to the many
776 Phase II papers produced by the collaboration.

777 **3.3.1 The Phase II CRES Apparatus**

778 **Magnet and Cryogenics**

779 The magnetic field for the the Phase II experiment is provided by a nuclear magnetic
780 resonance (NMR) spectroscopy magnet with a central bore diameter of 52 mm (see Figure
781 3.7). The magnet produces a background magnetic field with an average value of 0.959 T
782 and a 10 ppm variation across the bore diameter achieved using several shim coils built
783 into the magnet. Using an external NMR field probe the variation of the magnetic field

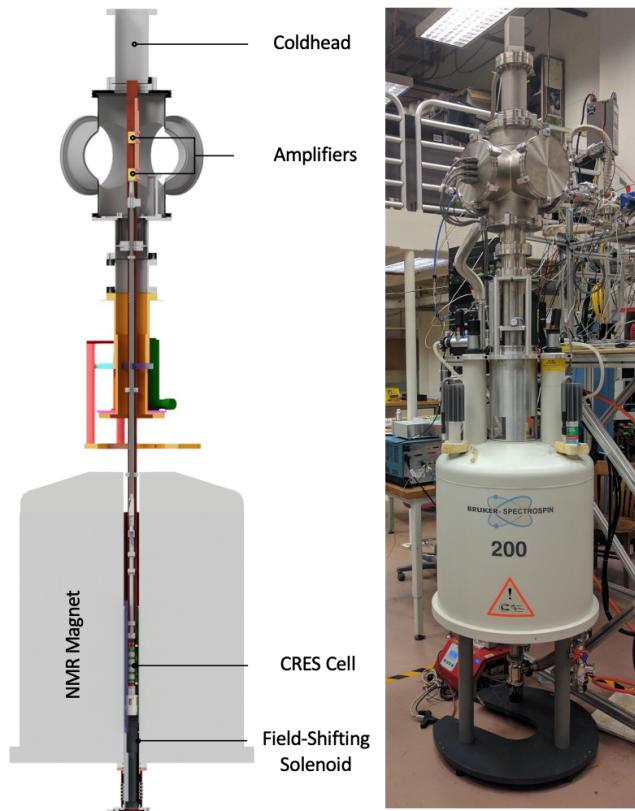


Figure 3.7: The Phase II CRES apparatus used to perform the first measurement of the tritium beta-decay spectrum using CRES.

784 along the vertical axis of the magnet bore was measured to obtain an accurate model of
 785 the magnetic field so that the CRES cell could be positioned for optimal magnetic field
 786 uniformity.

787 An external solenoid magnet was installed inside the magnet bore to provide the
 788 ability to shift the magnitude of the background magnetic field by values on the order of
 789 a few mT. The solenoid has inside diameter of 46 mm and a length of 350 mm, which
 790 terminates in a vacuum flange that allows it to be inserted into the NMR magnet bore
 791 from the bottom. By shifting the value of the magnetic field by a few mT, the cyclotron
 792 frequencies of electrons produced by the 17.8 keV ^{83m}Kr internal-conversion line can be
 793 shifted over a range of frequencies on the order of 100 MHz. This allows one to study the
 794 frequency dependent behavior of multiple CRES systematics such as detection efficiency
 795 that directly affect the measured shape of the tritium spectrum.

796 The inside of the magnet bore diameter was pumped down to a vacuum of less than
 797 10 μtorr using a turbomolecular pump, which allows for cryogenic cooling of the CRES

798 cell and RF system. Cooling power was supplied to the Phase II apparatus using a
 799 cryopump with its coldhead mounted above the primary magnet and CRES cell. This
 800 arrangement allowed for sufficient cooling power to be delivered to the amplifiers to cool
 801 them to a temperature of ≈ 40 K, while keeping the amplifiers far enough from the
 802 magnet so as not to be damaged by the large field strength. Thermal contact between
 803 the coldhead, amplifiers, RF system, and CRES cell is achieved using a copper bar that
 804 runs the full length of the apparatus. To prevent freeze-out of ^{83m}Kr on the walls of the
 805 CRES cell a separate heater was installed to keep the CRES cell near a temperature of
 806 85 K during the operation of the experiment.

807 CRES Cell

808 Located in the most uniform region of the magnetic field is the CRES cell, which is the
 809 region of the apparatus where radioactive decays of ^{83m}Kr and T_2 emit electrons that can
 810 be trapped and measured using CRES (see Figure 3.8). The CRES cell is manufactured

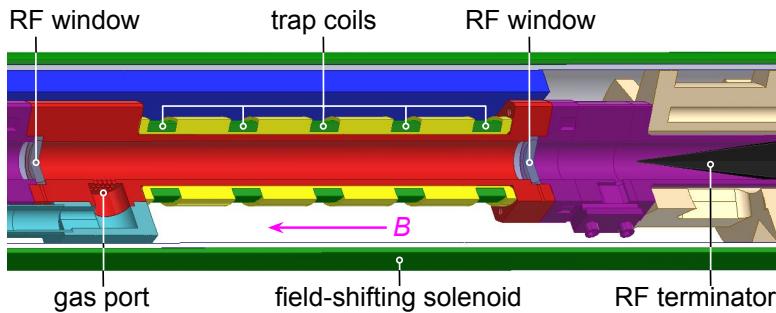


Figure 3.8: Diagram of the CRES cell portion of the Phase II apparatus.

810
 811 from a segment of cylindrical waveguide designed to operate at K-band frequencies
 812 near 26 GHz. The diameter of the waveguide determines which resonant modes of the
 813 waveguide will couple to the electron and transmit its radiation to the amplifiers. For
 814 Phase II a waveguide diameter of 1 cm was selected, which allows electrons to couple to
 815 the TE_{11} and TM_{01} cylindrical waveguide modes. To reduce complexity in modeling and
 816 analyzing the CRES data, it is ideal to select a diameter that prevents electrons from
 817 coupling to higher-order waveguide modes beyond the fundamental TE and TM modes.

818 Around the exterior of the cylindrical waveguide are several magnetic coils used
 819 to produce magnetic traps inside the CRES cell volume. Without a magnetic trap
 820 electrons produced from decays inside the CRES cell quickly impact the cell wall, which
 821 prevents a measurement of their cyclotron frequency using CRES. Each coil along the
 822 length of the waveguide produces a separate trap that is approximately harmonic in

shape. By independently controlling the currents provided to each coil the traps could be configured to have equal values of the magnetic field at the trap bottom despite a variable background magnetic field from the NMR magnet.

Two primary magnetic trap configurations were used during the Phase II experiment. The first was a shallow trap configuration used primarily for it's high energy resolution to study systematics using ^{83m}Kr decays, and the second was a deeper trap that could trap a higher percentage of pitch angles. The trade-off with this trap is that the higher trapping efficiency comes at the cost of lower energy resolution due to the greater variation in pitch angle. The deep trap was the trap used to measure the tritium beta-decay spectrum in Phase II.

The source gases were delivered into the CRES cell through a gas port located near the top end of the cylindrical waveguide. To prevent the gases from escaping the cell, vacuum tight RF transparent windows are needed to contain the tritium and krypton source gas across a 1 atm pressure differential, while still transmitting the cyclotron radiation without distortion. The crystalline material, CaF_2 , which has a thermal expansion coefficient similar to that of copper, was used for this purpose in the CRES cell. Two windows, each 2.4 mm thick, were used to seal off the ends of the CRES cell. The thickness of 2.4 mm corresponds to half of a cyclotron wavelength when one accounts for the permittivity of CaF_2 .

RF System

The RF system in the Phase II apparatus transferred the cyclotron radiation from the CRES cell to the receiver chain. The receiver chain performs the down-conversion and digitization required to obtain signals that can be analyzed to determine the cyclotron frequencies of electrons in the CRES cell (see Figure 3.9).

Below the CRES cell, at the bottom of the Phase II apparatus, is a tickler port and waveguide terminator. The tickler port is used to inject signals into the CRES cell and RF system for testing and calibration purposes. The waveguide terminator is designed to absorb cyclotron radiation emitted by electrons that transmits out of the bottom of the CRES cell. This lowers the total power received from electrons in the CRES cell, since all the energy radiated downwards is absorbed into the terminator. Earlier iterations of the Phase II apparatus used an RF short in this location that reflected this power up towards the amplifiers, however, interference between the upward traveling and reflected radiation led to a disappearance in the signal carrier that made reconstruction impossible.

Radiation traveling upward passes through the CaF_2 window passes through a $\lambda/4$

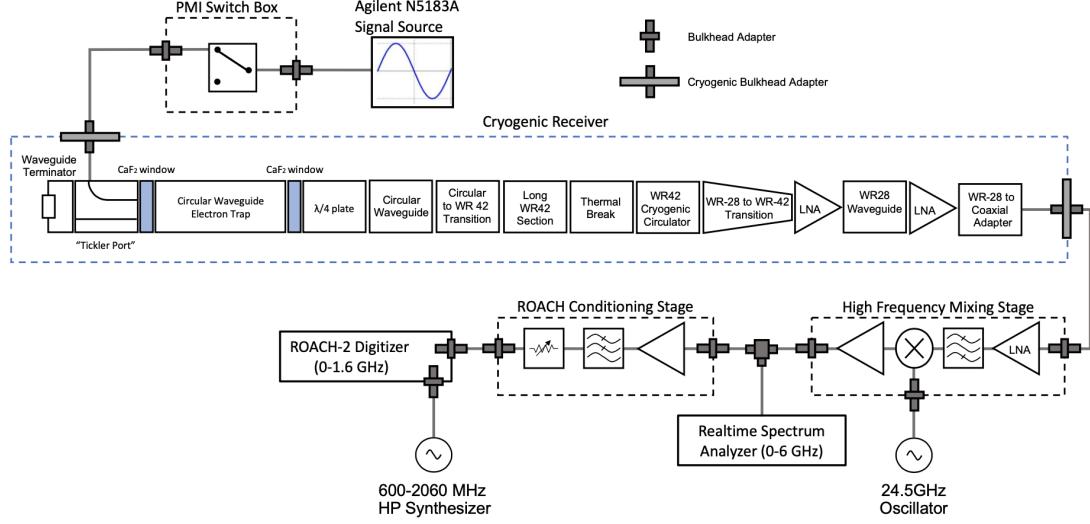


Figure 3.9: RF system diagram for the Phase II apparatus.

plate, which transforms the circularly polarized cyclotron radiation into linear polarization. The linearly polarized fields next travel through a segment of circular waveguide that transitions into a long segment of WR-42 waveguide that carries the fields out of the high magnetic field region. A thermal break segment is included, which consists of a segment of gold-plated stainless steel WR-42 waveguide, to help thermally isolate the relatively warm CRES cell from the colder amplifiers. The radiation then passes through a cryogenic circular, which prevents signals reflected from the amplifiers from interfering with the CRES cell before a WR-42 to WR-28 transition connects the waveguide to the first of the cryogenic amplifiers. The radiation passes through two cryogenic amplifiers before being coupled to a coaxial termination at the top of the Phase II apparatus.

The coaxial cable transfers the cyclotron radiation signals to a high-frequency mixing stage that performs an analog frequency down-conversion using a 24.5 GHz LO. Two forms of digitization can be used at this stage to readout the CRES data. One is a real-time spectrum analyzer that digitizes the CRES signal data in time-domain and computes the frequency spectrum in real-time, which allows for direct visualization of CRES signal spectrograms as the experiment is running. The real-time spectrum analyzer is most useful for taking small amount of streamed data for debugging and analysis of the system. The other method, which was used to collect the majority of the CRES data in Phase II, is a ROACH-2 FPGA and digitizer system. The ROACH system consists of a fast ADC that samples the CRES signal data at 3.2 GSps. Internal digital down-conversion stages implemented in the FPGA perform a mixing operation that reduces the bandwidth of the

878 CRES signals to 100 MHz. The FPGA implements a 8192 sample FFT and packetizes
879 time and frequency domain records in parallel. The packetized data is then transferred
880 from the ROACH to be analyzed by the data-processing pipeline.

881 **3.3.2 CRES Track and Event Reconstruction**

882 **Time-Frequency Spectrogram**

883 The online data-processing is intended to identify interesting data that could contain
884 CRES signals using a software real-time triggering algorithm. Interesting segments of
885 data identified by this algorithm are collected into files that are transferred to a server for
886 offline processing and analysis. The data files contain a continuous series of time-domain
887 samples, broken into a set of records, which are 4096 samples long. The time-series is
888 made up of 8-bit IQ samples acquired at 100 MHz.

889 Each time-series record is accompanied by an associated frequency spectrum consisting
890 of 4096 frequency bins approximately 24.4 kHz wide, which is represented as a power
891 spectral density. The individual frequency spectra can be organized temporally to create
892 a time-frequency spectrogram that represents the evolution of the cyclotron frequency
spectrum over the course of the CRES event (see Figure 3.10). The time-frequency

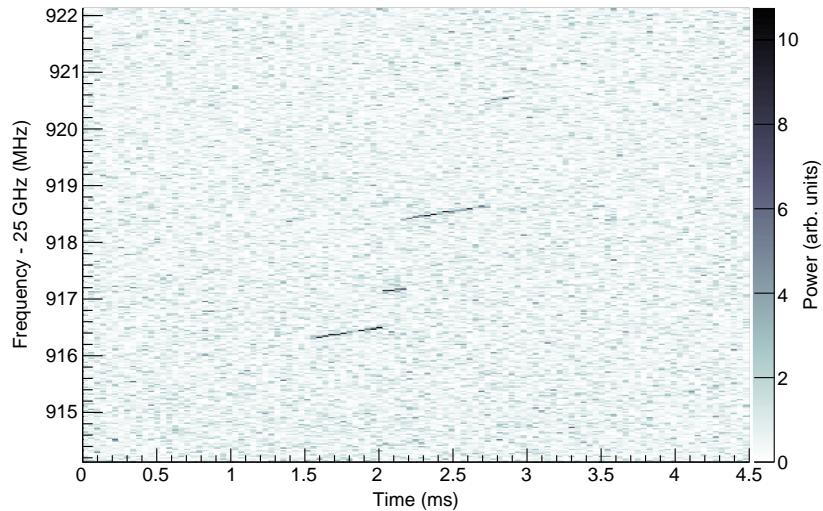


Figure 3.10: The time-frequency spectrogram of a tritium CRES event in the Phase II apparatus.

893 spectrogram is represented as a two-dimensional image where the color of each pixel is
894 proportional to the power spectral density. Each vertical slice of pixels in the image
895

896 represents a frequency spectrum, therefore, each horizontal bin represents the data
897 obtained over a duration of $4096 \times 0.01 \text{ MHz}^{-1} = 40.96 \mu\text{sec}$.

898 **CRES Event Data Features**

899 Phenomenologically, a CRES signal appears as a sinusoidal signal whose frequency slow
900 increases ("chirps") over time. Axial motion of the electron in the trap leads to the
901 formation of frequency sidebands that surround the more powerful carrier frequency, due
902 to doppler modulation of the electron's frequency as it bounces between the walls of the
903 magnetic trap. The critical piece of information that must be extracted from the track
904 and event reconstruction procedure is the carrier frequency, since it is this frequency
905 that gives the cyclotron frequency and thus the kinetic energy. While axial motion from
906 non- 90° pitch angles does change the average magnetic field experienced by an electron
907 and, therefore, changes the cyclotron frequency. We were not able to resolve sidebands
908 in Phase II, so a correction for the effect of the pitch angle on the cyclotron frequency
909 was not possible.

910 In the time-frequency spectrogram representation the chirping carrier frequency
911 appears as a linear track of high-power frequency bins (see Figure 3.10). The vertical
912 slope of the tracks is caused by the emission of energy from the electron in the form
913 of cyclotron radiation, therefore, the size of the slope parameter is directly proportional
914 to the Larmor power. The continuous track is periodically interrupted by random
915 jumps to higher frequency and lower energy caused by random inelastic collisions with
916 background gas molecules. The length of a track is an exponentially distributed variable
917 whose mean value is inversely proportional to the gas density. The size of the frequency
918 discontinuities is directly proportional to the energies of the rotational and vibrational
919 states of background gas species such as CO_2 .

920 A CRES event refers to the collection of tracks produced by a trapped electron until
921 it inevitably scatters into a pitch angle that can no longer be trapped. The goal of track
922 and event reconstruction is to first identify the set of tracks present in a time-frequency
923 spectrogram that represents a segment of data acquired in the Phase II apparatus. These
924 tracks must then be clustered into events from which we can determine the first track
925 produced by the electron and thus estimate its starting cyclotron frequency and kinetic
926 energy.

927 **Track Reconstruction**

928 The first step in this process is the identification of tracks in the time-frequency spectro-
929 gram, which is essentially an image processing feature identification task. The first step
930 in the track finding procedure is to normalize the power spectral density based on the
931 average noise power to obtain the time-frequency spectrogram in the form of normal-
932 ized, unitless power. Next a power threshold is applied to the normalized spectrogram
933 where only bins that have a signal-to-noise ratio greater than five are selected to build
934 tracks. In this case signal-to-noise ratio is defined as the ratio between the normalized,
935 unitless power of a bin divided by the average normalized power across the full frequency
936 spectrum.

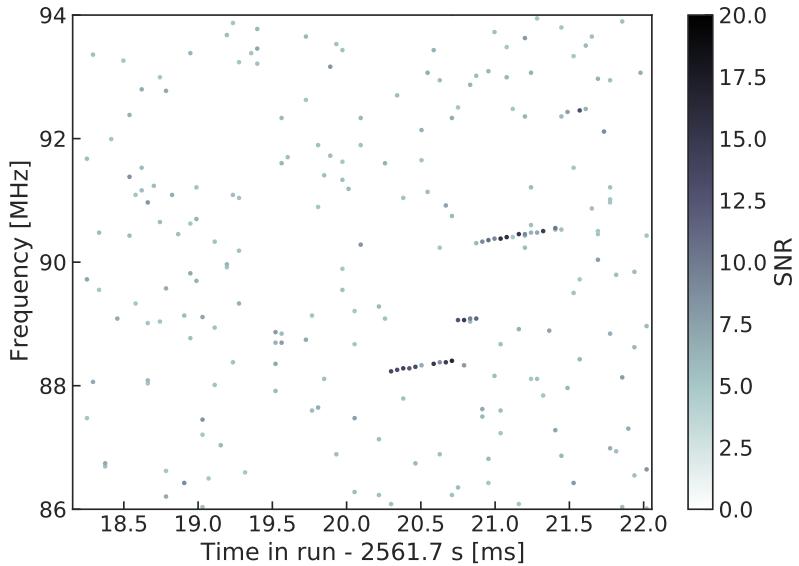


Figure 3.11: The sparse spectrogram obtained by placing a power cut on the raw spectrogram shown in Figure 3.10.

937 The spectrogram produced by this power cut, termed the sparse spectrogram, consists
938 only of a sparse collection of high-power frequency bins that could be part of a CRES
939 signal track (see Figure 3.11). In this form is it much easier to identify tracks "by eye",
940 however, for the Phase II analysis Project 8 developed it's own custom-made track finding
941 algorithm, called the sequential track finder (STF).

942 The STF algorithm processes the sparse spectrogram in sequential fashion, processing
943 each time-slice one-by-one until the end of the spectrogram is reached. Tracks are found
944 by searching for points in the sparse spectrogram that appear to fall on a straight line.
945 Multiple configurable parameters are built into the STF algorithm that allow the user to

946 tune the criteria for adding a point to an existing track or creating a brand new track.
947 These include parameters such as maximum time and frequency differences between
948 subsequent points in a track as well as minimum SNR values for the start and endpoints
949 of the track. Additionally, tracks are required to have a minimum length and slope to be
950 considered potential CRES tracks rather than random noise fluctuations.

951 The resulting output of the STF is a collection of track objects that consist of all of the
952 points that make up the track and their properties. The final step in track reconstruction
953 is to calculate the track properties and apply final cuts to reject the majority of false
954 tracks found by the STF. This involves the fitting of a line to the collection of track
955 points as well as the total and average power of the track obtained by computing the
956 sum and mean of the points powers. The starting frequency of the track is determined by
957 calculating the time coordinate that intersects with the linear fit. A cut is performed
958 to remove all tracks that do not have a specified average power over their duration, which
959 helps to remove the majority of noise fluctuations that have passed all previous cuts up
960 to this point.

961 Event Reconstruction

962 The final step is event reconstruction where the identified tracks are grouped into events
963 that contain all tracks likely caused by the same electron. This procedure simply attempts
964 to match tracks head to tail by checking if the start and end times of a pair of tracks
965 falls withing a certain tolerance. This tolerance is an additional configurable parameter
966 that can be tuned to an optimal value using monte carlo simulations of events in the
967 Phase II apparatus.

968 After the event building procedure has completed there is still a small likelihood that
969 false tracks have made it through to this stage in the reconstruction. Typically, cuts at
970 the track level are able to remove 95% of the false tracks identified by the STF, which
971 leads to a significant number of false tracks at the event building stage. However, the
972 additional event-level information makes it possible to reject events that contain these
973 false tracks with a high degree of confidence.

974 Two event level features are associated with events caused by real electrons — the
975 duration of the first track as well as the number of tracks in the event. Real electrons
976 tend to have event structures with longer first tracks and a higher number of total tracks.
977 Based on the values of these two criteria, a minimum threshold on the average power in
978 the first track was configured to reject false events. The average power in the first track
979 was chosen due to the critical nature of the starting frequency of the first track in an

980 event to the krypton and tritium spectrum analyses.

981 **3.3.3 Results from Phase II**

982 The primary result from Phase II is the first-ever measurement of the tritium beta-decay
983 spectrum using CRES, which lead to the first neutrino mass limit using the CRES
984 technique. However, Phase II also included a significant ^{83m}Kr measurement campaign
985 to understand important systematics relevant to the tritium spectrum measurement, but
986 also to understanding the fundamentals of the CRES technique itself. This required
987 high-resolution measurements of the ^{83m}Kr internal-conversion spectrum, which is an
988 interesting science result in its own right.

989 The results from Phase II represents a significant effort from the entire Project 8
990 collaboration over several years. Because the focus of my contributions to Project 8 is
991 directed towards the research and development efforts for the Phase III experiments, the
992 goal in this section is not to provide a detailed description of the the analyses that lead to
993 the Phase II results. Rather, I will provide brief descriptions of a few plots representative
994 of the main results from Phase II and direct the interested reader to the relevant Phase
995 II papers.

996 **Measurements with Krypton**

997 Measurements with krypton were a key calibration tool for Phase II of the experiment
998 and will most likely continue to be useful in future Phases of Project 8. In the context of
999 Project 8 krypton measurements refers to CRES measurements of the internal-conversion
1000 spectrum of the metastable state of krypton-83, ^{83m}Kr , produced by electron capture
1001 decays of ^{83}Rb . A supply of ^{83}Rb was built into the Phase II apparatus gas system that
1002 supplied the CRES cell with ^{83m}Kr via emanation.

1003 The ^{83m}Kr internal-conversion spectrum consists of several lines based on the orbital
1004 of the electron ejected during the decay. The conversion lines useful to Project 8 are
1005 those that emit electrons with kinetic energies that fall inside the detectable frequency
1006 bandwidth of the Phase II apparatus. These are the K; L2 and L3; M2 and M3; and N2
1007 and N3 lines with kinetic energies of 17.8 keV, \approx 30.4 keV, \approx 31.9 keV, and \approx 32.1 keV,
1008 respectively. The different energies of the lines allow us to test the linearity of the
1009 relationship between kinetic energy and frequency across the range of frequencies covered
1010 by the continuous tritium spectrum.

1011 By measuring the shape of the krypton spectrum we can characterize the effects of

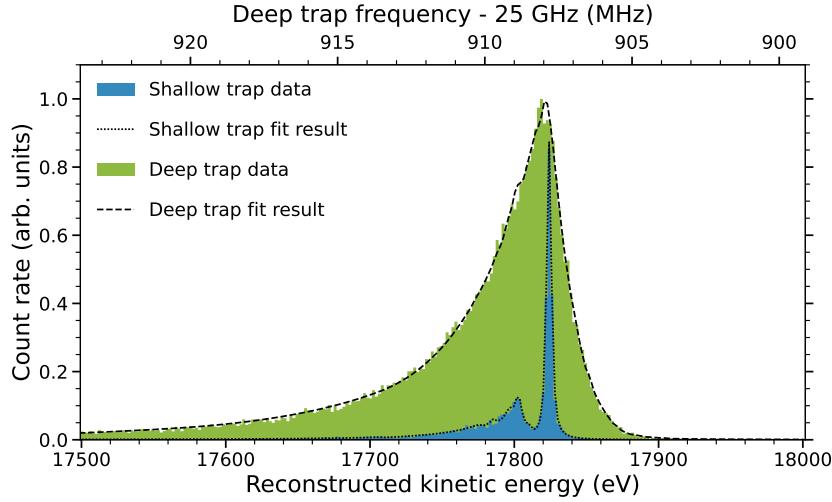


Figure 3.12: Fits to the measured 17.8-keV ^{83m}Kr conversion line using the deep and shallow trap configurations.

numerous detector related effects relevant to the tritium analysis. Specific examples include the variation in the magnetic field as a function of the radial position of the electron, variation in the magnetic field caused by the trap shape, variation in the average magnetic field for electron of different pitch angles, the effect of missing tracks due to scattering, among others. These spectrum shape measurements focused on the 17.8-keV krypton line and utilized different trap geometries based on the particular goal of the dataset (see Figure 3.12).

Krypton measurements with a shallow trap allow for high energy resolution, since variation in frequency due to pitch angle differences is sharply reduced in the shallow trap configuration. With this trap the main 17.8-keV peak of the conversion spectrum is clearly visible along with additional satellite peaks at lower energy, which correspond to the shakeup/shakeoff spectrum of the decay. The high accuracy of the fit demonstrates a high degree of understanding of the CRES systematics.

The broadening of the krypton spectrum seen for the deeper track is due to the higher range of electron pitch angles that can be trapped. Furthermore, with a deeper trap there is a larger parameter space of electron that could be produced with pitch angles that are trappable but not visible in the time-frequency spectrogram. These electrons live in the trap and can scatter multiple times before randomly scattering to a pitch angle that is now visible. This causes us to miss one to several of the electron's tracks earlier in the event, which leads us to mis-reconstruct the true starting frequency of an event. By measuring the krypton spectrum shape in the same deep trap used to detect tritium

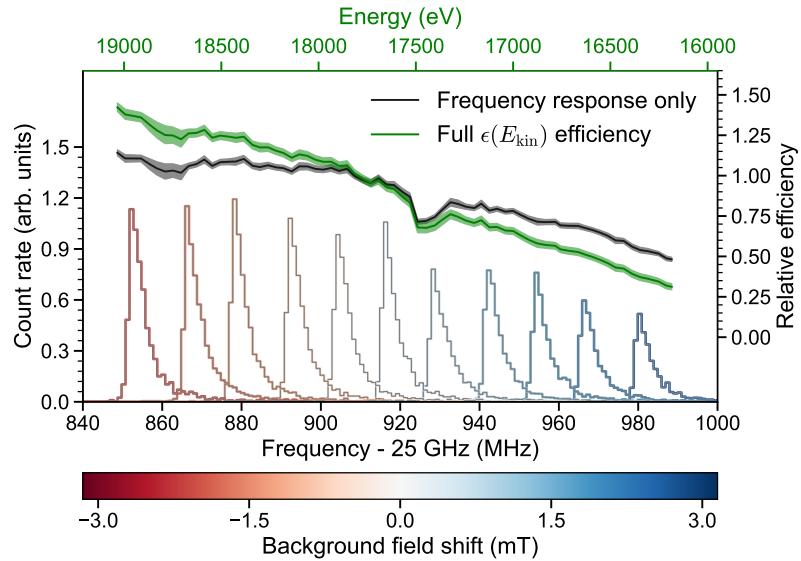


Figure 3.13: Measurements of the 17.8-keV ^{83m}Kr line using the deep trap configuration for different values of the magnetic field from the field shifting solenoid.

1033 events we can characterize the affect that this has on the spectrum shape to mitigate it's
 1034 impact on the tritium measurements.

1035 An additional systematic characterized with krypton is the calibration of the detection
 1036 efficiency of the Phase II apparatus as a function of frequency. Variations in the detection
 1037 efficiency as a function of frequency directly changes the measured shape of the continuous
 1038 tritium spectrum, which can lead to errors in the neutrino mass estimate if not modeled
 1039 appropriately. Using the field shifting solenoid the cyclotron frequency of the krypton
 1040 17.83 keV line was shifted across the full frequency range of the tritium spectrum data
 1041 (see Figure 3.13). Variations in the deep trap krypton spectrum shape can be used to
 1042 infer the detection efficiency as a function of frequency and correct for this affect in the
 1043 tritium measurements.

1044 Tritium Spectrum and Neutrino Mass Results

1045 The tritium measurement campaign resulted in the collection of 82 days of detector
 1046 live time during which 3770 total tritium events were detected. The track and event
 1047 reconstruction analysis extracted the starting frequencies of these tritium events, which
 1048 were used to build a frequency spectrum of tritium beta-decays. The resulting frequency
 1049 spectrum was then converted to an energy spectrum using the information gleaned from
 1050 the krypton measurement campaign to obtain the tritium beta-decay spectrum (see

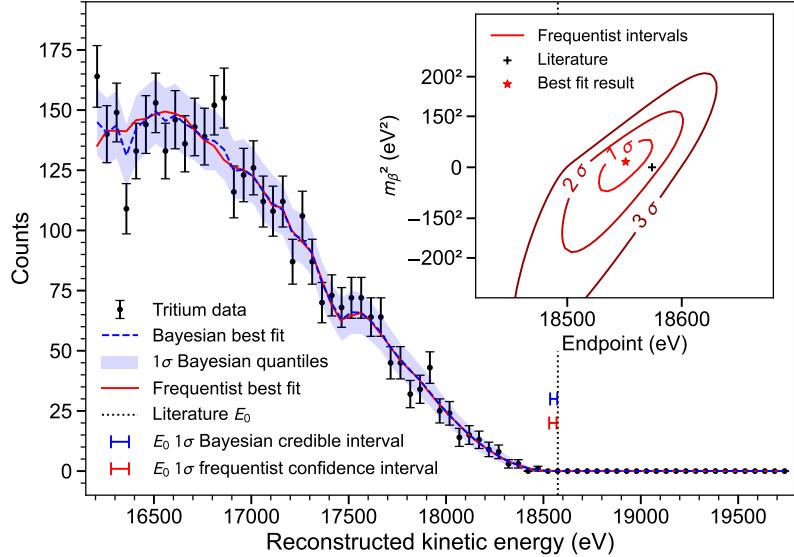


Figure 3.14: The measured tritium spectrum from Phase II with Bayesian and frequentist fits.

1051 Figure 3.14).

1052 CRES is inherently a very low background technique with the dominant source of noise
 1053 being random RF fluctuations. Monte carlo simulations backed up by measurements
 1054 of the RF noise background were used to set track and event characteristic cuts to
 1055 guarantee that zero false events would occur over the duration of the experiment with
 1056 90% confidence. Notably, the measured spectrum has zero events beyond the tritium
 1057 spectrum endpoint, which allows us to constrain the background rate in the Phase II
 1058 apparatus to less than 3×10^{-10} counts/ev/s. Achieving a low background is critical for
 1059 future neutrino mass experiments that seek to measure the neutrino mass with less than
 1060 100 meV sensitivity.

1061 Bayesian and frequentist based fits to the measured tritium spectrum, incorporating
 1062 information gained about CRES systematics from the krypton measurements, were
 1063 performed to extract upper limits on the tritium beta-decay spectrum endpoint as well as
 1064 the neutrino mass. The estimated spectrum endpoints are 18553^{+18}_{-19} eV for the Bayesian
 1065 analysis and 18548^{+19}_{-19} eV for the frequentist analysis. The quoted uncertainties are
 1066 1- σ , and both results are within 2- σ of the literature endpoint value of 15574 eV. The
 1067 estimated neutrino mass for both results is consistent with $m_\beta^2 = 0$. The 90% confidence
 1068 upper limits for the Bayesian analysis is $m_\beta < 155$ eV/c² and $m_\beta < 152$ eV/c for the
 1069 frequentist analysis.

Though the neutrino mass results from Phase II are not competitive with KATRIN it is a promising first step towards the development of more precise neutrino mass measurements using CRES. The low background and demonstrated high resolution with krypton measurements are promising features of the technique that were able to be demonstrated with the Phase II apparatus. As new technologies are developed to enable CRES measurements in larger volume, many of the lessons learned from Phase II will continue to influence the operation and design of the detectors.

3.4 Phase III R&D: Antenna Array CRES

The goal of Phase III in the Project 8 experimental program is to develop the technologies and expertise required to build an experiment that uses CRES to measure the neutrino mass with a target sensitivity of 40 meV. One of the key technologies is a method for performing high resolution CRES measurements in a large volume, which allows one to observe a sufficient quantity of tritium to measure the low-activity endpoint region of the tritium spectrum.

3.4.1 The Basic Approach

One possible approach, suggested in the original CRES publication, is to use many antennas to surround a volume of tritium gas in a magnetic field (see Figure 3.15). When a decay occurs the electron will begin to emit cyclotron radiation that can be collected by the array and used to perform CRES. Each antenna in the array collects only a small

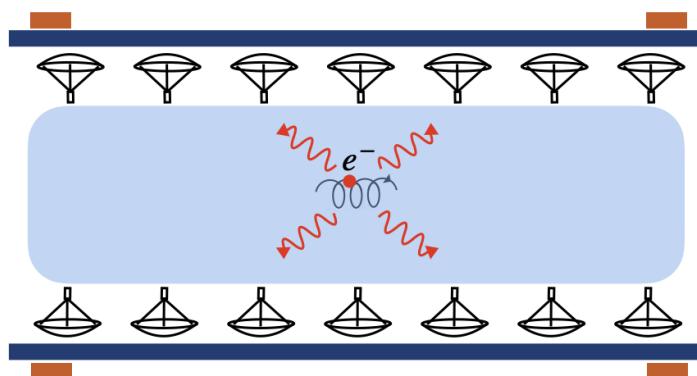


Figure 3.15: A cartoon illustration of the basics of the antenna array CRES technique.

fraction of the electron's signal power, which is less than 1 fW for a 18.6 keV kinetic energy electron in a 1 T magnetic field. Scaling to large volumes with the antenna

array approach is accomplished by increasing the number of antennas in the array, which increases the volume under observation proportionally, so that a sufficient population of tritium atoms can be observed to measure the tritium spectrum endpoint shape.

Several features of the antenna array approach make it an attractive candidate technology for a large volume experiment. One example is the accurate position reconstruction made possible by the multichannel nature of the array. Using techniques like digital beamforming it is possible to estimate the radial and azimuthal positions of the electron in the magnetic trap with a precision significantly less than the size of the cyclotron wavelength. This capability allows one to perform event-by-event estimations of the magnetic field experienced by an electron, which is crucial to achieving high energy resolution with the CRES technique.

The easy availability of position information with the antennas array approach is potentially a unique advantage that provides significant flexibility in the magnetic field uniformity requirements compared to other proposed approaches to large volume CRES (see Chapter 6). Spatial discrimination using digital beamforming leads to pileup reduction, which helps to reduce the potential of background events caused by missing tracks or by incorrectly clustering a group of tracks into an event. Limits on the background rate for a neutrino mass measurement with 40 meV sensitivity are stringent and the total activity of the tritium source for such an experiment is gigantic relative to the activity near the endpoint. Thus, pileup discrimination could be an important tool for a large scale CRES experiment.

Another beneficial quality of the antenna array approach is that the volume of the experiment can be scaled independent of frequency by simply adding more antennas to the array (see Figure 3.19). Resonant cavities, the proposed alternative large volume CRES technology, are ideally operated in magnetic fields that cause electrons to move with cyclotron frequencies near the fundamental cavity resonance, to avoid complex coupling of the electron to many cavity modes simultaneously. This leads to a coupling between the cavity volume and the magnetic field magnitude, which forces one to lower the magnetic field in order to increase the experiment scale. Whereas, for antenna arrays, in principle there is no physical limitation on the size of the antenna array that can be used at a particular magnetic field. However, the nature of scaling an antenna array based experiment leads to rapidly increasing cost and complexity due to the large number of antennas, amplifiers, and data streams that require substantial computer processing power to effectively analyze.

1125 **3.4.2 The FSCD: Free-space CRES Demonstrator**

1126 The complex collection of new experimental techniques and methods that come together in
1127 the antenna array CRES technique require the construction of a small scale demonstration
1128 experiment designed to develop an understanding of the principles of antenna array CRES
1129 measurements and the relevant systematics. Without operating such an experiment it
1130 is not possible to develop a design for a large scale CRES experiment with sufficient
1131 confidence that the experiment is capable of measuring the shape of the tritium spectrum
1132 endpoint to the degree of accuracy required for 40 meV sensitivity to the neutrino mass.
1133 Therefore, Phase III of the Project 8 experimental program is primarily focused on the
1134 development and operation of demonstrator experiments to inform the design of the final
1135 Phase IV experiment.

1136 Specifically for antenna array CRES, the associated demonstrator experiment in
1137 Phase III is called the Free-space CRES Demonstrator or FSCD. The goals of the FSCD
1138 include not only the development of antenna array CRES itself, but is also a capable
1139 neutrino mass measurement experiment in it's own right, with a target neutrino mass
1140 sensitivity of a few eV using a molecular tritium source.

1141 **Magnetic Field**

1142 The background magnetic field for the FSCD experiment is provided by a hospital-grade
1143 MRI magnet (see Figure 3.16). The magnet produces a magnetic field of approximately
1144 0.958 T, which corresponds to a tritium spectrum endpoint frequency of approximately
1145 25.86 GHz. The magnet is installed in the Project 8 laboratory located at the University of
1146 Washington, Seattle, and is shimmed to produce a uniform magnetic field with variations
1147 on the ppm scale. Measurements of the magnetic field non-uniformities were performed
1148 using a NMR probe and rotational gantry to capture measurements of the magnetic field
1149 around an elliptical surface in the center of the MRI magnet. During the operation of
1150 the FSCD an array of Hall or NMR magnetometers could be used to periodical measure
1151 the magnetic field in order to quantify its time stability.

1152 Inside the main magnetic field of the MRI magnet are additional magnets that provide
1153 the capability to shift the value of the background magnetic field as well as the magnets
1154 that produce the magnetic trap. Shifting the background value of the magnetic field on
1155 a scale of $O(\mu T)$ allows one to control the cyclotron frequencies of electrons with a fixed
1156 kinetic energy, which is key to effectively calibrating the FSCD. The preferred calibration
1157 method for the FSCD is a mono-energetic electron gun that can inject electrons into



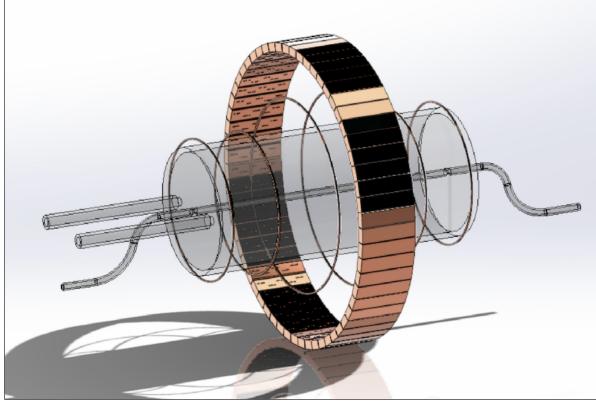
Figure 3.16: An image of the MRI magnet installed in the Project 8 laboratory at the University of Washington, Seattle.

1158 the magnetic trap with a known kinetic energy. In combination with the field shifting
 1159 magnet one can vary the cyclotron frequencies of the electrons to measure the response
 1160 of the antenna array as a function of the radiation frequency and electron position. This
 1161 procedure not only characterizes the response of the antenna array but also provides
 1162 further information on magnetic field uniformity, which important to achieving optimal
 1163 energy resolution.

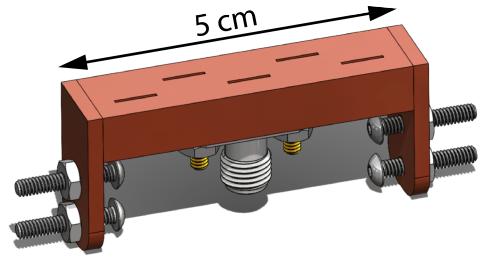
1164 Several additional magnetic coils will need to included inside the MRI magnet to
 1165 produce the magnetic trap. The ideal trap shape for CRES is the perfect magnetic box,
 1166 which has a flat bottom and step function walls. Any variation in the average magnetic
 1167 field experienced by an electron leads to changes in the cyclotron frequency that can
 1168 make determining the true starting kinetic energy more difficult. This includes changes
 1169 in the magnetic field caused by the walls of the magnetic trap as well as radial magnetic
 1170 field variations. The perfect box trap is completely uniform and has infinitely steep walls
 1171 that cause no change in the electron's cyclotron frequency as it is reflected from the
 1172 trap wall, however, such a trap cannot be made from any combination of magnetic coils
 1173 since it violates Maxwell's equations. The goal of magnetic trap design is to identify the
 1174 configuration of coils that produces a trap that approximates the perfect box trap as
 1175 closely as possible.

1176 **Antenna Array**

1177 The canonical antenna array design for a CRES experiment is a uniform cylindrical array
1178 of antennas that surrounds the magnetic trap volume. Since the FSCD is a demonstrator
1179 experiment, the antenna array design is the simplest form of the uniform cylindrical
1180 array, which is a single circular ring of antennas with a diameter of 20 cm (see Figure
1181 3.17). Along this circle are sixty slotted waveguide antennas that fully populate the



(a)



(b)

Figure 3.17: (a) A model of the FSCD antenna array, magnetic trap, and tritium containment vessel design.(b) A more detailed model of a prototype design for the 5-slot waveguide antenna design.

1181
1182 available space around the array circumference. In order to maximize the power collected
1183 from each electron it is optimal to cover as large a fraction of the solid angle around the
1184 magnetic trap as possible.

1185 The distance between antennas around the circumference of the array is proportional
1186 to the wavelength of the cyclotron radiation. Therefore, maximizing the solid angle
1187 coverage of the array, while minimizing channel count to keep the hardware and data
1188 acquisition costs manageable, biases one towards smaller array diameters. Antenna
1189 near-field effects limit the minimum diameter of the array for a given antenna design
1190 since the radiation from electrons that are too close to the array cannot be detected
1191 due to destructive interference caused by path-length differences from the electron to
1192 different points on the antenna surface.

1193 Slotted waveguide antennas are used in the FSCD antenna array due to their high
1194 efficiency and low loss, which comes from the lack of dielectric materials in the antenna
1195 structure. Coupling to the waveguide can be performed with a coaxial cable connected
1196 at the center or on either end of the waveguide. One of the drawbacks of waveguide

1197 antennas is the large amount of space required to fit them inside the limited MRI magnet
1198 volume. Alternative antenna designs, constructed from microstrip printed circuit boards
1199 require significantly less space at the cost of slightly higher energy loss in the antenna
1200 structure.

1201 The FSCD antenna design is a 5 cm long segment of WR-34 waveguide with 5 vertical
1202 slots cut into the side. The distance between slots along the length of the waveguide is
1203 a half wavelength for optimal power combination between the individual antenna slots.
1204 Each slot is offset from the center of the antenna face a small distance in order to most
1205 effectively couple the slot to waveguide modes inside the antenna.

1206 The passive power combination achieved by placing 5 slots in a single waveguide is a
1207 compromise intended to reduce the cost and complexity of the antenna array system.
1208 Each additional channel in the array requires it's own cryogenic amplifier and also increase
1209 the required computer power to process the raw data collected by digitizing each channel.
1210 Passive summation, achieved by combining antennas into arrays axially, reduces the array
1211 channel count at the cost of losses from imperfect passive combination. Imperfect passive
1212 combination is caused by effects such as re-radiation of energy from and destructive
1213 interference between slots in the waveguide antenna.

1214 Interference and re-radiation eventually limit the achievable the axial extent of passive
1215 power combination. The 5-slot designed developed for the FSCD is optimized to minimize
1216 the impact of these losses while achieving the maximum amount of axial coverage with a
1217 single ring of antennas. Scaling beyond the volume covered by a single ring of antennas is
1218 achieved by stacking additional rings of antennas together to cover a larger trap volume
1219 for a higher statistics measurement of the tritium spectrum endpoint region. A likely
1220 scenario for the FSCD experiment involves a staged experiment approach, where first
1221 a series of measurements is performed using only a single ring of antennas followed by
1222 experiments that add additional rings to the FSCD. The goal would be to first understand
1223 the principles of antenna array CRES using the simplest possible experiment, before
1224 attempting to scale the technique by expanding the antenna array size.

1225 **Tritium Source**

1226 While the primary purpose of the FSCD is as a technology demonstrator, it is unlikely
1227 for the collaboration to gain the required confidence in the antenna array CRES tech-
1228 nique to perform neutrino mass measurements at the 40 meV sensitivity level without
1229 an intermediate scale measurement of the neutrino mass using antenna array CRES.
1230 Therefore, the FSCD has an additional scientific goal of measuring the neutrino mass

1231 with a rough sensitivity goal of a few eV. This level of precision is achievable using a
1232 source of molecular tritium with a volume of approximately 1 L at a density comparable
1233 to potential Phase IV scenarios.

1234 Unlike previous CRES experiments, where the tritium source could be co-located
1235 with the receiving antenna inside a waveguide transmission line, the tritium source
1236 in the FSCD is thermally isolated from the antenna array to avoid freeze-out of the
1237 tritium molecules. The tiny radiation power emitted by electrons requires a system noise
1238 temperature of ≈ 10 K or less, in order to detect events at a high enough efficiency to
1239 reach the neutrino mass sensitivity goals of the experiment. Achieving a system noise of
1240 10 K requires that the antenna array and amplifiers operate at cryogenic, liquid helium
1241 temperatures of ≈ 4 K, which significantly lowers the vapor pressure of molecular tritium.
1242 By keeping the molecular tritium isolated in an RF-transparent vessel the tritium gas can
1243 be kept at a relatively warmer temperature in the range of 30 K to avoid the accumulation
1244 of tritium on the experiment surfaces.

1245 Data Acquisition and Reconstruction

1246 A fundamental change in the data acquisition system for the FSCD is the shift from
1247 single to multi-channel reconstruction. This transition results in a significant increase in
1248 the data-generation rate, which is linearly related to the number of independent channels
1249 in the array. The larger data volume coincides with an increased demand for computer
1250 processing power based on the need for more precise signal reconstruction algorithms
1251 driven by the FSCD and Phase IV sensitivity goals. Therefore, the data acquisition
1252 system for the FSCD is likely to represent a significantly larger fraction of the experiment
1253 cost and complexity than previous CRES experiments.

1254 Each antenna in the array is connected to a cryogenic amplifier and down-converted
1255 from the 26 GHz CRES frequency using an IQ-mixer to reduce the size of the analysis
1256 window in which the tritium spectrum is measured. Using an LO with a frequency of
1257 approximately 25.80 GHz the antenna array signals can be digitized at a rate of 200 MHz,
1258 which is sufficient bandwidth to resolve the complete sideband spectrum produced by
1259 axial oscillations of electrons in the FSCD magnetic trap.

1260 Direct storage of the raw FSCD antenna array data is undesirable, since the estimated
1261 amount of raw data generated is $O(1)$ exabyte per year. The management and storage
1262 of such a large dataset is infeasible for a demonstrator experiment on the scale of the
1263 FSCD and would represent a large fraction of the budget for a Phase IV scale antenna
1264 array based CRES experiment. Therefore, a sub-goal of the FSCD experiment is the

1265 development of real-time reconstruction methods that could reduce the raw data volume
1266 by detecting and reconstructing CRES events in real-time. The ultimate goal would be
1267 a complete real-time reconstruction pipeline that takes raw voltages samples from the
1268 antenna array and returns estimates for the starting kinetic energies of CRES events in
1269 the data.

1270 The feasibility of a real-time reconstruction pipeline rests on the development of
1271 computationally efficient algorithms that can be implemented without the need for
1272 enormous computing resources. One challenge with the antenna array approach is that
1273 the small radiation power of a single electron is distributed between each channel in
1274 the array, such that reconstruction using only the information in a single channel is not
1275 possible. Therefore, the simply performing the initial step in reconstruction — signal
1276 detection — requires orders of magnitude more computational power than previous CRES
1277 experiments. This operation will then be followed by other, potentially more expensive,
1278 reconstruction steps that are required in order to determine the kinetic energy of the
1279 electron.

1280 **3.5 Pilot-scale Experiments**

1281 **3.5.1 Choice of Frequency**

1282 The optimal CRES frequency for Project 8 is that which can reach our target sensitivity
1283 of 40 meV, while minimizing the cost and complexity of the overall experiment. Since the
1284 size of the background magnetic field determines the cyclotron frequency, which affects
1285 the entirety of the CRES detection system design, specifying the operating frequency of
1286 the CRES experiments is one of the first steps towards developing a full design.

1287 **Scaling Laws**

1288 In Phases I and II the background magnetic field was provided by an NMR magnet with
1289 a 0.959 T magnetic field. This magnetic field was selected primarily for convenience,
1290 however, the cyclotron frequencies for electrons near the tritium endpoint in a 0.959 T
1291 field ranges from 25 to 26 GHz, which is within the standard RF Ka-band. Therefore,
1292 microwave electronics specialized for these frequencies are easily obtainable for relatively
1293 low cost. Frequency choice for the upcoming large-scale experiments must be selected
1294 in a more rigorous manner than in the earlier phases due to the increasing scale and
1295 complexity of the systems and the 40 meV neutrino mass science goal.

1296 Naturally, for a larger volume experiment there is a bias towards lower frequencies, due
1297 to the direct relationship between wavelength and the physical size of the compatible RF
1298 components like antennas and cavities. With a longer wavelength a larger volume can be
1299 surrounded by an array with fewer antennas, which reduces hardware and data-processing
1300 costs. On the other hand, for a cavity experiment, the volume of the experiment is
1301 directly proportional to the wavelength since this sets the physical dimensions of the
1302 cavity. Furthermore, it is easier to engineer a magnet that provides a uniform magnetic
1303 field across several cubic-meters of space at a lower magnetic fields, which provides
1304 advantages in terms of cost-reduction as well as more uniform magnetic fields for CRES.

1305 A concern with lower magnetic fields and frequencies is the scaling of the Larmor
1306 power equation, which is proportional to the square of the frequency. Naively, one would
1307 predict that the SNR would decrease with lower fields, however, two additional scaling
1308 laws that affect the noise power also come into play. Noise power is directly proportional
1309 to the required bandwidth, which decreases linearly with the magnetic field. Furthermore,
1310 at lower frequencies it is possible to purchase amplifiers with lower noise temperatures
1311 until approximately 300 MHz at which point this relationship tends to flatten. Therefore,
1312 it is expected that the SNR remains approximately constant as the frequency decreases.

1313 The SNR directly impacts the overall efficiency of the experiment through its affects
1314 on CRES signal detection probabilities as well as energy resolution. Thus, the expectation
1315 that SNR remains the same at lower frequencies clearly biases large-scale experiments
1316 in this direction. One drawback of lower magnetic fields is the increased influence of
1317 external magnetic fields on the experiment. This includes magnetic fields from the
1318 building materials as well as variations in the earth's magnetic field. To deal with these
1319 affects a suitable magnetic field correction system will need to be devised, which includes
1320 constant monitoring of external fields.

1321 **Atomic Tritium Considerations**

1322 The pilot-scale experiments will be the first Project 8 experiments to combine CRES
1323 with atomic tritium, therefore, the optimal frequency should take into account the affect
1324 of the background magnetic field size on atom trapping. The primary influence of the
1325 background field magnitude is through the rate of dipolar spin-flips caused by a spin
1326 exchange interaction between trapped atoms.

1327 Atomic tritium is a simple quantum system with a hyperfine structure given by the
1328 addition of the nuclear and atomic spins. The addition of two spins leads to a hyperfine
1329 structure with four states in the (m_s, m_I) basis. The states with atomic spins directed

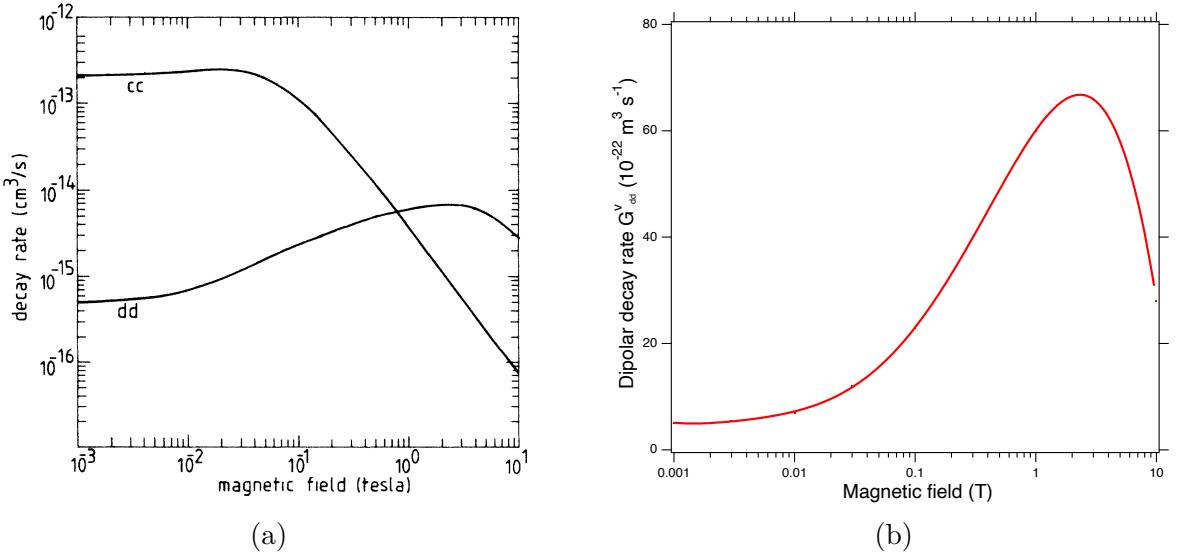


Figure 3.18: (a) A plot of the decay rate for the two-body dipolar spin exchange interaction for $c+c$ and $d+d$ state. (b) A plot of the decay rate of the dipolar spin exchange interaction for $d+d$ states as a function of magnetic field magnitude. Lowering the magnetic field is key for reducing the losses from this interaction.

1330 anti-parallel to the magnetic field have $m_s = -1/2$ and are labeled as the a and b states.
 1331 The a and b states are colloquially known as high-field seeking states, since their energy is
 1332 minimized when in regions of higher magnetic field. This leads to losses in the magnetic
 1333 trap as these atoms are drawn to higher fields away from the trap center. Alternatively,
 1334 the c and d states, with atomic spin $m_s = +1/2$, minimize their energy in low magnetic
 1335 fields because of the parallel alignment between spin and the magnetic field. Therefore,
 1336 these low-field seeking states tend to stay trapped significantly longer than the high-field
 1337 seeking states.

1338 Project 8 would do well to prepare the tritium atoms in purely c and d states before
 1339 trapping, however, even in this case losses still occur due to dipolar interactions between
 1340 pairs of c and d states leading to a flipped atomic spins and subsequent losses due
 1341 to high-field seeking atoms. The rate of these interactions depends on the magnitude
 1342 of the background magnetic field and is maximal for dd interactions around 1 T (see
 1343 Figure 3.18). The rate of losses from these interactions at 1 T requires atomic tritium
 1344 production at a rate two orders of magnitude larger than at 0.1 T, thus, requirements
 1345 on the whole atomic tritium system are significantly relaxed at lower magnetic fields,
 1346 which provides an additional argument for transitioning to lower frequencies with the
 1347 pilot-scale experiments.

1348 **3.5.2 Pilot-scale Experiment Concepts**

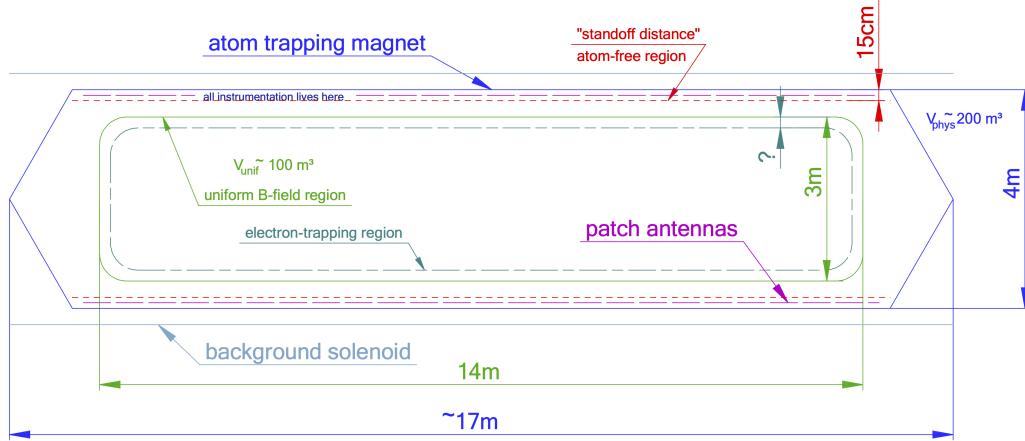


Figure 3.19: A conceptual sketch of a large-volume antenna array based CRES experiment to measure the neutrino mass.

1349 While the pilot-scale experiments are still in the early stages, enough is known to
1350 sketch the general features of these experiments at the cartoon level.

1351 **Pilot-scale Antenna Array CRES Experiment Concept**

1352 A conceptual design for an antenna-based CRES experiment is shown in Figure 3.19.
1353 A large solenoid magnet provides a uniform background magnetic field less than 0.1 T
1354 in magnitude. Inside this region is the atom trapping magnet that generates a high
1355 magnetic field at the walls, which decays exponentially towards the central region. Known
1356 magnet designs that produce suitable atom trapping fields include Ioffe-Prichard traps,
1357 which use conducting coils, as well as a Halbach array made from permanent magnets.
1358 Either magnet choice produces a region of high magnetic fields, which excludes atoms
1359 and allows for the placement of antennas inside the experiment.

1360 Inside this region an array of microstrip patch antennas is inserted to collect the
1361 cyclotron radiation without providing a surface for atomic tritium recombination. Due
1362 to the lower frequency of cyclotron radiation antennas of a larger size can be used,
1363 which lowers the total number of antennas required to observe the experiment volume.
1364 Because of this scaling, the lower frequency experiment uses a similar number of antennas
1365 compared to a much smaller demonstrator experiment with a 1 T magnetic field.

1366 The atomic tritium beamline that supplies fresh tritium atoms to the experiment is
1367 not shown in the figure. The general configuration would matches the one shown for the

₁₃₆₈ pilot-scale cavity experiment (see Figure 3.20).

₁₃₆₉ Pilot-scale Cavity CRES Experiment Concept

₁₃₇₀ The pilot-scale cavity experiment includes both an atomic tritium system and cavity
₁₃₇₁ CRES system. The atomic system consists of a thermal atom cracker located at the
₁₃₇₂ start of an evaporatively cooled atomic beamline. The atomic tritium system provides a
₁₃₇₃ supply of tritium atoms to the trap with temperatures on the order of a few mK. Atoms
₁₃₇₄ at this temperature can be trapped magneto-gravitationally, which is the reason for the
₁₃₇₅ vertical orientation of the cavity. At these low magnetic fields the trapping requirements
₁₃₇₆ for electrons and atoms differ enough such that it is advantageous to decouple the the
₁₃₇₇ trapping potentials to avoid radioactive heating of the tritium atoms from excess trapped
₁₃₇₈ electrons. Electron trapping is provided by a set of magnetic pinch coils at the top and
₁₃₇₉ bottom of the cavity and a multi-pole Ioffe or Halbach magnet serves to contain the
₁₃₈₀ atoms.

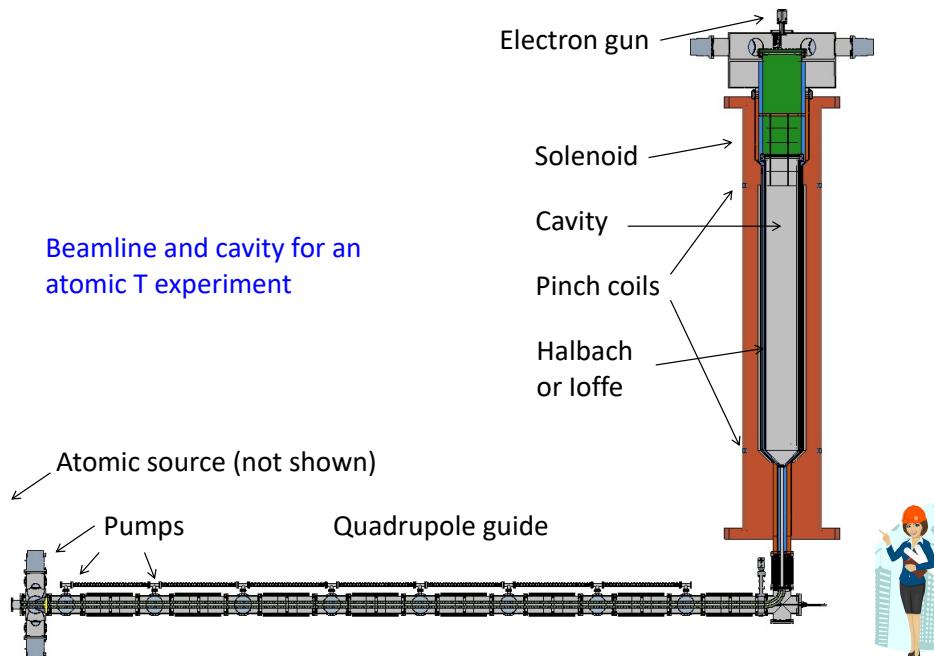


Figure 3.20: A conceptual sketch of a pilot-scale cavity CRES experiment with an atomic tritium beamline.

₁₃₈₁ The cavity design for the pilot-scale experiment consists of a large cylindrical cavity
₁₃₈₂ with a TE011 resonance of 325 MHz. Such a cavity is truly enormous, with a diameter
₁₃₈₃ of approximately 1.2 m and a height of 10 m. When an electron is produced inside
₁₃₈₄ the cavity with a cyclotron frequency that matches the TE011 resonant frequency it's

1385 cyclotron orbit couples the electron to the TE011, which drives a resonance in the cavity.
1386 These resonant fields can be read-out using an appropriate cavity coupling mechanism
1387 located at the center of the cavity. For more information on the cavity approach to
1388 CRES see Chapter 6.

1389 The bottom of the cavity has a cone termination to match the contour of the atom
1390 trapping magnet. This shape still allows for TE011 resonances with high internal Qs,
1391 which are required for good SNR in the cavity experiment. A small opening in the bottom
1392 of the cone serves as an entry point for the tritium atoms. To allow for calibration of
1393 the magnetic field inhomogeneities with an electron gun, the top of the cavity is left
1394 nearly completely open. Normally, this would drastically lower the Q-factor of the TE011
1395 mode, but a specially configured coaxial partition is inserted at the top. This termination
1396 scheme is designed to act as a perfect short for the TE011 mode since the circular shape
1397 of the partition matches the electric field boundary conditions for the TE011 mode.
1398 Simulations with HFSS have confirmed that this design results in a high quality TE011
1399 resonance despite the nearly completely open end.

1400 3.6 Phase IV

1401 The baseline CRES technology being pursued by the Project 8 collaboration are resonant
1402 cavities, which, due to their geometric properties, simple CRES signal structure, and low
1403 channel count, appear to be the better option for Phase IV. The current knowledge of the
1404 antenna array CRES approach reveals no technical obstacles that would preclude it as a
1405 baseline technology for Phase IV though it would most certainly be significantly more
1406 expensive. Therefore, antenna arrays represent a fallback approach if resonant cavities
1407 prove infeasible.

1408 The sensitivity of the pilot-scale atomic tritium experiment is estimated to be on
1409 the order of 0.1 eV, which means that increasing the sensitivity to reach the Phase IV
1410 goal will require a larger volume experiment. Because of the direct coupling between the
1411 RF characteristics of a cavity and its geometry, the baseline plan is to build multiple
1412 copies of the pilot-scale experiment (see Figure 3.21) to obtain the required amount of
1413 volume rather than increase the size of the cavity beyond the pilot-scale. The built-in
1414 redundancy of this approach is attractive in that the experiment has no single point of
1415 failure, additionally, building several copies of the a pilot-scale experiment will require
1416 minimal new engineering and design.

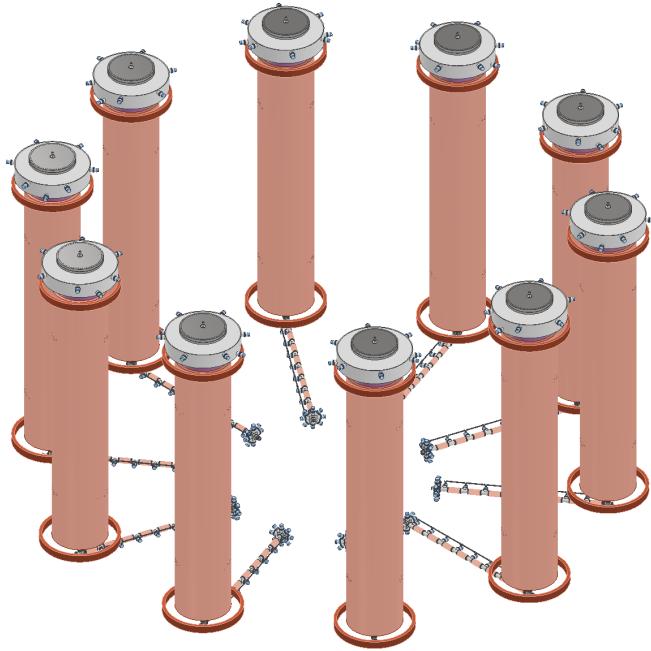


Figure 3.21: An illustration of a possible arrangement of ten pilot-scale cavity experiments for Phase IV. The experiments are arranged in a circle with an approximate diameter of 50 meters. Each atomic beamline connected to the bottom of each cavity is approximately 10 m in length. The cavities themselves are designed to operate at 325 MHz and are approximately 11 m tall. The circular arrangement of cavities has some advantages when it comes to cancellation of fringe fields from neighboring magnets, which is important due to the small magnetic field magnitudes consistent with these CRES frequencies. The advantage of ten independent atomic sources and cavities is that there is no single point of failure for the experiment. If an experiment goes down for repairs the other nine may continue running. Figure courtesy of Michael Huehn at UW-Seattle.

Chapter 4

Signal Reconstruction Techniques for Antenna Array CRES and the FSCD

4.1 Introduction

The transition from a waveguide CRES experiment to an antenna array CRES experiment introduces new challenges related to data acquisition, signal detection, and signal reconstruction caused by the multi-channel nature of the data. The development of signal reconstruction algorithms is crucial to the design of antenna array based experiments like the FSCD, because these algorithms directly influence the detection efficiency and energy resolution of the CRES experiment. In this Chapter I summarize my contributions to the development and analysis of signal reconstruction and detection algorithms for the FSCD experiment.

In Section 4.2 I discuss the primary tool for this work, which is the Locust simulations package developed by the Project 8 experiment. Locust is used to simulate CRES events in the detector. Locust uses Kassiopeia to calculate particle trajectory solutions for electrons in the magnetic trap. The trajectories are then used to calculate the response of the antenna array to the cyclotron radiation produced by the electron, which results in signals that can be used to analyze the performance of different signal reconstruction algorithms. More recently, Project 8 has developed CREsana, which is a new simulations package that takes a more analytical approach to CRES signal simulations for antenna arrays. Although CREsana signals were not used for the signal reconstruction algorithm development detailed here, we introduce the software as it plays a role in the antenna array measurements presented in Section 5.4.

In Section 4.3 I discuss the signal reconstruction and detection approaches analyzed for the FSCD experiment. In general there are two steps to signal reconstruction — detection and parameter estimation. With signal detection one is primarily concerned

1443 only with distinguishing between data that contains a signal versus data that contains only
1444 noise, whereas, with parameter estimation one is interested in extracting the kinematic
1445 parameters of the electron encoded in the cyclotron radiation signal shape. Due to
1446 the low signal power of electrons near the spectrum endpoint in the FSCD experiment,
1447 signal detection is a non-trivial problem. This is magnified by the need to maximize the
1448 detection efficiency of the experiment in order to achieve the neutrino mass sensitivity
1449 goals. My contributions to signal reconstruction analysis for the FSCD are focused on
1450 this signal detection component of reconstruction.

1451 After the discussion of various signal detection approaches, in Section 4.4 I present a
1452 more detailed analysis of the detection performance of three algorithms, which could be
1453 used to signal detection in the FSCD. This section was originally prepared for publication
1454 in JINST as a separate paper. The algorithms include a digital beamforming algorithm,
1455 a matched filter algorithm, and a neural network algorithm, which I analyze in terms of
1456 classification accuracy and estimated computational cost.

1457 **4.2 FSCD Simulations**

1458 Antenna array CRES and the FSCD requires a combination of different capabilities
1459 not often found in a single simulation tool. First of all, accurate calculations of the
1460 magneto-static fields produced by current-carrying coils are required in order to accurately
1461 model the magnetic trap and background magnets. The resulting magnetic fields must
1462 then be used to calculate the exact relativistic trajectory of electrons, which is required
1463 in order to calculate the electro-magnetic (EM) fields produced by the acceleration of
1464 the electron. Finally, the simulation has to model the interaction of the antenna and
1465 RF receiver chain with these EM-fields in order to produce the simulated voltage signals
1466 produced by the antenna array during the CRES event. At the time when Project 8 was
1467 developing this simulation capability, no single available simulation tool was known to
1468 adequately perform this suite of calculations, which prompted the development of custom
1469 simulation framework to simulate the FSCD. This simulation framework includes custom
1470 simulation tools developed by Project 8 as well as other open-source and proprietary
1471 software developed by third-parties.

¹⁴⁷² 4.2.1 Kassiopeia

¹⁴⁷³ Kassiopeia¹ is a particle tracking and static EM-field solver developed by the KATRIN
¹⁴⁷⁴ collaboration for simulations of their spectrometer based on magnetic adiabatic collimation
¹⁴⁷⁵ with an electrostatic filter [7]. Due to the measurement technique employed by the
¹⁴⁷⁶ KATRIN collaboration, Kassiopeia is not designed to solve for the EM-fields produced by
¹⁴⁷⁷ electrons in magnetic fields. However, it does provide efficient solvers for static electric
¹⁴⁷⁸ and magnetic fields and charged particle trajectory solvers. Because of this, Project 8
¹⁴⁷⁹ has incorporated parts of Kassiopeia into its own simulation framework.

¹⁴⁸⁰ Magnetostatic Field Solutions

¹⁴⁸¹ The solutions to the electric and magnetic fields generated by a static configuration of
¹⁴⁸² charges and currents is given by Maxwell's equations in the limit where the time-dependent
¹⁴⁸³ terms go to zero. In their static form Maxwell's equations [8] are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (4.1)$$

$$\nabla \times \mathbf{E} = 0 \quad (4.2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (4.4)$$

¹⁴⁸⁴ where we can see that the electric and magnetic fields are now completely decoupled
¹⁴⁸⁵ from each other. The solution for the magnetic field in this boundary value problem is
¹⁴⁸⁶ given by the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dr' \frac{r'^3 \mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|^3}, \quad (4.5)$$

¹⁴⁸⁷ which Kassiopeia uses a variety of numeric integration techniques to solve for a user
¹⁴⁸⁸ defined current distribution.

¹⁴⁸⁹ Kassiopeia Simulation of the FSCD Magnetic Trap

¹⁴⁹⁰ The trap developed for the FSCD experiment utilizes six current carrying coils, which
¹⁴⁹¹ surround a cylindrical tritium containment vessel (see Figure 4.1). Some critical aspects
¹⁴⁹² of the trap design include the total trapping volume, the maximum trap depth, the

¹<https://github.com/KATRIN-Experiment/Kassiopeia>

¹⁴⁹³ steepness of the trap walls, as well as the radial and azimuthal uniformity of the magnetic
¹⁴⁹⁴ fields.

¹⁴⁹⁵ The volume of the FSCD trap is a cylindrically shaped region with a radius of 5 cm
¹⁴⁹⁶ and a length of 15 cm resulting in a roughly 1 L total trap volume. The trap volume is an
¹⁴⁹⁷ important design feature, because it sets the volume of the experiment that is potentially
¹⁴⁹⁸ usable for CRES measurements. Trapping a larger volume allows one to observe a larger
¹⁴⁹⁹ number of tritium atoms, which increases the statistical power and sensitivity of the
¹⁵⁰⁰ neutrino mass measurement. Due to the cost of constructing magnets with large and
¹⁵⁰¹ uniform magnetic fields it is important that the trap use as much of the available volume
¹⁵⁰² as possible to limit the overall cost of the experiment.

| Coil | Radius (mm) | Z Pos. (mm) | Current (A × Turns) |
|------|-------------|-------------|---------------------|
| 1 | 50.0 | -92.3 | 750.0 |
| 2 | 50.1 | -56.9 | -220.3 |
| 3 | 68.5 | -19.5 | -250.0 |
| 4 | 68.5 | 19.5 | -250.0 |
| 5 | 50.1 | 56.9 | -220.3 |
| 6 | 50.0 | 92.3 | 750.0 |

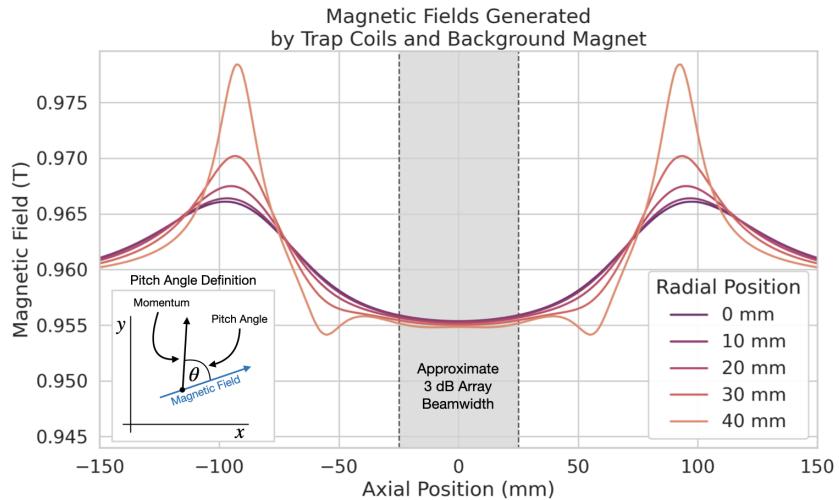
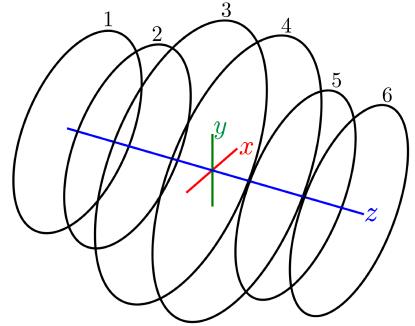


Figure 4.1: The geometry and parameters of the coils used to simulate the FSCD magnetic trap in Kassiopeia. Some axial profiles of the magnetic trap at different radial positions are show to demonstrate the shape of the magnetic field and trap depth as a function of position. Calculation of the magnetic field profiles was graciously done by René Reimann.

¹⁵⁰³ The depth of the FSCD trap is approximately 10 mT when measured along the
¹⁵⁰⁴ central axis, which is sufficient to trap electrons with pitch angles as small as 84°. The

trap depth factors into the efficiency of the experiment by directly controlling the range of electron pitch angles that can be trapped. If a higher fraction of pitch angles are trapped then, in principle, more decay events can be observed. However, the signals from electrons with small pitch angles are typically significantly harder to detect than larger pitch angles when using an antenna array, which increases the likelihood of not detecting the first track of the CRES event and harms the energy resolution of the experiment.

The steepness of the trap walls as well as any non-uniformities in the magnetic field contribute to the total energy resolution of the CRES measurement by causing uncertainty in the relationship between an electron's kinetic energy and it's cyclotron frequency. When an electron is trapped, it oscillates back and forth along the trap z-axis (see Figure 4.1) unless it is produced with a pitch angle of exactly 90° [9]. As the electron is reflected from the trap walls it experiences a change in the total magnetic field, which causes a modulation in the cyclotron frequency. This change in magnetic field from the trap introduces a correlation between the pitch angle and kinetic energy parameters of the electron that can reduce energy resolution. In order to mitigate this effect it is important to make the trap walls as steep as possible.

Particle Trajectory Solutions

The magnetic fields solved by direct integration of the electron's current density can be used by Kassiopeia to solve for the trajectory of electrons based on user specified initial conditions. Various distributions are available within Kassiopeia that can be sampled in order to replicate realistic event statistics, including uniform, Gaussian, and Lorentzian among others. In general, an electron has six kinematic parameters that define its trajectory, which are the three-dimensional coordinates of the initial position and the three components of the electron's momentum vector. However, when simulating CRES events it is more common to parameterize the electron's trajectory in terms of it's initial position, the kinetic energy, the pitch angle, and the initial direction of the component of the electron's momentum perpendicular to the magnetic field. This parameterization is completely equivalent to specify each component of the electrons initial position and momentum vectors.

From the initial parameters of the electron and the magnetic field, Kassiopeia solves for the trajectory of the electron. The direct approach proceeds by solving the motion of the electron using the Lorentz force equation, which takes the form of a set of differential

1537 equations

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\gamma m} \quad (4.6)$$

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{\gamma m}), \quad (4.7)$$

1538 where \mathbf{r} is the position of the electron, \mathbf{p} is the electron's momentum, e is the charge of
1539 the electron, m is the electron's mass, and γ is the relativistic Lorentz term. To account
1540 for kinetic energy losses from radiation Kassiopeia includes an additional term in the
1541 momentum differential equation, which calculates the change in the electron's momentum
1542 induced by synchrotron radiation. Kassiopeia solves this pair of differential equations
1543 using numerical integration, however, the exact trajectory can be computationally
1544 intensive to solve. If the adiabatic approximation can be applied, then Kassiopeia can
1545 make use of a simpler set of equations that can be more readily solved numerically.

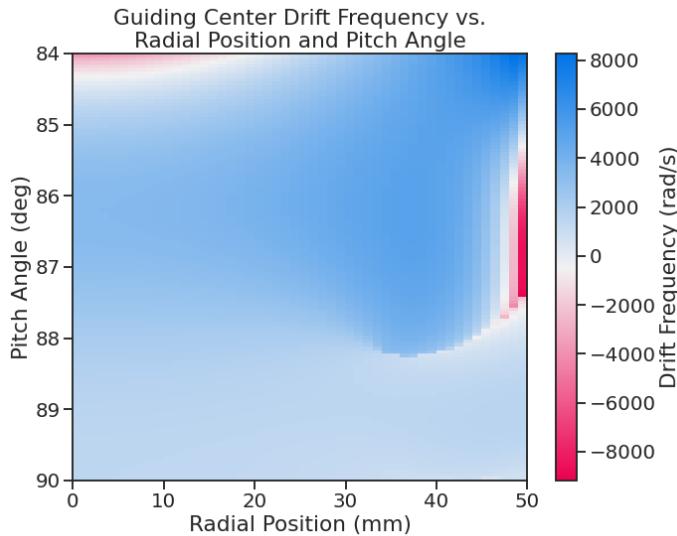


Figure 4.2: A map of the average ∇B -drift frequency for electrons trapped in the prototype FSCD trap shown in Figure 4.1. Negative drift frequencies indicate electrons that are drifting opposite to the standard direction, which means that they are close to escaping the magnetic trap.

1546 Even though Kassiopeia is not directly capable of simulating the cyclotron radiation,
1547 it is still an invaluable CRES simulation tool, due to the accurate trajectory solutions
1548 for electrons in magnetic traps. With Kassiopeia it is possible to test the efficiency of a
1549 particular trap design and analyze features of the electron trajectories that are important
1550 to the position, track, and event reconstruction algorithms (see Section 4.3). One example

of this for the FSCD is the analysis of the average ∇B -drift frequency as a function of
 the electrons radial position and pitch angle in the magnetic trap (see Figure 4.2). Radial
 gradients in the trap cause the guiding center of the electron to drift around the center of
 the magnetic trap with an average frequency on the order of 10^3 rad/s. This frequency,
 while slow compared to the length of a typical CRES time-slice, is large enough to cause
 a significant loss in efficiency of certain signal reconstruction algorithms. Therefore, it is
 important to model the drift of the electron in the reconstruction algorithm in order to
 mitigate the effects of this motion on the reconstruction.

4.2.2 Locust

The Locust² software package [10] is the primary simulation tool developed and used
 by the Project 8 collaboration for CRES experiments. Locust simulates the responses
 of antennas and receiver electronics chain to rapidly time-varying electric fields using
 a flexible approach that allows one to choose from a variety of electric field sources
 and antennas. Similarly, one can simulate the receiver chain using a series of modular
 generators that include standard signal processing operations such as down-mixing and
 fast Fourier transforms (FFT). Since the primary focus of this chapter is the application
 of Locust to analyses of the FSCD, we shall describe only the most relevant aspects of
 the software rather than provide a comprehensive description.

Cyclotron Radiation Field Solutions

Simulating CRES events in the FSCD requires that we calculate the electric fields
 produced by the acceleration of the electron. In the general case, this can be a complicated
 question to answer, due to back-reaction forces on the electron from its own electric fields
 that occur when the electron is surrounded by conductive material such as a waveguide
 or cavity. However, in the case of the FSCD it is possible to ignore such effects and
 approximate the electron as radiating into a free-space environment.

The equations that describe the electromagnetic fields from a relativistic moving
 point particle are the Liénard-Wiechert field equations [11, 12], which are obtained by
 differentiating the Liénard-Wiechert potentials. In their full form the Liénard-Wiechert
 field equations are

$$\mathbf{E} = e \left[\frac{\hat{n} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta} \cdot \hat{n})^3 |\mathbf{R}|^2} \right]_{t_r} + \frac{e}{c} \left[\frac{\hat{n} \times [(\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{n})^3 |\mathbf{R}|} \right]_{t_r} \quad (4.8)$$

²https://github.com/project8/locust_mc/tree/master

$$\mathbf{B} = [\hat{n} \times \mathbf{E}]_{t_r}, \quad (4.9)$$

where e is the charge of the particle, \hat{n} is the unit vector pointing from the particle to the position where the fields are calculated, β and $\dot{\beta}$ are the velocity and acceleration of the particle divided by the speed of light (c), \mathbf{R} is the distance from the particle to the field calculation position, and γ is the relativistic Lorentz term. The subscript t_r indicates that the equations must be evaluated at the retarded time so that the time-delay from the travel time of the electromagnetic radiation is correctly accounted for.

The only required input to calculate the electric field at the position of an FSCD antenna is the velocity and acceleration of the electron, which can be obtained from Kassiopeia simulations. Therefore, when simulating a CRES event Locust first runs a Kassiopeia simulation of the electron and calculates the electric field incident on the antenna. The only difficulty with this approach is the determination of the retarded time. The retarded time corresponds to the time that a photon, which has just arrived at an antenna at the space-time position (t, \mathbf{r}) , was actually emitted by the electron at the space-time position of $(t_r, \mathbf{r}_e(t_r))$. Defined in this way, finding the retarded time requires solving

$$c(t - t_r) = |\mathbf{r} - \mathbf{r}_e(t_r)|, \quad (4.10)$$

where the distance traveled by the photon between the measurement and retarded times is equal to the distance between the antenna and the electron at the retarded time. Locust solves Equation 4.10 using a built-in root finding algorithm to find the retarded time, and thus the electric field produced by the electron at the position of each antenna in the FSCD array.

1600 Antenna Response Modeling

With the electric field it is possible, in principle, to calculate the resulting voltages produced in the antenna. However, direct simulation of the antenna itself is computationally expensive since it would require the modeling of complex interactions of the electron's electric fields with charge carriers in the conductive elements of the antenna. Direct simulation of the antenna in Locust can be avoided by modeling the antenna response using the antenna factor, or antenna transfer function, approach. The antenna factor defines the voltage produced in the antenna terminal for an incident electric field [13],

$$A_F = \frac{V}{|\mathbf{E}|}, \quad (4.11)$$

1608 where V is the voltage and $|\mathbf{E}|$ is the magnitude of the incident electric field. To obtain the
 1609 antenna factor for the antennas developed for the FSCD Project 8 employs Ansys HFSS.
 1610 HFSS is a commercially available finite element method electromagnetic solver widely
 1611 used throughout the antenna engineering industry [14]. HFSS is capable of calculating
 1612 the antenna factor and gain patterns for complex antenna designs and outputting the
 1613 resulting quantities in the form of a text file that can be used as an input to the Locust
 1614 simulation.

1615 The antenna factor defines the steady-state response of the antenna to electromagnetic
 1616 plane waves and is a function of the frequency of the radiation. Therefore, in order to
 1617 apply the transfer function for the calculation of the antenna voltage response in the
 1618 time domain, Locust models the antenna as a linear time-invariant system [15]. In this
 1619 formalism the response of the system to the driving force is given by

$$y[n] = h * x = \sum_k h[k]x[n - k], \quad (4.12)$$

1620 where $y[n]$ is the discretely sampled response, x is the driving force stimulus, and h is
 1621 the finite impulse response (FIR) filter. When applied to the FSCD array, this formalism
 1622 calculates the voltage time-series produced in each antenna by convolving the electric
 1623 field time-series with the antenna FIR filter, which is obtained by performing a inverse
 1624 Fourier transform on the transfer function from HFSS.

1625 Radio-frequency Receiver and Signal Processing

1626 After obtaining the voltage time-series by computing the electron trajectory and antenna
 1627 response, Locust simulates the signal processing associated with the radio-frequency
 1628 receiver chain. The standard receiver chain used in Locust simulations of the FSCD
 1629 attempts to mimic the operations that would actually occur in hardware (see Figure 4.3).

1630 Frequency down-conversion is used in the FSCD to reduce the digitization bandwidth
 1631 required to read-out CRES data. According to the Nyquist sampling theorem [16], the
 1632 minimal sampling rate that guarantees no information loss for a signal with a bandwidth
 1633 Δf is given by

$$f_{\text{Nyq}} = 2\Delta f. \quad (4.13)$$

1634 The total bandwidth of CRES signal frequencies from tritium beta-decay ranges from 0
 1635 to 26 GHz in a 0.95 T magnetic field, therefore, direct digitization of CRES signals from
 1636 the FSCD would require sampling frequencies greater than 50 GHz, which is infeasible for

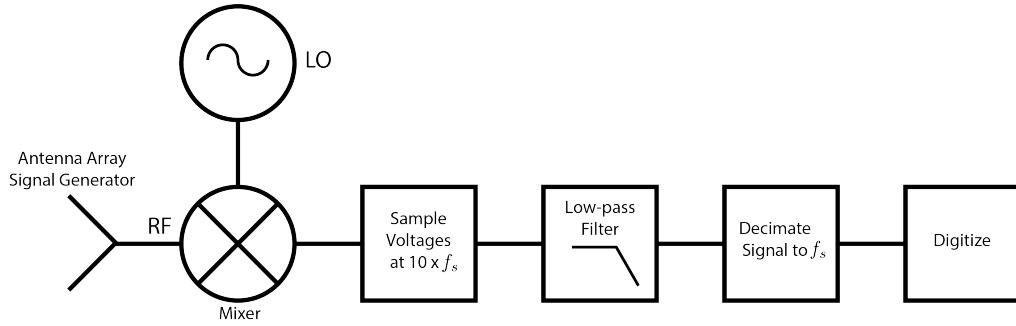


Figure 4.3: The receiver chain used by Locust when simulating CRES events in the FSCD.

1637 a real experiment. However, for the purposes of neutrino mass measurement we are only
 1638 interested in measuring the shape of the spectrum in the last 100 eV, which corresponds
 1639 to a frequency bandwidth of 5 MHz. Down-conversion is a technique for reducing the
 1640 base frequencies of signals in a bandwidth given by $[f_{\text{LO}}, f_{\text{LO}} + \Delta f]$ to the bandwidth
 1641 $[0, \Delta f]$, by performing the following multiplication

$$x(t) \rightarrow x(t)e^{-2\pi f_{\text{LO}} t}. \quad (4.14)$$

1642 In down-conversion the signal ($x(t)$) is multiplied by a sinusoidal signal with frequency
 1643 f_{LO} to reduce the absolute frequencies of the signals in the bandwidth. In the FSCD this
 1644 allows us to detect events in the last 100 eV of the tritium spectrum while sampling the
 1645 data far below 50 GHz. The standard bandwidth used in the FSCD is 200 MHz, which
 1646 allows for higher frequency resolution than the minimum sampling frequency for 100 eV
 1647 of energy bandwidth.

1648 Trying to directly simulate down-conversion with a frequency multiplication in Locust
 1649 would require the sampling of the electric fields at each antenna in the FSCD array with
 1650 a period of ≈ 20 ps, which is extremely slow computationally. To avoid this Locust
 1651 performs the down-conversion by intentionally under-sampling the electric fields with
 1652 a frequency of 2 GHz. Sampling below the Nyquist limit causes the higher frequency
 1653 components of the CRES signal to alias, however, Locust can remove these aliased
 1654 frequency peaks using a combination of low-pass filtering and decimation to recreate
 1655 frequency down-conversion. After filtering and decimation, Locust simulates digitization
 1656 by an 8-bit digitizer at a sampling frequency of 200 MHz to recreate the conditions of
 1657 the FSCD. The voltage offset and the digitizer range must be configured by the user
 1658 based on the characteristics of the simulation.

1659 **Data**

1660 The output of Locust simulations for the FSCD primarily consists of two data files. The
1661 first is the electron trajectory information calculated by Kassiopiea, which is output in
1662 the form of a `.root` file [17]. This file contains important kinematic information about
1663 the electron such as it's position and pitch angle as a function of time. The other file is
1664 produced by Locust and it contains the digitized signals acquired from each antenna in
1665 the FSCD array. The Locust output files conform to the Monarch specification developed
1666 by Project 8, which is based on the commonly used HDF5 file format, and matches the
1667 format of the files produced by the Project 8 data acquisition software. This makes it
1668 possible to use the same data analysis code to analyze both simulated and real data.

1669 **4.2.3 CRESana**

1670 Locust is the primary simulation tool used by Project 8 in the development and simulation
1671 of the FSCD. However, simulations of CRES events in larger antenna arrays (≥ 100
1672 antennas) using Locust can take several hours to complete, which is prohibitively long
1673 when one is performing a sensitivity analysis for a large scale antenna experiment. One
1674 of the reasons for Locust's slow operation is that the electric fields from the electron
1675 must be solved numerically for each time-step for each of the antennas in the array.
1676 These numerical solutions allow Locust to accurately simulate the electric fields from
1677 arbitrarily complicated electron trajectories at the cost of more computations and slower
1678 simulations. Therefore, an additional simulation tool that sacrifices some accuracy for
1679 computational efficiency would be extremely useful simulations and sensitivity analyses
1680 of larger antenna array experiments.

1681 To fill this need, Project has developed a new simulations package called CRESana³,
1682 specifically designed to perform analytical simulations of antenna array based CRES
1683 experiments. CRESana is not as flexible as Locust, but it provides a significant increase
1684 in simulation speed. It does this by using well-justified analytical approximations of the
1685 electrons motion in the magnetic field and the resulting electric fields from the electron's
1686 acceleration. The electric fields and signals generated by CRESana are consistent with
1687 theoretical calculations of the electron's radiation, and are test for accuracy using
1688 well-known test-case simulations and consistency checks.

³<https://github.com/MCFlowMace/CRESana>

1689 4.3 Signal Detection and Reconstruction Techniques for 1690 Antenna Array CRES

1691 Antenna Array CRES Signal Reconstruction

1692 A robust set of FSCD simulation tools are vital to the development of the analysis
1693 algorithms necessary for antenna array CRES to succeed. In order to perform CRES
1694 measurements using an antenna array, one must develop an algorithm that uses the
1695 multi-channel time-series obtained by digitizing the array to estimate the starting kinetic
1696 energies of electrons produced in the magnetic trap. This procedure consists of a multi-
1697 stage process of detecting a CRES signal then estimating the parameters of the electron
1698 that produced and is often referred to as simply CRES signal reconstruction.

1699 Compared with the signal reconstruction approaches of the Phase I and II CRES
1700 experiments, antenna array CRES requires a significantly different approach to signal
1701 reconstruction. In Phase I and II, CRES was performed using a waveguide gas cell that
1702 could be directly connected to a waveguide transmission line. The transmission line
1703 efficiently transmits the cyclotron radiation along its length to an antenna at either end
1704 of the waveguide. However, with an antenna array the electron is essentially radiating
1705 into free-space, therefore, the cyclotron radiation power collected by the array is directly
1706 proportional to the solid angle surrounding the electron that is covered with antennas.
1707 Because it is not practical to fully surround the magnetic trap with antennas, some of the
1708 cyclotron radiation power that would have been collected by the waveguide escapes into
1709 free-space. Furthermore, the power that is collected by the antenna array is split between
1710 every channel in the antenna array, which significantly lowers the signal-to-noise ratio
1711 (SNR) of CRES signals in a single antenna channel compared to a waveguide apparatus.
1712 Therefore, a suite of completely new signal reconstruction techniques are needed in order
1713 to perform CRES in the FSCD.

1714 Changes to the approach to CRES signal reconstruction are also motivated by the
1715 more ambitious scientific goals of the FSCD experiment. A measurement of the tritium
1716 beta-decay spectrum that is sensitive to neutrino masses as small as 40 meV requires that
1717 we measure the kinetic energies of individual electrons with a total energy broadening
1718 of 115 meV [18]. This resolution includes all sources of uncertainty in the electron's
1719 kinetic energy such as magnetic field inhomogeneities. This level of energy resolution is
1720 compatible only with an event-by-event signal reconstruction approach where the kinetic
1721 energies, pitch angles, and other parameters of the CRES events are estimated before

1722 constructing the beta-decay spectrum.

1723 The event-by-event approach is distinct from the analysis done for the Phase I and
1724 Phase II experiments where only the starting cyclotron frequency of the event was
1725 estimated by analyzing the tracks formed by the carrier frequency in the time-frequency
1726 spectrogram. These frequencies were then combined into a frequency spectrogram, which
1727 was converted to the beta-decay energy spectrum using an ensemble approach that
1728 averaged over all other event parameters. The ensemble approach to signal reconstruction
1729 results in poor energy resolution because other kinematic parameters such as pitch angle
1730 change the cyclotron carrier frequency due to changes in the average magnetic field
1731 experience by the electron, and it is therefore incompatible with the future goals of the
1732 Project 8 collaboration.

1733 Components of Reconstruction: Signal Detection and Parameter Estimation

1734 CRES signal reconstruction can be viewed as a two-step procedure consisting of signal
1735 detection followed by parameter estimation. In the former, one is concerned with
1736 identifying CRES signals in the data regardless of the signal parameters, whereas, in the
1737 latter one operates under the assumption that a signal is present and then estimates its
1738 parameters.

1739 More formally, signal detection is essentially a binary hypothesis test between the
1740 signal and noise data classes and parameter estimation describes a procedure of fitting a
1741 model to the observed data. While both of these processes are required for a complete
1742 reconstruction (see Figure 4.4), the focus of my work and this chapter is on the signal
1743 detection aspect of antenna array CRES signal reconstruction.

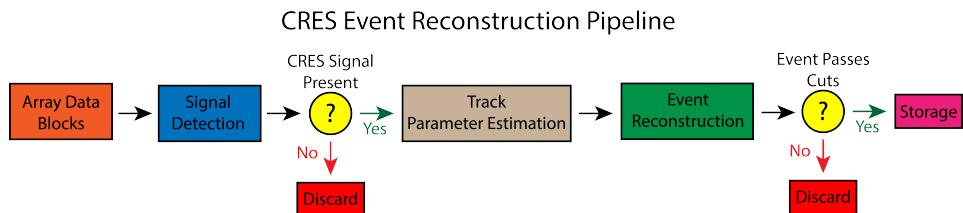


Figure 4.4: A high-level diagram depicting the process of CRES event reconstruction. The first step consists of identifying the presence of a signal in the data. This step is necessary to avoid the danger of performing a reconstruction of a false event, which would constitute a background contribution to the tritium spectrum measured by CRES.

1744 **Detection Theory**

1745 The problem of signal detection can be posed as a statistical hypothesis test [19]. For
1746 CRES signals, which are essentially vectors with added white Gaussian noise (WGN),
1747 one needs to choose between two hypotheses

$$\mathcal{H}_0 : \mathbf{y} = \boldsymbol{\nu} \quad (4.15)$$

$$\mathcal{H}_1 : \mathbf{y} = \mathbf{x} + \boldsymbol{\nu}, \quad (4.16)$$

1748 where \mathbf{y} is the CRES data vector, $\boldsymbol{\nu}$ is a sample of WGN, and \mathbf{x} represents the CRES
1749 signal. The hypothesis that the data contains only noise is labeled \mathcal{H}_0 and the hypothesis
1750 that the data contains a signal is labeled \mathcal{H}_1 .

1751 For illustrative purposes one can examine the case where one the first sample of
1752 data is used to distinguish between \mathcal{H}_0 and \mathcal{H}_1 . The value of the first data sample is
1753 distributed according to two gaussian distributions corresponding to \mathcal{H}_0 and \mathcal{H}_1 (see
1754 Figure 4.5). By setting a decision threshold on the value of this sample, one can choose
1755 the correct hypothesis with a probability given by the areas underneath the probability
1756 distribution curves. A true positive corresponds to correctly identifying that the data
1757 contains signal, whereas, a true negative means that one has correctly identified the data
1758 as noise. The rate at which the detector performs a true positive classification is given
1759 by the green region underneath $p(\mathbf{y}[0]; \mathcal{H}_0)$, and the rate at which the detector performs
1760 a true negative classification is given by the orange region underneath $p(\mathbf{y}[0]; \mathcal{H}_1)$. Two
1761 types of misclassifications are possible. Either we declare noise data as signal, which is
1762 call a false positive, or we declare signal data as noise, which is a false negative. Note
1763 that it is only possible to trade off these two types of errors by tuning the detection
1764 threshold. One cannot simultaneously reduce the rate of false positives without also
1765 increasing the rate of false negatives.

1766 The approach taken with CRES signals is to fix the rate of false positives by setting
1767 a minimum value for a detection threshold. The rate of false positives that is acceptable
1768 at the detection stage depends upon the rate of background events compatible with the
1769 sensitivity goals of the experiment. The ultimate goal of a neutrino mass measurement
1770 with 40 meV sensitivity in general has strict requirements on the number of background
1771 events, which requires a relatively high detection threshold to achieve. Consequently,
1772 the ideal signal detection algorithm is the one that achieves the maximum rate of true
1773 positives for a fixed rate of false positives, so that the detection efficiency of the experiment
1774 is maximized and potential sources of background are kept to a minimum.

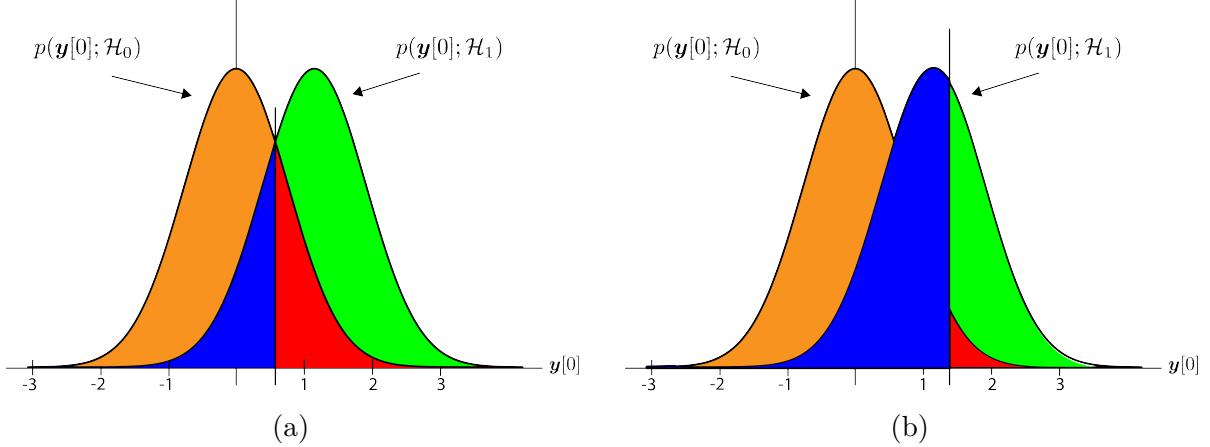


Figure 4.5: An illustration of two PDFs associated with a binary hypothesis test. The decision threshold is represented by the vertical line that partitions both distributions. The orange and red areas correspond to the true negative and false positive probabilities and the blue and green areas correspond to the false negative and true positive probabilities respectively. To decide between the two hypotheses we perform the likelihood ratio test specified by the Neyman-Pearson theorem. This approach achieves the highest true positive probability for a given false positive probability.

1775 According to the Neyman-Pearson theorem [20], the statistical hypothesis test that
 1776 maximizes the probability of detection for a fixed rate of false positives is the likelihood
 1777 ratio test, which is formed by computing the ratio of the signal likelihood to the noise
 1778 likelihood,

$$L(x) = \frac{P(\mathbf{y}; \mathcal{H}_1)}{P(\mathbf{y}; \mathcal{H}_0)} > \gamma. \quad (4.17)$$

1779 Here, the likelihood of the hypotheses \mathcal{H}_0 and \mathcal{H}_1 are described by the probability
 1780 distributions $P(\mathbf{y}; \mathcal{H}_0)$ and $P(\mathbf{y}; \mathcal{H}_1)$ respectively, and γ is the threshold for deciding \mathcal{H}_1 .
 1781 The decision threshold is determined by integrating $P(\mathbf{y}; \mathcal{H}_0)$ such that

$$P_{\text{FP}} = \int_{\gamma}^{\infty} P(\tilde{\mathbf{y}}; \mathcal{H}_0) d\tilde{\mathbf{y}} = \alpha, \quad (4.18)$$

1782 where α is the desired false positive detection rate given by the red colored areas shown
 1783 in Figure 4.5. The true positive detection rate is given by the similar integral

$$P_{\text{TP}} = \int_{\gamma}^{\infty} P(\tilde{\mathbf{y}}; \mathcal{H}_1) d\tilde{\mathbf{y}}, \quad (4.19)$$

1784 which corresponds to the green areas in Figure 4.5.

1785 Changing the decision threshold allows one to trade-off between P_{TP} and P_{FP} as

appropriate for the given situation. It is common to summarize the relationship between P_{TP} and P_{FP} using the receiver operating characteristic (ROC) curve, which is obtained by evaluating the true positive and false positive probabilities as a function of the decision threshold value (see Figure 4.6). The ROC curve provides a convenient way to compare

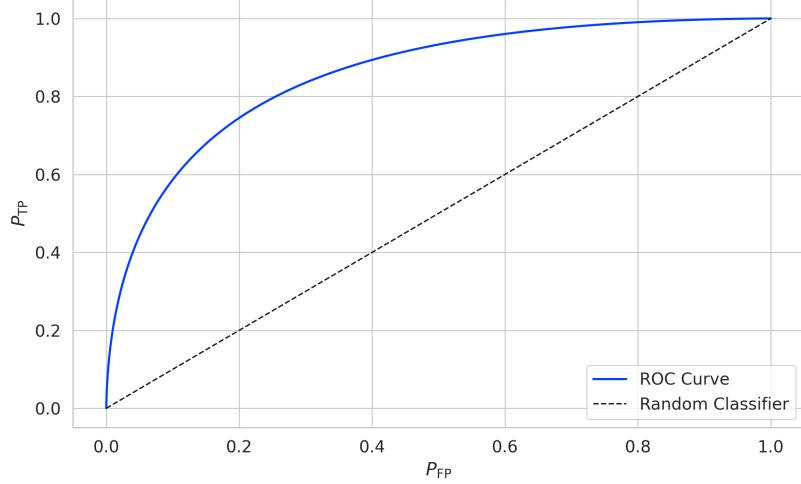


Figure 4.6: An example ROC curve formed by computing the P_{FP} and the P_{TP} for a given likelihood ratio test. As the decision threshold is increased P_{FP} decreases at the expense of a lower P_{TP} . The black dashed line indicates the lower bound ROC curve obtained by randomly deciding between \mathcal{H}_0 and \mathcal{H}_1 .

the performance of different signal detection algorithms. In general, a classifier with a higher the P_{TP} as a function of P_{FP} is desirable, which corresponds to a larger area underneath the respective ROC curve. A perfect classifier has an area underneath the curve of 1.0, however, such a classifier is almost never achievable in practice.

4.3.1 Digital Beamforming

Introduction to Beamforming

Beamforming refers to a suite of antenna array signal processing techniques that are designed to enhance the radiation or gain of the array in certain directions and suppress it in other direction [13]. Beamforming is of interest to Project 8 as a first level of signal reconstruction for the FSCD and other antenna array CRES experiments, which operates at the signal detection stage of reconstruction.

Beamforming is accomplished by performing a phased summation of the signals received by the antenna array. The beamforming phases are chosen such that the signals

1803 emitted by the array will constructively interfere at the point of interest (see Figure
 1804 4.7). As a consequence of the principle of reciprocity [21], when the array is operating in
 1805 receive mode, the signals emitted from a source at the same point will constructively
 interfere when summed. The origin of the phase delays in beamforming is the path-

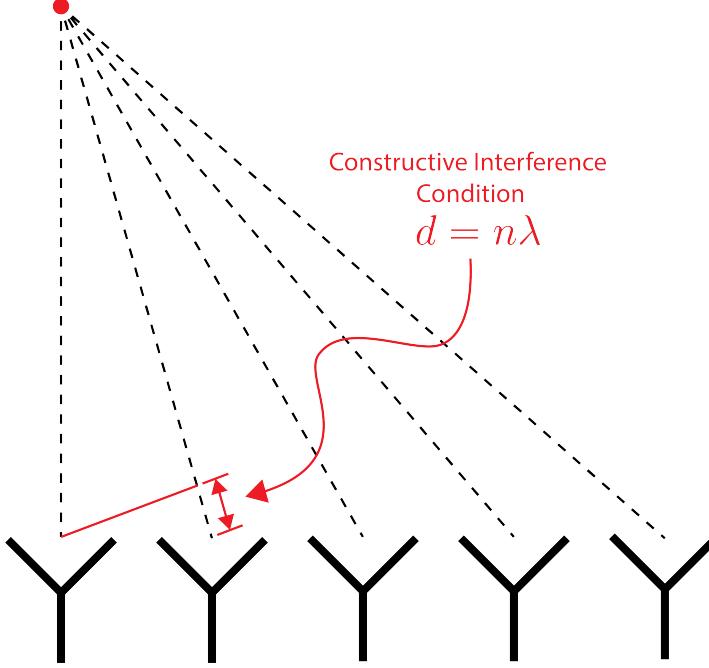


Figure 4.7: An illustration of the constructive interference condition which is the operating principle of digital beamforming using a uniform linear array as an example.

1806
 1807 length difference to the beamforming point between different antennas in the array. The
 1808 relationship between the phase delay and the path-length difference is given by the
 1809 familiar equation

$$\phi = \frac{2\pi d}{\lambda}, \quad (4.20)$$

1810 where ϕ is the phase delay, d is the path-length difference, and λ is the wavelength of
 1811 the radiation. In practice, one chooses the values of d by specifying the beamforming
 1812 positions of interest and then calculates the beamforming phases using Equation 4.20,
 1813 which is guaranteed to follow the constructive interference condition shown in Figure 4.7.

1814 Beamforming can be neatly expressed mathematically using the vector equation

$$y[n] = \Phi^T[n] \mathbf{x}[n], \quad (4.21)$$

1815 where $\mathbf{x}[n]$ is the array snapshot vector, $\Phi[n]$ is a vector of beamforming shifts, and
 1816 $y[n]$ is the resulting summed signal. The beamforming shifts consist of a set of complex

1817 numbers that contain the beamforming phase shift and an amplitude weighting factor,

$$\Phi[n] = [A_0[n]e^{-2\pi i \phi_0[n]}, A_1[n]e^{-2\pi i \phi_1[n]}, \dots, A_{N-1}[n]e^{-2\pi i \phi_{N-1}[n]}], \quad (4.22)$$

1818 where the set of magnitudes $A_i[n]$ are amplitude weighting factors and $\phi_i[n]$ are the phase
1819 shifts from the path-length differences. The index i is used to denote the antenna channel
1820 number. The amplitude weighting factor is the relative magnitude of the signal received
1821 by a particular antenna to the other antennas in the array, such that the antennas that
1822 receive signals with higher amplitude, due to being closer to the source, have more
1823 weight in the beamforming summation. The input and outputs signals beamforming
1824 are naturally expected to be functions of time as indicated by the index $[n]$, however, it
1825 is also possible to use time dependent beamforming phases that shift the beamforming
1826 position of the array over time.

1827 Digital beamforming is the type of beamforming algorithm of interest to Project 8 for
1828 CRES. Specifically, digital beamforming means that the beamforming phases are applied
1829 to the array signals in software rather than employing fixed beamforming phase shifts in
1830 the receiver chain hardware. The advantage of digital beamforming is that for a given
1831 series of array snapshots one can specify a large number of beamforming positions and
1832 effectively search for electrons by performing the beamforming summation associated
1833 with each point and applying a signal detection algorithm to identify the presence of a
1834 CRES signal.

1835 One of the most attractive features of digital beamforming is the spatial filtering
1836 effect, which is a direct consequence of the constructive interference condition used to
1837 define the beamforming phases. Spatial filtering allows for signals from multiple electrons
1838 at different positions in the trap to be effectively separated, because the constructive
1839 interference condition will force the signals from electrons at positions different from the
1840 beamforming position to cancel. This helps to reduce signal pile-up that could become
1841 an issue for large scale CRES experiments using a dense tritium source.

1842 The digital beamforming positions can be specified with arbitrary densities limited
1843 only by the available computational resources. This provides a very straight-forward way
1844 to estimate the position of the electron in the trap by using a dense grid of beamforming
1845 positions and maximizing the output power of the beamforming summation over this
1846 grid. This natural approach to position reconstruction is attractive due the requirements
1847 of an event-by-event signal reconstruction, which needs an accurate estimation of the
1848 exact magnetic field experienced by the electron in order to correctly estimate it's kinetic

1849 energy. Combined with an accurate map of the magnetic field inhomogeneities of the
1850 trap obtained from calibrations, beamforming allows one to apply this magnetic field
1851 correction with a spatial resolution that is a fraction of the cyclotron wavelength.

1852 **Laboratory Beamforming Demonstrations**

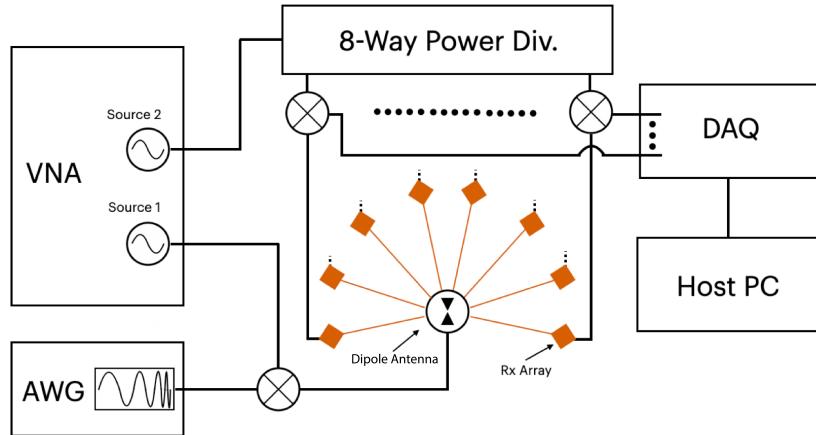


Figure 4.8

1853 **FSCD Beamforming Simulations**

1854 Using Locust simulations of the FSCD one can perform beamforming reconstruction
1855 studies using the simulated CRES signal data. As we mentioned in the previous section,

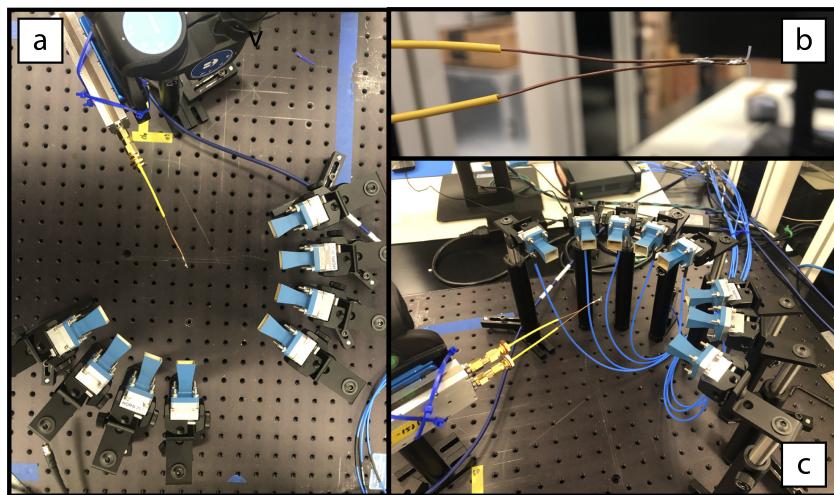


Figure 4.9

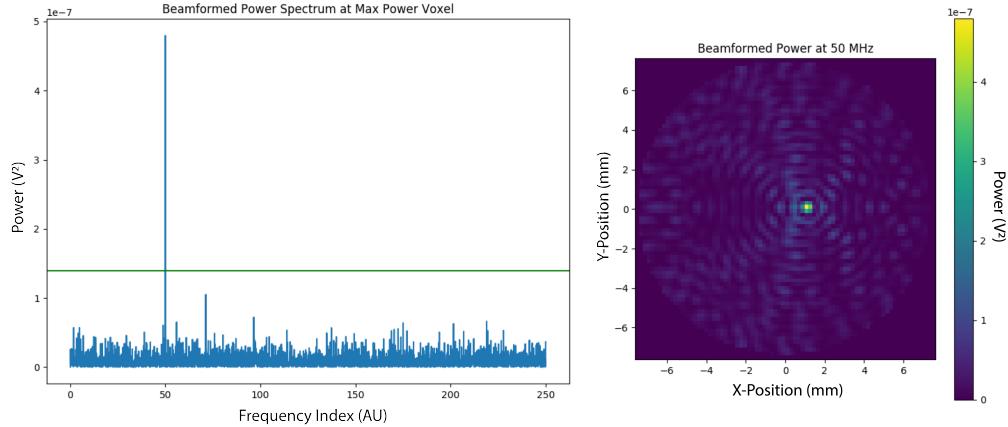


Figure 4.10

1856 the beamforming procedure begins by specifying a set of beamforming positions and
 1857 corresponding beamforming shifts. The beamforming positions form a grid that covers the
 1858 region of interest in the field of view of the antenna array. There are effectively an infinite
 1859 number of ways to specify the grid positions, however, uniform square grids are the most
 1860 commonly used due to their simplicity. In the FSCD experiment the number and pattern
 1861 of the grid positions would be optimized to cover the most important regions of the trap
 1862 volume to maximize detection efficiency while minimizing superfluous calculations.

1863 The beamforming grids used for signal reconstruction with the FSCD consist of a
 1864 set of points that cover a region of the two-dimensional plane formed by the perimeter
 1865 of the antenna array. The axial dimension is left out of the beamforming grid because
 1866 the electrons are assumed to occupy only an average axial position, which corresponds
 1867 to the center of the magnetic trap. This is because it is impossible to resolve the axial
 1868 position of the electron as a function of time due to the rapid axial oscillation frequencies
 1869 of trapped electrons relative to the FSCD time-slice duration.

1870 After beamforming, a summed time-series is obtained for each beamforming position
 1871 that can be evaluated for the presence of a signal using a detection algorithm. A beam-

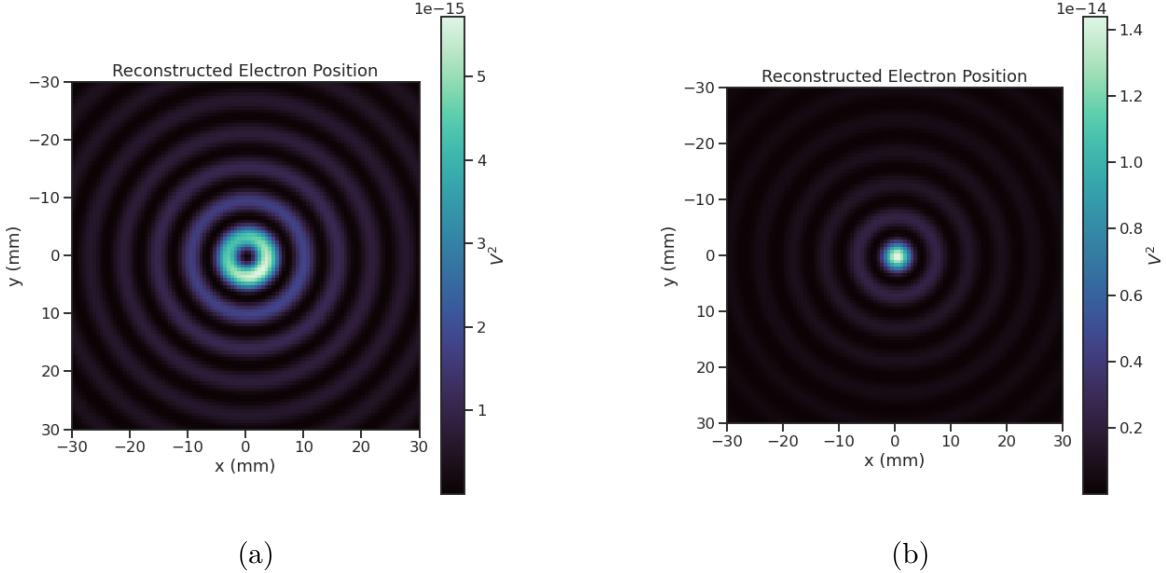


Figure 4.11: Beamforming images visualizing the reconstruction of an electron without (a) and with (b) the cyclotron phase correction. The images were generated using data from Locust simulations. The cyclotron phase refers to a phase offset equal to the relative azimuthal position of an antenna in the array. This phase offset is caused by the circular electron orbit and must be corrected for during reconstruction.

1872 forming image is a visualization method that is equivalent to arranging the beamforming
 1873 grid points according to their physical locations to form a three-dimensional matrix
 1874 where the first two dimensions encode the XY-position of the beamforming point and
 1875 the third dimension contains the summed time-series. The image is formed by taking the
 1876 time-averaged power (see Figure 4.11). Beamforming images are purely for the purposes
 1877 of visualization and are not particularly useful for signal detection or reconstruction.

1878 If the beamforming phases consist only of the spatial phase component from Equation
 1879 4.20, then the resulting beamforming image contains a relatively high-power ring-shaped
 1880 region that is centered on the position of the electron (see Figure 4.11a). The origin
 1881 of this shape is an additional phase offset particular to a cyclotron radiation source.
 1882 Essentially, the circular motion that produces the cyclotron radiation introduces a relative
 1883 phase offset to the electric fields that is equal to the azimuthal position of the field
 1884 measurement point. For example, if we have two antennas, one located at an azimuthal
 1885 position of 0° and another located at an azimuthal position of 90° , then the CRES signals
 1886 received by these antennas will be out of phase by 90° , which is the difference in their
 1887 azimuthal positions. This phase offset can be corrected by adding an additional term to
 1888 the beamforming phase equation that is equal to the azimuthal position of the antenna

1889 relative to the electron,

$$\phi_i[n] = \frac{2\pi d_i[n]}{\lambda} + \Delta\varphi_i[n], \quad (4.23)$$

1890 where $\Delta\varphi_i$ is difference between the azimuthal position of the electron and the i -th
 1891 antenna channel. Using the updated beamforming phases in the summation changes the
 1892 ring feature into a Bessel function peak whose maximum corresponds to the position of
 1893 the electron. Including this cyclotron phase correction significantly improves the signal
 1894 detection and reconstruction capabilities of beamforming by more than doubling the
 1895 summed signal power and shrinking the beamforming maxima feature size.

1896 The beamforming image examples in Figure 4.11 were produced using an electron
 1897 located on the central axis of the magnetic trap, which do not experience ∇B -drift.
 1898 However, for electrons produced at non-zero radial position the beamforming phases
 1899 must be made time-dependent in order to track the position of the electron's guiding
 1900 center over time. Without this correction the ∇B -drift causes the electron to move
 1901 between beamforming positions, which effectively spreads the cyclotron radiation power
 1902 over a wider area in the beamforming image (see Figure 4.12). This effect significantly

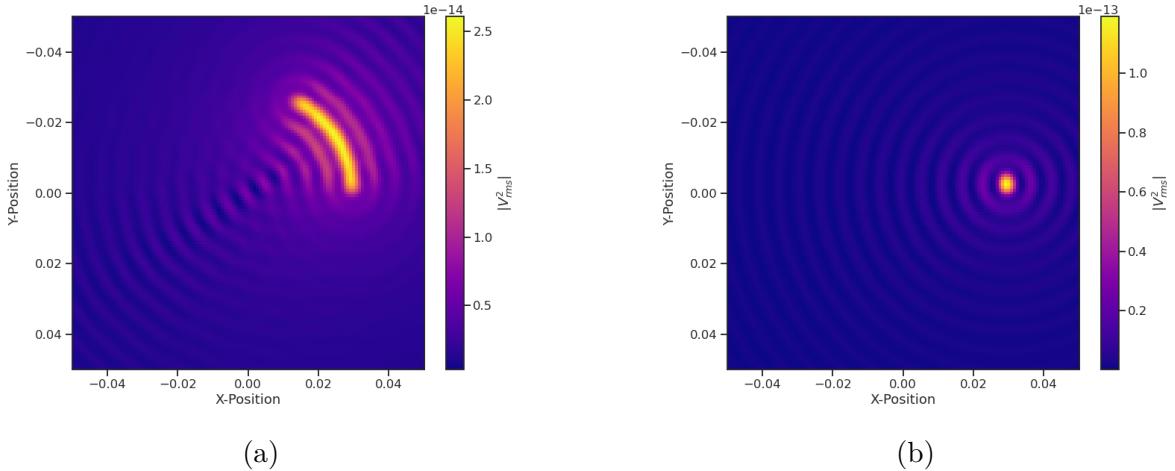


Figure 4.12: Beamforming images visualizing the reconstruction of an electron located off the central axis of the FSCD trap. In (a) we performing beamforming without the ∇B -drift correction, and in (b) we include the ∇B -drift correction.

1902
 1903 reduces the power of the beamforming maxima and increases the size of the beamforming
 1904 features, simultaneously harming detection efficiency and position reconstruction.

1905 The ∇B -drift correction simply adds a circular time-dependence to the beamforming
 1906 positions as a function of time,

$$r[n] = r_0 \quad (4.24)$$

$$\varphi[n] = \varphi_0 + \omega_{\nabla B} t[n], \quad (4.25)$$

where $\omega_{\nabla B}$ is the drift frequency and $t[n]$ is the time vector. In the ideal case the ∇B -drift frequencies from Figure 4.2 for the correct pitch angle and radial position would be used, however, it is not possible to know the electron's pitch angle a priori. In principle, one could perform multiple beamforming summations for a given beamforming position using different drift frequencies and choose the one that maximizes the summed power, but this approach leads to a huge computational burden that would be impractical for a real FSCD experiment. A compromise is to use an average value of $\omega_{\nabla B}$ obtained by averaging over the drift frequencies for electrons of different pitch angle at a particular radius. This approach keeps the computational cost of time-dependent beamforming to a minimum while still providing a significant increase in the detection efficiency of digital beamforming.

Signal Detection with Beamforming and a Power Threshold

Up to this point we have neglected any specific discussion of how digital beamforming is used for signal detection and reconstruction. This is because, strictly speaking, digital beamforming consists only of the phased summation of the array signals and cannot be used alone for signal detection. The example beamforming images shown in Figure 4.11 and Figure 4.12 were produced using simulated data that contained no noise, which significantly degrades the utility of analyzing the beamforming images for signal detection and reconstruction.

Digital beamforming as a detection algorithm is understood to mean digital beamforming plus a detection threshold placed on the amplitude of the frequency spectrum obtained by applying a fast Fourier transform (FFT) to the summed time-series (see Figure 4.13). This approach is most similar to the time-frequency spectrogram analysis employed in previous CRES experiments, however, in principle any signal detection algorithm could be used after the beamforming procedure. In Section 4.4 I analyze the signal detection performance of the power threshold approach in detail.

From the example frequency spectra in Figure 4.13 it is clear that without a reconstruction technique that coherently combines the signals from the full antenna our ability to detect CRES signals will be drastically reduced. Because the CRES signals are in-phase at the correct beamforming position the summed power increases as a function of N^2 compared to a single antenna channel, where N is the number of antennas. It is true that the noise power is also increased by beamforming, but, because the noise

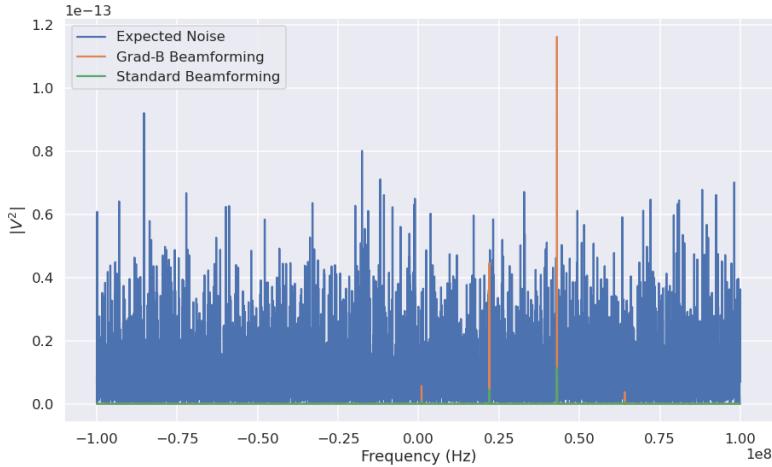


Figure 4.13: A plot of a typical frequency spectrum obtained by applying a Fourier transform to the time-series obtained from beamforming. The frequency spectra are plotted without noise on top of an example of a typical noise spectrum to visualize a realistic signal-to-noise ratio. In the example we see that without beamforming it would not be possible to detect anything since the signal amplitudes would be reduced by a factor of sixty relative to the noise. Additionally, we see that the ∇B -drift correction is needed to detect this electron since it comes from a simulation of an electron with a significant off-axis position.

1939 is incoherent, it's power only increases linearly. Consequently, the signal-to-noise ratio
 1940 (SNR) of the CRES signal increases linearly with the number of antennas, which greatly
 1941 improves detection efficiency compared to using only the information in a single antenna.

1942 The power threshold detection algorithm searches for high-power frequency bins that
 1943 should correspond to a frequency component of the CRES signal. In order to prevent
 1944 random noise fluctuations from being mistaken as CRES signals the power threshold
 1945 must be set high enough so that it is unlikely that random noise could be responsible. A
 1946 consequence of this is that many electrons that can be trapped will go undetected because
 1947 the modulation caused by axial oscillations leads to the cyclotron carrier power to falling
 1948 below the decision threshold. The time-dependent beamforming used to correct for the
 1949 ∇B -drift increases the volume of the magnetic trap where electrons can be detected,
 1950 but it is ineffective at increasing the range of detectable pitch angles (see Figure 4.14).
 1951 Fundamentally, this is because the power threshold only uses a fraction of the signal
 1952 power to detect electrons and ignores the power present in the frequency sidebands. In
 1953 the subsequent sections I examine two other signal detection algorithms that seek to
 1954 improve the detection efficiency of the FSCD by utilizing the more of the signal shape to
 1955 compute the detection test statistics.

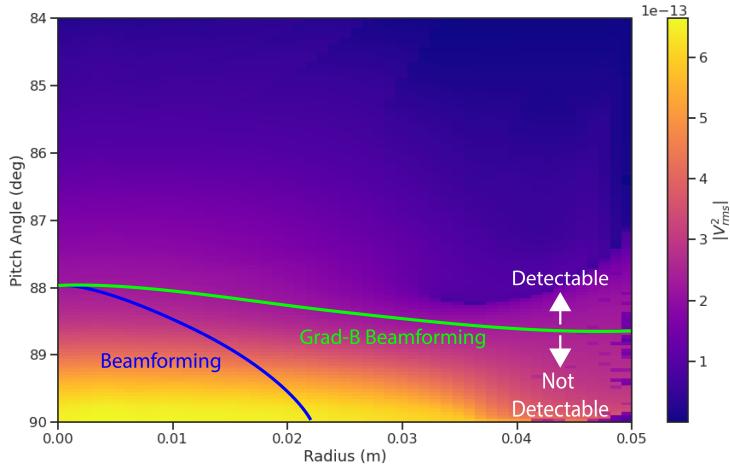


Figure 4.14: A plot of the total signal power received by the FSCD array from trapped electrons with different radial positions and pitch angles generated using Locust simulations. The lines on the plot indicate a 10 dB detection threshold above the mean value of the noise in the frequency spectrum. With static beamforming electrons with radial positions larger than about two centimeters are undetectable due to the change in the electron's position over time causing losses from beamforming phase mismatch. This is corrected by including ∇B -drift frequencies in the beamforming phases. Both beamforming techniques fail to detect electrons below $\approx 88.0^\circ$, since these signal are composed of several relatively weak sidebands that are comparable to the noise.

1956 4.3.2 Matched Filtering

1957 Introduction to Matched Filtering

1958 The problem of CRES signal detection is the problem of detecting a signal buried in
1959 WGN, which has been examined at great depth in the signal processing literature [19].
1960 For a fully known signal in WGN the optimal detector is the matched filter, which means
1961 that it achieves the highest true positive rate for a fixed rate of false positives. The
1962 matched filter test statistic is calculated by taking the inner product of the data with
1963 the matched filter template

$$\mathcal{T} = \left| \sum_n h^\dagger[n] y[n] \right|, \quad (4.26)$$

1964 where $h[n]$ is the matched filter template and $y[n]$ is the data. The matched filter test
1965 statistic defines a binary hypothesis test in which the data vector is assumed to be an
1966 instance of two possible data classes. By setting a decision threshold on the value of \mathcal{T} ,
1967 one can classify a given data vector as belonging to two distinct hypotheses. Under the
1968 first hypothesis the data is composed of pure WGN, and under the second hypothesis the

¹⁹⁶⁹ data is composed of the known signal with additive WGN. The matched filter template
¹⁹⁷⁰ is obtained by rescaling the known signal in the following way

$$h[n] = \frac{x[n]}{\sqrt{\tau \sum_n x^\dagger[n]x[n]}}, \quad (4.27)$$

¹⁹⁷¹ where τ is the variance of the WGN and $x[n]$ is the known signal. Strictly speaking,
¹⁹⁷² Equation 4.27 is only true for noise with a diagonal covariance matrix, however, in the
¹⁹⁷³ context of the FSCD we are justified in assuming this to be true. Defining the matched
¹⁹⁷⁴ filter templates in this way guarantees that the expectation value of \mathcal{T} is equal to one
¹⁹⁷⁵ when the data contains only noise, which is the standard matched filter normalization in
¹⁹⁷⁶ the signal processing literature.

¹⁹⁷⁷ Although matched filters are canonically formulated in terms of a perfectly known
¹⁹⁷⁸ signal, it is still possible to apply the matched filter technique given imperfect information
¹⁹⁷⁹ about the signal provided that the signal is deterministic. From our discussion of CRES
¹⁹⁸⁰ simulation tools for the FSCD (see Section 4.2) we know that the shape of CRES signals
¹⁹⁸¹ are completely determined by the initial parameters of the electron. The random collisions
¹⁹⁸² with background gas molecules which cause the formation of signal tracks are the only
¹⁹⁸³ stochastic component of the CRES event after the initial beta-decay, therefore, it is
¹⁹⁸⁴ possible to develop a matched filter for the detection of CRES signal tracks which are fully
¹⁹⁸⁵ determined by the parameters of the electron after the initial beta-decay or subsequent
¹⁹⁸⁶ collision events.

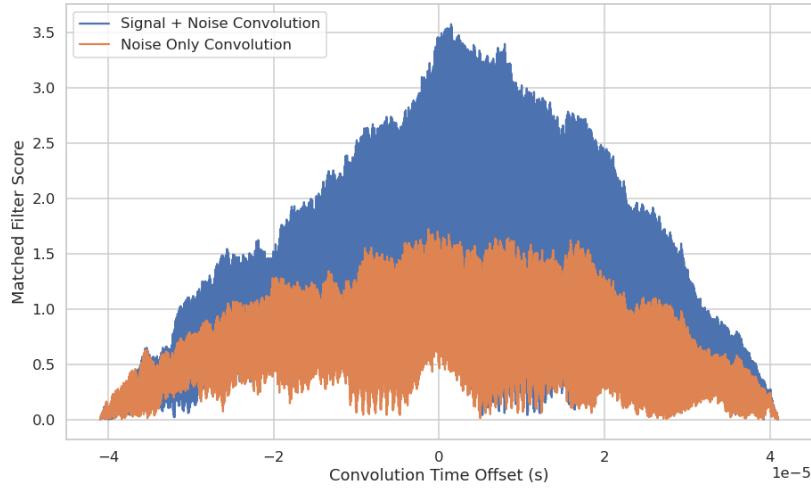


Figure 4.15

¹⁹⁸⁷ The matched filter test statistic for CRES signals is a modified version of Equation

$$\mathcal{T} = \max_{\mathbf{h}, m} |\mathbf{h} * \mathbf{y}| = \max_{\mathbf{h}, m} \left| \sum_k h^\dagger[k] x[m - k] \right|, \quad (4.28)$$

1989 where the matched filter inner product has been replaced with a convolution operation
 1990 and a maximization over the template and convolution delay (m). Replacing the inner
 1991 product with a convolution accounts for the fact that the start time of the CRES signal is
 1992 now an unknown parameter, in addition, we now perform a maximization of the matched
 1993 filter convolution over a number of different templates. Because the shape of the signal is
 1994 unknown we are forced to guess a number of different signal shapes to create a template
 1995 bank with which we can identify unknown signals by performing an exhaustive search.

1996 The template bank approach to matched filtering, while quite powerful, can quickly
 1997 become computationally intractable. This is especially true in the case of the FSCD
 1998 because of the large amount of raw data produced by the array that must be analyzed.
 1999 Specifically, the time-domain convolution specified by Equation 4.28 is particularly
 2000 computationally intensive and is a major barrier towards the implementation of a
 2001 matched filter for signal detection in an experiment like the FSCD. This can be avoided
 2002 by using the convolution theorem to replace the time-domain convolution with an inner
 2003 product in the frequency domain.

2004 The convolution theorem states that

$$\mathbf{f} * \mathbf{g} = \mathcal{F}^{-1}(\mathbf{F} \cdot \mathbf{G}) \quad (4.29)$$

2005 where \mathbf{f} and \mathbf{g} are discretely sampled time-series, \mathbf{F} and \mathbf{G} are the respective discrete
 2006 Fourier transforms, and \mathcal{F}^{-1} is the inverse discrete Fourier transform operator. The
 2007 convolution theorem allows us to perform the matched filter convolution by first com-
 2008 puting the Fourier transform of the template and data, then performing a point-wise
 2009 multiplication of the two frequency series, and finally performing the inverse Fourier
 2010 transform to obtain the convolution output. Because discrete Fourier transforms can be
 2011 performed extremely efficiently, the convolution theorem is almost always used in lieu of
 2012 directly computing the convolution.

2013 One thing to note here is that the convolution theorem for discrete sequences shown
 2014 here, is technically valid only for circular convolutions, which is not directly specified
 2015 in Equation 4.28. However, because typical CRES track lengths are much longer than
 2016 the Fourier analysis window and also that the frequency chirp rates are small compared
 2017 to the time-slice duration, it is relatively safe to use circular convolutions to evaluate

2018 matched filter scores for CRES signals, which allows us to apply the convolution theorem
2019 to compute matched filter scores using the frequency representation of the data and
2020 matched filter template.

2021 **Matched Filter Analysis of the FSCD**

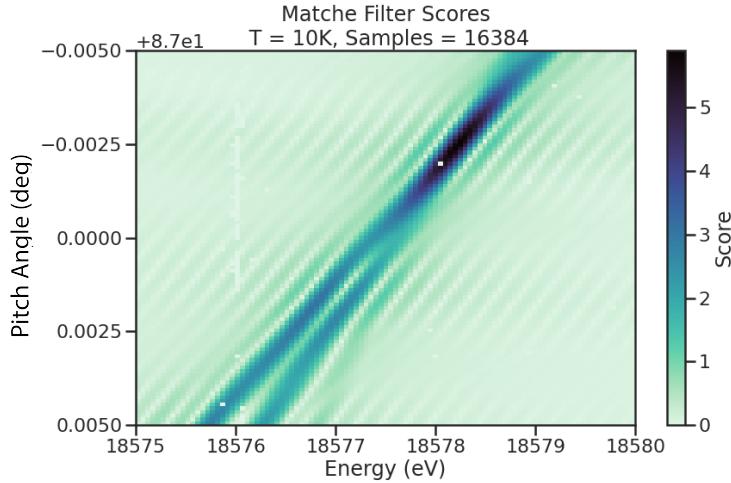


Figure 4.16

2022 The optimality provided by the matched filter makes it a useful algorithm for analysis
2023 of CRES experiment designs for sensitivity analyses, since it indicates the best possible
2024 detection efficiency achievable by an experiment configuration. The standard approach to
2025 performing these studies involves generating a large number of simulated electron signals
2026 that span the kinematic parameter space of electrons in the magnetic trap. In general,
2027 electrons have six kinematic parameters along with an additional start time parameter.

2028 In order to limit the number of simulations required to evaluate the detection efficiency
2029 the standard approach is to fix the starting axial position, starting azimuthal position,
2030 starting direction of the perpendicular component of the electron's momentum, and event
2031 start time to reduce the parameter space to starting radial position, starting kinetic
2032 energy, and starting pitch angle. The fixed variables are true nuisance parameters that do
2033 not affect the detection efficiency estimates for the FSCD design, because they manifest
2034 as phases which are marginalized during the calculation of the matched filter score.

2035 Across radial position, kinetic energy, and pitch angle we can define a regular grid of
2036 parameters and use Locust to simulate the corresponding signals. This grid of simulated
2037 signals can be used to estimate the likelihood of detecting signals in the FSCD by using
2038 the same set of signals as both the data and templates when evaluating the matched

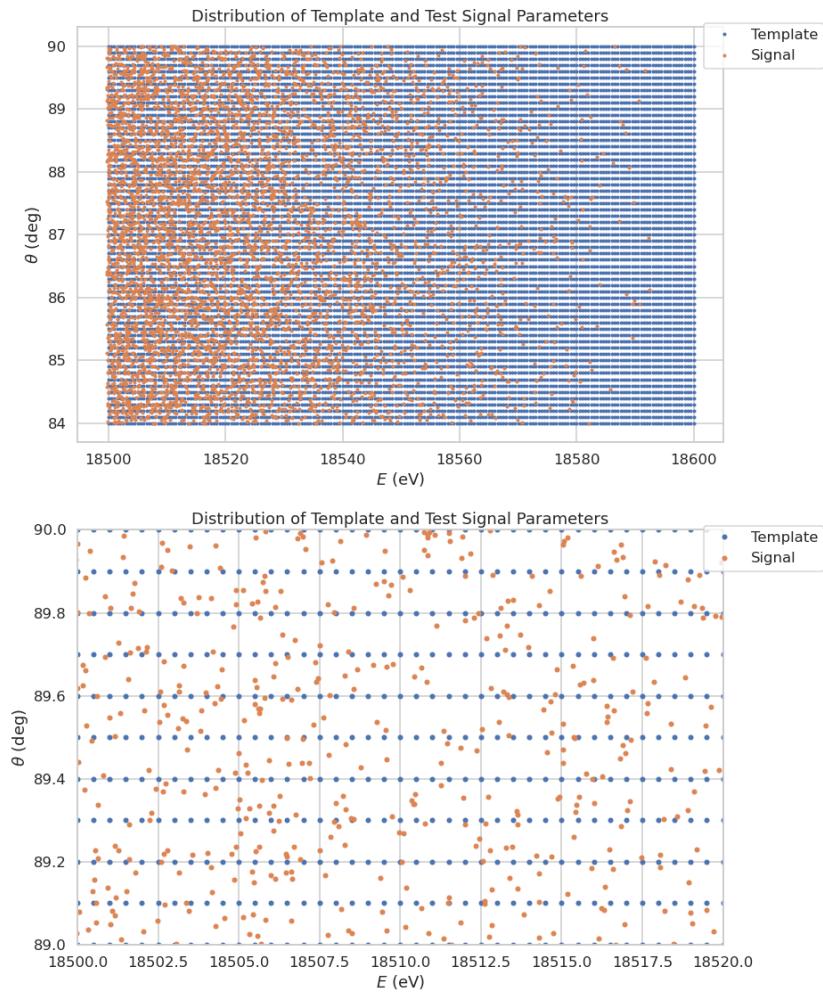


Figure 4.17

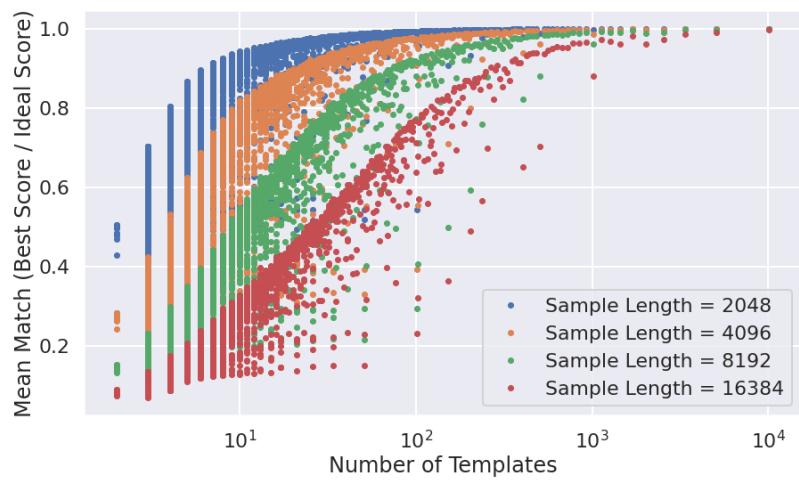


Figure 4.18

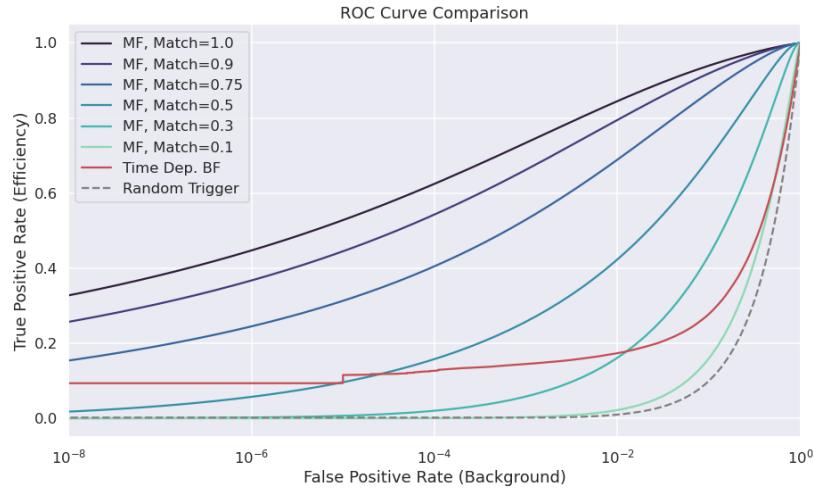


Figure 4.19

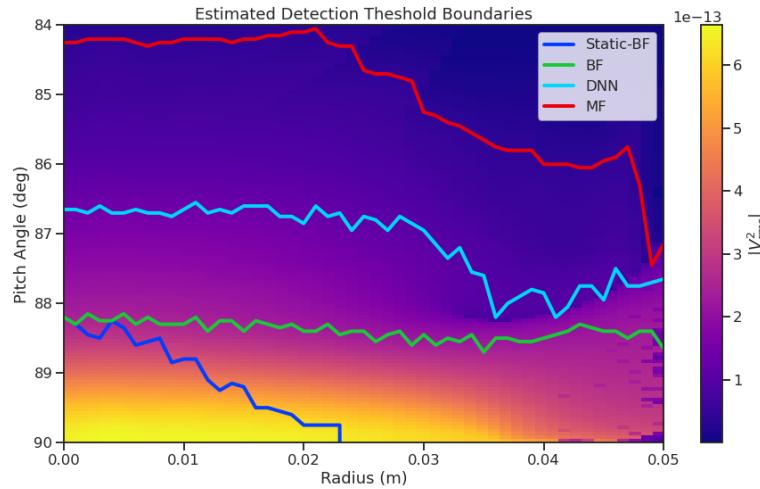


Figure 4.20

2039 filter scores. The matched filter score specifies the shape of the PDF that defines the
 2040 detection probability.

2041 Optimized Matched Filtering Implementation for the FSCD

2042 The biggest practical obstacle to the implementation of a matched filter template bank
 2043 detection approach is oftentimes the computational cost associated with exhaustively
 2044 calculating the matched filter scores of the template bank, and the FSCD is no exception
 2045 in this regard. At a basic level computing a matched filter score requires the convolution
 2046 of two vectors, which can be performed very efficiently by computers if the convolution

2047 theorem and fast Fourier transforms (FFT) are utilized. Furthermore, one can consider
 2048 applying digital beamforming as a pre-processing step to reduce the dimensionality of
 2049 the data before the matched filter is applied. In order to understand the relative gain
 2050 in computational efficiency offered by these optimizations we analyze the total number
 2051 of floating-point operations (FLOP) of several matched filter implementations in big O
 2052 notation that utilize different combinations of optimizations.

2053 A direct implementation of a matched filter as specified by Equation 4.28 involves
 2054 the convolution of N_{ch} signals of length N_s with template signals of length N_t . As a
 2055 uniform metric we shall compare the FLOP of the various matched filter implementations
 2056 on a per-template basis, since each implementation scales linearly with the number of
 2057 templates. The direct convolution approach to matched filtering costs

$$O(N_{\text{ch}}) \times O(N_s \times N_t) \quad (4.30)$$

2058 FLOP per-template, whose cost is dominated by the $O(M \times N)$ convolution operation.

2059 The computational cost of the direct matched filter approach can be significantly
 2060 reduced by exploiting the convolution theorem and FFT algorithms. If we restrict
 2061 ourselves to signals and templates that contain equal numbers of samples then the
 2062 convolution can be calculated by Fourier transforming both vectors, performing the
 2063 point-wise multiplication, and then performing the inverse Fourier transform to obtain
 2064 the convolution result. The FFT algorithm is able to compute the Fourier transform
 2065 utilizing only $O(N \log N)$ operations compared to $O(N^2)$ for a naive Fourier transform
 2066 implementation. This optimization results in a computational cost per-template of

$$O(N_{\text{ch}}) \times O(N_s \log N_s) \quad (4.31)$$

2067 A typical signal vector in the FSCD contains $O(10^4)$ samples in which case the FFT
 2068 reduces the computational cost of the matched filter by a factor of $O(10^3)$. This large
 2069 reduction in computational cost implies that a direct implementation of a matched filter
 2070 is completely infeasible in the FSCD due to resource constraints.

2071 Rather than relying solely on the matched filter it is tempting to consider using
 2072 digital beamforming as an initial step in the signal reconstruction for the purposes of
 2073 data reduction. The primary motivation is to reduce the dimensionality of the data by
 2074 a factor of N_{ch} by combining the array outputs coherently into a single channel. One
 2075 can view the beamforming operation as a partial matched filter, in the sense that the
 2076 matched filter convolution contains the beamforming phased summation along with a

2077 prediction of the signal shape. By separating beamforming from the signal shape one
2078 hopes to reduce the overall computational cost by effectively shrinking the number of
2079 templates and reducing the number of operations required to check each one.

2080 The nature of this optimization requires that we account for the number of templates
2081 used for pure matched filtering versus the hybrid approach. To first order, the total
2082 number of templates at the trigger stage is a product of the number of guesses for each
2083 of the electron's parameters

$$N_T = N_E \times N_\theta \times N_r \times N_\varphi, \quad (4.32)$$

2084 where N_E is the number of kinetic energies, N_θ is the number of pitch angles, N_r is the
2085 number of starting radial positions, and N_φ is the number of starting azimuthal positions.
2086 The starting axial position and cyclotron motion phase are not necessary to include in
2087 the template bank since these parameters manifest themselves as the starting phase of
2088 the signal, which is effectively marginalized when using a FFT to compute the matched
2089 filter convolution. Therefore, the total number of operations required by a matched filter
2090 to detect a signal in a segment of array data is on the order of

$$O(N_T) \times O(N_{\text{ch}}) \times O(N_s \log N_s) \quad (4.33)$$

2091 With the hybrid approach we attempt to remove the spatial parameters from the
2092 template bank by using beamforming to combine the array signals into a single channel.
2093 Beamforming explicitly assumes a starting position, which allows us to only use matched
2094 filter templates that span the two-dimensional space of kinetic energy and pitch angle.
2095 The total computational cost of the hybrid method is directly proportional to the number
2096 of beamforming positions. For the time-dependent beamforming defined in Section 4.3.1,
2097 the number of beamforming positions is given by

$$N_{\text{BF}} = N_r \times N_\varphi \times N_{\omega_{\nabla B}}, \quad (4.34)$$

2098 where N_r and N_φ are the same spatial parameters encountered in the pure matched
2099 filter template bank and $N_{\omega_{\nabla B}}$ is the number of ∇B -drift frequency assumptions. If a
2100 unique drift frequency is used for each pitch angle then the hybrid approach is effectively
2101 equivalent to a pure matched filter in the number of operations. The key efficiency gain
2102 of the hybrid approach is to exploit the relatively small differences in $\omega_{\nabla B}$ for electrons
2103 of different pitch angles by using only a small number of average drift frequencies.

2104 The total number of operations for the hybrid approach can be expressed as a sum of
 2105 the operations required by the beamforming and matched filtering steps,

$$O(N_{\text{BF}}) \times O(N_{\text{ch}}N_s) + O(N_{\text{BF}}) \times O(N_E N_\theta) \times O(N_s \log N_s). \quad (4.35)$$

2106 The first product in the sum is the number of operations required by beamforming,
 2107 which is simply the number of beamforming points times the computational cost of the
 2108 beamforming matrix multiplication, and the second product is the computational cost
 2109 of matched filtering the summed signal generated by each beamforming position. To
 2110 compare this to pure matched filtering we take the ratio of Equations 4.33 and 4.35 to
 2111 obtain

$$\Gamma_{\text{BFFM}} = \frac{O(N_{\omega_{\nabla B}})}{O(N_E N_\theta) \times O(\log N_s)} + \frac{O(N_{\omega_{\nabla B}})}{O(N_{\text{ch}})}. \quad (4.36)$$

2112 This expression can be simplified by observing that $O(N_E N_\theta) \times O(\log N_s) \gg O(N_{\text{ch}})$,
 2113 which means that the ratio of computational cost for the two methods can be reduced to

$$\Gamma_{\text{BFFM}} \approx \frac{O(N_{\omega_{\nabla B}})}{O(N_{\text{ch}})}. \quad (4.37)$$

2114 If we limit ourselves to a number of estimated drift frequencies of $O(1)$ then we see that
 2115 the estimated computational cost reduction of the hybrid approach is of $O(N_{\text{ch}})$. This is
 2116 quite a large reduction considering that the FSCD antenna array contains sixty antennas
 2117 in the baseline design.

2118 The main drawback of the hybrid approach is that the limited number of allowed
 2119 drift frequency guesses can lead to detection efficiency loss due to phase mismatch. The
 2120 degree of phase error from an incorrect drift frequency is proportional to the length of
 2121 the array data vector used by the signal detection algorithm. For signals with lengths
 2122 equal to the baseline FSCD Fourier analysis window of 8192 samples, typical phase errors
 2123 from using an average versus the exact ∇B -drift frequency are on the order of a few
 2124 percent in terms of the signal energy. This has a relatively small impact on the overall
 2125 detection efficiency, however, future experiments with antenna array CRES will want to
 2126 balance optimizations such as these during the design phase to keep experiment costs to
 2127 a minimum while still achieving scientific goals.

2128 **Kinetic Energy and Pitch Angle Degeneracy**

2129 More accurate modeling of a matched filter requires that we consider the effects of
 2130 mismatched signals and template, since this more accurately reflects the real-world usage

2131 of a matched filter where many incorrect templates are convolved with the data until the
 2132 matching template is found. One way to study this is to use the grid of simulated signals
 2133 to compute the matched filter scores between mismatched signals and templates and
 2134 evaluate the matched filter scores under this scenario. What one finds when performing
 2135 this analysis is that templates for kinetic energies and pitch angles that do not match
 2136 the underlying signal can have matched filter scores that are indistinguishable from the
 matched filter score of the correct template (see Figure 4.21 and Figure 4.21).

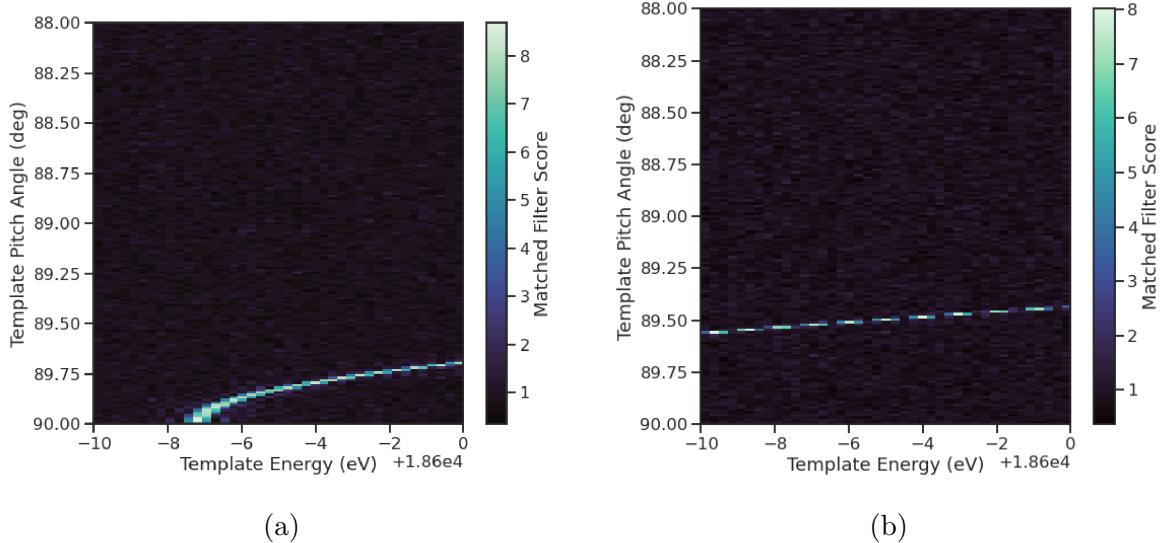


Figure 4.21: Two example illustrations of the correlation between kinetic energy and pitch angle imparted by the shape of the FSCD magnetic trap. The correlations manifest themselves as degeneracies in the matched filter score where multiple matched filter templates have the same matched filter for a particular signal. These degeneracies are a sign that the magnetic trap must be redesigned in order to break the correlation between pitch angle and kinetic energy.

2137
 2138 This degeneracy in matched filter score is the result of correlations between the kinetic
 2139 energy of the electron and the pitch angle caused by changes in the average magnetic field
 2140 experienced by an electron for different pitch angles. While in principle helpful for the
 2141 purposes of signal detection these correlations are unacceptable since they greatly reduce
 2142 the energy resolution of the experiment by causing electrons with specific kinetic energy
 2143 to templates across a wide range of energies. It is important to emphasize that this
 2144 degeneracy cannot be fixed by implementing a different signal reconstruction algorithm.
 2145 As revealed by the matched filter scores the shapes of the signals for different parameters
 2146 are identical. Resolving this degeneracy between pitch angle and energy requires the
 2147 design of a new magnetic trap with steeper walls so that the average magnetic field

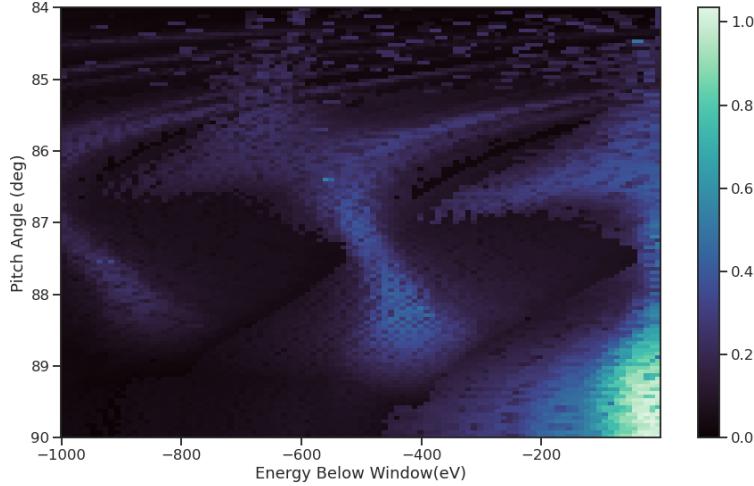


Figure 4.22: A visualization of the correlation between energy and pitch angle in the FSCD magnetic trap. The image is formed by computing the match of the best template from a grid consisting of pitch angles from 84 to 90 degrees in steps of 0.05 degrees, kinetic energies from 17574 to 18574 eV, located at 2 cm from the central axis, and simulated for a length of three FSCD time-slices. The signals used to compute the best matching template consisted of a grid from 84 to 90 degrees in steps of 0.05 degrees, kinetic energies from 18550 to 18575 eV in steps of 0.25 eV, located 2 cm from the central axis, and simulated for three FSCD time-slices. The colored regions of the plot show how well signals with lower energy can match those of higher energy for the FSCD magnetic trap, which is proportional to the achievable energy resolution of the FSCD design.

²¹⁴⁸ experienced by an electron is less dependent on pitch angle.

²¹⁴⁹ 4.3.3 Machine Learning

²¹⁵⁰ Machine learning is a vast and rapidly developing field of research [22]. In this Section
²¹⁵¹ we shall attempt to provided a brief introduction to some of the concepts and techniques
²¹⁵² of machine learning that were applied to CRES signal detection rather than attempt a
²¹⁵³ comprehensive overview.

²¹⁵⁴ Introduction to Machine Learning

²¹⁵⁵ Digitization of the FSCD antenna array generates large amounts of data that must be
²¹⁵⁶ rapidly processed to enable real-time signal detection and reconstruction. While digital
²¹⁵⁷ beamforming combined with a power threshold is relatively computationally inexpensive,
²¹⁵⁸ it is relatively ineffective at detecting CRES signal with small pitch angles, since it relies
²¹⁵⁹ on a visible frequency peak above the noise. On the other hand, a matched filter is able

2160 to detect signals with a significantly larger range of parameters, however, the exhaustive
2161 search of matched filter templates can be computationally expensive. Machine learning
2162 based triggering algorithms have been used successfully in many different high-energy
2163 physics experiments [23] and recent developments have shown success in the detection
2164 of gravitational wave signals using machine learning techniques [24, 25] in place of the
2165 more traditional matched filtering method. This motivates the exploration of machine
2166 learning as a potential CRES signal detection algorithm.

2167 There are several different approaches to machine learning, but the one most important
2168 to our discussion here is the supervised learning approach. In supervised machine learning
2169 one uses a differentiable model or function that is designed to map the input data to the
2170 appropriate label [22]. The data is represented as a multidimensional matrix of floating
2171 point values such as an image or a time-series, and the label is generally a class name
2172 such as signal or noise for classification problems or a continuous value like kinetic energy
2173 in the case of regression problems.

2174 In supervised learning the model is trained to map from the data to the correct label
2175 by evaluating the output of the model using a training dataset consisting of a set of
2176 paired data and labels. To evaluate the difference between the model output and the
2177 correct label a loss function is used to quantify the error between the model prediction
2178 and the ground truth. For example, a common loss function in regression problems is the
2179 squared error loss function, which quantifies error using the squared difference between
2180 the model output and label.

2181 Using the outputs of the loss function the next step in supervised learning is to
2182 compute the gradient of error with respect to the model parameters in a process called
2183 backpropagation. Using the model parameter gradients the last step in the supervised
2184 learning process is to perform an update of the parameter values in order to minimize
2185 the error in the model predictions across the whole dataset. This loop is performed many
2186 times while randomly shuffling the dataset until the error converges to a minimum value
2187 at which point the training procedure has finished. It is standard practice to monitor
2188 the training procedure by evaluating the performance of the model using a separate
2189 validation dataset that matches the statistical distribution of the training data and to
2190 check the performance of the model after training using yet another dataset called the
2191 test dataset. These practices help to guard against overtraining which is a concern for
2192 models with many parameters.

2193 **Convolutional Neural Networks**

2194 A popular class of machine learning models are neural networks. A neural network is
2195 essentially a function composed of a series of linear operations called layers which take a
2196 piece of data typically represented as a matrix, multiplies the elements of the data by a
2197 weight, and then sums these products to produce an output matrix. Neural networks
2198 composed of purely linear operations are unable to model complex non-linear behavior,
2199 therefore, non-linear activation functions are applied to the outputs of each of the layers
2200 to increase the ability of the neural network to model complex relationships between the
2201 data.

2202 Neural networks are typically composed of at least three layers, but with the present
2203 capabilities of computer hardware they more often contain many more than this. The
2204 first layer in a neural network is called the input layer, because it takes the data objects
2205 as input, and the last layer in a neural network is known as the output layer. The
2206 output layer is trained by machine learning to map the data to a desired output using
2207 the supervised learning procedure described in Section 4.3.3. In between the input and
2208 the output layer are typically several hidden layers that receive inputs from and transmit
2209 outputs to other layers in the neural network model. The term deep neural network
2210 (DNN) refers to those neural networks that have at least one hidden layer, which have
2211 proven to be extremely powerful tools for pattern recognition and function approximation.

2212 An important type of DNN are convolutional neural networks (CNN) that typically
2213 contain several layers which perform a convolution of the input with a set of filters. These
2214 convolution operations are typically accompanied by layers that attempt to down-sample
2215 the data along with the standard neural network activation functions. A standard CNN
2216 is composed of several convolutional layers at the beginning of the network and ends
2217 with a series of fully-connected neural network layers at the output. Intuitively, one
2218 can imagine that the convolutional layers are extracting features from the data that
2219 fully-connected layers use to perform the classification or regression task.

2220 **Deep Filtering for Signal Detection in the FSCD**

2221 CNNs have been extremely influential in the field of computer vision, particularly tasks
2222 such as image segmentation and classification, but have also been applied in numerous
2223 experimental physics contexts. Given the particular challenge posed by signal detection
2224 and reconstruction in the FSCD we are interested in exploring the potential of machine
2225 learning as an effective algorithm for real-time signal detection, since this application

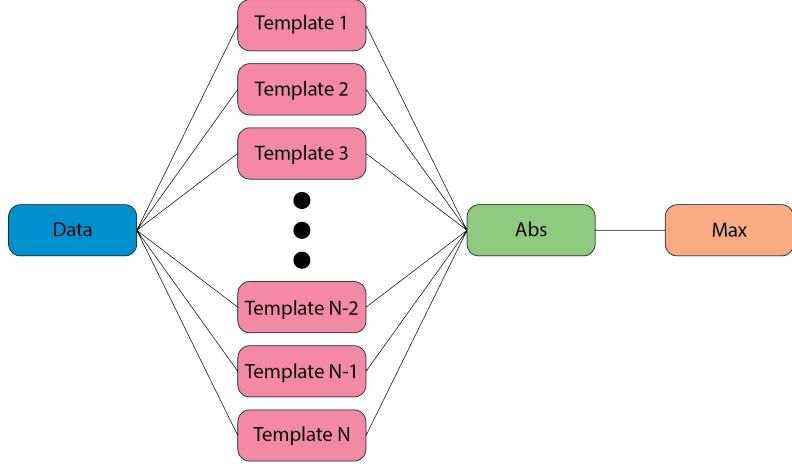


Figure 4.23: A representation of a matched filter template bank as a convolutional neural network. The network has a single layer composed of the templates, which act as convolutional filters. The activation of the neural network is an absolute value followed by a max operator.

2226 requires both high efficiency and fast evaluation.

2227 In the machine learning paradigm signal detection is equivalent to a binary classifi-
 2228 cation problem between the signal and noise data classes, and my investigation focuses
 2229 specifically on the application of CNNs to signal detection in the FSCD, which is moti-
 2230 vated by relatively recent demonstrations of CNNs achieving classification accuracies for
 2231 gravitational wave time-series signals comparable to a matched filter template bank. In
 2232 this framework it is possible to interpret the matched filter as a type of CNN composed
 2233 of a single convolutional layer with the templates making up the layer filters (see Figure
 2234 4.23). Since this neural network has no hidden layers, it is not a DNN like we have
 2235 been discussing so far, but we can attempt to construct a proper CNN that attempts to
 2236 reproduce the classification performance of the matched filter network.

2237 The name deep filtering refers to this scheme of replacing a matched filter template
 2238 bank with a DNN. The reason why one might want to do this is that it may be possible to
 2239 exploit redundancies and correlations between templates that may allow one to perform
 2240 signal detection with similar accuracy but with fewer computations, which is important
 2241 for real-time detection scenarios like the FSCD experiment. In Section 4.4 we perform a
 2242 detailed comparison of the signal detection performance of a CNN to beamforming and a
 2243 matched filter template bank.

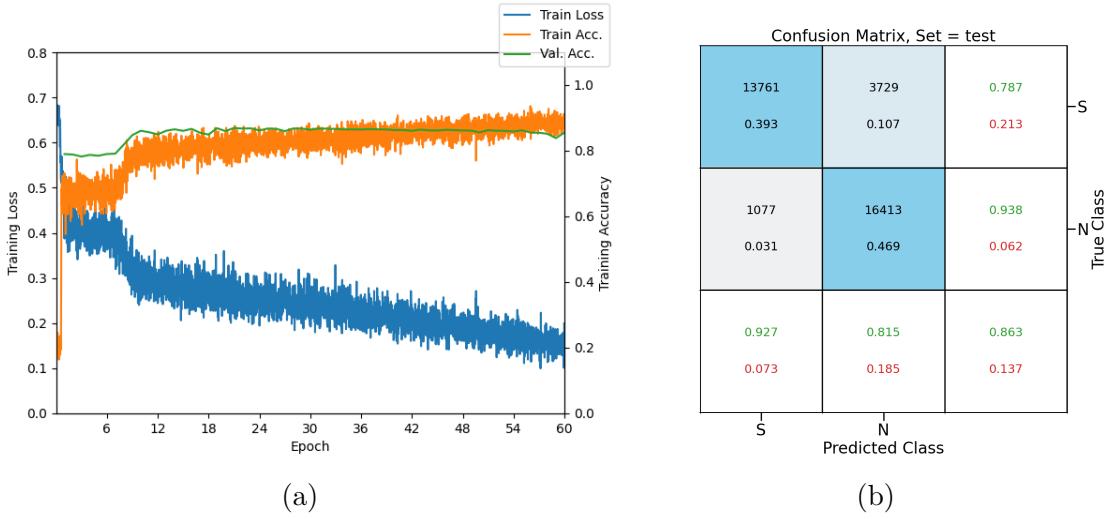


Figure 4.24

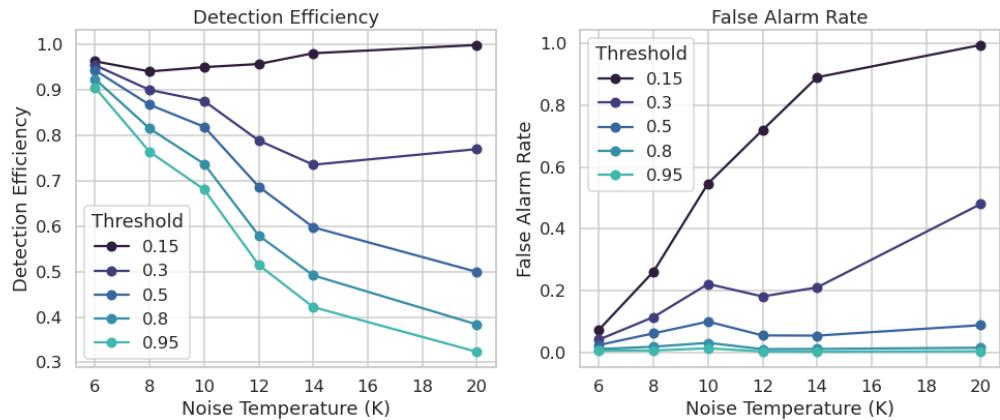


Figure 4.25

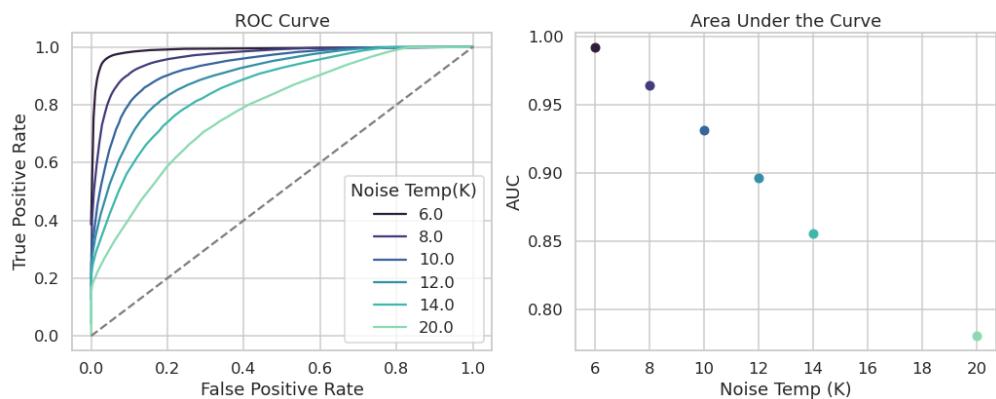


Figure 4.26

2244 **4.4 Analysis of Signal Detection Algorithms for the An-** 2245 **tenna Array Demonstrator**

2246 This section contains an early version of the manuscript for the triggering paper prepared
2247 for publication in JINST. In it I present a relatively detailed analysis of the signal
2248 detection performance of the three signal detection approaches discussed so far using a
2249 population of simulated CRES signals generated with Locust. The focus of the paper is
2250 on the performance of the signal detection algorithms for pitch angles below 88.5° where
2251 the beamforming power threshold begins to fail.

2252 **4.4.1 Introduction**

2253 Cyclotron Radiation Emission Spectroscopy (CRES) is a technique for measuring the
2254 kinetic energies of charged particles by observing the frequency of the cyclotron radiation
2255 that is emitted as they travel through a magnetic field [2]. The Project 8 Collaboration
2256 is developing the CRES technique as a next-generation approach to tritium beta-decay
2257 endpoint spectroscopy for neutrino mass measurement. Recently, Project 8 has used
2258 CRES to perform the first ever tritium beta-decay energy spectrum and neutrino mass
2259 measurement [4, 5].

2260 Previous CRES measurements have utilized relatively small volumes of gas that are
2261 directly integrated with a waveguide transmission line, which transmits the cyclotron
2262 radiation emitted by the trapped electrons to a cryogenic amplifier. While this technology
2263 has had demonstrable success, it is not a feasible option for scaling up to significantly
2264 larger measurement volumes. In particular, the goal of the Project 8 Collaboration
2265 is to use CRES combined with atomic tritium to measure the neutrino mass with a
2266 40 meV sensitivity. Achieving this sensitivity goal will require a multi-cubic-meter scale
2267 measurement volume in order to obtain the required event statistics in the tritium
2268 beta-spectrum endpoint region; hence, there is a need for new techniques to enable large
2269 volume CRES measurements for future experiments.

2270 One approach is to surround a large volume with an array of antennas that together
2271 collect the cyclotron radiation emitted by trapped electrons [3, 26]. A promising array de-
2272 sign is an inward-facing uniform cylindrical array that surrounds the tritium containment
2273 volume. Increasing the size of the antenna array, by adding additional rings of antennas
2274 along vertical axis, allows one to grow the experimental volume until a sufficient amount
2275 of tritium gas can be observed by the array. A challenging aspect of this approach is

2276 that the total radiated power emitted by an electron near the tritium spectrum endpoint
2277 is on the order of 1 fW or less, which is then distributed between all the antennas in
2278 the array. Consequently, detecting the presence of a CRES signal and determining the
2279 electron's kinetic energy requires reconstructing the entire antenna array output over the
2280 course of the CRES event, posing a significant data acquisition and signal reconstruction
2281 challenge.

2282 Project 8 has developed a triggering system to enable real-time identification of CRES
2283 events using an antenna array [27]. Previous measurements with the CRES technique
2284 have utilized a threshold on the frequency spectrum formed from a segment of CRES
2285 time-series data. This algorithm relies on the detection of a frequency peak above the
2286 thermal noise background, which limits the kinematic parameter space of detectable
2287 electrons. Due to the limitations of this power threshold, Project 8 has been investigating
2288 alternative signal identification approaches, including both matched filtering and machine
2289 learning based classifiers, to improve the detection efficiency of the experiment. In
2290 order to evaluate the relative gains in detection efficiency that come from utilizing
2291 these alternative algorithms, we develop analytical models for the power threshold and
2292 matched filter signal classifier performance applicable to an antenna array based CRES
2293 detector. In addition, we implement and test a basic convolutional neural network (CNN)
2294 as a first step towards the development of neural-network based classifiers for CRES
2295 measurements. These results allow us to compare the estimated detection efficiencies of
2296 each of these methods, which we weigh against the associated computational costs for
2297 real-time applications.

2298 The outline of this paper is as follows. In Section 4.4.2 we give an overview of a
2299 prototypical antenna array CRES experiment, and describe the major steps involved
2300 in the proposed approach to real-time signal identification. In Section 4.4.3 we develop
2301 models for the power threshold and matched filter algorithms, and introduce the machine
2302 learning approach and CNN architecture. In Section 4.4.4 we describe our process for
2303 generating simulated CRES signal data and the details of training the CNN. Finally,
2304 in Section 4.4.5 we perform a comparison of the signal classification accuracy of the
2305 three approaches and discuss the relevant trade-offs in terms of detection efficiency and
2306 computational cost.

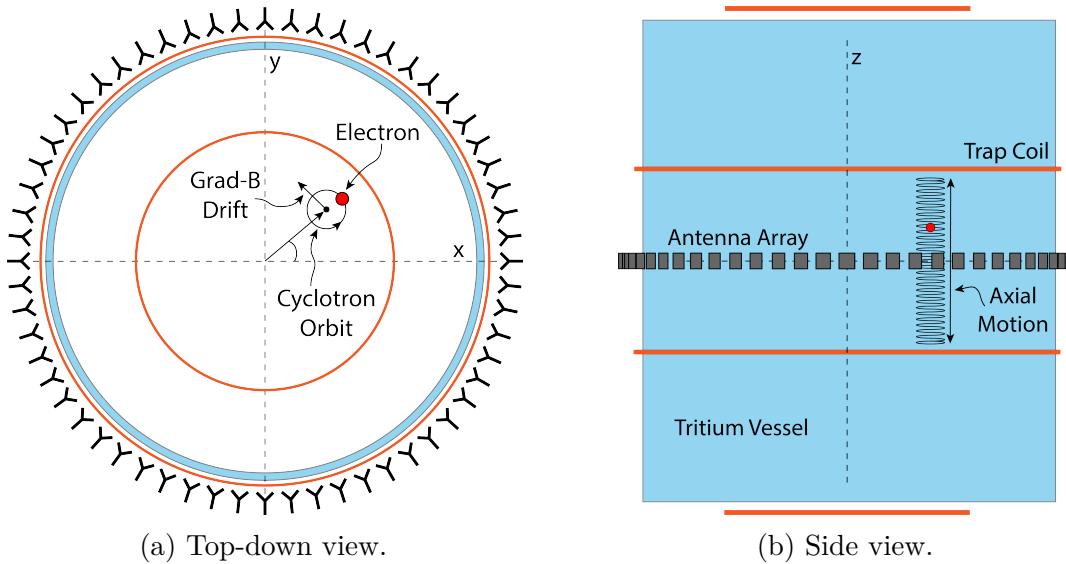


Figure 4.27: An illustration of the conceptual design for an antenna array CRES tritium beta-decay spectrum measurement. The antenna array geometry consists of a 20 cm interior diameter with 60 independent antenna channels arranged evenly around the circumference. The nominal antenna design is sensitive to radiation in the frequency range of 25-26 GHz, which corresponds to the cyclotron frequency of electrons emitted near the tritium beta-spectrum endpoint in a 1 T magnetic field. The array is located at the center of the magnetic trap produced by a set of current-carrying coils. The nominal magnetic trap design is capable of trapping electrons up to 5 cm away from the central axis of the array and traps electrons within an approximately 6 cm long axial region centered on the antenna array.

²³⁰⁷ 4.4.2 Signal Detection with Antenna Array CRES

²³⁰⁸ 4.4.2.1 Antenna Array and DAQ System

²³⁰⁹ In order to explore the potential of antenna array CRES for neutrino mass measurement,
²³¹⁰ the Project 8 Collaboration has developed a conceptual design for a prototype antenna
²³¹¹ array CRES experiment [3, 26], called the Free-space CRES Demonstrator or FSCD,
²³¹² which could be used as a demonstration of the antenna array measurement technique
²³¹³ (see Figure 4.27). The FSCD utilizes a single ring of antennas, which is the simplest
²³¹⁴ form of a uniform cylindrical array configuration, to surround a radio-frequency (RF)
²³¹⁵ transparent tritium gas vessel. A prototype version of this antenna array has been built
²³¹⁶ and tested by the Project 8 collaboration to validate simulations of the array radiation
²³¹⁷ pattern and beamforming algorithms [6]. In the FSCD the antenna array is positioned
²³¹⁸ at the center of the magnetic trap formed by a set of electro-magnetic coils that are
²³¹⁹ designed to produce a magnetic trap with flat central region and steep walls both radially

2320 and axially.

2321 When a beta-decay electron is trapped its motion consists of three primary components.
2322 The component with the highest frequency is the cyclotron orbit whose frequency is
2323 determined by the size of the background magnetic field. The FSCD design assumes
2324 a background magnetic field value of approximately 0.96 T, which results in cyclotron
2325 frequencies for electrons with kinetic energies near the tritium beta-spectrum endpoint
2326 from 25 to 26 GHz. The component with the next highest frequency is the axial oscillation
2327 experienced by electrons with pitch angles of less than 90° [9]. The flat region of the
2328 FSCD magnetic trap extends approximately 3 cm above and below the antenna array
2329 plane causing electrons to move back and forth as they are reflected from the trap walls.
2330 Typical oscillation frequencies are on the order of 10's of MHz, which results in an
2331 oscillation period that is $O(10^3)$ smaller than the duration of a typical CRES event.
2332 Therefore, when reconstructing CRES events we treat the electron as occupying only an
2333 average axial position at the center of the magnetic trap, since we are not able to resolve
2334 the axial position as a function of time. The component of motion with the smallest
2335 frequency is ∇B -drift caused by radial field gradients in the trap, producing an orbit of
2336 the electron around the central axis of the trap with a frequency on the order of a few
2337 kHz, dependent on the pitch angle and the radial position of the electron.

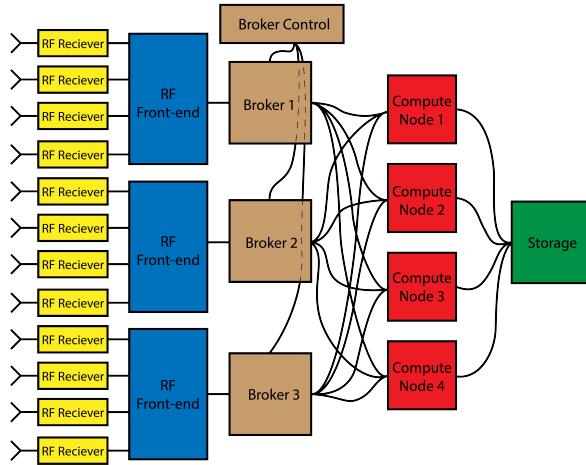


Figure 4.28: A high-level diagram of the DAQ system architecture envisioned for the FSCD.

2338 The data acquisition (DAQ) system digitizes the signals from the antenna array and
2339 combines three data streams into a time-ordered matrix of array snapshots that can be
2340 used by the reconstruction algorithms. The FSCD DAQ system design [27] is divided into
2341 three layers 4.28. The first layer is the RF front-end, which includes the antenna array,

2342 the RF receiver boards, and the digitization electronics. The receiver board contains an
 2343 amplifier, RF mixer, and bandpass filter to enable down-conversion, and is followed by
 2344 the digitization electronics that sample the CRES signals at 200 MHz. In order to achieve
 2345 an adequate signal-to-noise ratio to detect CRES events, the DAQ system for the antenna
 2346 array demonstrator must have a total system noise temperature of ≈ 10 K, which we
 2347 can achieve by using low-noise amplifiers and operating at cryogenic temperatures. After
 2348 digitization, the array data must be reorganized from individual data streams sorted
 2349 by channel into array snapshots sorted by time. In order to solve this data transfer
 2350 and networking problem the second layer of the DAQ system consists of a set of broker
 2351 computer nodes that reorganize the array data into time-ordered chunks. This approach
 2352 allows us accommodate different data transfer requirements by scaling the number of
 2353 broker nodes in this layer accordingly. Next, the broker layer distributes these chunks
 2354 of array data to the final layer of the DAQ system, which consists of a set of identical
 2355 reconstruction nodes that perform the calculations required for CRES reconstruction.
 2356 Similar to the broker layer, the number of reconstruction nodes can be increased or
 2357 decreased depending on the amount of computer power required for real-time CRES
 2358 reconstruction.

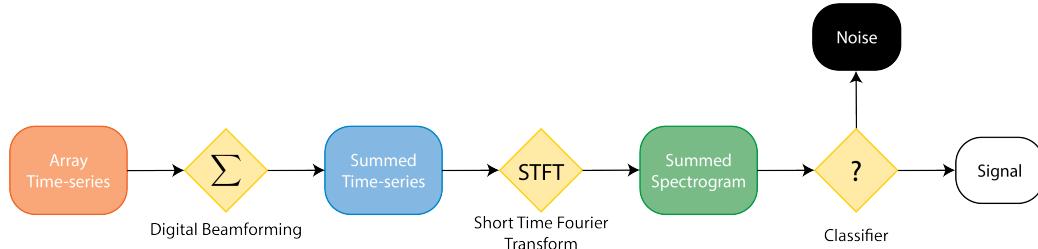


Figure 4.29: A block diagram illustration of the real-time triggering algorithm proposed for antenna array CRES reconstruction.

2359 The design of the FSCD DAQ system is intended to enable a significant portion of
 2360 the CRES event reconstruction to occur in real-time. The motivation for this comes from
 2361 the fact that the FSCD antenna array generates approximately 1 exabyte of raw data
 2362 per year of operation. Therefore, in order to reduce the data-storage requirements, it is
 2363 ideal to perform at least some of the CRES event reconstruction in real-time so that it
 2364 is possible to save a reduced form of the data for offline analysis. The first step of the
 2365 real-time reconstruction would be a real-time signal detection algorithm, which is the
 2366 focus of this paper. Our approach consists of three main operations performed on the
 2367 time-series data blocks including digital beamforming, a short time Fourier transform

2368 (STFT), and a binary classification algorithm to distinguish between signal and noise
2369 data (see Figure 4.29).

2370 **4.4.2.2 Real-time Signal Detection**

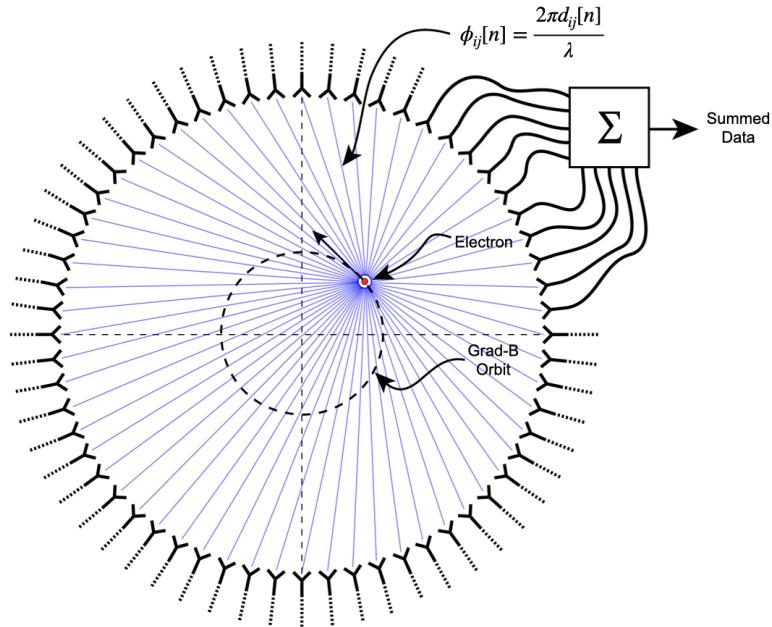


Figure 4.30: An illustration of the digital beamforming procedure. The blue lines indicate the various distances from the beamforming position to the antenna. In the situation depicted the actual position of the electron matches the beamforming position, so we should expect constructive interference when the phase shifted signals are summed. To prevent the electron's ∇B -motion from moving the electron off of the beamforming position, the beamforming phase include a time-dependence to follow the trajectory of the electron in the magnetic trap.

2371 The first step in the real-time detection algorithm is digital beamforming, which is
2372 a phased summation of the signals received by individual antennas in the array (see
2373 Figure 5.21). The phase shifts correspond to the path length differences between a spatial
2374 position and each individual antenna such that, when there is an electron located at
2375 the beamforming position, all the signals received by the array constructively interfere.
2376 Since we do not know ahead of time where an electron will be produced in the detector,
2377 we define a grid of beamforming positions that cover the entire region where electrons
2378 can be trapped and perform a phased summation for each of these points for every
2379 time-step in the array data block. As we saw in Section 4.4.2.1, the axial oscillation
2380 of the electrons prevents us from resolving it's position along the Z-axis of the trap,

2381 therefore our beamforming grid need only cover the possible positions of the electron in
 2382 the two-dimensional plane defined by the antenna array.

2383 The equation defining digital beamforming can be expressed as

$$\mathbf{y}[n] = \Phi^T[n]\mathbf{x}[n], \quad (4.38)$$

2384 where $\mathbf{x}[n]$ is array snapshot vector at the sampled time n , $\Phi[n]$ is the matrix of
 2385 beamforming phase shifts, and $\mathbf{y}[n]$ is summed output vector that contains the voltages
 2386 for each of the summed channels that correspond to a particular beamforming position.
 2387 The elements of the beamforming phase shift matrix can be expressed as a weighted
 2388 complex exponential

$$\Phi_{ij}[n] = A_{ij}[n] \exp(2\pi i \phi_{ij}[n]), \quad (4.39)$$

2389 where the indices i and j label the beamforming and antenna positions respectively. The
 2390 weight A_{ij} accounts for the relative power increase for antennas that are closer to the
 2391 position of the electron, and ϕ_{ij} is the total beamforming phase shift for the j -th antenna
 2392 at the i -th beamforming position.

2393 The beamforming phase shift is a sum of two terms

$$\phi_{ij}[n] = \frac{2\pi d_{ij}[n]}{\lambda} + \theta_{ij}[n], \quad (4.40)$$

2394 where the first term is the phase shift originating from the path length difference ($d_{ij}[n]$)
 2395 between the beamforming and antenna positions, which are represented by the vectors
 2396 (r_j, θ_j) and $(r_i, \theta_i[n])$, and the second term is the angular separation ($\theta_{ij}[n]$) of the two
 2397 positions. The angular separation enters into the beamforming phase due to an effect
 2398 caused by the circular orbit of the electron that produces radiation whose phase is linearly
 2399 dependent on the relative azimuthal position of the antenna [28, 29]. The time-dependence
 2400 of the beamforming phases is intended to correct for the effects of ∇B -drift, which cause
 2401 the guiding centers of electrons to orbit the center of the magnetic trap. By including a
 2402 linear time-dependence in the azimuthal beamforming position,

$$\theta_i[n] = \omega_{\nabla B} t[n] + \theta_{i,0}, \quad (4.41)$$

2403 where $\omega_{\nabla B}$ is the azimuthal grad-B drift frequency, $t[n]$ is the time vector and, $\theta_{i,0}$ is the
 2404 starting azimuthal position, we can configure the beamforming phases to effectively track
 2405 the XY-position of the guiding center over the event duration. Predicting accurate values
 2406 of $\omega_{\nabla B}$ for a specific trap and set of kinematic parameters will be done by simulations,

2407 which are performed using the Kassiopeia software package [7] by Project 8.

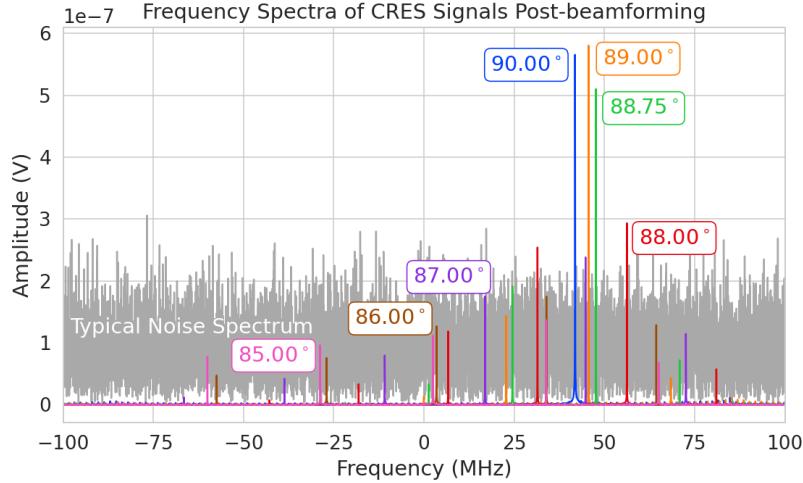


Figure 4.31: Frequency spectra of simulated CRES signals post-beamforming. The signal of a 90° electron consists of a single frequency component that is easy to detect with a power threshold on the frequency spectrum. This power threshold is still effective for signals with relatively large pitch angles such as 89.0° and 88.75° , which are composed of a main carrier and a few small sidebands. Signals with smaller pitch angles, below about 88.5° , tend to be dominated by sidebands such that no single frequency component can be reliably distinguished from the noise with a power threshold.

2408 After digital beamforming, we apply a short-time Fourier transform (STFT) to the
2409 summed time-series to obtain the frequency spectrum representation of the signals (see
2410 Figure 4.31). From the detection perspective, the frequency representation of the CRES
2411 data is advantageous compared to the time domain, because the frequency spectra of
2412 CRES signals are well-approximated by a frequency and amplitude modulated sinusoidal
2413 whose carrier frequency increases as a linear chirp. The modulation is caused by the axial
2414 oscillation of the electron in the magnetic trap and produce frequency spectra that are
2415 well-described by a small number of frequency components. The linear chirp is caused
2416 by the energy loss due to cyclotron radiation, which results in a relatively slow increase
2417 in the frequency components of the CRES signal over time. During the standard Fourier
2418 analysis window for the FSCD of $40.96 \mu\text{sec}$, we expect a typical CRES signal to increase
2419 in frequency by approximately 15 kHz, which is smaller than the frequency bin width
2420 given the 200 MHz sample rate. Therefore when considering a single frequency spectrum
2421 it is justifiable to neglect the effects of the linear frequency chirp.

2422 In the cases where the electron's pitch angle is $\gtrsim 88.5^\circ$, the majority of the signal
2423 power is contained in a single frequency component, with the remaining signal power

contained in a small number of sidebands proportional to the electron's axial modulation
 (see Figure 4.31). In these cases detection is relatively straight-forward by implementing
 a power threshold on the STFT, since the amplitude of the main signal peak is distinct
 from the thermal noise spectrum. However, as the pitch angle of the electron is decreased
 below 88.5° , the modulation index of the signal increases causing the maximum amplitude
 of the frequency spectrum to be comparable to typical noise fluctuations. At this point,
 the power threshold trigger is no longer able to distinguish between signal and noise
 leading to a reduction in detection efficiency. The neutrino mass sensitivity of the FSCD
 is directly linked to the overall detection efficiency. And, because the distribution of
 electron pitch angles is effectively uniformly distributed across the range of pitch angles
 that can be trapped, the overall detection efficiency is directly influenced by the range of
 pitch angles that have detectable signals. Therefore, utilizing a signal detection algorithm
 that can more effectively identify signals with pitch angles less than 88.5° will improve
 both detection efficiency and ultimately the neutrino mass sensitivity of the FSCD and
 other CRES experiments.

Modeling the detection performance of alternative signal detection algorithms for
 the FSCD requires that we pose the signal detection problem in a consistent manner.
 The approach we take is to perform a binary hypothesis test on the frequency spectra
 generated by the STFT. Mathematically, this is expressed as,

$$\mathcal{H}_0 : y[n] = \nu[n] \quad (4.42)$$

$$\mathcal{H}_1 : y[n] = x[n] + \nu[n]. \quad (4.43)$$

Where under hypothesis \mathcal{H}_0 , the vector representing the frequency spectrum ($y[n]$) is composed of pure white Gaussian noise (WGN) represented by $\nu[n]$, and under hypothesis \mathcal{H}_1 the frequency spectrum is composed of a CRES signal ($x[n]$) with added WGN. The dominant source of noise in a FSCD-like experiment is expected to be thermal Nyquist-Johnson noise, which is well approximated by a WGN distribution. In order to decide between these two hypotheses we follow the standard Neyman-Pearson approach by performing a log-likelihood ratio test between the probability distributions of the signal classifier output under \mathcal{H}_1 and \mathcal{H}_0 [19]. The output of the log-likelihood ratio test is called the test statistic, which is used to assign the data as belonging to the noise (\mathcal{H}_0) or signal (\mathcal{H}_1) classes by setting a decision threshold on the value of the test statistic.

In practice, we select the decision threshold by finding the value of the test statistic that guarantees an acceptable rate of false positives and then attempt to maximize

2455 the signal detection probability under that fixed false positive rate. Because the signal
 2456 classifier will be used to evaluate the spectra of $O(10^2)$ beamforming positions every
 2457 40.96 μ sec, we will require the signal classifiers to operate with decision thresholds that
 2458 provide false positive rates significantly smaller than 1%. This reduces the burden placed
 2459 on later stages of the CRES reconstruction chain to reject these false positives and
 2460 decreases the overall likelihood of reconstructing a false event. Below, we calculate the
 2461 probability distributions that allow us characterize how different detection algorithms
 2462 will perform for CRES signals in an FSCD experiment.

2463 **4.4.3 Signal Detection Algorithms**

2464 **4.4.3.1 Power Threshold**

2465 The power threshold detection algorithm uses the maximum amplitude of the frequency
 2466 spectra as the detection test statistic. To model the performance of this approach,
 2467 consider first the case where the signal is pure WGN. For a single bin in the frequency
 2468 spectrum, the probability that the amplitude falls below a specific threshold value is
 2469 given by the Rayleigh cumulative distribution function (CDF),

$$\text{Ray}(x; \tau) = 1 - \exp(-|x|^2/\tau), \quad (4.44)$$

2470 where the complex amplitude of the frequency bin is x , and τ is the WGN variance.
 2471 Because the noise samples for each frequency bin are independent and identically dis-
 2472 tributed (IID), the probability that every bin in the frequency spectrum falls below the
 2473 threshold is the joint CDF formed by the product of each individual frequency bin CDF,

$$F_0(x; \tau, N_{\text{bin}}) = \text{Ray}(x; \tau)^{N_{\text{bin}}}. \quad (4.45)$$

2474 The PDF for the power threshold classifier can then be obtained by differentiating the
 2475 CDF.

2476 The probability distribution for the power threshold classifier under \mathcal{H}_1 is formed in
 2477 a similar way, but the frequency bins that contain signal must be treated separately. For
 2478 a frequency bin that contains both signal and noise we can describe the probability that
 2479 the amplitude of the bin will fall below our threshold using the Rician CDF,

$$\text{Rice}(x; \tau, \nu) = 1 - Q_1 \left(\frac{|\nu|}{\sqrt{2\tau}}, \frac{|x|}{\sqrt{2\tau}} \right), \quad (4.46)$$

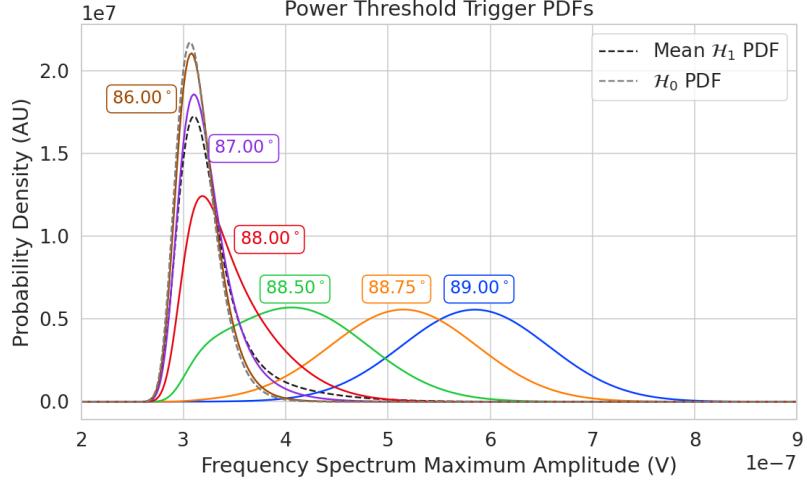


Figure 4.32: PDFs of the power threshold test statistic for CRES signals with various pitch angles as well as the PDF for the noise-only signal case. The average PDF computed for pitch angles ranging from 85.5 to 88.5° is also shown. As the pitch angle is decreased the signal PDF converges towards the noise PDF which indicates that the power threshold trigger is unable to distinguish between small pitch angle signals and noise.

where the parameter $|\nu|$ defines the noise-free amplitude of the signal and Q_1 is the Marcum Q-function. This time the CDF that describes the probability that the entire spectrum falls below the decision threshold is the product of both signal and noise CDFs,

$$F_1(x; \tau, \nu, N_{\text{bin}}, N_s) = \text{Ray}(x; \tau)^{N_{\text{bin}} - N_s} \prod_{k=0}^{N_s} \text{Rice}(x; \tau, \nu_k). \quad (4.47)$$

The first half of Equation 4.47 is the contribution from the bins in the frequency spectrum that contain only noise, and the second half is the product of the Rician CDFs for the frequency bins that contain signal peaks with a noise-free amplitude of $|\nu_k|$. In Figure 4.32 we show plots of example PDFs under \mathcal{H}_1 and \mathcal{H}_0 .

4.4.3.2 Matched Filtering

The shape of a CRES signal is completely determined by the initial conditions of the electron as it is emitted from beta-decay, which implies that it is possible to apply matched filtering as a signal detection algorithm. With a matched filter one uses the shape of the known signal, which is called a template, to filter the incoming data by computing the convolution between the signal and the data [19]. For cases where the signal is buried in WGN, the matched filter is the optimal detector in that it achieves

2494 the maximum probability of a true detection for a fixed false positive rate. Since CRES
 2495 signals have an unknown shape but are deterministic, we can apply a matched filter by
 2496 using simulations to generate a large number of signal templates called a template bank,
 2497 which spans the parameter space of possible signals. Then at detection time, we use the
 2498 template bank to identify signals by performing the matched filter convolution for each
 2499 template in an exhaustive search.

2500 As we saw from the frequency spectra in Figure 4.31, CRES signals are highly periodic
 2501 in nature. In such cases, it is advantageous to utilize the convolution theorem to replace
 2502 the matched filter convolution with an inner product in the frequency-domain. With the
 2503 convolution theorem, the matched filter test statistic that describes the detection of a
 2504 signal buried in WGN using a matched filter template bank is given by

$$\mathcal{T} = \max_{\mathbf{h}} \left| \sum_{n=0}^{N_{\text{bin}}} h^\dagger[n] y[n] \right|, \quad (4.48)$$

2505 where $h^\dagger[n]$ is the complex conjugate of the signal template. For the case when our
 2506 template bank consists of only a single template it is possible to derive an exact analytical
 2507 form for the PDF describing the matched filter test statistic. First, we derive PDF under
 2508 the signal hypothesis, where the equation describing the matched filter test statistic, also
 2509 known as the matched filter score, becomes

$$\mathcal{T} = \left| \sum_{n=0}^{N_{\text{bin}}} h^\dagger[n] y[n] \right|. \quad (4.49)$$

2510 Each noisy frequency bin represented by $y[n]$ is the sum between value of the signal
 2511 at that bin and complex WGN, which means that $y[n]$ is itself Gaussian distributed.
 2512 Therefore, the value of the inner product between the template and the data is also a
 2513 complex Gaussian variable; and, since the matched filter score is the magnitude of this
 2514 inner product, it must follow a Rician distribution.

2515 We can derive the equation for the Rician PDF by expressing the matched filter
 2516 template \mathbf{h} in terms of the corresponding simulated signal, which we write as \mathbf{x}_h to
 2517 distinguish from the signal in the data. Using the standard normalization and assuming
 2518 uncorrelated WGN, the matched filter templates can be written as

$$\mathbf{h} = \frac{\mathbf{x}_h}{\sqrt{\tau |\mathbf{x}_h|^2}} \quad (4.50)$$

2519 where τ is the noise variance. Inserting this into Equation 4.48 and expressing the data
 2520 as a sum between a signal and a WGN vector yields,

$$\mathcal{T} = \frac{1}{\sqrt{\tau|\mathbf{x}_h|^2}} \left| \sum_{n=1}^{N_{\text{bin}}} x_h[n] (x[n] + \nu[n]) \right|. \quad (4.51)$$

2521 Next, we transform the expression by isolating the randomly distributed components
 2522 giving

$$\mathcal{T} = \frac{\left| \sum_{n=1}^{N_{\text{bin}}} x_h[n] x[n] \right|}{\sqrt{\tau|\mathbf{x}_h|^2}} + \frac{1}{\sqrt{\tau|\mathbf{x}_h|^2}} \left| \sum_{n=1}^{N_{\text{bin}}} x_h[n] \nu[n] \right|. \quad (4.52)$$

2523 The first term of 4.52 can be simplified by using the Cauchy-Schawrz inequality to express
 2524 the magnitude of the inner product in terms of the magnitudes of the signal and template
 2525 as well as an orthogonality constant which we call "match" (Γ). Using this we obtain,

$$\mathcal{T} = |\mathbf{h}| |\mathbf{x}| \Gamma + \frac{1}{\sqrt{\tau|\mathbf{x}_h|^2}} \left| \sum_{n=1}^{N_{\text{bin}}} x_h[n] \nu[n] \right|. \quad (4.53)$$

2526 The second term is a sum of Gaussian distributed variables, which we should expect also
 2527 follows a Gaussian distribution. Each of the samples $\nu[n]$ is described by

$$\nu[n] \sim \mathcal{N}(0, \tau), \quad (4.54)$$

2528 where $\mathcal{N}(0, \tau)$ is a complex Gaussian distribution with zero mean and variance τ . There-
 2529 fore,

$$\frac{x_h[n]}{\sqrt{\tau|\mathbf{x}_h|^2}} \nu[n] \sim \mathcal{N}\left(0, \frac{x_h[n]^2}{|\mathbf{x}_h|^2}\right), \quad (4.55)$$

$$\sum_{n=1}^{N_{\text{bin}}} \frac{x_h[n]}{\sqrt{\tau|\mathbf{x}_h|^2}} \nu[n] \sim \mathcal{N}\left(0, \frac{\sum_{n=1}^{N_{\text{bin}}} x_h[n]^2}{|\mathbf{x}_h|^2}\right) = \mathcal{N}(0, 1), \quad (4.56)$$

$$|\mathbf{h}| |\mathbf{x}| \Gamma + \sum_{n=1}^{N_{\text{bin}}} \frac{x_h[n]}{\sqrt{\tau|\mathbf{x}_h|^2}} \nu[n] \sim \mathcal{N}(|\mathbf{h}| |\mathbf{x}| \Gamma, 1). \quad (4.57)$$

2530 We see that \mathcal{T} is magnitude of a complex variable with mean $|\mathbf{h}| |\mathbf{x}| \Gamma$ and variance one. In
 2531 order to simply the expression a bit further, we define the quantity $\mathcal{T}_{\text{ideal}} = |\mathbf{h}| |\mathbf{x}| \Gamma$, which
 2532 we call the ideal matched filter score, because it represents the value of the matched
 2533 filter inner product that we would expect if no noise was present in the signal. We can

2534 write the matched filter test statistic as the magnitude of a two-dimensional vector in
 2535 the complex plane

$$\mathcal{T} = |(\mathcal{T}_{\text{ideal}} + n_r, n_i)|, \quad (4.58)$$

2536 where n_r and n_i are the real and imaginary components of the noise each with variance
 2537 $1/2$, which is modeled by a Rician distribution with shape factor $\mathcal{T}_{\text{ideal}}$. Therefore, the
 2538 probability distribution of the matched filter test statistic is given by,

$$P_1(x; \mathcal{T}_{\text{ideal}}) = 2x \exp(- (x^2 + \mathcal{T}_{\text{ideal}}^2)) I_0(2x\mathcal{T}_{\text{ideal}}), \quad (4.59)$$

2539 where I_0 is the zeroth-order modified Bessel function.

2540 The shape of the matched filter score distribution is controlled by the parameter
 2541 $\mathcal{T}_{\text{ideal}}$, which is effectively the value of the matched filter score if the data contained no
 2542 noise. Without noise, the data vector reduces to the signal, \mathbf{x} , in which case Equation
 2543 4.49 becomes the magnitude of an inner product between two vectors. We can write
 2544 the magnitude of an inner product in terms of the lengths of the individual vectors and
 2545 a constant that describes the degree of orthogonality between them. Applying this to
 2546 Equation 4.49, we obtain

$$\mathcal{T}_{\text{ideal}} = |\mathbf{h}^\dagger \cdot \mathbf{x}| = |\mathbf{h}| |\mathbf{x}| \Gamma \quad (4.60)$$

2547 where Γ describes the orthogonality between \mathbf{h} and \mathbf{x} . From the point of view of matched
 2548 filtering, we can interpret Γ as describing how well the template matches the underlying
 2549 signal in the data.

2550 The matched filter score PDF under the noise hypothesis can be readily obtained
 2551 from Equation 4.59 by setting the value of $\mathcal{T}_{\text{ideal}}$ to zero, since the data contains no signal
 2552 in the noise case. Doing this, we obtain the Rayleigh distribution that describes the
 2553 matched filter score under \mathcal{H}_0 ,

$$P_0(x) = 2x \exp(-x^2). \quad (4.61)$$

2554 Equations 4.59 and 4.61 describe the behavior of the matched filter test statistic
 2555 under \mathcal{H}_0 and \mathcal{H}_1 for a single template. However, defining a PDF that describes the
 2556 matched filter test statistic in the case of multiple templates is in general a mathematically
 2557 intractable problem, since there is no guarantee of orthogonality between matched filter
 2558 templates. This leads to correlations between the matched filter scores of different
 2559 templates because only one sample of noise is used to compute the matched filter scores
 2560 of the template bank. In order to proceed, we need to make the simplifying assumption

that we can treat the matched filter scores as IID variables, which allows to ignore correlations between templates. The overall effect of this will be an underestimate of the performance of the matched filter, since we are under counting the number of templates that could contribute a detectable score.

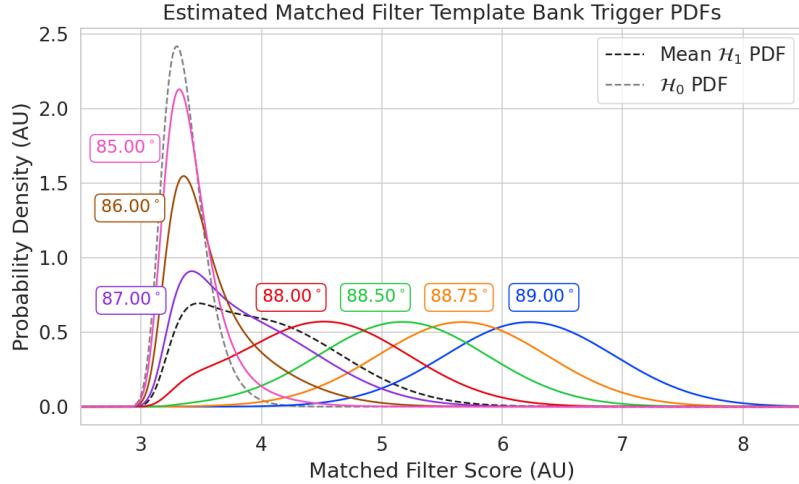


Figure 4.33: Plots of the estimated PDFs for the matched filter template bank test statistic for CRES signals with various pitch angles as well as the estimated PDF for the noise only signal case. We assume an estimated number of templates of 10^5 and perfect match between signal and template i.e. $\Gamma_{\text{best}} = 1$. The mean PDF includes signals ranging from $85.5 - 88.5^\circ$ in pitch angle. There is a much larger distinction between the signal PDFs at small pitch angle compared to the power threshold indicating a higher detection efficiency for these signals.

For \mathcal{H}_0 we model the probability that the matched filter score falls below our threshold using the CDF obtained by integrating Equation 4.61. Because we are assuming that the matched filter scores using different templates are independent, the probability that the matched filter score for all templates falls below a threshold value is the joint CDF formed by multiplying the CDF for each template. Under \mathcal{H}_0 this is

$$F_0(x) = \left(1 - e^{-x^2}\right)^{N_t}, \quad (4.62)$$

where x is the matched filter score threshold and N_t is the number of templates. We should expect that the distribution describing the matched filter template bank maximum score depends on N_t , because with more templates there is a greater chance of a random match between the template and data.

For \mathcal{H}_1 , we start by denoting the CDF of the best matching template as $F_{\text{best}}(x; \mathcal{T}_{\text{best}})$,

and treat the matched filter scores for all other templates as negligible ($\mathcal{T}_{\text{ideal}} \approx 0$). Then we form the joint CDF by combining the distributions for all templates used during detection. Since we are exhaustively checking the matched filter scores, the number of templates checked will be a randomly distributed variable that ranges from zero to the total number of available templates. If we assume that signals are uniformly distributed across the parameter space spanned by the template bank then on average we check $(N_t - 1)/2 \approx N_t/2$ templates for each inference. Therefore, the estimated CDF under \mathcal{H}_1 is

$$F_1(x; \mathcal{T}_{\text{best}}) = F_{\text{best}}(x; \mathcal{T}_{\text{best}}) \left(1 - e^{-x^2}\right)^{N_t/2}. \quad (4.63)$$

In Figure 4.33 we show plots of the estimated matched filter template bank classifier PDFs under both \mathcal{H}_0 and \mathcal{H}_1 .

4.4.3.3 Machine Learning

In this paper we focus on Convolutional Neural Networks (CNN) as an example of a machine learning based signal classifier. CNNs are constructed using a series of convolutional layers, each composed of a set of filters that are convolved with the input data. The individual convolutional filters can be viewed as matched filter templates that are learned from a set of simulated data rather than being directly generated. This opens the possibility of finding a more efficient representation of the matched filter templates during the training process that can potentially reduce computational cost at inference time while still offering good classification performance.

The machine learning approach is distinct from both the power threshold and matched filtering in that we do not attempt to manually engineer a test statistic that is computed from the data for classification. Instead, we attempt calculate the test statistic by constructing a differentiable function that maps the complex frequency series generated by the STFT to a binary classification as either signal or noise. The test statistic for the machine learning classifier can be expressed as

$$\mathcal{T} = G(\mathbf{y}; \boldsymbol{\Omega}) \quad (4.64)$$

where \mathbf{y} is the noisy data vector and $G(\mathbf{y}; \boldsymbol{\Omega})$ is the machine learning model parameterized by the weights $\boldsymbol{\Omega}$. By using supervised learning on a labeled set of training signals, we can modify the function parameters to learn the mapping from the data to the likelihood of \mathbf{y} belonging to either \mathcal{H}_1 or \mathcal{H}_0 .

The CNN architecture used for this work is summarized by Table 4.1. No strategic

Table 4.1: A summary of the CNN model layers and parameters. The output of each 1D-Convolution and Fully Connected layer is passed through a LeakyReLU activation function and re-normalized using batch normalization before being passed to the next layer in the model. The output of the final Fully Connected layer in the model is left without activation so that the model outputs can be directly passed to the Binary Cross-entropy loss function used during training.

| Layer | Type | Input Channels | Output Channels | Parameters |
|-------|-----------------|----------------|-----------------|--|
| 1 | 1D-Convolution | 2 | 15 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 1)$ |
| 2 | Maximum Pooling | 15 | 15 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 4)$ |
| 3 | 1D-Convolution | 15 | 20 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 1)$ |
| 4 | Maximum Pooling | 20 | 20 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 4)$ |
| 5 | 1D-Convolution | 20 | 25 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 1)$ |
| 6 | Maximum Pooling | 25 | 25 | $(N_{\text{kernel}} = 4, N_{\text{stride}} = 4)$ |
| 7 | Fully Connected | 3200 | 512 | NA |
| 8 | Fully Connected | 512 | 64 | NA |
| 9 | Fully Connected | 64 | 2 | NA |

hyper-parameter optimization approach was implemented beyond the manual testing of different CNN architecture variations, so this particular model is best viewed as a proof-of-concept rather than a rigorously optimized design. Numerous model variations were tested, some with significantly more layers and convolutions filters per layer, as well as others that were even smaller than the architecture in Table 4.1. Ultimately, the model architecture choice was driven by the motivation to find the minimal model whose classification performance was still comparable to the larger CNN’s tested, because of the importance of minimizing computational cost in real-time applications. It is possible that more sophisticated machine learning models could improve upon the classification results achieved here, but we leave this investigation for future work.

4.4.4 Methods

4.4.4.1 Data Generation

To test the triggering performance of the classifiers, simulated CRES signals were generated using the Locust simulations package [10, 28] developed by the Project 8 collaboration. Locust uses the separately developed Kassiopeia package to calculate the magnetic fields produced by a user defined set of current carrying coils along with any specified background magnetic fields, resulting in a magnetic trap. Next, Kassiopeia calculates the trajectory of an electron in this magnetic field starting from a set of user

2623 specified initial conditions. The Locust software then uses the electron trajectories from
2624 Kassiopeia to calculate the resulting electromagnetic fields using the Liénard-Wiechert
2625 equations, and determine the voltages generated in the antenna array with the antenna
2626 transfer function. Locust then simulates the down-conversion, filtering, and digitization
2627 steps resulting in the simulated CRES signals for an electron.

2628 The shape of the received CRES signal is determined by the initial kinematic param-
2629 eters, including the starting position of the electron, the starting kinetic energy of the
2630 electron, and the pitch angle. For the studies performed here we constrain ourselves to a
2631 single initial electron position located at $(x, y, z) = (5, 0, 0)$ mm, and using this starting
2632 position we generate two datasets by varying the initial kinetic energy and the starting
2633 pitch angle. The first dataset consists of a two-dimensional square grid of kinetic energy
2634 and pitch angle spanning an energy range from 18575-18580 eV with a spacing of 0.1 eV,
2635 and pitch angles from 85.5-88.5° with a spacing of 0.001°, resulting in 153051 signals with
2636 a unique energy-pitch angle combination. This dataset is intended to represent a matched
2637 filter template bank. The second dataset was generated by randomly sampling uniform
2638 probability distributions covering the same parameter space to produce approximately
2639 50000 signals randomly parameterized in energy and pitch angle. This dataset provides
2640 the training and test data for the machine learning approach, and acts as a representative
2641 sample of signals to evaluate the performance of the matched filter template bank.

2642 Each signal was simulated for a duration of 40.96 μ s, which is equivalent to 8192
2643 samples at the FSCD digitization rate, and begins at time $t = 0$ s for all simulations.
2644 This duration represents a single frequency spectrum generated by the STFT. The output
2645 of the Locust simulation is a matrix of array snapshots with size given by the number of
2646 channels times the event length ($N_{\text{ch}} \times N_{\text{sample}}$), which we pre-process using the digital
2647 beamforming summation and STFT described in Section 4.4.2.2. The ∇B -drift correction
2648 uses the exact value of $\omega_{\nabla B}$, obtained from the Kassiopeia simulation of that electron.
2649 In practice, an average value for $\omega_{\nabla B}$ could be used, because there is limited variation in
2650 drift frequency across this parameter space.

2651 4.4.4.2 Template Number and Match Estimation

2652 The estimated PDF for the matched filter template bank depends on the score of the
2653 best matching template or equivalently the match of the best template (Γ_{best}) as well
2654 as the number of templates. One expects that with a higher number of templates the
2655 average value of Γ_{best} will increase, however, there is a point of diminishing returns at
2656 which more templates will not significantly increase match, but will still increase the

2657 likelihood of false positives. Therefore, it is desirable to use the minimum number of
2658 templates that provide an acceptable mean value of Γ_{best} .

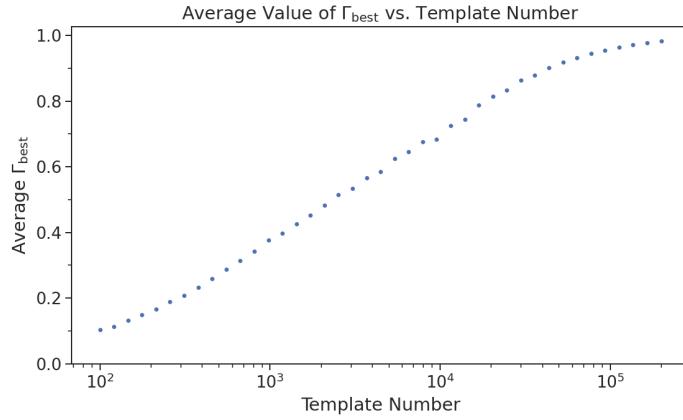


Figure 4.34: The mean match of the matched filter template bank to a test set of randomly parameterized signals as a function of the number or density of templates. The parameter space includes pitch angles from $85.5 - 88.5^\circ$ and energies from 18575 – 18580 eV.

2658
2659 To quantify the relationship between match and template number, we calculated
2660 the mean match of the random dataset to a selection of templates obtained from the
2661 regularly spaced dataset. The results are shown in Figure 4.34, where we find that the
2662 average value of Γ_{best} is an exponential function of the number of templates. From this
2663 plot we select the desired value of mean match at which we would like to evaluate the
2664 matched filter PDF and can infer the required number of templates.

2665 4.4.4.3 CNN Training and Data Augmentation

2666 To prepare the data for training the model, we split the random dataset in half to create
2667 distinct training and test datasets. Additionally, a randomly selected 20% of the training
2668 data is isolated for use as a validation set during the training loop. The size of the
2669 training, validation, and test datasets are then tripled by appending two additional copies
2670 of the data to increase the sample size of the dataset after data augmentation. The
2671 data is loaded with no noise, which is added to each data batch during the training
2672 phase by generating a new noise sample from a complex WGN distribution. In order to
2673 ensure an even split between signal and noise data we append to the noise-free signals an
2674 equal number of empty signals composed of all zeros. Therefore, as the data is randomly
2675 shuffled during training, on average an equal number of empty signals will be included
2676 with the training signals. After adding the sample of WGN to the data batch, the empty

signals represent the noise-only data that the model must distinguish from signal data.

As the training signals are loaded we apply a unique random phase shift as the first form of data augmentation. Since the data is generated using the same initial axial position and cyclotron orbit phase, the randomization is an attempt to prevent overtraining on these features. During each training epoch the data is randomly shuffled and split into batches of 2500 signals. Each batch of signals is then circularly shifted by a random number of frequency bins to simulate a kinetic energy shift from -75 to 20 eV to simulate a training dataset with a larger energy range. Next, a sample of complex WGN, consistent with the expected 10 K Nyquist-Johnson noise expected for the FSCD, is generated and added to the signal, which prevents overtraining on noise features. As a final step, the data is renormalized by the standard deviation of the noise so that the range of values in the data is close to $[-1, 1]$, which helps ensure well-behaved back-propagation.

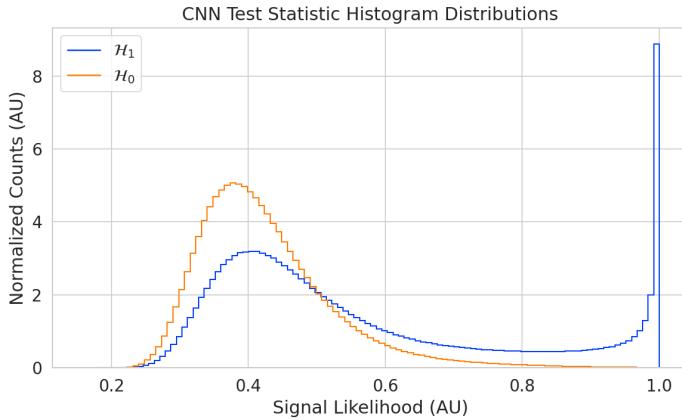


Figure 4.35: Histograms of the trained CNN model output from the test dataset. The blue histogram shows the model outputs for signal data. The oddly shaped peak near the end is the result of the softmax function mapping the long tail of the raw output distribution to the range $[0, 1]$.

The Binary Cross-entropy loss function is used to compute the loss for each batch of data and the model weights are updated using the ADAM optimizer with a learning rate of 5×10^{-3} . After each training epoch, the loss and classification accuracy of the validation dataset are computed to monitor for overtraining. It was noticed that the relatively high noise power and the fact that a new sample of noise was used for each batch together provided a strong form of regularization, since no evidence of over-training was observed even after several thousand epochs. Typically, the loss and classification accuracy of the model converged after a few hundred training epochs, but the training loop was

2698 extended to 3000 epochs to attempt to achieve the best possible performance. The
2699 training procedure generally took about 24 hrs using a single NVIDIA V100 GPU [30].

2700 After training the model, we use it to classifying the test dataset and generate
2701 histograms of the model outputs for both classes of data. The data augmentation
2702 procedure for the evaluation of the test data mirrors the training procedure without
2703 the validation split. Since a random circular shift and a new sample of WGN is added
2704 to each batch, the testing evaluation loop is run for 100 epochs to get a representative
2705 sample of noise and circular shifts. The model outputs for each batch are passed through
2706 a softmax activation and then combined into histograms, which we show in Figure 4.35.

2707 4.4.5 Results and Discussion

2708 4.4.5.1 Trigger Classification Performance

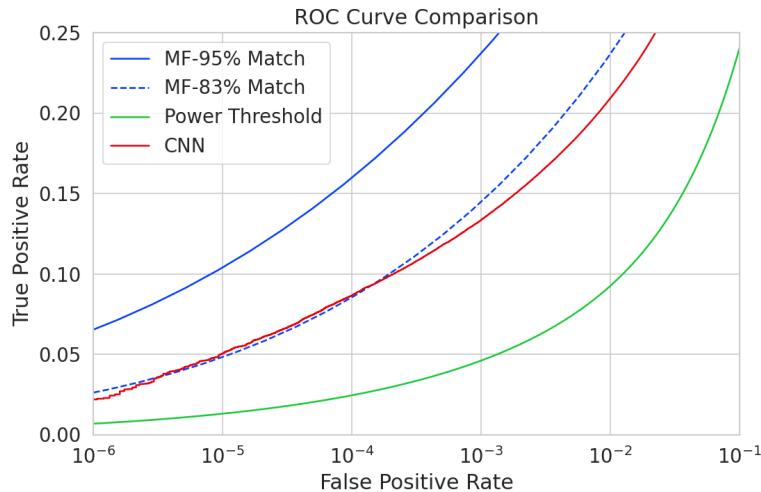


Figure 4.36: ROC curves describing the detection efficiency or true positive rates for the three signal classification algorithms examined in this paper.

2709 Using the matched filter and power threshold CDFs, along with the classification
2710 results from the CNN, we compare detection performance by computing receiver operating
2711 characteristic (ROC) curves. Specifically, we compare the detection performance averaged
2712 over the full signal parameter space in order to get a measure of the overall detection
2713 efficiency achieved by each algorithm. For the power threshold and matched filter
2714 algorithms, we obtain the mean ROC curve by taking the average over all signals in the
2715 regularly spaced dataset. In the case of the matched filter, we examine two cases using
2716 different numbers of templates, which have different values of mean match. The ROC

2717 curve describing the CNN is obtained by forming a histogram of the network outputs
2718 for each class of signal and from this computing the estimated CDFs and ROC curve.
2719 In Figure 4.36, we show the ROC curves obtained for each of the detection algorithms,
2720 visualized in terms of true positive rate and false positive rate.

2721 The true positive rate of a signal classifier is equivalent to its detection efficiency, and
2722 we see that for the population of signals with pitch angles $< 88.5^\circ$ the power threshold
2723 has a consistently lower detection efficiency than the CNN and the matched filter. This
2724 result could have been predicted from the visualization of signal spectra in Figure 4.31,
2725 where we see that there is no way to distinguish between a noise peak and a signal peak
2726 with high confidence at small pitch angles. The CNN offers a significant and consistent
2727 increase in detection efficiency over the power threshold approach, with the relative
2728 improvement in detection efficiency increasing as the false positive rate decreases. If
2729 we compare the CNN to the matched filter, we see that the performance of the tested
2730 network is roughly equivalent to a matched filter detector with an average match of
2731 about 83%, which uses approximately 20000 matched filter templates. The overall best
2732 detection efficiency is achieved by the matched filter classifier if a large enough template
2733 bank is used. We show in the plot the ROC curve for a matched filter template bank
2734 with 95% average match, which is achieved with approximately 100000 templates. Since
2735 the matched filter is known to be statistically optimal for detecting a known signal in
2736 WGN, it is somewhat expected that this algorithm has the highest detection efficiency.

2737 A potentially impactful difference between the matched filter and CNN algorithms is
2738 that the CNN relies upon convolutions as its fundamental calculation mechanism, whereas
2739 our implementation of a matched filter utilizes an inner product. Since convolution is
2740 a translation invariant operation, the detection performance of CNN can be extended
2741 to a wider range of CRES event kinetic energies with less cost than the matched filter,
2742 a feature that we exploited during the CNN training by including circular translations
2743 of the CRES frequency spectra in the training loop. Increasing the range of kinetic
2744 energies detectable by a matched filter requires a proportional increase in the number of
2745 templates, which directly translates into increased computational and hardware costs.
2746 From a practical perspective, the detection algorithm is always limited by the available
2747 computational hardware, so estimating the relative costs is a key factor in determining
2748 their feasibility. Below we perform a more detailed analysis of the relative costs of each
2749 of the detection algorithms.

2750 **4.4.5.2 Computational Cost and Hardware Requirements**

2751 In the process of investigating triggering approaches for an antenna array CRES exper-
2752 iment, we have uncovered a strong tension between detection efficiency and computational
2753 resources. To relate the computational cost estimates to actual costs, we compare the
2754 theoretical amount of computer hardware required to implement the signal classifiers
2755 for real-time detection in an FSCD experiment. To do this we shall utilize order of
2756 magnitude estimates of the theoretical peak performance values for currently available
2757 Graphics Processing Units (GPUs) as a metric. This approach will underestimate the
2758 amount of required hardware, since it is unlikely that any CRES detection algorithm
2759 could reach the theoretical peak performance of the hardware.

2760 Of the three detection algorithms tested, the power threshold classifier is the least
2761 expensive. It requires that we check whether the amplitude of each frequency bin in
2762 the STFT is below or above our decision threshold. The STFT combined with digital
2763 beamforming produces $N_{\text{bin}}N_b$ frequency bins that must be checked every N_{bin}/f_s seconds.
2764 This requires approximately $O(10^{10})$ FLOPS to check in real-time. Current generations of
2765 GPUs have peak theoretical performances in the range of $O(10^{13}) - O(10^{14})$ FLOPS [31],
2766 dependent on the required floating-point precision of the computation. Therefore, the
2767 entire computational needs of a real-time triggering system using a power threshold
2768 classifier, including digital beamforming and generation of the STFT, could be met by a
2769 single high-end GPU or a small number of less powerful GPUs. Since triggering is only
2770 one step of the full real-time signal reconstruction approach, limiting the computational
2771 cost of this stage is ideal. However, we have seen that the power threshold classifier does
2772 not provide sufficient detection efficiency across the entire range of possible signals,
2773 which is the primary motivation for exploring more complicated triggering solutions.

2774 As discussed, the computational cost of the matched filter approach requires counting
2775 the number of templates that must be checked for each frequency spectra produced by the
2776 STFT. Computing the matched filter scores requires $O(N_bN_tN_{\text{bin}})$ operations, since for
2777 each of the N_b beamforming positions we must multiply N_t templates with a data vector
2778 that has length N_{bin} . The time within which we must perform this calculation is equal
2779 to N_{bin}/f_s to keep up with the data generation rate. To cover the 5 eV kinetic energy
2780 range spanned by the template bank, we saw that 10^4 to 10^5 templates are required in
2781 order to match or exceed the detection efficiency of the CNN. If the number of templates
2782 scales linearly with the kinetic energy range of interest as expected, then we would
2783 require 10^5 to 10^6 matched filter templates with this more realistic range of energies.
2784 Considering this, the estimated computational cost of the matched filter is between

2785 $O(10^{15})$ to $O(10^{16})$ FLOPS, which is $O(10^2)$ to $O(10^3)$ high-end GPUs.

2786 Lastly, we have the CNN classifier. To estimate the computational cost we simply
2787 sum the number of convolutions and matrix multiplications specified by the network
2788 architecture shown in Table 4.1. Each convolutional layer consists of $N_{\text{in}}N_{\text{out}}N_{\text{kernel}}L_{\text{input}}$
2789 floating-point operations, where N_{in} is the number of input channels, N_{out} is the number
2790 of output channels, N_{kernel} is the size of the convolutional kernel, and L_{input} is the length
2791 of the input vector, and the fully connected layers each contribute $N_{\text{in}}N_{\text{out}}$ operations.
2792 Summing all the neural network layers we estimate that the CNN would require $O(10^6)$
2793 floating point operations for each frequency spectra; therefore, the total computation
2794 cost of the CNN trigger is this cost times the number of beamforming positions per the
2795 data acquisition time, which is $O(10^{13})$ FLOPS or $O(10^0)$ GPUs.

2796 Compared with the matched filter approach the CNN requires $O(100)$ to $O(1000)$
2797 fewer GPUs to implement, dependent on the exact number of templates used in the
2798 template bank. The 100 eV kinetic energy range is motivated by the application of these
2799 detection algorithms to an FSCD-like neutrino mass measurement experiment. However,
2800 if a significantly larger range of kinetic energies is required, a CNN may be the preferred
2801 detection approach despite the lower average detection efficiency due to computational
2802 cost considerations. The low estimated computational cost of the CNN is directly related
2803 to the small network size.

2804 Additional experiments with larger CNNs, generated by increasing the depth and
2805 width of the neural network, and we observed that these changes provided minimal
2806 ($\lesssim 1\%$) improvement in the classification accuracy of the model. A potential reason
2807 for this could be the sparse nature of the signals in the frequency domain and the low
2808 SNR which makes for a challenging dataset to learn from. Future work could investigate
2809 modifications to the neural network architecture such as sparse convolutions, which may
2810 improve the classification accuracy of the model or further reduce the computational
2811 costs of this approach. Alternatively, more complicated CNN architectures such as a
2812 ResNet [32] or VGG model [33] may provide improved classification performance over a
2813 basic CNN. An additional promising area of investigation are recurrent neural networks,
2814 which may be able to exploit the time-ordered features of the STFT for more accurate
2815 signal detection if the electron signals last for multiple Fourier transform windows.

2816 Our estimate of the computational cost of the matched filter is somewhat naive if
2817 we notice that the majority of the values that make up a CRES frequency spectra are
2818 zero (see Figure 4.31). Therefore, the majority of operations in the matched filter inner
2819 product are unnecessary, and we could instead evaluate the matched filter inner product

2820 using only the $\lesssim 10$ frequency peaks that make up CRES signal. This optimization
2821 reduces the number of operations required to check each template by a factor of $O(100)$
2822 to $O(1000)$, which brings the estimated computational cost of the matched filter in
2823 line with the CNN. Although this level of sparsity results in a multiplication with very
2824 low arithmetic complexity, the resulting sparse matched filter algorithm is still likely
2825 to be constrained by memory access speed rather than compute speed. Ultimately, the
2826 comparison of the relative computational and hardware costs between the matched filter
2827 and CNN will depend on the efficiency of the software implementation and hardware
2828 support for neural network and sparse matrix calculations.

2829 **4.4.6 Conclusion**

2830 Increasing the detection efficiency and overall event rate of the CRES technique represents
2831 a key developmental path towards new scientific results and broader applications of the
2832 CRES technique. It is what motivates both the antenna array detection approach and
2833 the development of real-time signal reconstruction algorithms. We have demonstrated
2834 that significant gains in the detection efficiency of the CRES technique are achievable
2835 by utilizing triggering algorithms that account for the specific shape of CRES signals in
2836 the detector. These algorithms emphasize the need for accurate and fast methods for
2837 CRES simulation, since they directly contribute to the success of matched filter methods
2838 by providing a way to generate expected signal templates and also serve as a source of
2839 training data for machine learning approaches.

2840 The improvements in detection efficiency offered by these alternative approaches to
2841 triggering are crucial to the success of efforts to develop scalable technologies for CRES
2842 measurement, since they provide a significant increase in the detectable parameter space
2843 of CRES events, which allows for a better utilization of the larger detection volume.
2844 While we have focused on the real-time detection of CRES signals from antenna arrays,
2845 these same signal classifiers could be used in CRES experiments utilizing a different
2846 detector technologies, since the same principles of signal detection will apply. For example,
2847 previous CRES measurements by the Project 8 collaboration that utilized a waveguide
2848 gas cell, could have improved their detection efficiency by employing a matched filter
2849 or neural network classifier to identify trapped electrons with pitch angles that are too
2850 small to be detected by the power threshold approach. Furthermore, alternative CRES
2851 detector technologies such as resonant cavities [3] could also see similar improvements
2852 in detection efficiency, which is of crucial importance to future efforts by the Project 8
2853 collaboration to utilize CRES to measure the neutrino mass.

2854 Chapter 5 |

2855 Antenna and Antenna Measurement Sys- 2856 tem Development for the Project 8 Ex- 2857 periment

2858 5.1 Introduction

2859 The FSCD and antenna array CRES represent an innovative approach to beta-decay
2860 spectroscopy. While much can be learned from simulations about the systematics of
2861 CRES with antenna arrays, laboratory measurements and demonstrations provide critical
2862 inputs to sensitivity and simulation models as well as provide a means for calibration and
2863 commissioning of the experiment. Therefore, a robust program of antenna and antenna
2864 measurement hardware development is important to the success of the FSCD and the
2865 development of antenna array CRES more broadly.

2866 In this chapter we summarize the development of an antenna measurement system
2867 at Penn State to implement and test the techniques of antenna array CRES on the
2868 bench-top, in order to support the efforts of the Project 8 collaboration. In Section 5.2
2869 we provide an introduction to some fundamental parameters and concepts related to
2870 antenna measurements as well as an overview of the Penn State antenna measurement
2871 system hardware. In Section 5.3 we include the manuscript of a paper [29] which
2872 details the design and characterization of a specialized antenna developed to mimic the
2873 electric fields emitted by an electron in a CRES experiment. This antenna, called the
2874 Synthetic Cyclotron Antenna (SYNCA), is intended as a calibration tool for antenna
2875 arrays developed for CRES measurements. Lastly, in Section 5.4 we summarize a set
2876 of prototype FSCD antenna array measurements with the SYNCA [6], which we use to
2877 validate the simulated performance of the antenna array and estimate systematic errors
2878 associated with the antenna array.

2879 5.2 Antenna Measurements for CRES experiments

2880 5.2.1 Antenna Parameters

2881 Antenna characterization measurements are intended to validate simulations of the
2882 antenna array performance, which ultimately informs the neutrino mass sensitivity of
2883 the experiment. In this section, I shall summarize a few fundamental concepts relating
2884 to antennas and antenna measurement, before introducing how Project 8 uses antenna
2885 measurement for the development of antenna array CRES.

2886 5.2.1.1 Radiation Patterns

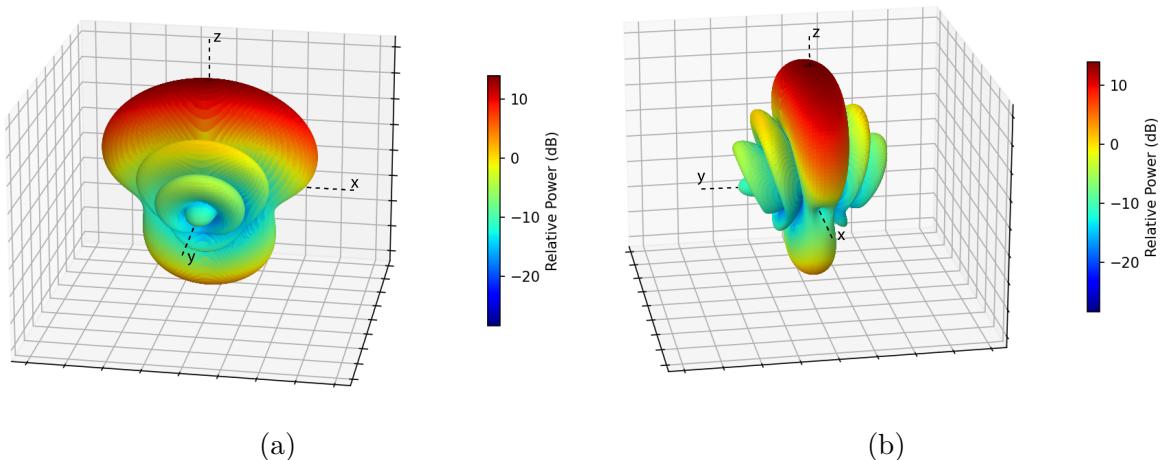


Figure 5.1: An example radiation pattern generated using HFSS simulations. The color and radial distance of the surface from the origin indicate the relative magnitude of radiation power emitted by the antenna in that direction. The primary goal of most antenna measurements is typically to measure the antenna pattern, which is used to derive many useful antenna performance parameters.

2887 Antennas are conductive structures designed to carry alternating electric currents
2888 in order to transmit energy in the form of electro-magnetic (EM) waves [13]. Perhaps
2889 the most fundamental way to characterize an antenna, is to map out the radiated power
2890 density as a function of position, which is called the radiation pattern (see Figure 5.1).
2891 We find the radiation power density by calculating the time-averaged Poynting vector for
2892 all positions surrounding the antenna, which in equation form is

$$\mathbf{W}(x, y, z) = \langle \mathbf{E}(x, y, z, t) \times \mathbf{H}^*(x, y, z, t) \rangle_t, \quad (5.1)$$

2893 where $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$ are the time-dependent electric and magnetic fields
 2894 produced by the antenna [8]. The radiation power density has units of W/m^2 and is
 2895 more typically called the energy flux density in physics applications, since it is a measure
 2896 of the amount of energy passing through a unit area over time.

2897 Because the radiation power density is a measure of power per unit area, its value
 2898 in a particular direction will depend on the distance from the antenna at which we are
 2899 measuring. This is undesirable for practical applications A related quantity, which is
 2900 distance independent, is the energy flux per unit solid angle or radiation intensity, which
 2901 is computed directly from the radition power density by multiplying by the squared
 2902 distance from the antenna. Specifically,

$$U = r^2 W(x, y, z), \quad (5.2)$$

2903 where r is the distance from the antenna to the field measurement point. The radiation
 2904 intensity is typically defined in regions where the Poynting vector consists only of a radial
 2905 component where it is safe to treat as a scalar quantity.

2906 5.2.1.2 Directivity and Gain

2907 Since the radiation intensity is a measure of average power per unit solid angle, it is
 2908 independent of distance and more useful as feature for antenna measurement. However,
 2909 most antenna measurements are performed in terms of the directly related directivity
 2910 and gain quantities. Directivity is defined as the ratio between the radiation intensity at
 2911 particular point on the radiation pattern to the average radiation intensity computed
 2912 over all solid angles [13]. The equation that relates the radiation intensity to directivity
 2913 is

$$D = \frac{U}{U_0} = \frac{4\pi U}{P_{\text{rad}}}, \quad (5.3)$$

2914 where U_0 is the average radiation intensity over all solid angles, which simply the total
 2915 radiated power (P_{rad}) divided by 4π . Closely related to directivity is concept of gain,
 2916 which accounts for energy losses that occur inside then antenna when attempting to
 2917 transmit or receive a signal. The antenna gain is given by

$$G = \frac{4\pi U}{P_{\text{in}}}, \quad (5.4)$$

2918 where P_{in} is the total power delivered to the antenna. Gain can be thought of as the ratio
 2919 of the antenna's radiation intensity to that of a hypothetical isotropic, lossless radiator.

2920 The maximum values of gain and directivity exhibited by the main lobe of the antenna
 2921 pattern as well as the ratio between the gain of the main lobe and any side-lobes are
 2922 important figures of merit used to evaluate antenna designs.

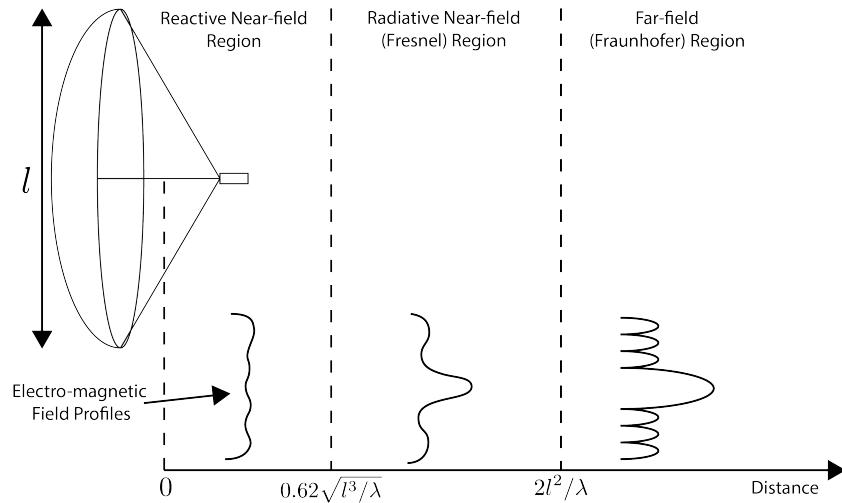


Figure 5.2: An illustration of the three field regions important for the analysis of an antenna system. Very close to the antenna the electric fields are primarily reactive so there is no radiation. If a receiving antenna were placed in this region most of the energy would be reflected back to the transmitter. Outside of the reactive near-field is the radiative near field. At these distances the antenna does radiate, but the radiation pattern is not well-defined since it changes based on the distance of the receiving antenna. It is only in the far-field region where the radiation pattern becomes constant as a function of distance, which is where the majority of antenna engineering is assumed to take place. The antenna arrays developed by Project 8 for CRES measurements operate in the radiative near-field due to the importance of limiting power loss from free-space propagation, which complicates the design of the antenna system.

2923 5.2.1.3 Far-field and Near-field

2924 Radiation patterns are only well-defined in regions where the shape of the radiation
 2925 pattern is independent of distance. The region where this approximation is valid is called
 2926 the "far-field", and in this region we can treat the EM fields from the antenna as spherical
 2927 plane waves. A rule of thumb for antennas is that the far-field approximation is valid
 2928 when the condition

$$R > \frac{2l^2}{\lambda} \quad (5.5)$$

2929 is met. In this expression, R is the distance from the antenna, l is the largest characteristic
 2930 dimension of the antenna, and λ is the wavelength of the radiation (see Figure 5.2).

2931 The region very close to the antenna is called the reactive near-field, because in this
2932 region the reactive component of the EM field is dominant. Unlike radiative electric
2933 fields, the reactive electric and magnetic fields are out of phase from each other by
2934 90°, since they are the result of electrostatic and magnetostatic effects coming from the
2935 self-capacitance and self-inductance of the antenna. The reactive fields are unable to
2936 transfer energy a significant distance from the antenna and are thus completely negligible
2937 for most antenna applications. The limit of the reactive near-field for an electrically-large
2938 antenna is typically taken to be

$$R < 0.62\sqrt{l^3/\lambda}. \quad (5.6)$$

2939 The unique application of antennas by Project 8 is somewhat limited by reactive near-
2940 field effects in the form of a maximum radial position for electrons inside the uniform
2941 cylindrical antenna array. If electrons are too close to the edge of the array than reactive
2942 near-field effects leads to a large reduction in the received power and consequently
2943 detection efficiency. This leads to a significant volume inside of the antenna array that
2944 is unsuitable for CRES lowering the volumetric efficiency of the antenna array CRES
2945 technique relative to a cavity experiment.

2946 In between the reactive near-field and the far-field is the radiative near-field region.
2947 In this region the fields are primarily radiative, however we are still too close to the
2948 antenna for the spherical plane wave approximation to apply. Therefore, interference
2949 effects between EM waves emitted from different points on the antenna occur causing the
2950 shape of the radiation pattern to change as a function of distance from the antenna. If we
2951 evaluate the far-field distance limit for the FSCD one finds an estimated far-field distance
2952 of 43 cm, which is a factor of four larger than the radius of the antenna array designed for
2953 the experiment. Consequently, we expect near-field effects to influence the performance
2954 of the antenna array highlighting the importance of calibration and characterization
2955 measurements.

2956 5.2.1.4 Polarization

2957 The polarization of an EM wave defines the spatial orientation of the electric field
2958 oscillations in the plane perpendicular to the direction of the propagation, and is defined
2959 in terms of orthogonal polarization components. In our application, one analyzes the
2960 properties of radiation propagating along the radial (\hat{r}) direction away from the antenna,
2961 which implies that the electric fields can be described as a linear combination of orthogonal

2962 polarization components

$$\mathbf{E}_{\text{tot}} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}, \quad (5.7)$$

2963 in Cartesian coordinates, or

$$\mathbf{E}_{\text{tot}} = E_\theta \hat{\theta} + E_\phi \hat{\phi}, \quad (5.8)$$

2964 in spherical coordinates.

2965 In general, one defines partial radiation patterns, directivities, and gains so that the
2966 performance of the antenna for the desired polarization can be analyzed. The radiation
2967 pattern defined in terms of partial patterns is

$$U_{\text{tot}} = U_\phi + U_\theta, \quad (5.9)$$

2968 where U_ϕ and U_θ are the radiation intensities in a particular direction for the respective
2969 polarization components. Similarly, a quantity such as gain can be written in terms of
2970 partial gains,

$$G_{\text{tot}} = G_\phi + G_\theta = \frac{2\pi U_\phi}{P_{\text{in}}} + \frac{2\pi U_\theta}{P_{\text{in}}}. \quad (5.10)$$

2971 If we view an electron performing a circular orbit in the XY-plane from the side, that
2972 is, along the X or Y axes, then we would observe the electron to be performing a linear
2973 oscillation perpendicular to the viewing axis. From this intuitive picture, we can predict
2974 that the primary polarization of electric fields from CRES events to be linearly polarized
2975 in the $\hat{\phi}$ direction when viewed with an antenna positioned in the XY-plane.

2976 5.2.1.5 Antenna Factor and Effective Aperture

2977 A useful way to characterize the performance of an antenna is to measure the electric
2978 field magnitude required to produce a signal with an amplitude of one volt in the antenna
2979 terminals. This ratio between the magnitude of the incoming electric field and the
2980 magnitude of the signal produced by the antenna is called the antenna factor, which is
2981 written as

$$A_F = \frac{|\mathbf{E}_{\text{in}}|}{V_{\text{ant}}}, \quad (5.11)$$

2982 where A_F is the antenna factor, E_{in} is the incoming electric field, and V_{ant} is the magnitude
2983 of the voltage produced by the antenna.

2984 The antenna factor can be expressed in terms of the antenna's gain through a related
2985 quantity called the effective aperture. The effective aperture defines for a given incident
2986 radiation power density (W/m^2) the power that is received by the antenna. Therefore,

2987 the effective aperture gives the equivalent area of the antenna,

$$A_{\text{eff}} = \frac{P_{\text{rec}}}{P_{\text{in}}} = \frac{\lambda^2}{4\pi} G, \quad (5.12)$$

2988 where the received power is P_r and the total incoming power is P_{in} .

2989 If we express the incident power in terms of the magnitude of the Poynting vector,
2990 then

$$|\mathbf{S}_{\text{in}}| = |\mathbf{E}_{\text{in}}|^2 / \eta_0, \quad (5.13)$$

2991 where η_0 is the impedance of free-space, which relates the magnitudes of the electric and
2992 magnetic fields in a vacuum, and is defined by

$$\eta_0 = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\epsilon_0}{\mu_0}}. \quad (5.14)$$

2993 The total received power by the antenna can therefore be expressed as

$$P_{\text{rec}} = |\mathbf{S}_{\text{in}}| A_{\text{eff}} = |\mathbf{S}_{\text{in}}| \frac{\lambda^2}{4\pi} G = \frac{|\mathbf{E}_{\text{in}}|^2 \lambda^2 G}{4\pi \eta_0}. \quad (5.15)$$

2994 To relate this to the antenna factor recall that we can relate the voltage produced by
2995 the antenna to the received power with

$$P_{\text{rec}} = \frac{V_{\text{ant}}^2}{Z} = \frac{|\mathbf{E}_{\text{in}}|^2}{A_F^2 Z}, \quad (5.16)$$

2996 where Z is the system impedance. Setting Equations 5.15 and 5.16 equal to each other,
2997 we obtain the following expression for antenna factor in terms of gain

$$A_F = \sqrt{\frac{4\pi\eta_0}{ZG\lambda^2}} = \frac{9.73}{\lambda\sqrt{G}}. \quad (5.17)$$

2998 The second expression in Equation 5.17 is obtained by evaluating the constant terms
2999 assuming a system impedance of 50Ω .

3000 We have gone through the effort of expressing the antenna factor in terms of gain
3001 to highlight that the majority of antenna parameters that we care to measure for a
3002 CRES experiment can be obtained from the radiation or gain pattern of the antenna.
3003 The antenna factor is a particularly important parameter for CRES measurements
3004 due to its relevance to antenna array simulations with the Locust software [10, 28].
3005 Specifically, Locust simulates the trajectory of an electron in a magnetic trap by running

3006 the Kassiopeia software package [7] and then uses the Liénard-Wiechert equations [11, 12]
3007 to calculate the electric fields that are incident on the antenna.

3008 To compute the response of the antenna to the electric field, Locust relies upon linear
3009 time-invariant system theory [15], which computes the response of the antenna (i.e. the
3010 voltage time series generated by the antenna) using a convolution between the electric field
3011 time-series and the antenna impulse response. This approach is necessary for correctly
3012 modeling the antenna response to the electric field due to the broadband and non-
3013 stationary nature of the electric fields from CRES events. Since antenna measurements
3014 take place under steady-state conditions, parameters such as the radiation pattern, gain,
3015 and antenna factor are defined in the frequency domain. However, by performing an
3016 inverse Fourier transform on the antenna factor we can obtain the antenna impulse
3017 response, which allows us to simulate CRES events in the antenna array demonstrator
3018 experiment.

3019 **5.2.2 Antenna Measurement Fundamentals**

3020 **5.2.2.1 Friis Transmission Equation**

3021 The antenna factor, sometimes called the antenna transfer function, is used to model
3022 how the antenna will respond to electric fields emitted from a CRES event. Therefore,
3023 being able to measure the antenna transfer function of the antenna array is a key step
3024 in the commissioning and calibration phases of an antenna array CRES experiment. A
3025 common approach to antenna characterization is to perform a two antenna transmit-
3026 receive measurement where an antenna with a known gain is used to characterize the
unknown gain of the antenna under test (see Figure 5.3).

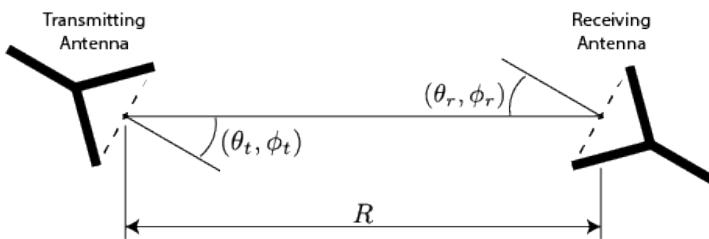


Figure 5.3: An illustration of the Friis measurement technique commonly used for antenna characterization measurements.

3027
3028 To analyze this two antenna setup we seek to calculate the amount of power from
3029 the transmitting antenna that we will detect with the receiving antenna. Using our
3030 understanding of antenna gain, we can calculate the power density transmitted by an

3031 antenna in a direction (θ_t, ϕ_t) at frequency f and distance R , which is given by

$$w_t = \frac{P_t}{4\pi R^2} G_t(\theta_t, \phi_t, f). \quad (5.18)$$

3032 Here, P_t is the total power delivered to the transmitting antenna and $G_t(\theta_t, \phi_t, f)$ is
3033 the value of the transmitting antenna gain. The power density is the power per unit
3034 area, so to calculate the total power delivered to the receiving antenna we multiply the
3035 transmitted power density by the effective area of the receiving antenna,

$$P_r = w_t A_{eff,r} = P_t \frac{G_t(\theta_t, \phi_t, f) G_r(\theta_r, \phi_r, f) c^2}{(4\pi R f)^2}, \quad (5.19)$$

3036 where $G_r(\theta_r, \phi_r, f)$ is the gain of the receiving antenna. Equation 5.19 is called the
3037 Friis transmission equation [34, 35], which is of fundamental importance for antenna
3038 measurements, since it allows one to measure the gain of an unknown antenna by
3039 measuring the power received from an antenna with a known gain pattern. Alternatively,
3040 if no antenna with a known gain pattern is available, two identical antennas with unknown
3041 gain patterns can be used.

3042 5.2.2.2 S-Parameters and Network Analyzers

3043 Instead of directly measuring the power received by the antenna under test, it is more
3044 common to measure the ratio of the received power to the transmitted power,

$$\frac{P_r}{P_t} = \frac{G_t(\theta_t, \phi_t, f) G_r(\theta_r, \phi_r, f) c^2}{(4\pi R f)^2}. \quad (5.20)$$

3045 This power ratio can be easily measured using a vector network analyzer (VNA), which
3046 automates a significant fraction of the measurement process. Network analyzers are
3047 used to measure the scattering or S-parameters of a multi-port RF device [36], which
3048 describes how waves are scattered between the device ports. The antenna measurements
3049 we have been considering can be modeled as a two-port microwave device that we can
3050 characterize by measuring how incident voltage waves are transmitted or reflected (see
3051 Figure 5.4). We can write the scattered waves (V_1^- and V_2^-) in terms of the incident (V_1^+
3052 and V_2^+) waves using the scattering matrix

$$\begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix}, \quad (5.21)$$

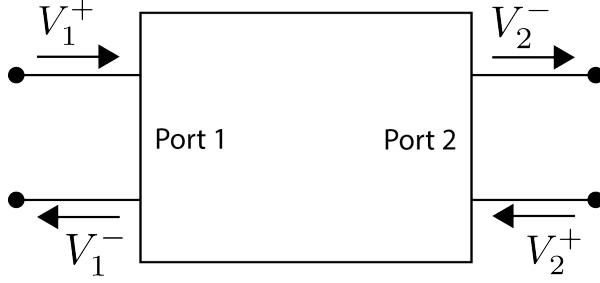


Figure 5.4: Illustration of a two-port S-parameter measurement setup. S-parameters characterize how incoming waves of voltage or power scatter off of the RF device under test. This allows you to measure important properties of the device. In particular, we can use this framework to model a two antenna radiation pattern measurement, which we can then automate using a VNA.

3053 where the elements of the matrix are the device S-parameters. It is assumed that,
 3054 when exciting the device from a particular port, that all other ports in the network are
 3055 terminated at the system impedance. This ensures that the incident waves from other
 3056 ports in the network are zero. Therefore, the S-parameters are the ratios between the
 3057 scattered and incident waves,

$$S_{ij} = \frac{V_i^-}{V_j^+}. \quad (5.22)$$

3058 Alternatively, S-parameters can be defined as the ratio of the scattered and incident
 3059 power, which is proportional to the ratio of the squared voltage waves. Returning to
 3060 our antenna measurement setup, we see that measuring the ratio of the received to the
 3061 transmitted power is equivalent to measuring the ratio of power being scattered from port
 3062 1 to port 2 in a RF network. Therefore, measuring an antenna's gain can be accomplished
 3063 quite easily, by using a VNA to perform a two port S_{21} measurement.

3064 5.2.2.3 Antenna Array Commissioning and Calibration Measurements

3065 Up to this point we have been discussing calibration and commissioning measurements
 3066 as they apply to a single antenna. While these measurements play an important role
 3067 in validating the radiation patterns of the individual array elements, the ultimate goal
 3068 is to use a phased array of these antennas. Therefore, we must also consider antenna
 3069 measurement techniques that apply to the whole array system.

3070 By measuring the gain of each individual array element we can predict the features of
 3071 the signals received during a CRES event using the antenna factor (see Section 5.2.1.5).
 3072 However, unpredictable changes to the antenna performance can be introduced by the

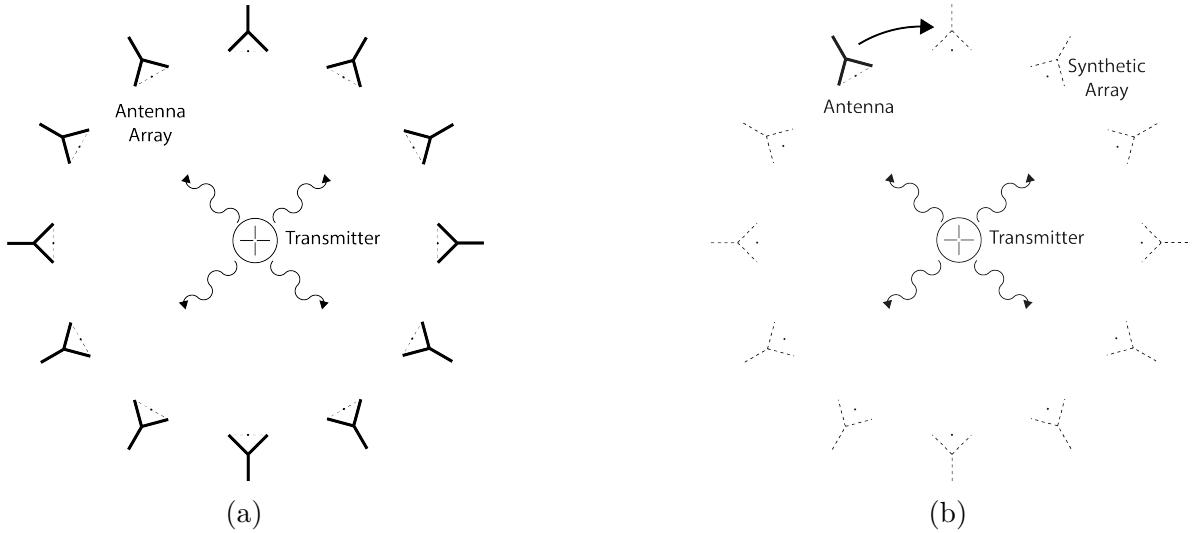


Figure 5.5: Two measurement approaches to characterizing an antenna array for CRES measurements. The full-array approach (a) requires a complete antenna array with all the associated hardware. The synthetic array approach (b) utilizes a single antenna and a set of rotation/translation stages to reposition the transmitter or the receiving antenna to synthesize the signals that would be received by the full-array. This approach reduces the cost and complexity of array measurements. A down-side of the synthetic array approach is that multi-channel effects such as reflections cannot be measured. Utilizing both the full-array and the synthetic array is a powerful way to quantify the impact of errors from the multi-channel array.

incorporation of the antennas into the circular array geometry, therefore, we employ both individual antenna and full-array measurements in the commissioning of the FSCD to account for these effects.

There are two main approaches to array measurements that could be used for characterization and calibration (see Figure 5.5). One approach is to construct the complete array and use an omni-directional transmitting antenna to measure the power received by each channel in the antenna array. In Section 5.3 we describe the development of an omni-directional transmitter that also mimics the radiation phase characteristics of a CRES event, which is useful because the entire array can be tested without repositioning. Alternatively, a full antenna array can be synthesized by repeatedly moving and measuring a single array element. This approach is ideal for identifying if different channels in the antenna array are affecting each other through multi-path interference by comparing the measurement results of the synthetic array to the real array.

5.2.3 The Penn State Antenna Measurement System

The development of antenna array based CRES requires the capability to test and calibrate different antenna array designs to validate the performance of the as-built antenna array before and during the experiment. With these aims in mind we developed an antenna measurement system at Penn State specifically designed to mimic the characteristics of the antenna experiment designed for demonstration of the antenna array CRES technique by the Project 8 collaboration.

The Penn State antenna measurement system utilizes a two antenna measurement configuration with a stationary reference antenna and a test antenna mounted on a set of motorized translation and rotation stages (see Figure 5.6). The antenna measurement system can be operated in two distinct modes, one focused on the characterization of the radiation patterns of prototype antennas and the other focused on the validation of data-acquisition (DAQ) and CRES signal reconstruction techniques to bridge the gap between real measurements and simulation. In both measurement configurations it is critical to isolate the antennas from the environment so that multi-path reflections do not negatively influence the measurement results. For this reason we surround the measurement volume with microwave absorber foam (AEMI AEC-1.5) [37] specifically designed to attenuate microwave radiation near the 26 GHz measurement range of the system.

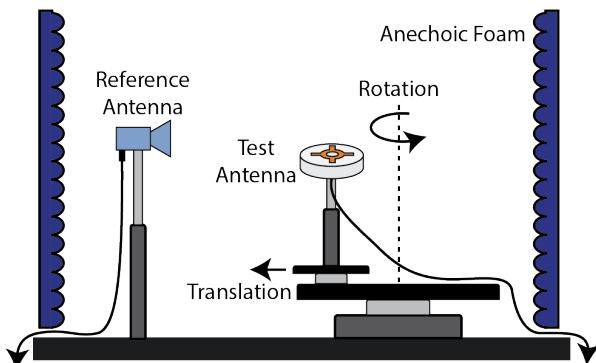


Figure 5.6: Illustration of the antenna measurement system developed for the Project 8 Collaboration. The reference and test antennas can be connected to different data acquisition configurations depending on the measurement goals. The reference antenna is typically a standard horn antenna and the test antenna is mounted on a set of translation stages for positioning. Automated translation stages allow for relatively painless data-taking enabling synthetic antenna array measurements using only a single receiving antenna. Anechoic form designed to mitigate RF reflections surrounds the setup.

In the first measurement configuration the reference antenna is typically a well-

3106 characterized horn antenna as pictured, since horn antennas have well-known and stable
 3107 radiation patterns making them ideal as standard references. For characterization
 3108 measurements, the test antenna represents the antenna-under-test whose pattern we wish
 3109 to characterize. Mounting the test antenna on motorized rotation and translation stages
 3110 allows us to automate the procedure significantly speeding up the radiation pattern
 3111 measurement process.

3112 In the second measurement configuration one is interested in recreating the conditions
 3113 of an antenna array CRES experiment as it concerns the antenna array and DAQ system.
 3114 In this case, the reference antenna is a prototype FSCD antenna, which will be used to
 3115 construct the antenna array in the FSCD experiment, and the test antenna is a specially
 3116 designed synthetic cyclotron antenna (SYNCA) as picture in Figure 5.6. The SYNCA is
 3117 designed such that the radiation pattern mimics that of a CRES electron so that the
 3118 signals received by the prototype CRES array antenna mimic what is expected for a real
 3119 CRES experiment.

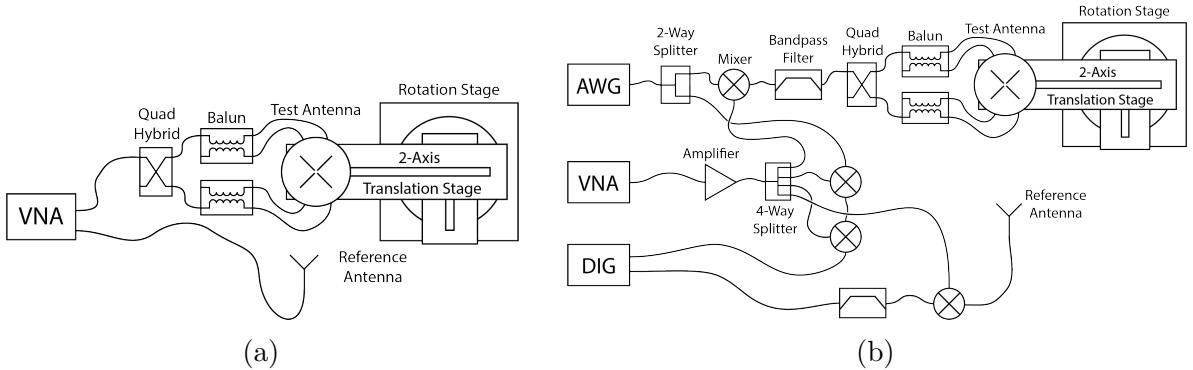


Figure 5.7: Diagrams of two measurement system configurations. Configuration (a) utilizes a VNA and is more suited to antenna characterization. Configuration (b) utilizes an AWG and VNA as a signal generation system and digitizer to collect measurement data, which is more suited to simulating CRES measurements. The transmission chain utilizes a quadrature hybrid and a pair of baluns to drive the cross-dipole variant test antenna developed for synthetic CRES measurements.

3120 In Figure 5.7 we show two high-level system diagrams of the Penn State antenna
 3121 measurement system that depict the important system components and the connections
 3122 between them. The two configurations of the measurement system utilize different
 3123 hardware. For characterization and radiation pattern measurements, one prefers the
 3124 configuration shown in Figure ???. In this case a vector network analyzer (VNA) is used
 3125 as both the transmission source and data acquisition system and it is relatively easy to
 3126 calibrate over a wide range of frequencies. Whereas, if one is more interested in recreating

what would take place in the FSCD experiment then the configuration shown in Figure ?? is preferable, since this system effectively mimics the receiver chain envisioned for the FSCD experiment.

The characterization configuration utilizes a network analyzer (Keysight N5222A) [38, 39] with two independent sources and four measurement ports as the primary measurement tool. A standard reference antenna is connected to one measurement port, and the test antenna is connected to a separate port. The typical reference antenna used for these studies is a Pasternack PF9851 horn antenna [40]. In the measurement shown, the test antenna represents a SYNCA antenna, which requires a transmission chain consisting of quadrature hybrid coupler [41, 42] (Marki QH-0226) connected to two baluns [43] (Marki BAL-0026) to generate feed signals with the appropriate phases. The VNA measures the radiation pattern by performing a transmission S-parameter measurement, which can be used with the knowledge of the reference antenna's radiation pattern to determine the radiation pattern of the test antenna (see Section 5.2.1).

The second configuration is more complicated and incorporates more hardware components in order to more closely mimic the DAQ system envisioned for the FSCD experiment. The basic approach is to produce CRES-like radiation and use an antenna combined with a realistic RF receiver chain to acquire the signals. On the transmit side, an arbitrary waveform generator [44] (AWG, RIGOL DG5252) is used to generate a waveform that mimics a CRES signal at a baseband frequency up to 250 MHz. This frequency is then up-converted to the CRES signal frequency band of 25.8 to 26.0 GHz using a mixer [45] (Marki MM1-0832L) and a bandpass filter (K&L Microwave 3C62-25900/T200-K/K) to reject unwanted mixing components outside out of the 200 MHz CRES signal band. The local oscillator signal for mixing is provided by one of the VNA sources configured to run in a continuous wave setting. On the receive side, a prototype antenna is used to detect the radiation emitted by the test antenna, which is down-converted and filtered using the same mixer and bandpass filter as the transmission chain. Lastly, data acquisition is performed using a 14-bit ADC sampling at 500 MSa/s [46] (CAEN DT530) to digitize the down-converted signals.

In order to distribute the LO to all mixers a 4-way power splitter (MiniCircuits ZC4PD-18263-S+) along with an amplifier (Marki APM-6848) is used to drive the four mixers used in the measurement system. A limitation of using the VNA as an LO source is that there is no control of the LO phase when a measurement is triggered by the control script, which leads to a random phase offset between acquisitions. This makes it impossible to perform synthetic array measurements, which require strict control over

3162 the starting phase of the transmitted signal. In order to monitor the random phase of the
3163 LO, a 2-way power splitter (MiniCircuits Z99SC-62-S+) is used to split the signal from
3164 the AWG between the transmission path and a LO monitoring path. The LO monitoring
3165 path consists of an up-conversion and down conversion using two mixers connected by a
3166 coaxial cable, and monitors the relative phase of the LO using a channel on the digitizer
3167 to sample this path. A phase shift in the LO will lead to a proportional phase shift in
3168 the mixed signal, which is measured and removed from the received signals.

3169 The test antenna is mounted on a set of motorized stages, which are identical for
3170 both measurement configurations. A rotational stage (ThorLabs PRMTZ8) is used as
3171 the base layer with additional translation stages mounted on top of this. The rotational
3172 stage is ideal for measuring a complete azimuthal scan of the test antenna's radiation
3173 pattern as well as for moving a SYNCA antenna in circular motion to recreate the
3174 symmetry of the FSCD antenna array. On top of the rotational stage we mount two
3175 linear translation stages (ThorLabs MTS50-Z8 and MTS25-Z8) in a cross-wise manner
3176 so that the test antenna can be moved along two perpendicular axes. Using the linear
3177 stages in combination with the rotational stage allows one to fine-tune the positioning of
3178 the test antenna so that it can be perfectly aligned with the central axis of the array.
3179 A LabView script was developed to automate the measurement of a full 360° radiation
3180 pattern and control the measurement electronics. Data from these acquisitions is stored
3181 on university provided cloud storage.

3182 **5.3 Development of a Synthetic Cyclotron Antenna (SYNCA)** 3183 **for Antenna Array Calibration**

3184 This section is the manuscript of the publication [29] detailing the development of a
3185 Synthetic Cyclotron Antenna (SYNCA) for antenna array characterization measurements
3186 by the Project 8 collaboration.

3187 **5.3.1 Introduction**

3188 Neutrinos are the most abundant standard model fermions in our universe, but due to
3189 weak interaction cross-sections with other particles, neutrinos are particularly difficult
3190 to study. Consequently, many fundamental properties of neutrinos are still unknown
3191 including the absolute scale of the neutrino mass [47]. Direct, kinematic measurements of
3192 the neutrino mass are particularly valuable due to their model independent nature [48].

3193 To date the most sensitive direct neutrino mass measurements have been performed by
 3194 the KATRIN collaboration [49], which measures the molecular tritium β -decay spectrum
 3195 to infer the neutrino mass. Current data from neutrino oscillation measurements [47]
 3196 allow for neutrino masses significantly smaller than the design sensitivity of the KATRIN
 3197 experiment; therefore, there is a need for new technologies for performing direct neutrino
 3198 mass measurements to probe lower neutrino masses.

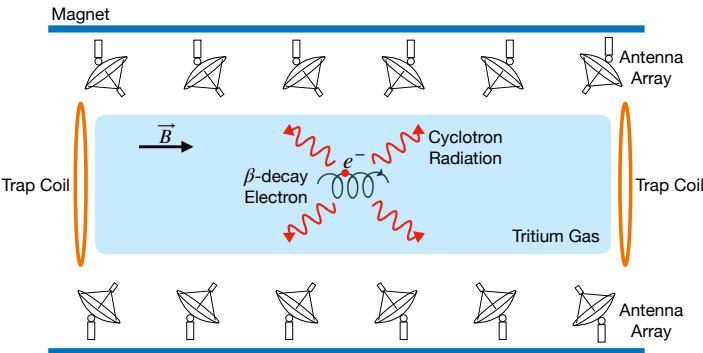


Figure 5.8: A sketch of an antenna array large-volume CRES experiment. Electrons from β -decays are confined in a magnetic field using a set of trap coils. The cyclotron radiation produced by the motion of the trapped electrons can be detected by a surrounding antenna array to determine the electron energies. Measuring the energies of many electrons produces a β -decay spectrum.

3199 The Project 8 collaboration is developing new methods for neutrino mass measurement
 3200 based on Cyclotron Radiation Emission Spectroscopy (CRES) [50–53], with the goal of
 3201 measuring the absolute scale of the neutrino mass with a 40 meV/c² sensitivity [?, 48].
 3202 This sensitivity goal will require the development of two separate technical capabilities.
 3203 First is the development of an atomic tritium source, which avoids significant spectral
 3204 broadening due to molecular final states [54]. Second is the technology for performing
 3205 CRES in a multi-cubic-meter experimental volume with high combined detection and
 3206 reconstruction efficiency, which is required in order to obtain sufficient event statistics
 3207 near the tritium spectrum endpoint.

3208 One approach for a large-volume CRES experiment is to use an array of antennas,
 3209 which surrounds a volume of tritium gas, to detect the cyclotron radiation produced
 3210 by the β -decay electrons when they are trapped in a background magnetic field using a
 3211 set of magnetic trapping coils (see Figure 5.8). Project 8 has developed a conceptual
 3212 experiment design to study the feasibility of this approach. The design consists of a
 3213 single circular array of antennas with a radius of 10 cm and 60 independent channels
 3214 positioned around the center of the magnetic trap. The motivation behind this antenna

array design is to first develop an understanding of the antenna array approach to CRES
 with a small scale experiment before attempting to scale the technique to large volumes
 by using multiple antenna rings to construct the full cylindrical array. The development
 of the antenna array approach to CRES has largely proceeded through simulations using
 the Locust software package [28, 55], which is used to model the fields emitted by CRES
 events and predict the signals received by the surrounding antenna array. To validate
 these simulations, a dedicated test stand is being constructed to perform characterization
 measurements of the prototype antenna array developed by Project 8 (see Figure 5.9)
 and benchmark signal reconstruction methods using a specially designed transmitting
 calibration probe antenna.

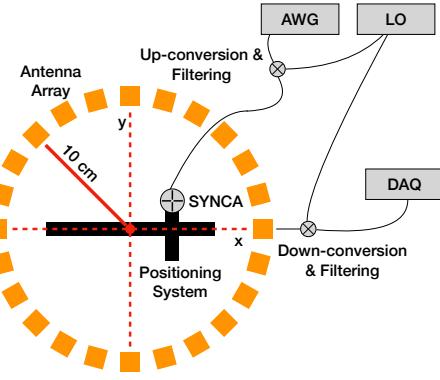


Figure 5.9: A schematic of the antenna array test stand. The circular antenna array has a radius of 10 cm with 60 independent channels (limited number shown for clarity). The test stand includes an arbitrary waveform generator (AWG), local oscillator (LO), and data acquisition (DAQ) hardware. Finally, a specialized Synthetic Cyclotron Antenna (SYNCA) is used to inject signals to test the antenna array.

We call this probe antenna the Synthetic Cyclotron Antenna or SYNCA. The SYNCA
 is a novel antenna design that mimics the cyclotron radiation generated by individual
 charged particles trapped in a magnetic field, which will be used in the antenna test
 stand to perform characterization measurements, simulation validation, and reconstruc-
 tion benchmarking. This paper provides an overview of the design, construction, and
 characterization measurements of the SYNCA performed in preparation for its usage as
 a transmitting calibration probe.

In Section 5.3.2 we provide a description of the cyclotron radiation field characteristics
 that we recreate with the SYNCA. In Section 5.3.3 we give an overview of the simulations
 performed to develop an antenna design that mimics the characteristics of cyclotron
 radiation. In Section 5.3.4 we outline characterization measurements to validate that

3236 the fields generated by the SYNCA match simulation, and finally in Section 5.3.5 we
 3237 demonstrate an application of the SYNCA to test phased array reconstruction techniques
 3238 on the bench-top.

3239 5.3.2 Cyclotron Radiation Phenomenology

3240 To understand the cyclotron radiation phenomenology that the SYNCA should mimic,
 3241 we consider a charged particle moving at relativistic speed in the presence of an external
 3242 magnetic field (see Figure 5.10). In the special case we shall examine, the entirety of
 3243 the electron's momentum is directed perpendicular to the magnetic field; therefore, the
 3244 trajectory of the electron is confined to the cyclotron orbit plane. Because the momentum
 3245 vector is oriented perpendicular to the magnetic field, electrons with these trajectories
 3246 are said to have pitch angles of 90°.

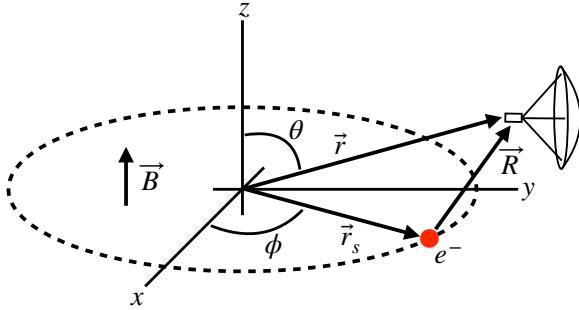


Figure 5.10: An electron (red dot) performing cyclotron motion in the x-y plane. The resulting cyclotron radiation is observed by an antenna located at the field point of interest.

3247 The cyclotron radiation fields generated by this circular trajectory are those which
 3248 we aim to reproduce with the SYNCA. We can describe the electromagnetic (EM) fields
 3249 using the Liénard-Wiechert equations [8, 28], which in non-covariant form express the
 3250 electric field as

$$\vec{E} = e \left[\frac{\hat{n} - \vec{\beta}}{\gamma^2 (1 - \vec{\beta} \cdot \hat{n})^3 |\vec{R}|^2} \right]_{t_r} + \frac{e}{c} \left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \hat{n})^3 |\vec{R}|} \right]_{t_r}, \quad (5.23)$$

3251 where e is the particle's charge, $\hat{n} = (\vec{r} - \vec{r}_s)/|\vec{r} - \vec{r}_s|$ is the unit vector pointing from the
 3252 electron to the field measurement point, $\vec{\beta} = \dot{\vec{r}}_s/c$ is the velocity of the particle divided
 3253 by the speed of light, and γ is the relativistic Lorentz factor. The equation is meant to
 3254 be evaluated at the retarded time as indicated by $t_r = t - |\vec{R}|/c$, which accounts for the

3255 time delay due to the finite speed of light between the point where the field was emitted
 3256 and the point where the field is detected.

3257 We would like to simplify Equation 5.23 it at all possible. As a first step we analyze
 3258 the relative magnitudes of the electric field polarization components. Consider an electron
 3259 following a circular cyclotron orbit in a uniform magnetic field whose guiding center
 3260 is positioned at the origin of the coordinate system. The equation of motion can be
 3261 expressed as

$$\vec{r}_s = (r_c \cos \omega_c t_r) \hat{x} + (r_c \sin \omega_c t_r) \hat{y}. \quad (5.24)$$

3262 For single antenna located along the y-axis at position $\vec{r} = r_a \hat{y}$ we are interested in the
 3263 incident electric fields from the electron. The electric field is given by Equation 5.23,
 3264 which we evaluate in the regime where $r_a \gg r_c$. This limit can be justified by comparing
 3265 the radius of the cyclotron orbit for an electron with the tritium beta-spectrum endpoint
 3266 energy of 18.6 keV in a 1 T magnetic field to the typical ($r_a \simeq 100$ mm) radial position
 3267 of the receiving antenna. We find that the cyclotron orbit has a radius of 0.46 mm which
 3268 is approximately a factor of 200 smaller than the typical antenna radial position. In this
 3269 regime we can make the approximation $\vec{R} \simeq r_a \hat{y}$ and the expression for the electric field
 3270 at the antenna's position becomes

$$\vec{E} = \frac{e}{\gamma^2 r_a^2} \frac{\hat{x} \left(\frac{r_c \omega_c}{c} \sin \omega_c t_r \right) + \hat{y} \left(1 - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)}{\left(1 - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)^3} - \frac{e}{cr_a} \frac{\hat{x} \left(\frac{r_c^2 \omega_c^3}{c^2} - \frac{r_c \omega_c^2}{c} \cos \omega_c t_r \right)}{\left(1 - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)^3}. \quad (5.25)$$

3271 Since the receiving antenna is part of a circular array of antennas, it is useful to rewrite
 3272 Equation 5.25 in terms of the azimuthal ($\hat{\phi}$) and radial (\hat{r}) polarizations. Making use of
 3273 the fact that for an antenna located at $R = r_a \hat{y}$ that $\hat{\phi} = -\hat{x}$ and $\hat{r} = \hat{y}$ we find

$$\vec{E} = \hat{\phi} E_\phi + \hat{r} E_r \quad (5.26)$$

$$E_\phi = \frac{e}{\left(1 - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)^3} \left[-\frac{\frac{r_c \omega_c}{c} \sin \omega_c t_r}{\gamma^2 r_a^2} + \frac{\omega_c \left(\frac{r_c^2 \omega_c^2}{c^2} - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)}{cr_a} \right] \quad (5.27)$$

$$E_r = \frac{e \left(1 - \frac{r_c \omega_c}{c} \sin \omega_c t_r \right)}{\gamma^2 r_a^2 \left(1 - \frac{r_c \omega_c}{c} \cos \omega_c t_r \right)^3}. \quad (5.28)$$

3274 For the purposes of designing a synthetic cyclotron radiation antenna we are interested
 3275 in the dominant electric field polarization emitted by the electron. The antenna is being
 3276 designed to mimic the cyclotron radiation produced by electrons with kinetic energies of
 3277 approximately 18.6 keV in a 1 T magnetic field [54]. Since the relativistic beta factor for

3278 an electron with this kinetic energy is $|\vec{\beta}| \simeq \frac{1}{4}$, the approximations $\gamma \simeq 1$ and $\frac{r_c \omega_c}{c} \simeq \frac{1}{4}$ are
 3279 justified. Inserting these expressions into the equations for the electric field components
 3280 above simplifies the comparison of the magnitudes of the two components. Additionally,
 3281 we compare the time-averaged magnitudes to evaluate the root mean squared electric
 3282 field ratio. The time-averaged ratio of the radial and azimuthally polarized electric fields
 3283 with the above simplifications is given by

$$\frac{\langle |E_r| \rangle}{\langle |E_\phi| \rangle} = \frac{8 - \sqrt{2}}{\left| 1 - \frac{r_a}{r_c} \frac{1-2\sqrt{2}}{8} \right|} \simeq \frac{r_c}{r_a} \frac{8(8 - \sqrt{2})}{2\sqrt{2} - 1} = 0.13, \quad (5.29)$$

3284 where we have made use of the fact that for these magnetic fields and kinetic energies
 3285 the cyclotron radius is much smaller than the radius of the antenna array.

3286 From Equation 5.29 we see that the time-averaged azimuthal polarization is larger than
 3287 the radial polarization by about a factor of 8, which makes it the dominant contribution
 3288 to the electric fields at the position of the antenna. We must also consider the directivity
 3289 of the receiving antenna which can have a gain that is disproportionately large for a
 3290 specific polarization component. Because the E_ϕ component is dominant, the receiving
 3291 antenna array is designed with an azimuthal polarization, which negates the voltages
 3292 induced in the antenna from the radially polarized fields. Therefore, we conclude that
 3293 for the purpose of designing the SYNCA antenna it is acceptable to approximate the
 3294 electric fields from Equation 5.23 as purely azimuthally or ϕ -polarized. The simplified
 3295 expression for the electric field received by an antenna becomes

$$\vec{E} = E_\phi \hat{\phi} = \frac{e \frac{r_c \omega_c}{c}}{4r_a r_c} \left[\frac{\frac{r_c \omega_c}{c} - \cos \omega_c t - \frac{4r_c}{r_a} \sin \omega_c t}{(1 - \frac{r_c \omega_c}{c} \cos \omega_c t)^3} \right]_{t_r} \hat{\phi}, \quad (5.30)$$

3296 where the radius of the cyclotron orbit is called r_c , the cyclotron frequency is called ω_c ,
 3297 and the radial position of the receiving antenna is called r_a . Equation 5.30 has been
 3298 evaluated in the non-relativistic limit where $\gamma \simeq 1$, which is justified by the fact that
 3299 $|\vec{\beta}| \simeq \frac{e}{4}$ for an electron with an 18.6 keV kinetic energy in a 1 T magnetic field.

3300 This rather complicated expression can be simplified using Fourier analysis. Assuming
 3301 a background magnetic field of 1 T and a kinetic energy of 18.6 keV we calculate
 3302 numerically the electric field using Equation 5.30 and apply a discrete Fourier Transform
 3303 to visualize the frequency spectrum (see Figure 5.11).

3304 We observe that the azimuthally polarized electric field is periodic with a base cyclotron
 3305 frequency of 25.898 GHz corresponding to the highest power frequency component in

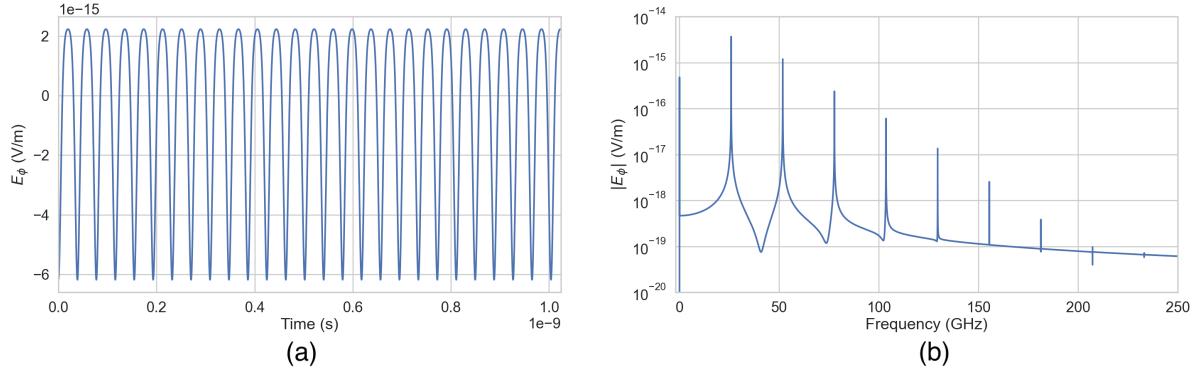


Figure 5.11: A plot of the numeric solution to Equation 5.31. The time-domain representation of the signal (a) is composed of a zero frequency term and a series of harmonics separated by the main cyclotron frequency as shown in the plot of the frequency spectrum (b). We can see that the relative amplitude of the harmonics beyond $k = 7$ are smaller than the main carrier by a factor of about 10^{-5} and are completely negligible.

3306 Figure 5.11. The frequency spectrum reveals that the signal is composed of a constant
 3307 term with zero frequency and a series of harmonics separated by 25.898 GHz. Therefore,
 3308 we can represent the azimuthal electric fields from the electron as a linear combination
 3309 of pure sinusoids with frequencies given by $\omega_k = k\omega_c$ ($k \in 0, 1, 2, \dots$) and amplitudes
 3310 extracted from the Fourier representation. Using this representation we can transform
 3311 the equation for the azimuthally polarized electric fields in Equation 5.30 into

$$E_\phi = \frac{e^{\frac{r_c \omega_c}{c}}}{4r_a r_c} \sum_{k=0}^7 A_k e^{i\omega_k t_r}, \quad (5.31)$$

3312 where we have truncated the sum over harmonics at the 7th order for completeness. The
 3313 amplitudes A_k are dimensionless complex numbers, which encode the relative powers of
 3314 the harmonics as well as the starting overall phase of the cyclotron radiation. Because
 3315 magnitude of the relative amplitudes exponentially decreases for higher harmonics, it is
 3316 usually sufficient to consider only the terms up to $k = 4$ where the relative amplitude
 3317 of the harmonics has decreased from the main carrier by a factor of approximately 100.
 3318 However, for completeness we include harmonics up to 7th order in Equation 5.31. The
 3319 range of frequencies to which the receiving antenna array in the antenna test stand is
 3320 sensitive is defined by the antenna's transfer function. The receptive bandwidth for
 3321 the antennas used in the test stand is a range of frequencies with a bandwidth on the
 3322 order of a few GHz centered around the main cyclotron carrier frequency of 25.898 GHz.
 3323 Therefore, the higher order harmonics as well as the zero frequency term can be ignored

3324 when considering only the signals that will be received by the antenna array.

3325 Considering only the 1st order harmonic term from Equation 5.31, which represents
3326 the portion of the electric field that will be detected by the array, and evaluating this at
3327 the retarded time we obtain the following for the ϕ -polarized electric fields

$$E_\phi \propto \cos \left(\omega_c \left(t - |\vec{R}|/c \right) - \Delta \right), \quad (5.32)$$

3328 where the arbitrary phase Δ is defined by $A_k = |A_k|e^{i\Delta}$. We are interested in the
3329 characteristics of the amplitude of the electric field as a function of the radial distance
3330 component ($|\vec{R}|$) of the retarded time. In particular, the maximum of E_ϕ occurs when
3331 the argument of the cosine function is equal $n\pi$ where $n \in \{0, \pm 2, \pm 4, \dots\}$; however, the
3332 solutions where n is negative can be discarded since they represent unphysical negative
3333 overall phases. Applying this condition to Equation 5.32 gives a condition on the radial
3334 position of the maximum of E_ϕ

$$\omega_c(t - |\vec{R}|/c) - \Delta = n\pi, \quad (5.33a)$$

$$|\vec{R}| = \frac{c}{\omega_c} ((\omega_c t - \Delta) - n\pi), \quad (5.33b)$$

3335 which is a function of time in the frame of the moving electron (t). Equation 5.33 can
3336 be further simplified by noticing that the azimuthal position of the electron ($\phi_e(t)$) as a
3337 function of time is defined by $\phi_e(t) = \omega_c t - \Delta$ which reduces Equation 5.33 to

$$|\vec{R}| = \frac{c}{\omega_c} (\phi_e(t) - n\pi). \quad (5.34)$$

3338 Equation 5.34 represents an archimedean spiral which is formed when plotting the
3339 amplitude of E_ϕ in the x-y plane. The solution where $n = 0$ represents the leading edge
3340 of the radiation spiral which propagates outward from the electron at the speed of light.
3341 The additional solutions for $n > 0$ represent the persistent spiral at radii inside the
3342 leading edge of the radiated fields that have not yet been detected by the receiver at the
3343 current time. In Figure 5.12a we show the expected spiral pattern for the maxima of the
3344 cyclotron radiation.

3345 In particular, we note that for the circular array geometry of the test stand, depicted
3346 as the series of circles in Figure 5.12a, each antenna receives a linearly polarized wave
3347 with a phase offset that corresponds to the azimuthal angle for that antenna element.
3348 Therefore, as we show in Figure 5.12b, when the relative phase of the received signal is
3349 plotted as a function of the receiving antenna's azimuthal position the result is also an

3350 Archimedean spiral.

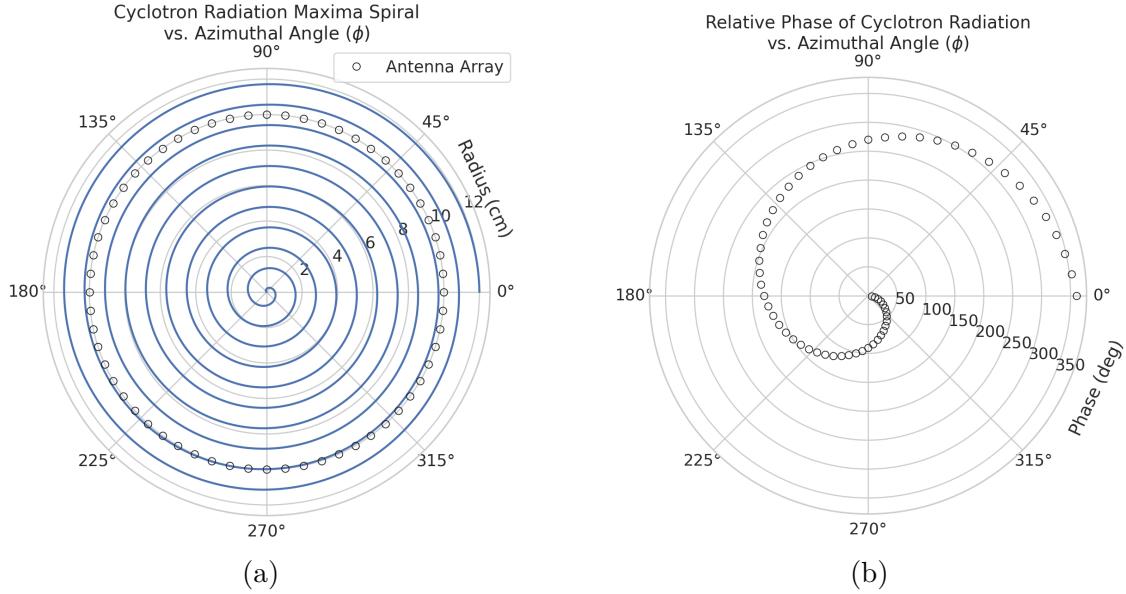


Figure 5.12: The amplitude maxima of the cyclotron radiation form an Archimedean spiral as the radiation propagates outward from the cyclotron orbit center (a). A circular antenna array located at a fixed radius from the orbit center will receive electric fields with equal magnitude in each of its channels, but the phase of the electric field incident on each array channel will be linearly out of phase from its neighbor antennas by an amount equal to the angular separation of the two channels (b).

3351 Based on these analytical calculations we can characterize the magnitude, polarization,
3352 and phase of the signals received by the antenna array using three criteria. These criteria
3353 are the basis of comparison for the radiation produced by the SYNCA and cyclotron
3354 radiation emitted by electrons and will be used to evaluate the performance of antenna
3355 designs. The criteria are:

- 3356 1. Electric fields that are ϕ -polarized near $\theta = 90^\circ$
- 3357 2. Uniform time-averaged electric field magnitudes around the circumference of a
3358 circle centered on the antenna
- 3359 3. Electric fields whose phase is equal to the azimuthal angle at the point of measure-
3360 ment plus a constant

3361 The Locust simulation package [55] can be used to directly simulate the EM fields
3362 generated by electrons performing cyclotron motion to validate the analytical calculations.
3363 Locust simulates the EM fields by first calculating the trajectory of the electrons in

3364 the magnetic trap using the Kassiopeia software package [56]. The trajectory can then
 3365 be used to solve for the EM fields using the Liénard-Wiechert equations directly with
 3366 no approximations. The resulting electric field solutions drive a receiving antenna by
 3367 convolving the time-domain fields with the finite-impulse response filter of the antenna
 3368 or they can be examined directly to study the field characteristics that the SYNCA must
 3369 reproduce. In the next section we compare the radiation field patterns for electrons
 3370 simulated with Locust to patterns from a SYNCA antenna design.

3371 **5.3.3 SYNCA Simulations and Design**

3372 One potential SYNCA design is the crossed-dipole antenna [57]. A crossed-dipole antenna
 3373 consists of two dipole antennas, one of which is rotated 90° with respect to the other,
 3374 which are fed with signals that are out of phase from the opposite dipole by 90° (see
 Figure 5.13). This arrangement causes the signals fed to each arm of the dipole to be

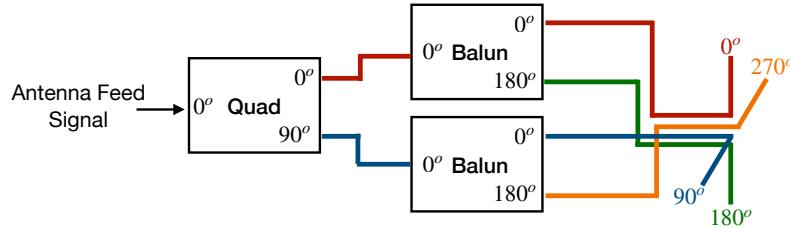


Figure 5.13: An idealized crossed-dipole antenna consists of two electric dipole antennas oriented perpendicular to each other and is fed with four signals with a quadrature phase relationship. An example antenna feed circuit is shown which is composed of a chained combination of a quadrature hybrid-coupler (Quad) and two baluns.

3375
 3376 out of phase from each of the neighboring arms by 90°, which mirrors the spatial phase
 3377 relationship of cyclotron radiation fields.

3378 A potential drawback of this design is that standard crossed-dipole antennas do not
 3379 radiate uniform electric fields near the $\theta = \pi/2$ plane. Typical crossed-dipole antennas
 3380 use dipole arm lengths equal to $\lambda/4$ or larger [57], where λ is the wavelength at the
 3381 desired operating frequency. Such large arm lengths cause the electric field magnitude
 3382 to vary significantly around the circumference of the antenna. However, making the
 3383 antenna electrically small by shrinking the arm length can improve the antenna pattern
 3384 uniformity.

3385 In general, the criterion for an electrically small antenna is that the largest dimension
3386 of the antenna (D) obey $D \lesssim \lambda/10$ [13]. In our application, we are attempting to mimic
3387 the cyclotron radiation emitted by electrons produced from tritium β -decay with energies
3388 near the spectrum endpoint. For a background magnetic field of 1 T, the corresponding
3389 cyclotron frequency of tritium endpoint electrons is approximately 26 GHz. Therefore, the
3390 electrically small condition would require that the largest dimension of the crossed-dipole
3391 antenna be smaller than 1.2 mm.

3392 A crossed-dipole antenna with an overall size of 1.2 mm is challenging to fabricate due
3393 to the small dimensions of the dipole arms that, in practice, are fragile and unsuitable
3394 for use as a calibration probe. To mitigate some of the challenges with the fabrication
3395 of such a small antenna, a variant crossed-dipole antenna design using printed circuit
3396 board (PCB) technology (see Figure 5.14) was developed in partnership with an antenna
3397 prototyping company, Field Theory Consulting ¹.

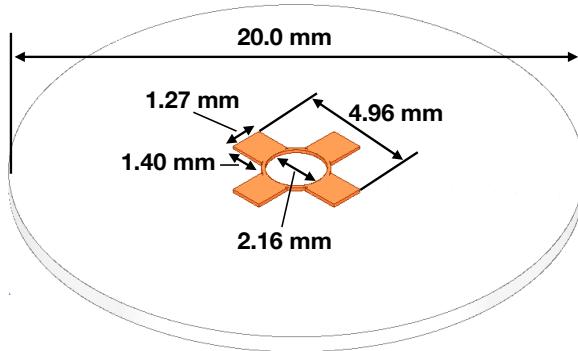


Figure 5.14: A model of the PCB crossed-dipole antenna with dimensions. The design has an inside diameter of 2.16 mm for the central circular trace, which is 0.13 mm wide. The dipole arms each have a width of 1.27 mm and protrude beyond the circular trace by 1.40 mm, which gives an overall width of 4.96 mm for the length of the antenna PCB trace from end-to-end. The overall size of the antenna is 20.0 mm the majority of which is the PCB dielectric material. This design was observed in simulation to maintain the field characteristics of the idealized crossed-dipole while being simpler to fabricate due to the increased size of the antenna.

3397 The PCB crossed-dipole design uses four rectangular pads to represent the dipole arms,
3398 which are connected by a thin circular trace. The circular trace both adds mechanical
3399 stability to the antenna and improves the azimuthal uniformity of the electric fields
3400 compared to a more standard crossed-dipole geometry. Furthermore, the circular trace
3401 allows for a greater separation between dipole arms than standard crossed-dipoles, which

¹<https://fieldtheoryinc.com/>

3403 is required to accommodate the coaxial connections to each pad. The pads each contain
 3404 a through-hole solder joint to connect coaxial transmission lines using hand soldering.
 3405 The antenna PCB has no ground plane on the bottom layer as this was observed in
 3406 simulation to significantly distort the radiation pattern in the plane of the PCB. The
 3407 only ground planes present in the model are the outer conductors of the four coaxial
 3408 transmission lines which feed the antenna. These are left unterminated on the bottom of
 3409 the PCB dielectric material.

3410 The antenna design development utilized a combination of Locust electron simula-
 3411 tions and antenna simulations using ANSYS HFSS [14], a commercial finite-element
 3412 electromagnetic simulation software. Two antenna designs were simulated: an idealized
 3413 electrically small crossed-dipole antenna with an arm length of 0.40 mm and an arm
 3414 separation of 0.05 mm, as well as a PCB crossed-dipole antenna with the dimensions
 3415 shown in Figure 5.14. Plotting the magnitude of the electric fields generated by the
 3416 antennas across a 10 cm square located in the same plane as the respective antennas
 3417 reveals the expected cyclotron spiral pattern (see Figure 5.15) which closely matches
 3418 the prediction for simulated electrons. The spiral pattern demonstrates that the electric
 3419 fields have the appropriate phases to mimic cyclotron radiation, which fulfills SYNCA
 criterion 3 identified in Section 5.3.2.

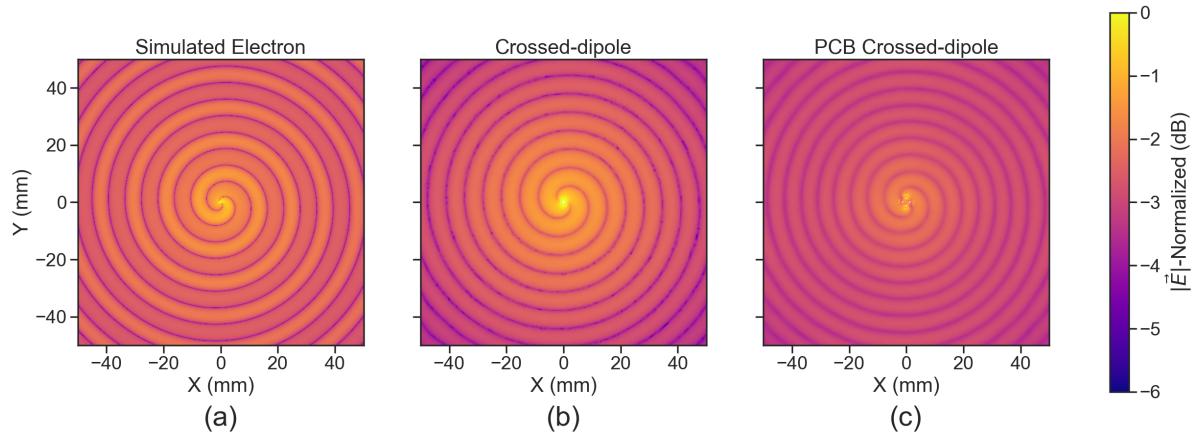


Figure 5.15: A comparison of the electric field magnitudes, normalized by the maximum value of the electric field in each simulation, plotted on a 10 cm square to visualize the Archimedean spirals formed by the electron (a), the crossed-dipole antenna (b), and a PCB crossed-dipole antenna (c). The matching patterns indicate that the electric fields have similar phase characteristics. These images were generated using Locust simulations for the electron and ANSYS HFSS for both antennas.

3420

3421 As we can see from Figure 5.16, the crossed-dipole antenna, which uses an idealized

3422 geometry, exhibits good agreement with simulation. The antenna has a maximum
 3423 deviation from a simulated electron of approximately 0.5 dB in the total electric field, 1
 3424 dB for the ϕ -polarized electric field and 1 dB for the θ -polarized electric field.

In comparison, the pattern of the PCB crossed-dipole antenna, because the simulation incorporates the geometry of the coax transmission lines, exhibits some distortion from the idealized cross-dipole simulations. The vertically oriented ground planes of the coax lines introduce more θ -polarized electric fields than are observed for simulated electrons near $\theta = 90^\circ$. The significant θ -polarized field minimum is still present but shifted to approximately $\theta = 65^\circ$. The θ -polarized field deviations of the PCB crossed-dipole

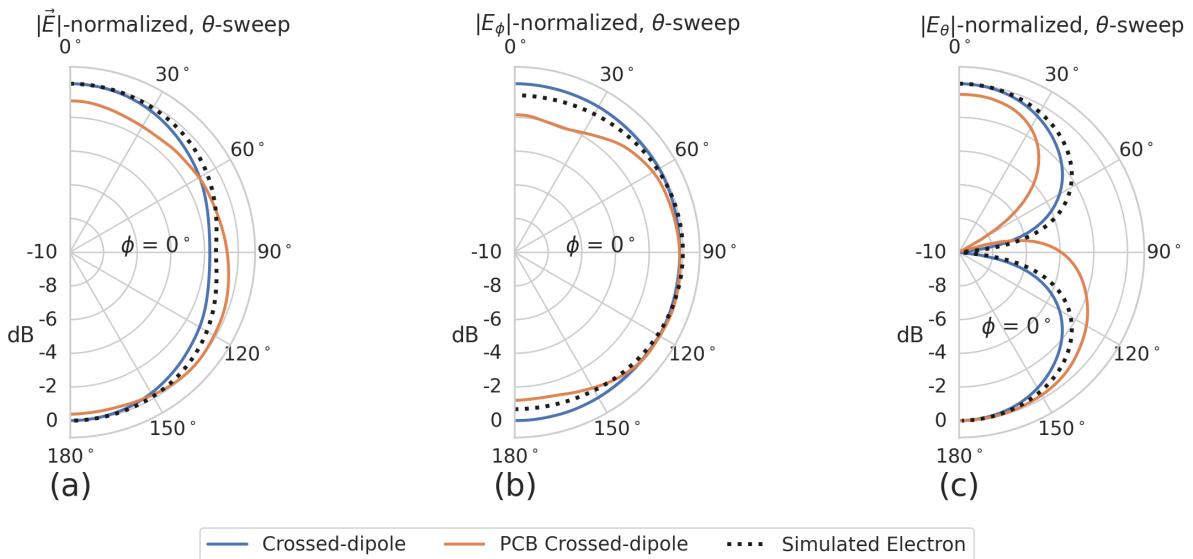


Figure 5.16: A comparison of the normalized electric field magnitudes for the ideal crossed-dipole, PCB crossed-dipole, and a simulated electron as a function of the polar angle (θ). (a) Shows the total electric field, (b) shows the ϕ -polarized electric field component, and (c) shows the θ -polarized electric field component. These images were generated using Locust simulations for the electron and ANSYS HFSS for both antennas.

3430 antenna should not greatly impact the performance of the antenna because the receiving
 3431 antenna array is primarily ϕ -polarized. Therefore deviations in the θ -polarized fields
 3432 will be suppressed due to the polarization mismatch. More importantly, the ϕ -polarized
 3433 electric field pattern generated by the PCB crossed-dipole closely matches simulated
 3434 electrons across the polar angle range of $50^\circ < \theta < 150^\circ$. In this region the PCB crossed-
 3435 dipole differs by less than 0.5 dB from simulated electrons. This range greatly exceeds
 3436 the beamwidth of the receiving antenna array which is designed to be most sensitive
 3437 to fields produced near $\theta = 90^\circ$. Therefore, we conclude that the PCB crossed-dipole
 3438

3439 antenna generates a ϕ -polarized radiation pattern that fulfills SYNCA criterion 1 from
3440 Section 5.3.2.

3441 The final SYNCA criterion is related to the uniformity of the electric fields when
3442 measured azimuthally around the antenna. As we saw for real electrons in Section 5.3.2
3443 it is expected that the magnitude of the electric field be completely uniform as a function
3444 of the azimuthal angle due to the symmetry of the cyclotron orbit. In Figure 5.17 we plot
3445 the total electric field as a function of azimuthal angle for an electron, the crossed-dipole
antenna, and the PCB crossed-dipole antenna. The crossed-dipole antenna exhibits

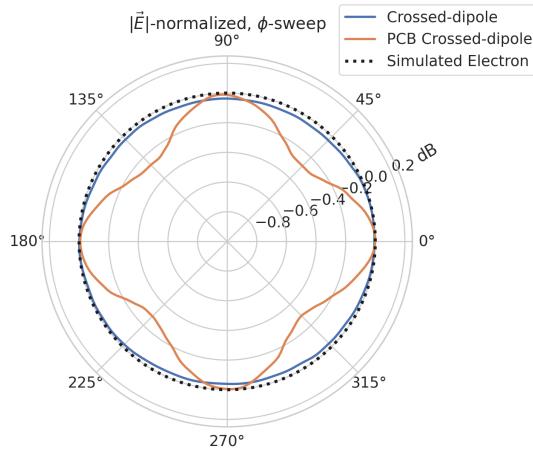


Figure 5.17: A comparison of the normalized electric field magnitudes for the crossed-dipole, PCB crossed-dipole, and a simulated electron as a function of the azimuthal angle (ϕ) evaluated at $\theta = 90^\circ$. This image was generated using Locust simulations for the electron and ANSYS HFSS for both antennas.

3446
3447 perfect uniformity around the azimuthal angle, whereas the PCB crossed-dipole has a
3448 small periodic deviation with a maximum difference of 0.3 dB caused by the coaxial
3449 transmission lines below the PCB. Such a small deviation from uniformity is acceptable
3450 since it is smaller than the expected variation in uniformity caused by imperfections in
3451 the antenna fabrication process, which modifies the antenna shape in an uncontrolled
3452 manner by introducing solder blobs with a typical size of a few tenths of a millimeter on
3453 the dipole arms (see Figure 5.18). Additionally, the SYNCA will be separately calibrated
3454 to account for azimuthal differences in the electric field magnitude. Therefore we see
3455 from the simulated performance of the PCB crossed-dipole antenna that this antenna
3456 design meets all three of the SYNCA criteria.

5.3.4 Characterization of the SYNCA

Two SYNCAs were manufactured using the PCB crossed-dipole design (see Figure 5.18). The antenna PCB (Matrix Circuit Board Materials, MEGTRON 6) is connected to four 2.92 mm coaxial connectors (Fairview Microwave, SC5843) using semi-rigid coax (Fairview Microwave, FMBC002), which also physically support the antenna PCB. The antenna PCB consists only of two layers which correspond to the copper antenna trace and the PCB dielectric. Each coax line is connected to the associated dipole arm using through-hole soldering and phase matched to ensure that the electrical length of each of the transmission lines is identical at the operating frequency. The antenna PCB is further reinforced using custom cut polystyrene foam blocks, which have an electrical permittivity nearly identical to air. A custom 3D printed mount is included at the base of the antenna to support the coax connectors and to provide a sturdy mounting base.

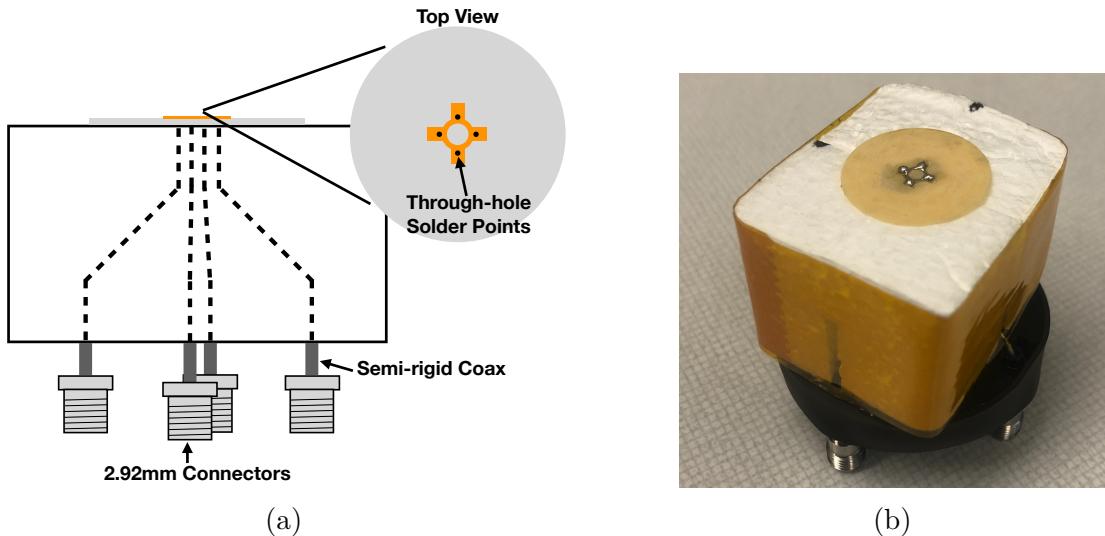


Figure 5.18: (a) A cartoon schematic which highlights the routing of the semi-rigid coax transmission lines. (b) A photograph of a SYNCA constructed using the modified crossed-dipole PCB antenna design. Visible in the photograph of the SYNCA are four blobs of solder which are an artifact of the SYNCA's hand-soldered construction. These solder blobs are the most significant deviation from the SYNCA design shown in Figure 5.14 and are responsible for a significant fraction of the irregularities seen in the antenna pattern.

Characterization measurements were performed using a Vector Network Analyzer (VNA) to measure the electric field magnitude and phase radiated by the SYNCA to verify the radiation pattern (see Figure 5.19). The VNA is connected to the SYNCA



Figure 5.19: A schematic of the VNA characterization measurements (a). This setup allows for antenna gain and phase measurements across a full 360° of azimuthal angles using a motorized rotation stage and control of the radial position of the SYNCA using a translation stage. A photo of the setup in the lab is shown in (b).

3472 at one port through a hybrid-coupler whose outputs are connected to two baluns to
 3473 generate the signals with the appropriate phases to feed the SYNCA (see Figure 5.13).
 3474 The other port of the VNA is connected to a single reference horn antenna that serves
 3475 as a field probe. To position the SYNCA, a combination of translation and rotation
 3476 stages are used to characterize the antenna's fields across the entire radiation pattern
 3477 circumference. This measurement scheme is equivalent to measuring the fields generated
 3478 by the SYNCA using a full circular array of probe antennas.

3479 The antenna measurement space is surrounded by RF anti-reflective foam to isolate
 3480 the measurements from the lab environment (see Figure 5.19b) and remaining reflections
 3481 are removed using the VNA's time-gating feature. The SYNCA is affixed to the stages
 3482 by a custom RF transparent mount made of polystyrene foam. The coaxial cables deliver
 3483 the antenna feed signals generated by the VNA to the SYNCA while still allowing
 3484 unrestricted rotation. The horn antenna probe is nominally positioned in the plane
 3485 formed by the antenna PCB ($\theta = 90^\circ$ or $z = 0$ mm) at a distance of 10 cm from the
 3486 SYNCA, to match the expected position of the antenna array relative to the SYNCA in
 3487 the antenna array test stand. The horn antenna can be manually raised or lowered to
 3488 different relative vertical positions to characterize the radiation pattern at different polar
 3489 angles.

3490 Several 360° scans were performed with probe vertical offsets of -10.0 mm, -5.0 mm,
 3491 0.0 mm, 5.0 mm, and 10.0 mm relative to the antenna PCB plane. These probe offsets

3492 cover a 2 cm wide vertical region centered on the SYNCA PCB, approximately equal to
 3493 ± 6 degrees of polar angle. The measurements show that the SYNCA is generating fields
 3494 with nearly isotropic magnitude across the probed region. The standard deviation of the
 3495 electric field magnitude measured around the antenna circumference is approximately
 3496 2.9 dB for a typical rotational scan. The presence of a significant pattern null is noted
 3497 near 45° (see Figure 5.20), which we attribute to small imperfections in the antenna
 3498 PCB that could be introduced from the hand soldered terminations connecting the coax
 3499 cables to the antenna. There is no significant difference in the radiation pattern when
 3500 measured across the 2 cm vertical range. The measured relative phases closely follow
 3501 the expectation for an electron, being linear with the measurement rotation angle and
 3502 forming the expected spiral pattern. Other than the small phase imperfections there is
 3503 a slight sinusoidal bias to the phase data, which we determined is the result of a small
 3504 ($\lesssim 1$ mm) offset of the antenna's phase center from the rotation axis of the automated
 3505 stages.

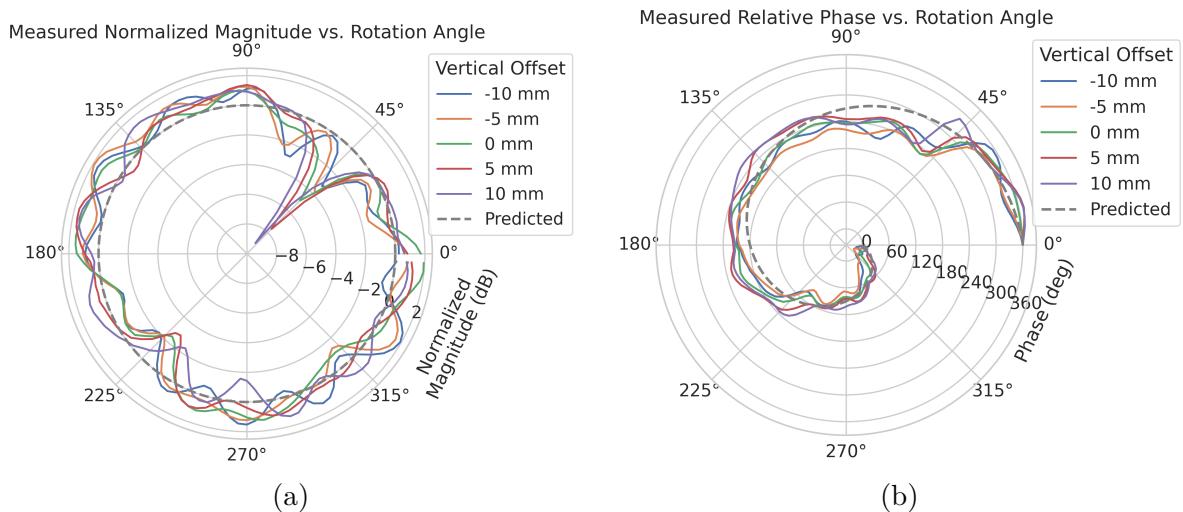


Figure 5.20: Linear interpolations of the measured electric field magnitude (a) and phase (b). The data was acquired using a VNA at 120 points spaced by 3 degrees from 0 to 357 degrees of azimuthal angle. The different color lines indicate the vertical offset of the horn antenna relative to the SYNCA PCB and the dashed line shows the expected shape from electron simulations. No significant difference in the antenna pattern is observed for the measured vertical offsets.

3506 The characterization measurements confirm the simulated performance of the SYNCA.
 3507 As expected the fields generated by the antenna are nearly isotropic in magnitude, ϕ -
 3508 polarized, and are linearly out of phase around the circumference of the antenna as

3509 predicted for cyclotron radiation in Section 5.3.2. Small imperfections in the magnitude
 3510 and phase of the antenna are expected, particularly at the antenna's high operating
 3511 frequency of 26 GHz where small geometric changes can have significant impacts on
 3512 electrical properties. However, calibration through careful characterization measurements
 3513 can be used to remove the majority of these pattern imperfections, including the relatively
 3514 large pattern null near 45°, which will allow for the usage of the SYNCA as a test source
 3515 for free-space CRES experiments utilizing antenna arrays. In the next section we use the
 3516 VNA measurements obtained here as a calibration for signal reconstruction using digital
 3517 beamforming.

3518 **5.3.5 Beamforming Measurements with the SYNCA**

3519 Digital beamforming is a standard technique for signal reconstruction using a phased
 3520 array [58]. The SYNCA, since it exhibits the same cyclotron phases as a trapped electron,
 3521 can be used to perform simulated CRES digital beamforming reconstruction experiments
 3522 on the bench-top without the need for the magnet, cryogenics, and vacuum systems
 3523 required by a full CRES experiment. The fields received by the individual elements
 3524 of the antenna array will have phases dependent on the spatial position of the source
 3525 relative to the antennas. Therefore, a simple summation of the received signals will fail
 3526 to reconstruct the signal due to destructive interference between the individual channels
 3527 in the array. However, applying a phase shift associated with the source's spatial position

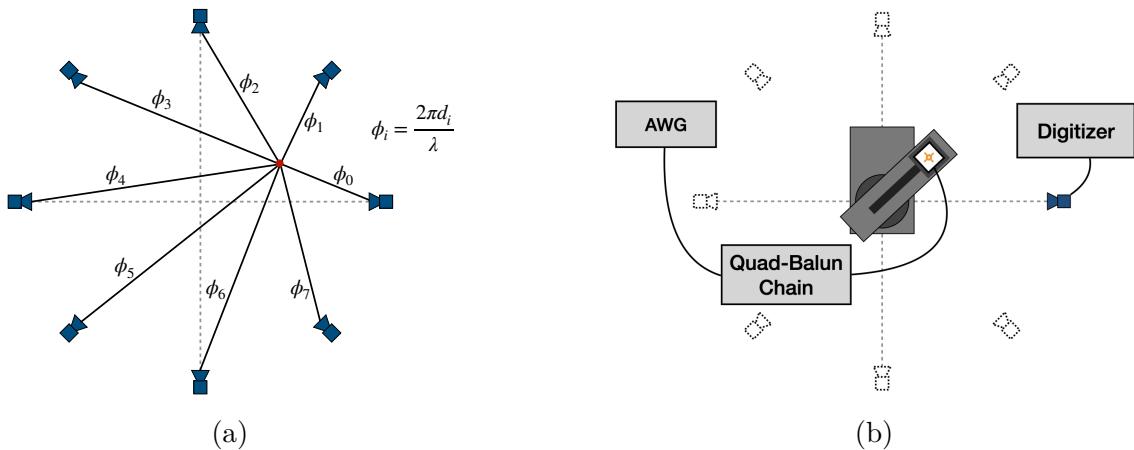


Figure 5.21: (a) A depiction of the relative phase differences for signals received by a circular antenna array from an isotropic source. The phases correspond to a unique spatial position. (b) A schematic of the setup used to perform digital beamforming.

3528 removes phase differences and results in a constructive summation of the channel signals
 3529 (see Figure 5.21). We can summarize the digital beamforming operation succinctly using
 3530 the following equation

$$y[t_n] = \sum_{m=0}^{N-1} x_m[t_n] A_m e^{i\phi_m}, \quad (5.35)$$

3531 where $y[t_n]$ represents the summed array signal at time t_n , $x_m[t_n]$ is the signal received
 3532 by channel m at time t_n , ϕ_m is the phase shift applied to the signal received at channel
 3533 m , and A_m is an amplitude weighting factor that accounts for the different signal power
 3534 received by individual channels. By changing the digital beamforming phases, the point
 3535 of constructive interference can be scanned across the sensitive region of the array to
 3536 search for the location of a radiating source, which is identified as the point of maximum
 3537 summed signal power above a specified threshold. The digital beamforming phases consist
 3538 of two components,

$$\phi_m = 2\pi d_m / \lambda + \theta_m, \quad (5.36)$$

3539 where d_m is the distance from the m -th array element to the source, and θ_m is the
 3540 relative angle between the source position and the m -th antenna. The first component is
 3541 the standard digital beamforming phase that corresponds to the spatial position of the
 3542 source, and the second component is the cyclotron phase that corresponds to the relative
 3543 azimuthal phase offset.

3544 With a small modification to the hardware used to characterize the SYNCA (see
 3545 Figure 5.19), we can perform a digital beamforming reconstruction of a synthetic CRES
 3546 event. By replacing the VNA with an arbitrary waveform generator (AWG), the SYNCA
 3547 can be used to generate cyclotron radiation with an arbitrary signal structure, which
 3548 can then be detected by digitizing the signals received by the horn antenna. Rotational
 3549 symmetry allows us to use the rotational stage of the positioning system to rotate the
 3550 SYNCA to recreate the signals that would have been received by a complete circular
 3551 array of antennas.

3552 Using this setup, signals from a 60 channel circular array of equally spaced horn
 3553 antennas were generated with the SYNCA positioned 10 mm off the central array axis,
 3554 reconstructed using digital beamforming, and compared to Locust simulation (see Figure
 3555 5.22). When the cyclotron spiral phases are not used, which is equivalent to setting θ_m
 3556 in Equation 5.36 to zero, the SYNCA's position is reconstructed as a relatively faint ring
 3557 as predicted by simulation. However, when the appropriate cyclotron phases are used
 3558 during the beamforming procedure, both the simulated electron and the SYNCA appear

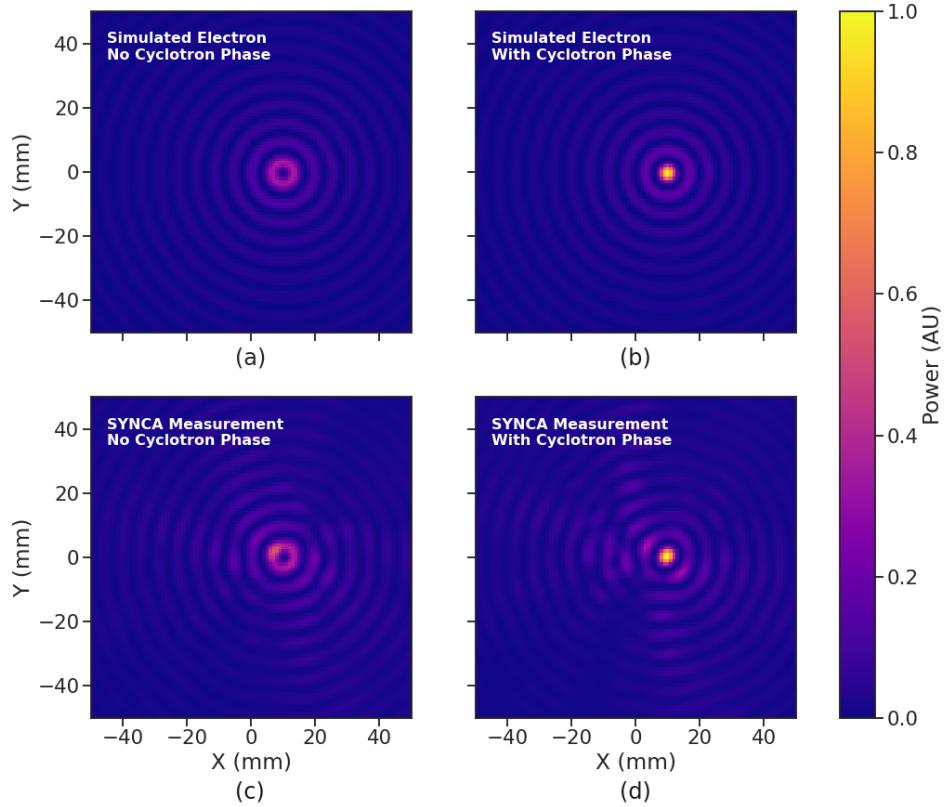


Figure 5.22: Digital beamforming maps generated using a simulated 60 channel array and electron simulated using the Locust package. (a) and (b) show the beamforming maps for simulated electrons without the cyclotron spiral phases and with the cyclotron spiral phases respectively. (c) and (d) show the beamforming maps produced from SYNCA measurements. We observe good agreement between simulated electrons and the SYNCA measurements.

as a single peak of high relative power corresponding to the source position. Therefore, we observe good agreement between the simulated and SYNCA reconstructions. While it may seem that for the case with no cyclotron phase corrections the ring reconstructs the position of the electron as effectively as beamforming with the cyclotron phase corrections, it is important to note that the simulations and measurements were generated without a realistic level of thermal noise. The larger maxima region and lower signal power, which occurs without the cyclotron phase corrections, significantly reduce the probability of detecting an electron in a realistic noise background.

To bound the beamforming capabilities of the synthetic array of horn antennas, we performed a series of beamforming reconstructions where the SYNCA was progressively moved off the central axis of the array (see Figure 5.23). To extract an estimate of the

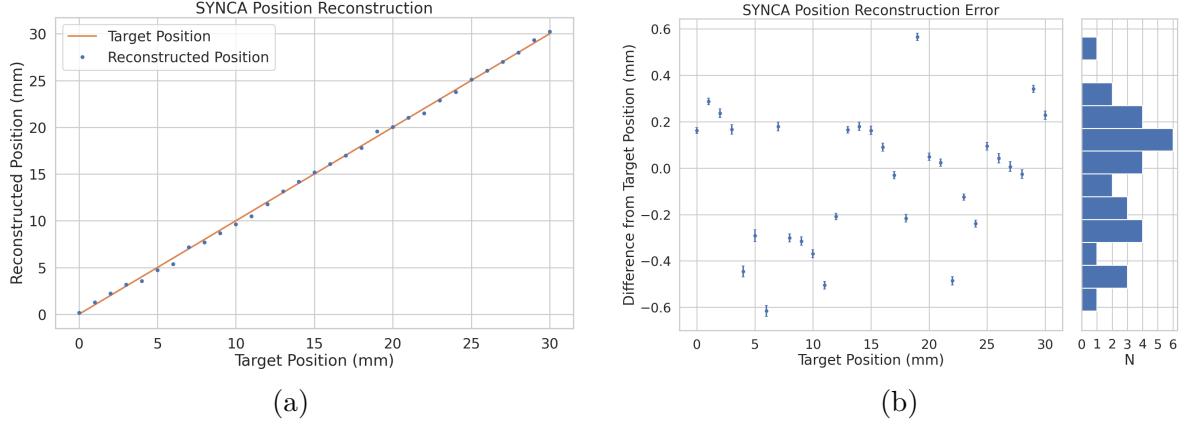


Figure 5.23: A plot of the SYNCA’s reconstructed position using the synthesized horn-antenna array and digital beamforming. (a) Shows the reconstructed position of the SYNCA compared with the target position indicated by the positioning system readout. (b) Shows the reconstruction error, which is the difference between the target and reconstructed positions. The error bars in (b) are the uncertainty in the mean position of the 2D Gaussian used to fit the digital beamforming reconstruction peak obtained from the fit covariance matrix. The mean fit position uncertainty of 0.02 mm is an order of magnitude smaller than the typical reconstruction error of 0.3 mm obtained by calculating the standard deviation of the difference between the reconstructed and target position.

position of the SYNCA using the digital beamforming image we apply a 2-dimensional (2D) Gaussian fit to the image data and extract the estimated centroid value. We find that the synthetic horn antenna array reconstructs the position of the SYNCA with a 1σ -error of 0.3 mm with no apparent trend across the 30 mm measurement range. This reconstruction error is an order of magnitude larger than mean fit position uncertainty of 0.02 mm indicating that systematic effects related to the SYNCA positioning system could be contributing additional uncertainty to the measurements. Note that the current mean reconstruction error of 0.3 mm is a factor of 20 smaller than the full width at half maximum of the digital beamforming peak (6 mm), which could be interpreted as a naive estimate of the position reconstruction performance of this technique. Because these measurements are intended as a proof-of-principle demonstration, we do not investigate potential sources of systematic errors further; however, we expect that a similar and more thorough investigation will be performed using the Project 8 antenna array test stand, where typical reconstruction errors can be used to estimate the energy resolution limits of antenna array designs.

3585 **5.3.6 Conclusions**

3586 In this paper we have introduced the SYNCA, which is a novel antenna design that
3587 emits radiation that mimics the unique properties of the cyclotron radiation generated by
3588 charged particles moving in a magnetic field. The characterization measurements of the
3589 SYNCA validated the simulated performance of the PCB crossed-dipole antenna design.
3590 Additionally, the SYNCA was used to estimate the position reconstruction capabilities
3591 of a synthesized array of horn antennas and experimentally reproduced the simulated
3592 digital beamforming reconstruction of electrons.

3593 While the SYNCA performs well, there exist discrepancies in the phase and magnitude
3594 of the radiation pattern compared to the simulated SYNCA design that are related to
3595 the small geometric differences in the soldered connections. Future design iterations that
3596 replace the soldered connections with a fully surface mount design could improve the
3597 radiation pattern at the cost of some complexity and expense. Furthermore, improving
3598 the design of the antenna PCB and mounting system would allow the antenna to be
3599 inserted into a cryogenic and vacuum environment where in-situ antenna measurement
3600 calibrations could be performed.

3601 The discrepancies in the radiation pattern and phases exhibited by the as-built
3602 SYNCA should not greatly impact its performance as a calibration probe. Both magni-
3603 tude and phase variations can be accounted by applying the SYNCA characterization
3604 measurements as a calibration to the data collected by the antenna array test stand. The
3605 separate calibration of the SYNCA radiation does not impact the primary goals for the
3606 antenna array test stand which are array calibration and signal reconstruction algorithm
3607 performance characterization, because it can be performed with standard reference horn
3608 antennas with well understood characteristics.

3609 The SYNCA antenna technology advances the CRES technique by providing a
3610 mechanism to characterize free-space antenna arrays for CRES measurements without
3611 the need for a magnet and cryogenics system, which would be required for calibration
3612 using electron sources. Both the Project 8 collaboration as well as future collaborations
3613 which are developing antenna array based CRES experiments can make use of SYNCA
3614 antennas as an important component of their calibration and commissioning phases.

3615 **5.4 FSCD Antenna Array Measurements with the SYNCA**

3616 **5.4.1 Introduction**

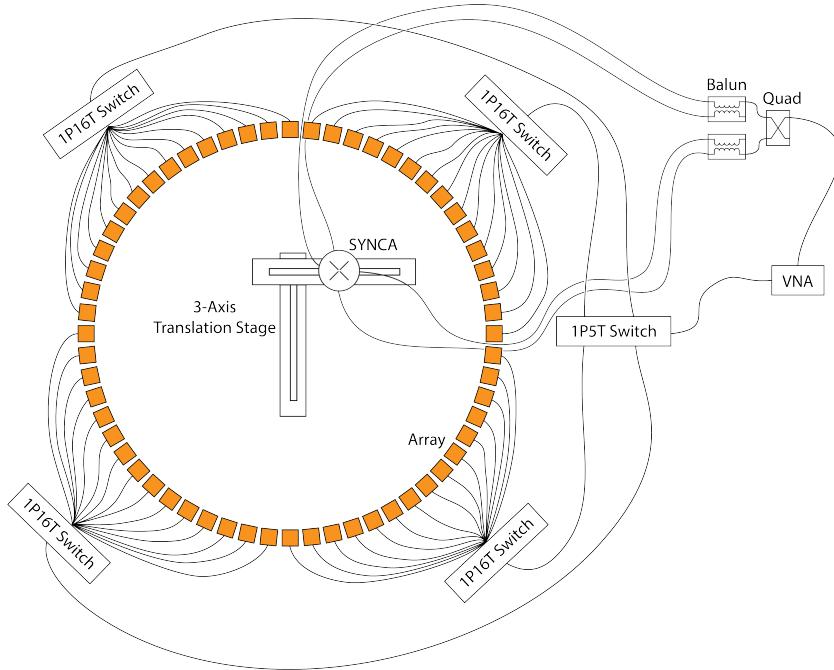


Figure 5.24: A diagram of the array measurement system used to test the prototype FSCD antenna array. A VNA is used as the primary measurement tool, which is connected to the array through a series of switches. The other port of the VNA connects to the SYNCA through the quad-balun chain used to provide the SYNCA feed signals. During measurements the SYNCA is positioned inside the center of the antenna array and translated to different radial and axial positions using a 3-axis manual translation stage setup.

3617 Using the SYNCA we can perform full-array measurements of prototype versions
3618 of the FSCD antenna array to test its performance with a realistic cyclotron radiation
3619 source (see Figure 5.24). The goal is to check how the measured power received by
3620 the array compares to FSCD simulations as a function of the radial and axial position
3621 of the SYNCA. These measurements are intended to validate the antenna research
3622 and development by Project 8, which has been driven primarily by simulations with
3623 Locust [10] and CREsana (see Section 4.2.3), and identify any discrepancies with these
3624 simulations tools. This knowledge will provide confidence in the simulations necessary
3625 for the analysis of the sensitivity of larger antenna array based CRES experiment designs
3626 to the neutrino mass.

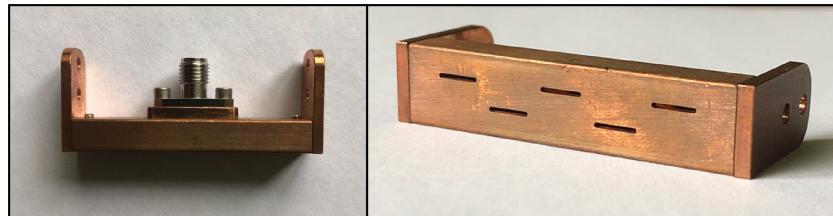
3627 As shown in Section 5.3, the SYNCA does have some radiation pattern imperfections
3628 that complicate the comparison between measurement and simulation data. One way to
3629 disentangle some of the effects of these imperfections is to perform an additional set of
3630 measurements using a synthetic antenna array setup along with the SYNCA antenna.
3631 Since the synthetic array setup uses only a single array antenna, the data should be
3632 free of errors associated with individual antenna differences and multi-path interference,
3633 which are two error sources being tested with the full-array setup. By comparing the
3634 synthetic array data to the FSCD array data and to simulation data one can evaluate the
3635 significance of these effects relative to the errors introduced by SYNCA imperfections.

3636 **5.4.2 Measurement Setups**

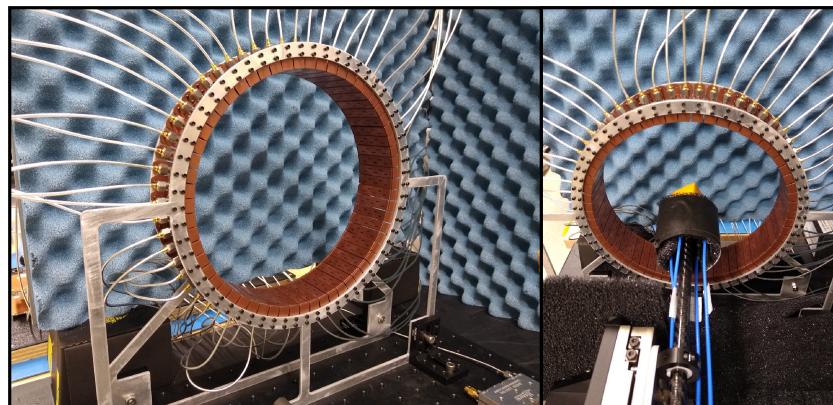
3637 **5.4.2.1 FSCD Array Setup**

3638 The antenna design that composes the array is the 5-slot waveguide antenna developed
3639 for the FSCD experiment (see Figure 5.25a). The antenna is 5 cm long and is constructed
3640 out of WR-34 waveguide with a 2.92 mm coax connector located at the center of the
3641 antenna. Copper flanges located on both ends of the antenna are used to mount the
3642 antenna in the array support structure. The antennas are supported by two circular steel
3643 brackets that can be bolted to both ends of the waveguide to construct the circular array
3644 (see Figure 5.25b). The antenna array consists of sixty identical waveguide antennas
3645 with a radius of 10 cm. The array is mounted perpendicular to an optical breadboard
3646 surface using a pair of the steel brackets, which provide sufficient space for the coaxial
3647 cable connections and allows for easy positioning of the SYNCA antenna. The SYNCA is
3648 mounted on the end of a carbon fiber rod attached to a set of manual translation stages,
3649 which are used to move the SYNCA antenna to different positions inside the array (see
3650 Figure 5.25c). The stages allow for independent motion in three different axes and can
3651 position the SYNCA at radial distances up to 5 cm from the center.

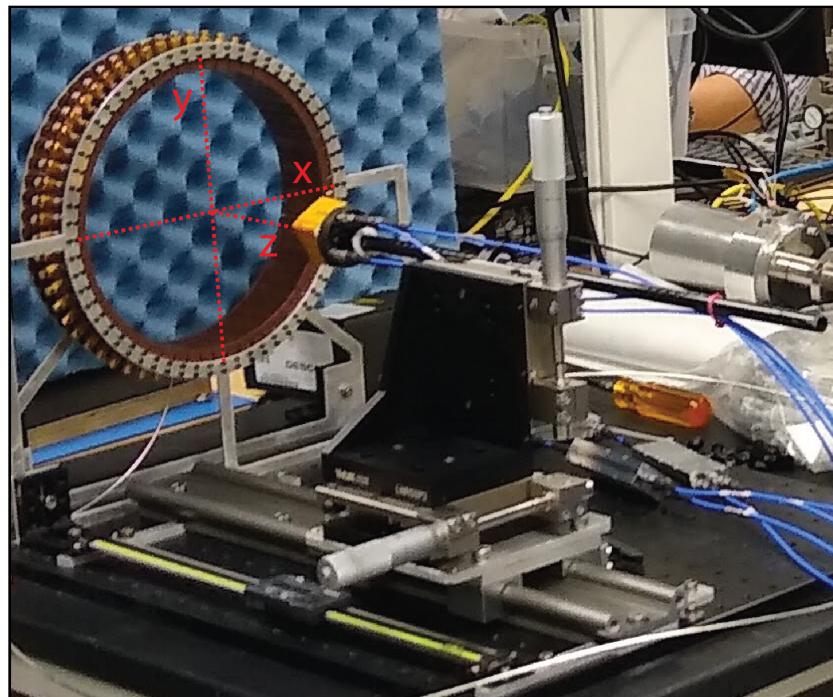
3652 Data acquisition is accomplished using a two-port VNA in combination with a series
3653 of microwave switches that allow the VNA to connect to each channel in the array . The
3654 first port of the VNA is connected to the quad-balun chain used to feed the SYNCA (see
3655 Section 5.3), and the second port of the VNA connects to a 1P5T microwave switch. The
3656 1P5T switch is connected to four separate 1P16T switch boards that connect directly
3657 to the array. The data acquisition is controlled by a python script running on a lab
3658 computer, which is connected to the VNA and an Arduino board programmed to control
3659 the microwave switches. The script uses the switches to iteratively connect each of the



(a)



(b)



(c)

Figure 5.25: Photos of the prototype FSCD antenna (a), the FSCD array and SYNCA (b), and the translation stages and coordinate system used to position the SYNCA (c).

3660 antennas in the array to the VNA. The VNA is configured to load a specific calibration
3661 file for each antenna channel and performs the measurements of all available S-parameters.
3662 The separate calibration files is an attempt to remove phase and magnitude errors caused
3663 by different propagation through the RF switches. Array measurements were performed
3664 for the set of SYNCA positions consisting of radial (x-axis) positions from 0 to 50 mm in
3665 5 mm steps and axial (z-axis) positions from 0 to 50 mm in 5 mm steps resulting in 121
3666 array measurements. At each SYNCA position we measured the two-port S-parameter
3667 matrix using a linear frequency sweep from 25.1 to 26.1 GHz with 101 discrete frequencies.

3668 **5.4.2.2 Synthetic Array Setup**

3669 A photograph of the setup used to perform the synthetic array measurements is shown
3670 in Figure 5.26. One important difference between this setup and the FSCD array setup
3671 is that the synthetic array measurements were performed with a waveform generator and
3672 digitizer instead of a VNA. The electronics configuration is identical to the diagram in
3673 Figure 5.7b. Despite the differences, one is still able to compare the measured phases of
3674 the synthetic array and the relative magnitude of the power, since the digitized signal
3675 power is directly proportional to S21.

3676 The arbitrary waveform generator in the setup is configured to produce a 64 MHz
3677 sine wave signal that is up-converted to 25.864 GHz using a mixer and the VNA source.
3678 This signal is passed through a bandpass filter and fed to the SYNCA quad-balun chain.
3679 A single FSCD antenna is positioned 10 cm from the SYNCA and aligned vertically so
3680 that the center of the 5-slot waveguide is in the plane of the SYNCA PCB (see Figure
3681 5.26). This position corresponds to $z = 0$ in Figure 5.25c. The SYNCA is rotated
3682 in three degree steps to synthesize an antenna array with 120 channels. This channel
3683 count is more than could physically fit in a 10 cm radius array, but there is no cost to
3684 over-sampling. Additionally, over-sampling allows for a check of the smoothness of the
3685 antenna array radiation pattern. The signals from the FSCD antenna are down-converted
3686 using the second mixer connected to the VNA source before being digitized at 250 MHz
3687 and saved to disk. Several synthetic array measurement scans were performed by using
3688 the linear translation stage to change the radial position of the SYNCA. In total eight
3689 scans were taken from 0 to 35 mm using a radial position step size of 5 mm.

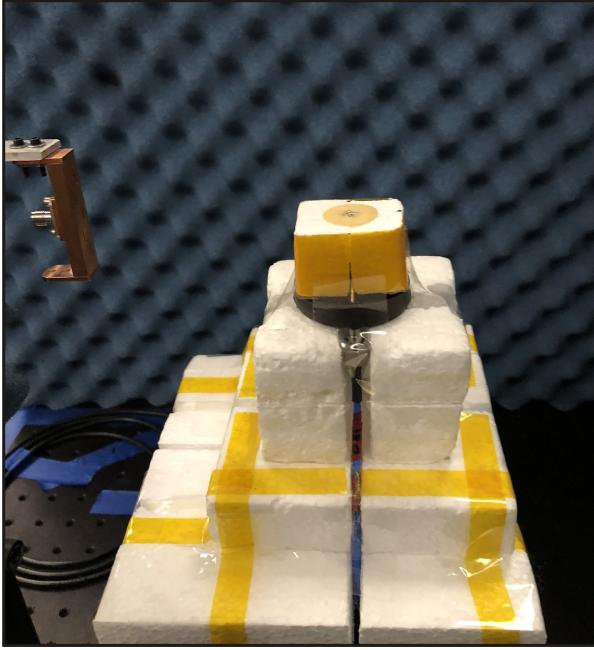


Figure 5.26: A photo of the FSCD antenna and the SYNCA in the synthetic array measurement setup at Penn State.

3690 **5.4.3 Simulations, Analysis, and Results**

3691 The Locust and CRESana simulation packages utilize the antenna transfer functions
3692 to calculate the power that would be received by each antenna from a CRES electron.
3693 The equivalent quantity in the measurement setup is the S21 matrix element, which
3694 indicates the ratio of the power received by an antenna in the array to the amount of
3695 power delivered to the SYNCA. Therefore, the analysis focuses on comparing the relative
3696 magnitudes and phase of the S21 parameters measured by the VNA as a function of
3697 the array channel and the SYNCA position. Additionally, we apply a beamforming
3698 reconstruction to the S21 data to evaluate how the summed power and beamforming
3699 images change as a function of the position of the SYNCA.

3700 **5.4.3.1 Simulations**

3701 Simulations for the FSCD array measurements were performed using CRESana, which
3702 performs analytical calculations of the EM-fields produced by an electron at the position
3703 of the antennas. At each sampled time CRESana computes the electric field vector at the
3704 antenna positions, which is projected onto the antenna polarization axis to obtain the
3705 co-polar electric field. The magnitude of the co-polar electric field is then multiplied by

3706 a flat antenna transfer function to calculate the corresponding voltage signal. CRESana
 3707 simulations exploit the flat transfer functions of the FSCD antennas, which allows the
 3708 electric field to be multiplied by the antenna transfer function rather than performing
 3709 the full FIR calculation. These calculations produce a voltage time-series for each of the
 3710 antennas in the array that can be compared to the laboratory measurements.

3711 CRESana was configured to simulate a 90° electron in a constant background magnetic
 3712 field of ≈ 0.958 T with a kinetic energy of 18.6 keV. These parameters were chosen
 3713 in order to mimic a CRES event near the tritium beta-decay spectrum endpoint in
 3714 the FSCD experiment. The constant background magnetic field guarantees that the
 3715 guiding center of the electron is stationary across the duration of the simulation which is
 3716 consistent with the SYNCA in the laboratory measurements. Simulations were performed
 3717 with the electron's guiding center at radial positions from 0 to 45 mm in steps of 1 mm
 3718 and axial positions from 0 to 30 mm in steps of 1 mm. The simulations generated time
 3719 series consisting of 8192 samples at 200 MHz for the sixty channel FSCD antenna array
 3720 geometry.

3721 5.4.3.2 Phase Analysis

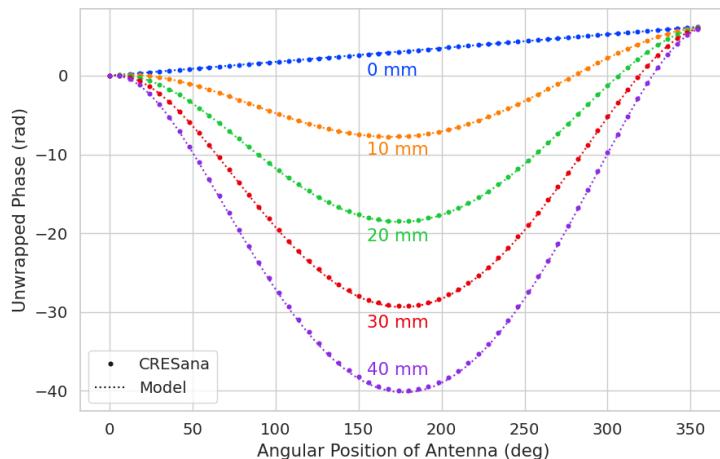


Figure 5.27: The unwrapped phases of signals received by the FSCD antenna array from an electron with a 90° pitch angle located in the plane of the antenna array. The data points indicated the phases extracted from simulation and the dashed lines show the model predictions.

3722 Correct modeling of the signal phases is fundamental to reconstruction for both
 3723 beamforming and matched filter approaches. The beamforming reconstruction relies on

3724 a signal phase model developed from Locust simulations, which allows one to predict the
3725 relative signal phases for a specific magnetic trap and electron position. The equation
3726 for the model is

$$\phi_{ij}(t) = \frac{2\pi d_{ij}(t)}{\lambda} + \theta_{ij}(t), \quad (5.37)$$

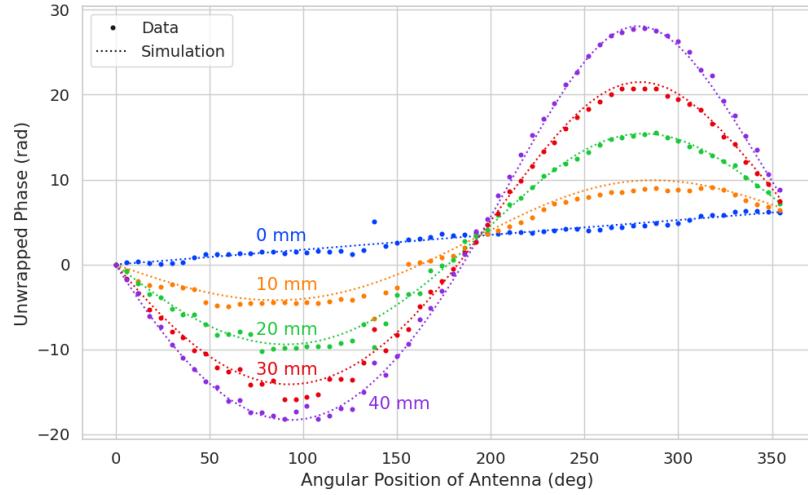
3727 where $d_{ij}(t)$ is distance between the assumed electron position and the antenna position,
3728 and $\theta_{ij}(t)$ is the angular separation between the electron and antenna positions. For
3729 details on the components of the phase model see Section 5.3.2. In Figure 5.27 we
3730 compare the phases predicted by Equation 5.37 to phases extracted from CREsana
3731 simulations of an electron located in the plane of the antenna array at a series of radial
3732 positions. One observes excellent agreement between the model and simulation.

3733 The measured signal phases from the FSCD array and synthetic array are shown
3734 in Figures 5.28a and 5.28b compared to the signal phase model. The axial position of
3735 the SYNCA in both plots is $z = 0$ mm, such that the plane of the PCB is aligned with
3736 the center of the FSCD antenna. The data shown in Figure 5.28a corresponds to the
3737 S-parameters measured at 25.80 GHz which is the frequency closest to the one used in
3738 the synthetic array setup. The different slope and sinusoidal phases exhibited by Figure
3739 5.28a and 5.28b reflects differences in the coordinate system for each setup. In general,
3740 we see that the phase model predicts the large scale features of the phases quite well,
3741 but there are some small scale deviations or errors from the phase model that do not
3742 appear to be present in simulation.

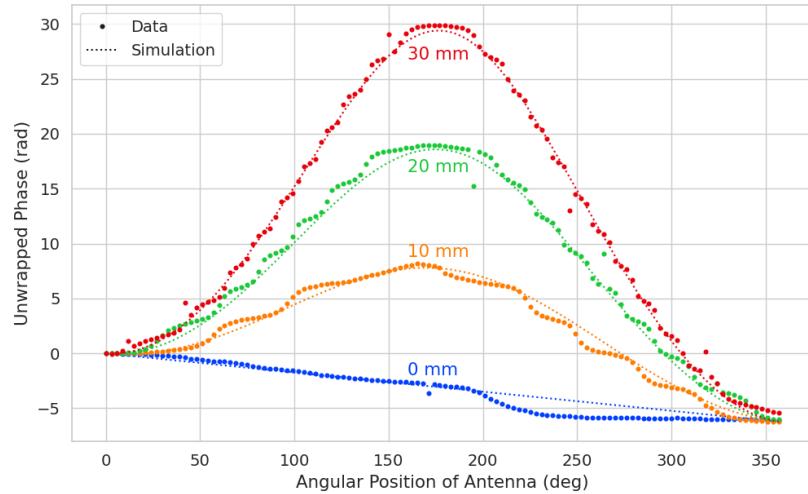
3743 A comparison of the phase errors, which are the difference between measurement and
3744 model is shown in Figure 5.29. The FSCD array data is referred to as the JUGAAD
3745 data in the plot legend, which is an alternative name for the FSCD array setup.

3746 The phase error at $R = 0$ in Figure 5.29 forms a smooth curve, with the exception of
3747 an outlier data point caused by a bug in the data acquisition script. One can attribute
3748 the observed phase error at this position to imperfections in the antenna pattern of the
3749 SYNCA. As the SYNCA is moved away from $R = 0$ mm one observes that the phase
3750 error exhibits oscillations whose frequency increases as a function of the radial position
3751 of the SYNCA. These oscillations have the appearance of a diffraction pattern, which
3752 is particularly clear for the radii ≥ 15 mm, due to the bilateral symmetry of the phase
3753 error peaks around 180° .

3754 One can observe a higher average variance in the phase errors measured for the FSCD
3755 array compared to the synthetic array. This is best seen by comparing the curves at
3756 $R \leq 15$ mm where the smooth synthetic array curves are distinct from the relatively
3757 noisy FSCD array errors. The extra noise in the FSCD array is most likely caused by



(a)



(b)

Figure 5.28: Plots of the measured unwrapped phases from the FSCD array (a) and the synthetic array (b) compared to the model predictions for a series of radial positions. The different phases of the sinusoidal phase oscillations in the two plots reflects differences in the coordinate systems of the measurements.

3758 differences in the radiation patterns of the antennas that make up the array as well as
 3759 differences in the transmission lines through the switch network that introduce additional
 3760 phase errors into the measurement. Since the synthetic array measurements use only
 3761 a single antenna, these extra error terms are not present, which explains the relatively
 3762 smoother phase error curves. Despite the extra phase errors in the FSCD array, it is still
 3763 possible to observe a similar phase error oscillation effect as the SYNCA is moved away

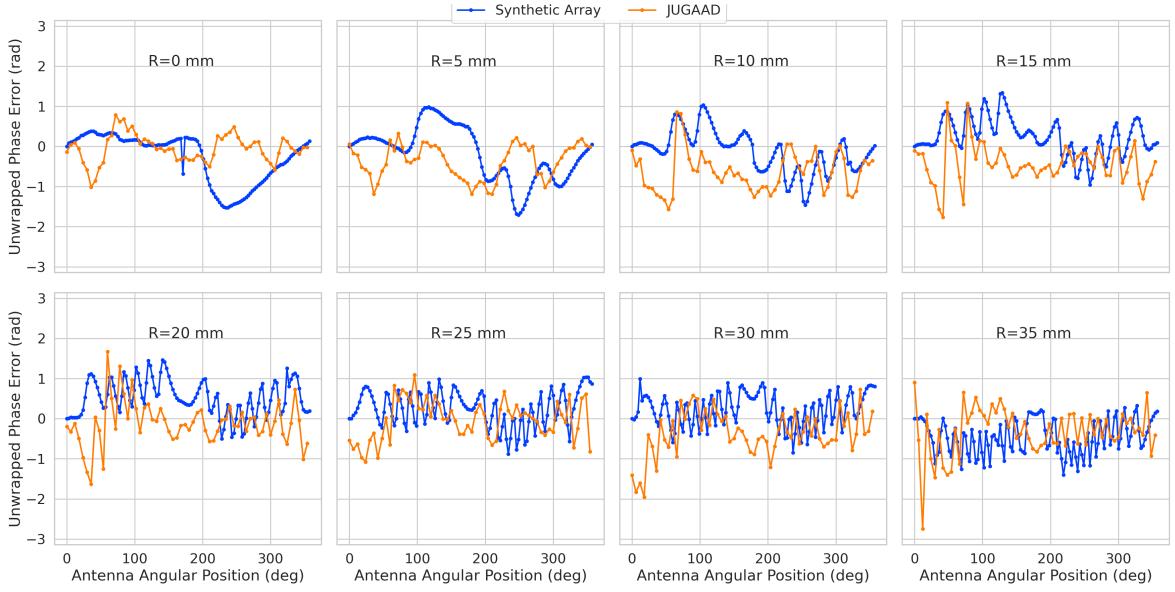


Figure 5.29: The phase errors between the measurement and model for the synthetic array (blue) and the FSCD array (orange) for a series of radial positions. The label JUGAAD refers to an alternative name for the FSCD array setup. As the SYNCA is translated off-axis phase errors with progressively higher oscillation frequency enter into the measurements.

3764 from $R = 0$ mm.

3765 The diffraction pattern exhibited by the phase error oscillations is more easily observed
 3766 by plotting the phase errors in a two-dimensional map, which is done in Figures 5.30a and
 3767 5.30b. For the synthetic array data ones observes a relatively clear diffraction pattern
 3768 that emerges as the SYNCA is moved radially. The bilateral symmetry of the diffraction
 3769 patterns is due to the bilateral symmetry of the circular synthetic array around the
 3770 translation axis of the SYNCA. A similar pattern is also visible in the FSCD array data,
 3771 although, it is obscured by the additional phase error that results from the multi-channel
 3772 array.

3773 The physical origin of the phase error diffraction pattern is attributed to interference
 3774 effects arising from path-length differences between the individual slots in the FSCD
 3775 antenna and the SYNCA transmitter. Since we are operating in the radiative near-field of
 3776 the FSCD antenna, the path length differences between the slots introduces a significant
 3777 change in the summation of the signals that occurs inside the waveguide, which causes
 3778 the radiation pattern of the antenna to change as a function of distance. Therefore, when
 3779 the SYNCA is positioned off-axis the different path-lengths from the SYNCA to each
 3780 antenna results in different radiation patterns leading to the observed diffraction pattern.

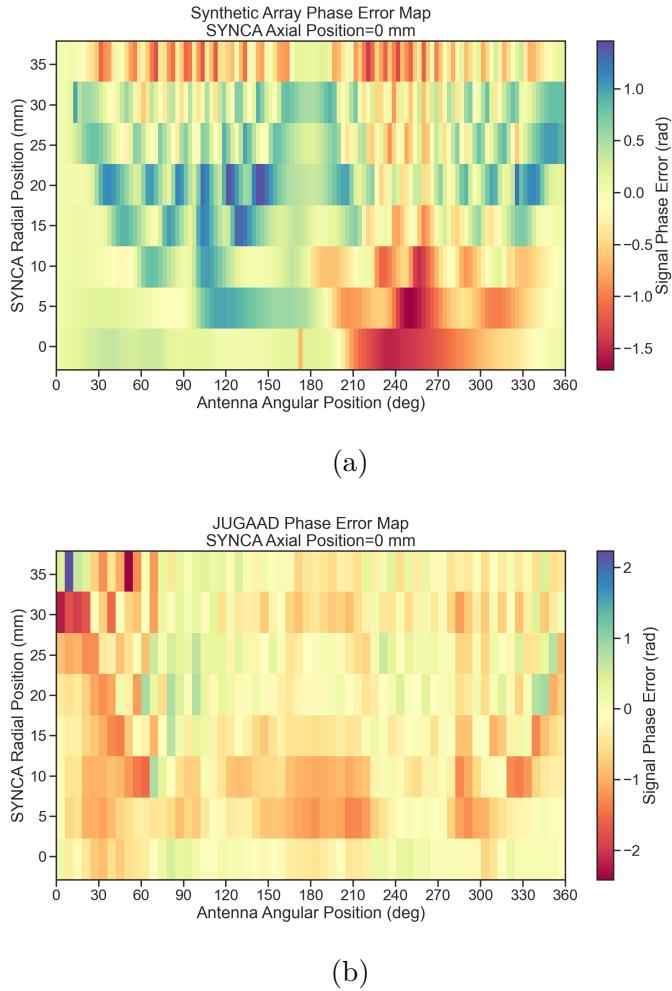


Figure 5.30: Two dimensional plots of the phase errors for the synthetic array (a) and the FSCD (JUGAAD) array (b). In both plots we observe evidence of a similar diffraction pattern with bilateral symmetry, but the FSCD array measurements have an additional phase error contribution from the different antennas and paths through the switch network.

This near-field effect is not present in simulations, because in order to simplify the calculations we assume that the far-field approximation can be applied to the FSCD antennas. This means that the radiation pattern and antenna transfer functions are independent of the distance between the transmitter and the receiving antenna. In principle, we can account for these near-field effects with a more detailed simulation of the FSCD antennas either in CRESana or Locust, which would result in an additional term in the beamforming phase model. However, this would increase the computational intensity of the simulation software. In the next section we briefly discuss the impact of

3789 these near-field effects on the measured magnitude of the signals.

3790 **5.4.3.3 Magnitude Analysis**

3791 Exactly as for the signal phase, one can use simulations to construct a model that
3792 describes the magnitude of the signals received by each channel in the antenna array.
3793 By examining the results of simulations or by analyzing the Liénard-Wiechert equation
3794 one can show that radiation pattern from a 90° pitch angle electron in a magnetic field
3795 is omni-directional. Therefore the relative magnitudes of the signals received by each
3796 channel will be determined by the free-space power loss, which is proportional to the
3797 inverse distance between the assumed electron position and the antenna.

3798 A consequence of this is that the signals produced in the array for electrons off the
3799 central axis will have larger amplitudes for the antennas closer to the electron compared
3800 to those which are further away. The amplitudes of the signals received by the array
from an electron located at a series of radial positions are shown in Figure 5.31.

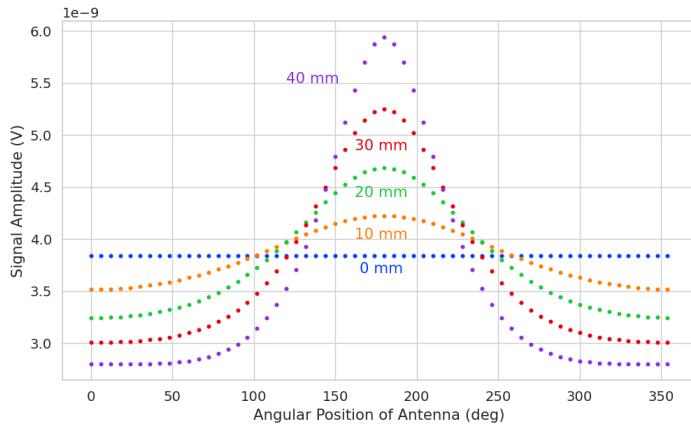


Figure 5.31: The amplitude of the signals from CREsana for the FSCD array from a 90° electron. As the electron is moved from $R = 0$ the signals begin to have unequal amplitudes depending on the distance from the electron to the antenna.

3801
3802 One expects to see a similar trend in the signal magnitudes in both the FSCD and
3803 synthetic arrays. The normalized signal magnitudes extracted from the full and synthetic
3804 array setups for a series of radial SYNCA positions are shown in Figure 5.32. The data
3805 corresponds to a SYNCA axial position of $z = 0$ mm and at a frequency 25.86 GHz. One
3806 complication is that the radiation pattern of the SYNCA is not perfectly omni-directional,
3807 which causes the measured magnitudes at $R = 0$ mm to diverge from the perfectly flat
3808 behavior exhibited by electrons.

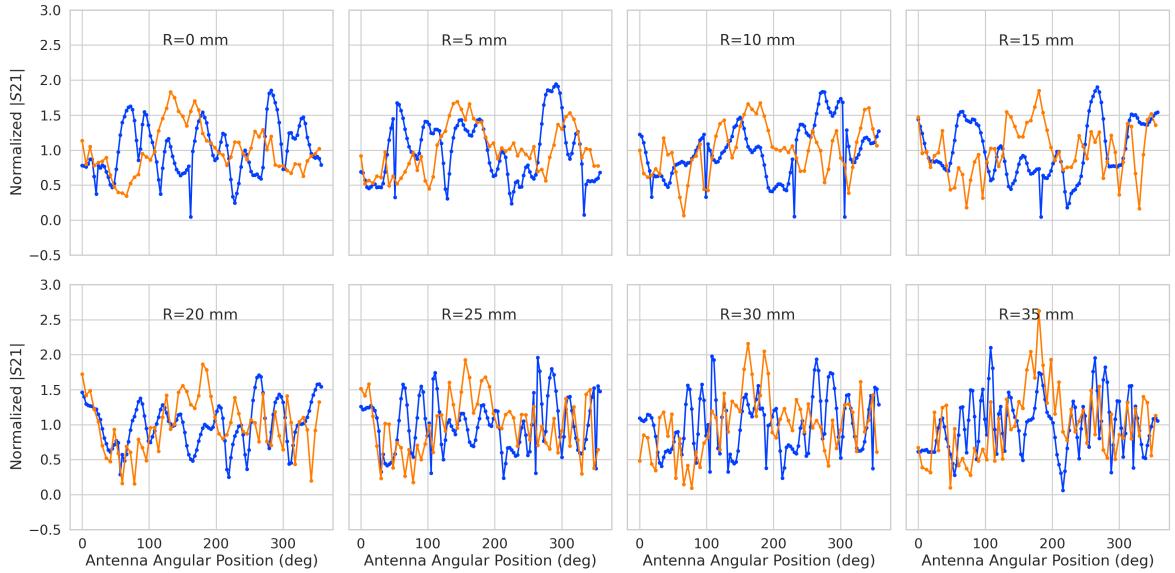
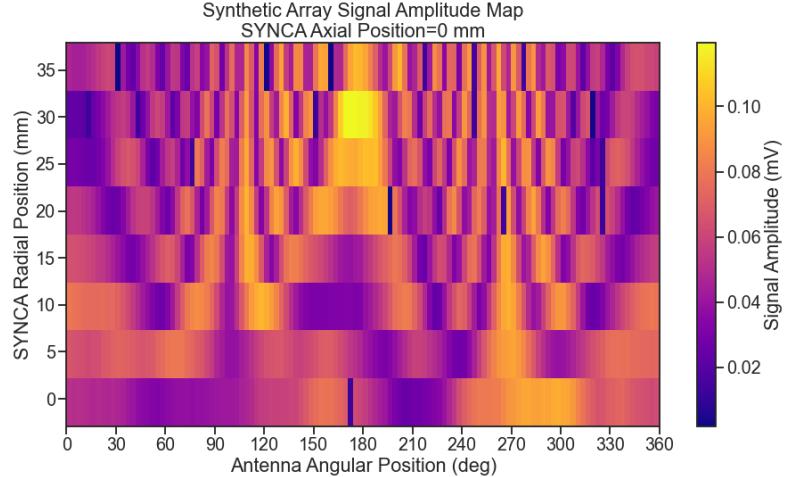


Figure 5.32: The normalized magnitudes of the S21 parameters measured in the FSCD (orange) and synthetic array (blue) setups. The dominant observed behavior as a function of radius is the increase in the number of magnitude peaks, which was noted in the phase error curves. There does not appear to be a strong change in the relative amplitude of a group of antennas as predicted by CREsana.

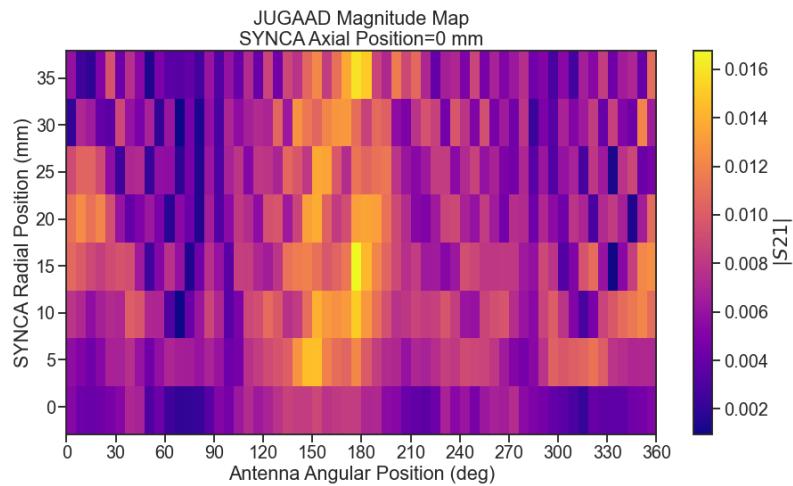
As the SYNCA is moved off-axis one observes a similar increase in the number of magnitude peaks in the synthetic array data that one would expect from a diffraction pattern, although this trend is not as stark compared to the phase data. Noticeably, there does not appear to be a set of channels with disproportionately larger amplitude at large R , which would be expected based on the trends from CREsana.

Comparing the magnitudes of the synthetic array to the FSCD array in Figure 5.32 we see that there is a similar amount of variability in the magnitudes at $R = 0$ mm, although there is potentially more small scale error in the magnitude curve caused by channel differences in the FSCD array. We observe a similar trend in the number of magnitude error peaks in the FSCD array data to the synthetic array data, which mirrors the diffraction effect observed in the phase data. The diffraction effect can be visualized more clearly by plotting a similar two-dimensional map of the magnitudes (see Figure 5.33).

The fact that one observes a similar diffraction pattern in the signal magnitudes as a function the SYNCA position reinforces the conclusions from the phase analysis that near-field effects are having a significant impact on the radiation pattern of the FSCD array. These near-field effects lead to changes in the magnitude and phase of the



(a)



(b) The two-dimensional maps showing the diffractive pattern exhibited by the FSCD and synthetic array signal magnitudes.

Figure 5.33

radiation pattern of the FSCD antenna as a function of distance. If left uncorrected these errors reduce detection efficiency by causing power loss in the beamforming or matched filter reconstruction due to phase mismatch. We explore the impact of these phase and magnitude errors on beamforming in the next section.

5.4.3.4 Beamforming Characterization

Errors in the signal magnitudes and phases lead to errors in signal reconstruction. For example, a matched filter reconstruction requires accurate knowledge of the signals in

3833 each channel to achieve optimal performance. Uncorrected errors leads to mismatches
 3834 between the template and signal, which reduces detection efficiency and introduces
 3835 uncertainty in the parameter estimation. In this section, we analyze the beamformed
 3836 signal amplitude as a function of the position of the SYNCA to quantify the impact of
 3837 the phase and magnitude errors on signal reconstruction. Because of the imperfections
 3838 in the SYNCA source, it is inappropriate to directly compare the beamformed signal
 3839 amplitude of the FSCD array or synthetic array. Such a comparison would not allow
 3840 one to disentangle losses that occur because of the antenna array from those that occur
 3841 because of the source. Therefore, we focus on comparing the beamforming of the FSCD
 3842 array to the synthetic array.

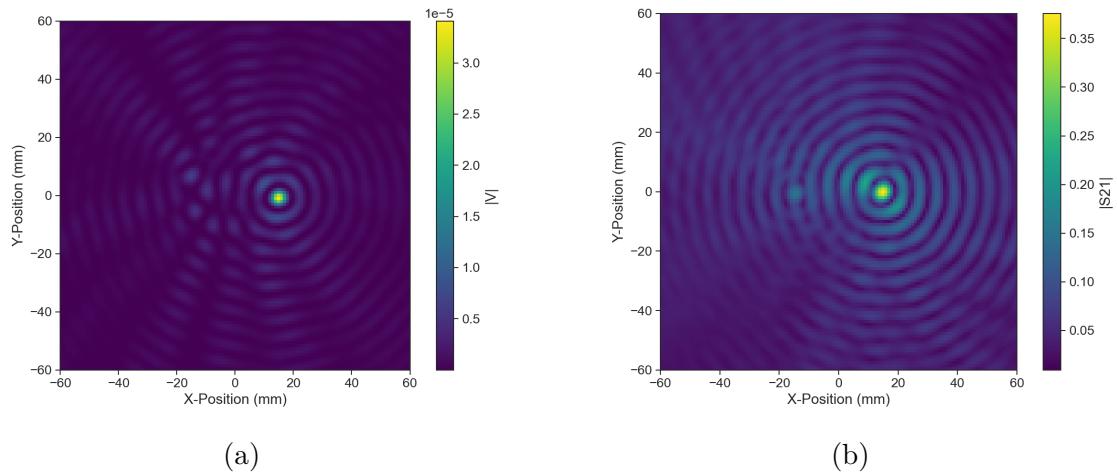


Figure 5.34: Beamforming images from the synthetic array (a) and FSCD array (b) setups with the SYNCA positioned 15 mm off the central axis. In both images we see a clear maxima that corresponds to the true SYNCA position. However, in the FSCD array there is an additional faint peak located at the opposite position of the beamforming maximum. This additional peak is the mirror of the true peak and is the result of reflections between antennas in the FSCD array.

3843 The first method of comparison is to analyze the images generated by applying the
 3844 beamforming reconstruction specified in Section 4.3.1 to the FSCD and synthetic array
 3845 data (see Figure 5.34). The beamforming grid consisting of a square 121×121 grid
 3846 spanning a range of -60-mm to 60 mm in the x and y dimensions. The beamforming
 3847 images formed from the synthetic array produces a three-dimensional matrix where each
 3848 grid position contains a summed time series. A single beamforming image is formed from
 3849 this data matrix by taking the mean over the time dimension. In the case of the FSCD
 3850 array, the VNA generates frequency domain data such that each grid position contains a
 3851 summed frequency series produced by the VNA sweep. For this data a single image is

3852 formed by averaging in the frequency domain.

3853 There is a clear difference between the synthetic and FSCD array beamforming images,
 3854 which is the additional faint beamforming maxima located directly opposite the maxima
 3855 corresponding to the SYNCA position. The images in Figure 5.34 were generated with
 3856 data collected at a SYNCA radial position of 15 mm, which agrees well with the observed
 3857 beamforming maximum in both images. We observe that the faint beamforming peak is
 3858 located directly opposite of the true beamforming maximum similar to a mirror image.
 3859 Therefore, the origin of this additional feature appears to be reflections between the two
 3860 sides of the circular antenna array that are not present for the synthetic array since only
 3861 a single physical antenna is used.

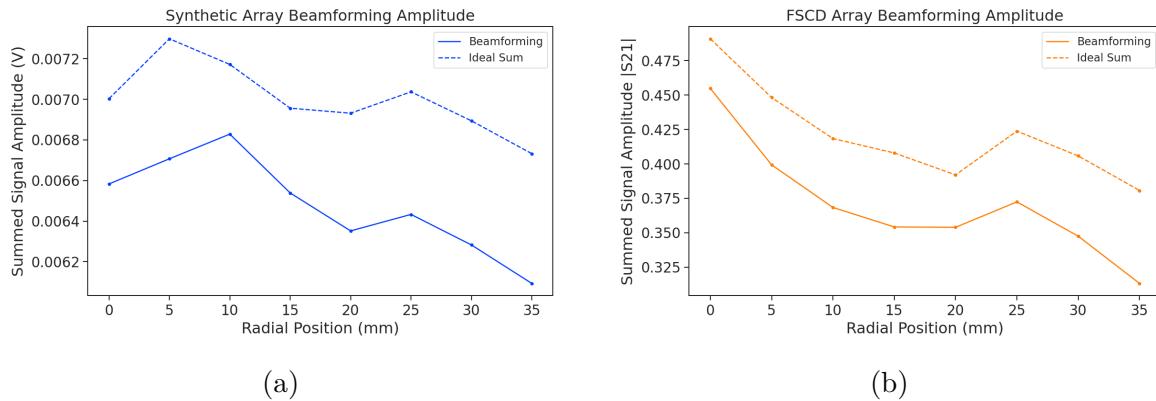


Figure 5.35: A comparison of the maximum signal amplitude obtained by beamforming to the signal amplitude obtained with an ideal summation as a function of the radial position of the SYNCA. The amplitudes for the synthetic array are shown in (a) and the FSCD array are shown in (b). In both setups we observe that the signal amplitudes obtained from beamforming are smaller than the signal amplitude that could be attained with the ideal summation without phase mismatch.

3862 From the beamforming images we extract the maximum amplitude, which we plot
 3863 as a function of the radial position of the SYNCA (see Figure 5.35). The phase errors
 3864 we observed in the FSCD and synthetic arrays leads to power loss at the beamforming
 3865 stage due to phase mismatches between the signals at different channels. This power
 3866 loss can be quantified by comparing the signal amplitude obtained from beamforming to
 3867 the amplitude which would be obtained from an ideal summation. We perform the ideal
 3868 summation by phase shifting each array channel to the same phase and then summing.
 3869 The comparison between the beamforming and ideal sums is shown in Figure 5.35, where
 3870 we observe that both the synthetic and FSCD arrays experience power losses from the
 3871 beamforming summation.

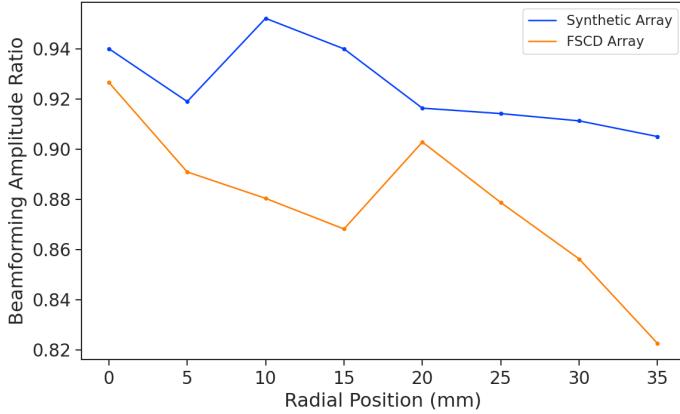


Figure 5.36: The ratio of the beamforming signal amplitude to the ideal signal amplitude for the FSCD and synthetic arrays. We see that the FSCD array has a larger power loss from phase error compare to the synthetic array which indicates that calibration errors associated with the multiple channels as well as reflections are impacting the signal reconstruction.

3872 The beamforming power loss can be quantified using the ratio of the beamforming to
 3873 ideal signal amplitudes. Computing this ratio as a function of SYNCA radial position
 3874 radius for the FSCD and synthetic arrays we find that the FSCD array has a uniformly
 3875 smaller beamforming amplitude ratio, which means that the FSCD array has a larger
 3876 beamforming power loss (see Figure 5.36). The primary contributions to the beamforming
 3877 power loss in the synthetic array are phase errors from the SYNCA and phase errors
 3878 from the FSCD antenna near-field. Both of these phase errors contribute to beamforming
 3879 losses in the FSCD array, but there are clearly additional phase errors in the FSCD array
 3880 measurements contributing to the smaller ratio. Two potential error sources include phase
 3881 differences in the different antenna channels that could not be corrected by calibration as
 3882 well as reflections between antennas in the array. The total effect of these additional phase
 3883 errors is to reduce the beamforming amplitude ratio by about 5% from the beamforming
 3884 ratio of the synthetic array. Therefore, we estimate that if no effort is made to correct
 3885 these phase errors in an FSCD-like experiment, then we expect approximately a 10%
 3886 total signal amplitude loss from a beamforming signal reconstruction.

3887 **5.4.4 Conclusions**

3888 The estimated power loss of a beamforming reconstruction obtained from this analysis
 3889 provides valuable inputs to sensitivity calculations of a FSCD-like antenna array exper-
 3890 iment to measure the neutrino mass, since it helps to bound systematic uncertainties

3891 from the antenna array and reconstruction pipeline. This power loss lowers the estimated
3892 detection efficiency of the experiment since some of the signal power is lost due to
3893 improper combining between channels and also increases the uncertainty in the electron's
3894 kinetic energy by contributing to errors in the estimation of the electron's cyclotron
3895 frequency.

3896 If these reconstruction losses prove unacceptable there are steps that can be taken
3897 to mitigate their effects. Some examples include the development of a more accurate
3898 antenna simulation approach that can reproduce the observed near-field interference
3899 patterns of the FSCD antennas and the implementation of a calibration approach that
3900 allows for the relative phase delays of the array to be measured without changing or
3901 disconnecting the antenna array configuration.

Chapter 6

Development of Resonant Cavities for Large Volume CRES Measurements

6.1 Introduction

The cavity approach was originally an alternative CRES measurement technology under consideration by the Project 8 collaboration for the Phase IV experiment. After pursuing an antenna array based CRES demonstrator design for several years the increasing costs and complexity of the antenna arrays led to a reconsideration of the baseline technology for the ultimate CRES experiment planned by Project 8. Currently, a cavity based CRES experiment is the preferred technology choice for future experiments by the Project 8 collaboration including the Phase IV experiment.

In this chapter I provide a brief summary of resonant cavities and sketch out the key features of a cavity based CRES experiment. In Section 6.2 I provide a brief introduction to cylindrical resonant cavities and the solutions for the electromagnetic fields in the cavity volume.

In Section 6.3 I describe the main components of a cavity based CRES experiment, including the background and trap magnets, cavity geometry and design, and cavity coupling considerations. I also discuss some relevant trade-offs between an antenna array and cavity CRES experiment, and highlight some reasons for the transition of Project 8 to the development of a cavity based experiment.

Finally, in Sections 6.4 and 6.5, I present the design and development of an open mode-filtered cavity that could be used in a cavity based CRES experiment with atomic tritium. The results of the cavity simulations are confirmed by laboratory measurements of a proof-of-principle prototype that demonstrates key features of the design.

3926 6.2 Cylindrical Resonant Cavities

3927 Resonant cavities are sealed conductive containers, which allows us to describe the
3928 electromagnetic (EM) fields contained in the cavity volume as a superposition of resonant
3929 modes [36]. The field shapes of the resonant modes are determined by Maxwell's equations
3930 and the boundary conditions enforced by the cavity geometry. Of interest to Project 8
3931 for CRES measurements are cylindrical cavities due to their ease of construction and
3932 integration with atom and electron trapping magnets.

3933 6.2.1 General Field Solutions

3934 Consider a long segment of conducting material with a cylindrical cross-section (see
3935 Figure 6.1). A geometry such as this can be used as a waveguide transmission line to
3936 transfer EM energy from point to point, or, if conducting shorts are inserted on both
3937 ends of the cylinder, the waveguide becomes a resonant cavity.

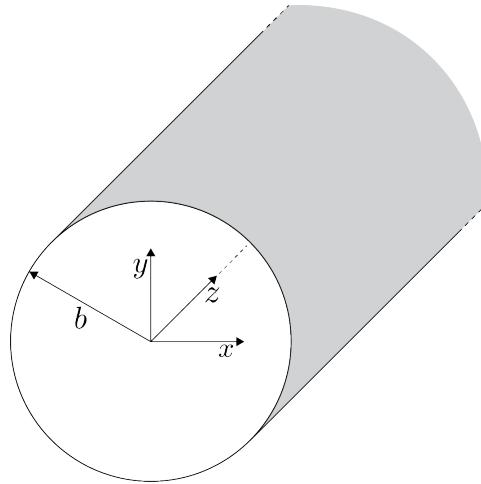


Figure 6.1: Geometry of a cylindrical waveguide with radius b .

3938 The fields allowed inside a cylindrical cavity are determined by the boundary conditions
3939 of the cylindrical geometry. The general approach to solving the fields begins by assuming
3940 solutions to Maxwell's equations of the form

$$\mathbf{E}(x, y, z) = (\mathbf{e}(x, y) + \hat{z}e_z(x, y))e^{-i\beta z}, \quad (6.1)$$

$$\mathbf{H}(x, y, z) = (\mathbf{h}(x, y) + \hat{z}h_z(x, y))e^{-i\beta z}. \quad (6.2)$$

3941 The solutions assume a harmonic time dependence of the form $e^{i\omega t}$ and propagation

3942 along the positive z-axis. The functions $\mathbf{e}(x, y)$ and $\mathbf{h}(x, y)$ represent the transverse
 3943 (\hat{x}, \hat{y}) components of the electric and magnetic fields respectively, and $e_z(x, y)$, $h_z(x, y)$
 3944 represent the longitudinal components. The version of Maxwell's equations in the case
 3945 where there are no source terms can be written as a pair of coupled differential equations,

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \quad (6.3)$$

$$\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E}, \quad (6.4)$$

3946 where ϵ and μ are the permittivity and permeability of the material inside the waveguide
 3947 or cavity. Using the field solutions from Equations 6.1 and 6.2 one can solve for the
 3948 transverse components of the fields in terms of the longitudinal fields. Because we
 3949 are interested in cylindrical cavities it is advantageous to write the field solutions in
 3950 cylindrical coordinates. After performing this transformation the set of four equations
 3951 for the transverse field components are,

$$H_\rho = \frac{i}{k_c^2} \left(\frac{\omega\epsilon}{\rho} \frac{\partial E_z}{\partial\phi} - \beta \frac{\partial H_z}{\partial\rho} \right), \quad (6.5)$$

$$H_\phi = \frac{-i}{k_c^2} \left(\omega\epsilon \frac{\partial E_z}{\partial\rho} + \frac{\beta}{\rho} \frac{\partial H_z}{\partial\phi} \right), \quad (6.6)$$

$$E_\rho = \frac{-i}{k_c^2} \left(\beta \frac{\partial E_z}{\partial\rho} + \frac{\omega\mu}{\rho} \frac{\partial H_z}{\partial\phi} \right), \quad (6.7)$$

$$E_\phi = \frac{i}{k_c^2} \left(-\beta \frac{\partial E_z}{\partial\phi} + \omega\mu \frac{\partial H_z}{\partial\rho} \right), \quad (6.8)$$

3952 where k_c is the cutoff wavenumber defined by $k_c^2 = k^2 - \beta^2$ with $k = \omega\sqrt{\mu\epsilon}$ being the
 3953 wavenumber of the EM radiation.

3954 This set of equations can be used to solve for a variety of different modes that can be
 3955 obtained by setting conditions on E_z and H_z . For cylindrical cavities two types of modes
 3956 are allowed, which correspond to solutions where $E_z = 0$ and $H_z = 0$ respectively.

3957 6.2.2 TE and TM Modes

3958 The TE family of modes corresponds to the case where $E_z = 0$. This implies that H_z is
 3959 a solution to the Helmholtz wave equation

$$(\nabla^2 + k^2)H_z = 0. \quad (6.9)$$

3960 For solutions of the form $H_z(\rho, \phi, z) = h_z(\rho, \phi)e^{-i\beta z}$, Equation 6.9 can be solved using
 3961 the standard technique of separation of variables. Rather than reproduce the derivation
 3962 here we shall simply quote the solutions for the transverse fields [36], which are

$$H_\rho = \frac{-i\beta}{k_{c_{nm}}} (A \sin n\phi + B \cos n\phi) J'_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.10)$$

$$H_\phi = \frac{-i\beta n}{k_{c_{nm}}^2 \rho} (A \cos n\phi - B \sin n\phi) J_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.11)$$

$$E_\rho = \frac{-i\omega\mu n}{k_{c_{nm}}^2 \rho} (A \cos n\phi - B \sin n\phi) J_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.12)$$

$$E_\phi = \frac{i\omega\mu}{k_{c_{nm}}} (A \sin n\phi + B \cos n\phi) J'_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}. \quad (6.13)$$

3963 One can observe that the solutions have a periodic dependence on ϕ , and radial profiles
 3964 given by the Bessel functions of the first kind. The integer indices n and m arise from
 3965 continuity conditions on the EM fields in the azimuthal and radial directions. For the
 3966 TE modes $n \geq 0$ and $m \geq 1$. $k_{c_{nm}}$ is the cutoff wavenumber for the TE_{nm} mode given by

$$k_{c_{nm}} = \frac{p'_{nm}}{b}, \quad (6.14)$$

3967 where b is the radius of the cavity or waveguide and p'_{nm} is the m -th root of the derivative
 3968 of the n -th order Bessel function (see Table 6.1).

Table 6.1: A table of the values of p'_{nm} .

| n | p'_{n1} | p'_{n2} | p'_{n3} |
|-----|-----------|-----------|-----------|
| 0 | 3.832 | 7.016 | 10.174 |
| 1 | 1.841 | 5.331 | 8.536 |
| 2 | 3.054 | 6.706 | 9.970 |

3969 The TM mode family corresponds to the case where $H_z = 0$, and $(\nabla^2 + k^2)E_z = 0$.
 3970 Again, we assume solutions of the form $E_z(\rho, \phi, z) = e_z(\rho, \phi)e^{-i\beta z}$, for which the general
 3971 form of the solutions is the same as for the TE modes. However, the different boundary
 3972 conditions for the TM modes results in particular solutions with a different form, which
 3973 we shall quote here without derivation. The transverse fields of the TM modes are given
 3974 by

$$H_\rho = \frac{-i\omega\epsilon n}{k_{c_{nm}}^2 \rho} (A \cos n\phi - B \sin n\phi) J_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.15)$$

$$H_\phi = \frac{-i\omega\epsilon}{k_{c_{nm}}}(A \sin n\phi + B \cos n\phi) J'_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z} \quad (6.16)$$

$$E_\rho = \frac{-i\beta}{k_{c_{nm}}}(A \sin n\phi + B \cos n\phi) J'_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.17)$$

$$E_\phi = \frac{-i\beta n}{k_{c_{nm}}^2 \rho}(A \cos n\phi - B \sin n\phi) J_n(k_{c_{nm}}\rho) e^{-i\beta_{nm}z}, \quad (6.18)$$

which one may notice are the same solutions as the TE modes with H and E flipped.
 The cutoff wavenumber for the TM modes is given by, $k_{c_{nm}} = p_{nm}/b$, where the values of
 p_{nm} correspond to the m -th zero of the n -th order Bessel function (see Table 6.2).

Table 6.2: A table of the values of p_{nm} .

| n | p_{n1} | p_{n2} | p_{n3} |
|-----|----------|----------|----------|
| 0 | 2.405 | 5.520 | 8.654 |
| 1 | 3.832 | 7.016 | 10.174 |
| 2 | 5.135 | 8.417 | 11.620 |

6.2.3 Resonant Frequencies of a Cylindrical Cavity

A cylindrical cavity is constructed by taking a section of cylindrical waveguide and shorting both ends with conductive material. This means that the electric fields inside a cylindrical cavity are exactly those we derived in Section 6.2.2 with the additional condition that the electric fields must go to zero at $z = 0$ and $z = L$ (see Figure 6.2).

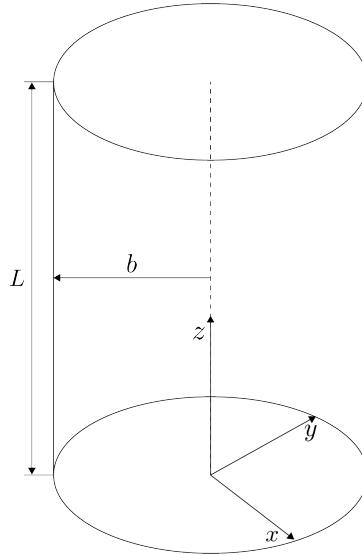


Figure 6.2: The geometry of a cylindrical cavity with length L and radius b .

3982

3983 The transverse electric field solutions for a cylindrical waveguide are of the form

$$\mathbf{E}(\rho, \phi, z) = \mathbf{e}(\rho, \phi) (A_+ e^{-i\beta_{nm}z} + A_- e^{i\beta_{nm}z}), \quad (6.19)$$

3984 where A_+ and A_- are arbitrary amplitudes of forward and backward propagating waves.

3985 In order to enforce that \mathbf{E} is zero at both ends of the cavity we require that

$$\beta_{nm}L = 2\pi\ell, \quad (6.20)$$

3986 where $\ell = 0, 1, 2, 3, \dots$. Using this constraint on the propagation constant we can solve

3987 for the resonant frequencies of the TE_{nml} and the TM_{nml} modes in a cylindrical cavity.

3988 For the TE modes the resonant frequencies are

$$f_{nml} = \frac{c}{2\pi\sqrt{\mu_r\epsilon_r}} \sqrt{\left(\frac{p'_{nm}}{b}\right)^2 + \left(\frac{\ell\pi}{L}\right)^2}, \quad (6.21)$$

3989 and the frequencies of the TM modes are

$$f_{nml} = \frac{c}{2\pi\sqrt{\mu_r\epsilon_r}} \sqrt{\left(\frac{p_{nm}}{b}\right)^2 + \left(\frac{\ell\pi}{L}\right)^2}. \quad (6.22)$$

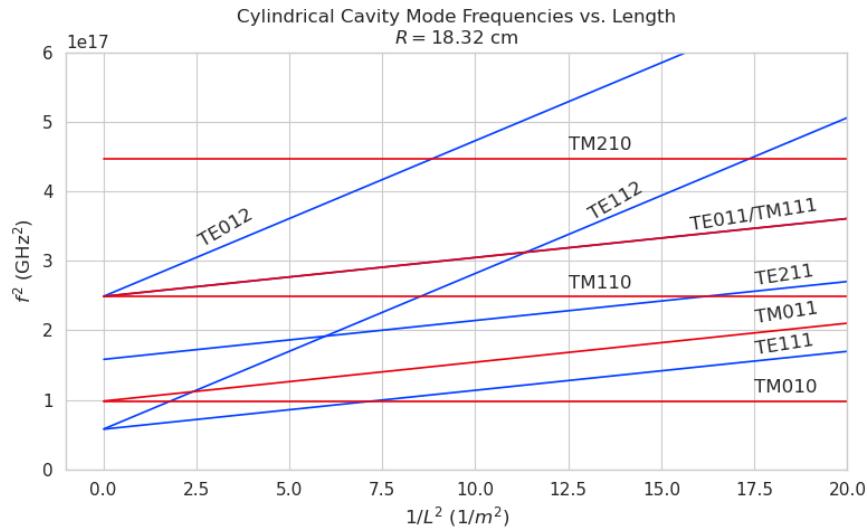


Figure 6.3: Relation of mode frequency to cavity length for a cylindrical cavity with a radius of 18.32 cm.

3990 6.2.4 Cavity Q-factors

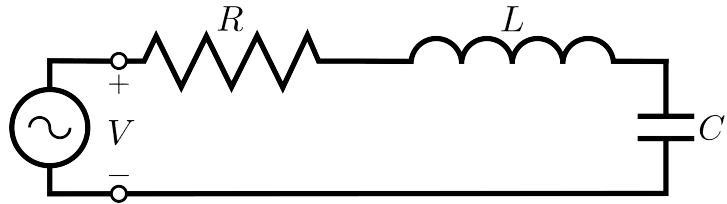


Figure 6.4: A series RLC circuit.

3991 The resonant behavior of cylindrical cavities can be modeled as a series RLC circuit
 3992 (see figure 6.4). The input impedance of the circuit can be obtained by applying
 3993 Kirchhoff's laws to calculate the impedance of the equivalent circuit. For a series RLC
 3994 circuit the input impedance is

$$Z_{\text{in}} = \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C \right). \quad (6.23)$$

3995 The resistance in the circuit represents all sources of loss in the cavity, which is primarily
 3996 caused by the finite conductivity of the cavity walls. The inductor and capacitor represent
 3997 the energy stored in the cavity in the form of electric and magnetic fields. If the circuit
 3998 is being driven by an external power source we can write the input power in terms of the
 3999 circuit input impedance and the source voltage

$$P_{\text{in}} = \frac{1}{2} Z_{\text{in}} |I|^2 = \frac{1}{2} |I|^2 \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C \right). \quad (6.24)$$

4000 The resistor introduces a loss into the system with a power given by

$$P_{\text{loss}} = \frac{1}{2} |I|^2 R, \quad (6.25)$$

4001 and the capacitor and inductor store energies given by

$$W_e = \frac{1}{4} \frac{|I|^2}{\omega^2 C}, \quad (6.26)$$

$$W_m = \frac{1}{4} |I|^2 L, \quad (6.27)$$

4002 respectively. Using these expressions we can write the input power and input impedance

4003 expressions in terms of the lost power and stored energy

$$P_{\text{in}} = P_{\text{loss}} + 2i\omega(W_m - W_e), \quad (6.28)$$

$$Z_{\text{in}} = \frac{P_{\text{loss}} + 2i\omega(W_m - W_e)}{\frac{1}{2}|I|^2}. \quad (6.29)$$

4004 The condition for resonance in the RLC circuit is that the stored magnetic energy
 4005 is equal to the stored electric energy ($W_e = W_m$). When this occurs $Z_{\text{in}} = R$, which is a
 4006 purely real impedance, and $P_{\text{in}} = P_{\text{loss}}$. The resonant frequency of the circuit can be
 4007 determined from the condition $W_e = W_m$ from which one finds that

$$\omega_0 = \frac{1}{\sqrt{LC}}. \quad (6.30)$$

4008 An important performance parameter for any resonant system is the Q-factor, which
 4009 quantifies the quality of the resonator as the ratio of the stored energy multiplied by the
 4010 resonant frequency to the average energy lost per second. For the series RLC circuit, the
 4011 Q-factor is given by the expression

$$Q_0 = \omega \frac{W_e + W_m}{P_{\text{loss}}} = \frac{1}{\omega_0 RC}, \quad (6.31)$$

4012 from which one observes that as the resistance of the RLC circuit is decreased the quality
 4013 factor of the resonator increases. From the perspective of cylindrical cavities this implies
 4014 that as one decreases the resistance of the cavity walls it is expected that the Q-factor of
 4015 the cavity should increase, which is indeed the case. In certain applications where a high
 4016 Q is desireable it is possible to manufacture a cavity out of superconducting materials in
 4017 order to minimize the power losses of the system.

4018 The Q-factor of the resonator also determines with bandwidth (BW) of the system.
 4019 A cavity with a high Q-factor will resonant with a smaller range of frequencies than a
 4020 cavity with a low Q-factor. To see this we can examine the behavior of the RLC circuit
 4021 when driven by frequencies near the resonance. For a frequency $\omega = \omega_0 + \Delta\omega$, where
 4022 $\Delta\omega = \omega - \omega_0 \ll \omega_0$, we can write the input impedance as

$$Z_{\text{in}} = R + i\omega L \left(\frac{\omega^2 - \omega_0^2}{\omega^2} \right), \quad (6.32)$$

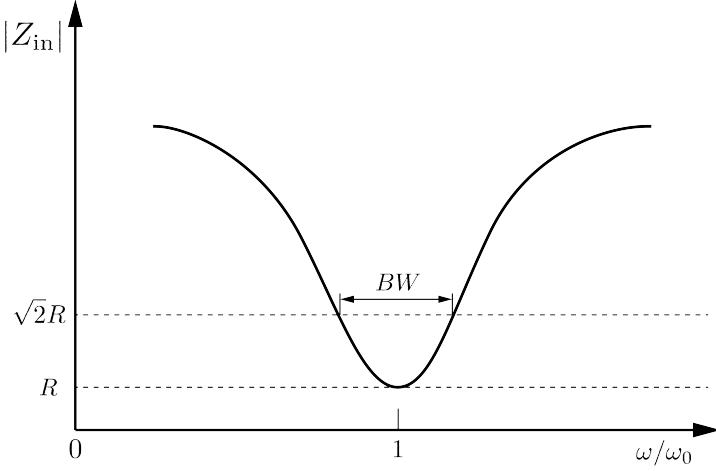


Figure 6.5: Illustration of the behavior of the input impedance of the series RLC circuit as a function of the driving frequency. The BW is proportion to the width of the resonance, which is inversely proportional to Q.

and by expanding $(\omega^2 - \omega_0^2)/\omega^2$ to first order in $\Delta\omega$, we obtain

$$Z_{\text{in}} \approx R + i \frac{2RQ_0\Delta\omega}{\omega_0}. \quad (6.33)$$

Therefore, the magnitude of the input impedance near the resonance is given by

$$|Z_{\text{in}}| = R \sqrt{1 + 4Q_0^2 \frac{\Delta\omega^2}{\omega^2}}, \quad (6.34)$$

from which we observe that for the series RLC circuit the input impedance is minimized at the resonant frequency, which corresponds to the maximum input power (see Figure 6.5). The half-power BW is the range of frequencies over which the input power drops to half the input power on resonance. This occurs when $|Z_{\text{in}}| = \sqrt{2}R$, which corresponds to $\Delta\omega/\omega = \text{BW}/2$. Using Equation 6.34 one can find that

$$2R^2 = R^2(1 + Q_0^2\text{BW}^2), \quad (6.35)$$

which implies

$$\text{BW} = \frac{1}{Q_0} \quad (6.36)$$

It is important to emphasize that the Q-factor defined here, Q_0 , is technically the unloaded Q. It reflects the quality of the cavity or resonant circuit without the influence of any external circuitry. In practice, however, a cavity is invariably coupled to an

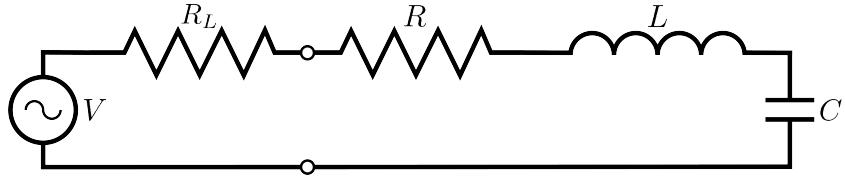


Figure 6.6: A series RLC circuit coupled to an external circuit with input impedance R_L .

4034 external circuit to drive a cavity resonance or to measure the energy of a resonant mode.
 4035 Coupling a cavity to an external circuit changes the Q by loading the equivalent cavity
 4036 RLC circuit (see Figure 6.6). The Q-factor of the cavity when it is loaded by an external
 4037 circuit is called the loaded Q, which is the quantity that one actually measures when
 4038 exciting a resonance in the cavity. Using the series RLC circuit model one can see that
 4039 the load resistor in Figure 6.6 will add in series with the resistor in the circuit for a total
 4040 equivalent resistance of $R + R_L$. Therefore, the loaded Q is given by

$$Q_L = \frac{1}{\omega_0(R + R_L)C}, \quad (6.37)$$

4041 from which one observes that the loaded Q is always less than the intrinsic Q of the
 4042 cavity.

4043 The amount of coupling that is desireable depends on the specific application of
 4044 the resonator. If one wants a resonator that is particular frequency selective than it
 4045 makes sense to limit the amount of coupling to the cavity to maintain a small BW,
 4046 alternatively, if a larger BW is need one can increase the cavity coupling by tuning the
 4047 input impedance of the external circuit. The critical point, where maximum power is
 4048 transferred between the cavity and the external circuit, occurs when the input impedance
 4049 of the cavity matches the input impedance of the external transmission line. For the
 4050 series RLC circuit on resonance, this matching condition corresponds to

$$Z_0 = Z_{in} = R, \quad (6.38)$$

4051 where Z_0 is the impedance of the transmission line. The loaded Q at this critical point
 4052 is, therefore,

$$Q_L = \frac{1}{2\omega_0 Z_0 C} = \frac{Q_0}{2}. \quad (6.39)$$

4053 One can described the degree of coupling between the cavity and an external circuit by

4054 defining a coupling factor, g , such that,

$$g = \frac{Q_0}{Q_L} - 1. \quad (6.40)$$

4055 When $g = 1$ then $Q_L = Q_0/2$, and the cavity is said to be critically coupled as we
4056 described. If $Q_L < Q_0/2$, then the cavity is undercoupled to the transmission line,
4057 corresponding to $g < 1$. Alternatively, if $Q_L > Q_0/2$, then $g > 1$, and the cavity is
4058 overcoupled to the transmission line. Various specialized circuits can be used to tune the
4059 input impedance of the external circuit as seen by the cavity to achieve a wide range of
4060 different coupling factors based on the desired application of the cavity.

4061 6.3 The Cavity Approach to CRES

4062 6.3.1 A Sketch of a Molecular Tritium Cavity CRES Experiment

4063 Resonant cavities can be used to perform CRES measurements, and they represent the
4064 current preferred technology by the Project 8 collaboration. The basic approach to a
4065 neutrino mass measurement using a resonant cavity and molecular tritium beta-decay
source is illustrated by Figure 6.7.

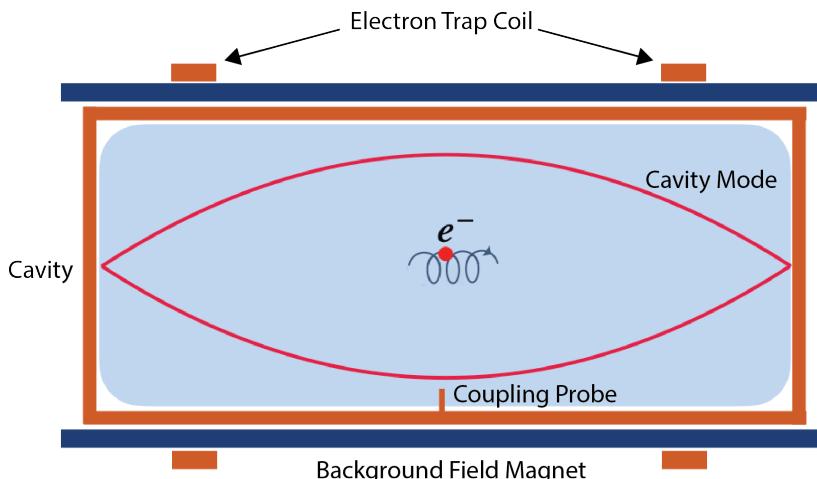


Figure 6.7: A cartoon depiction of a cavity CRES experiment. A metallic cavity filled with tritium gas is inserted into a uniform background magnetic field to perform CRES measurements. Electrons from beta-decays inside the cavity can be trapped and used to excite a resonant mode(s). By coupling to the cavity mode with a suitable probe one can measure the cyclotron frequency of the electron and perform CRES.

4066

4067 At the core of the experiment is a large resonant cavity filled with tritium gas. The
4068 filled cavity is then placed in a uniform magnetic field provided by a primary magnet
4069 that provides the background magnetic field. The value of the background magnetic field
4070 sets the range of cyclotron frequencies for electrons emitted near the tritium spectrum
4071 endpoint. When a beta-decay electron is produced in the cavity it is trapped using a set
4072 of magnetic pinch coils that keep electrons inside the cavity volume.

4073 Electrons trapped inside the cavity do not radiate in the same way as electrons
4074 in free-space. Effectively, the same boundary conditions that were used to derive the
4075 resonant modes of a cylindrical cavity in Section 6.2 apply to the radiation of the electron
4076 as well. The coupling of an electron performing cyclotron motion in a cavity has been
4077 studied in detail for measurements of the electron’s magnetic moment [59–61] If an
4078 electron is emitted with a kinetic energy that corresponds to a cyclotron frequency that
4079 matches a resonant frequency of the cavity, then energy radiated by the electron excites
4080 a corresponding resonance in the cavity. The strength of the electron’s coupling to the
4081 cavity is given to first order by the dot product between the electrons trajectory and
4082 the electric field vector of the resonant mode. Additional effects, such as the Purcell
4083 enhancement [62], alter the emitted power from the free-space Larmor equation [63]. If an
4084 electron is moving with a cyclotron frequency that is far from any resonant modes in the
4085 cavity, then radiation from the electron is suppressed. One can interpret this somewhat
4086 surprising effect as the metallic walls of the cavity reflecting the radiated energy back to
4087 the electron.

4088 Detecting an electron in the cavity is accomplished by coupling the cavity to an
4089 external transmission line that leads to an amplifier and RF receiver chain [64]. The
4090 coupling of the cavity resonance to the amplifier occurs through a coupling probe or
4091 aperture designed to read-out the excitation of the mode(s) excited by the electron. For
4092 CRES measurements, the placement of a wire antenna coupling probe inside the cavity
4093 volume leads to unacceptable losses of tritium atoms due to recombination to molecular
4094 tritium on the antenna surface, therefore, apertures are the preferred coupling method
4095 for cavity CRES experiments.

4096 One of the attractive features of the CRES technique for neutrino mass measurement
4097 is the gain in statistics that comes from the differential nature of the tritium spectrum
4098 measurement. Initially, this seems incompatible with cavities, due to the narrow reso-
4099 nances of cavity modes giving relatively small bandwidth. However, by intentionally
4100 over-coupling to a single cavity mode one can achieve bandwidths of a few 10’s of MHz
4101 (see Section 6.2), which is sufficient for a measurement of the tritium spectrum endpoint

4102 region.

4103 **6.3.2 Magnetic Field, Cavity Geometry, and Resonant Modes**

4104 **Magnetic Field and Volume Scaling**

4105 For a CRES experiment, cylindrical cavities are a natural choice since they match
4106 the geometry of standard solenoid magnets, which are needed in order to produce the
4107 background magnetic field for CRES measurements. Furthermore, the cylindrical shape is
4108 compatible with a Halbach array, which is the leading choice of atom trapping technology
4109 for future atomic tritium experiments by the Project 8 collaboration. Cylindrical
4110 cavities also benefit from well-established machining practices that are able to achieve
4111 high geometric precision at large lengths scales. More exotic cavity designs are under-
4112 consideration and there are on-going efforts to investigate the potential advantages these
4113 may have over the standard cylindrical geometry.

4114 As we saw in Section 6.2, the physical dimensions of the cavity are directly coupled
4115 to the resonant frequencies of the cavity. This dependency links the size of the cavity to
4116 the magnitude of the background magnetic field, because the magnetic field determines
4117 the cyclotron frequencies of trapped electrons. Specifically, as the size of the cavity is
4118 increased to accommodate larger volumes of tritium gas, the frequencies of the resonant
4119 modes decrease proportionally. This requires that the magnetic field also decrease in
4120 order to maintain coupling between electrons and the desired cavity mode.

4121 The required cavity size is ultimately determined by the required statistics in the
4122 tritium spectrum endpoint region. Because the gas density must be kept below a certain
4123 level to ensure that electrons have sufficient time to radiate before scattering, larger
4124 volumes become the only way to achieve higher event statistics. To achieve the sensitivity
4125 goals of Phase III and IV cavity volumes on the order of several cubic-meters are required,
4126 which pushes one towards frequencies in the range of 100's of MHz.

4127 **Single-mode Cavity CRES**

4128 It is tempting to consider maintaining a high magnetic field, while still increasing the size
4129 of the cavity, in order to increase the radiated power from trapped electrons for better
4130 SNR. However, if one were to maintain the same magnetic field while increasing the
4131 size of the cavity, the electrons would begin to couple to higher order modes with more
4132 complicated transverse geometries. The danger with this approach is that a complicated
4133 mode structure could introduce systematic errors into the CRES signals. Example

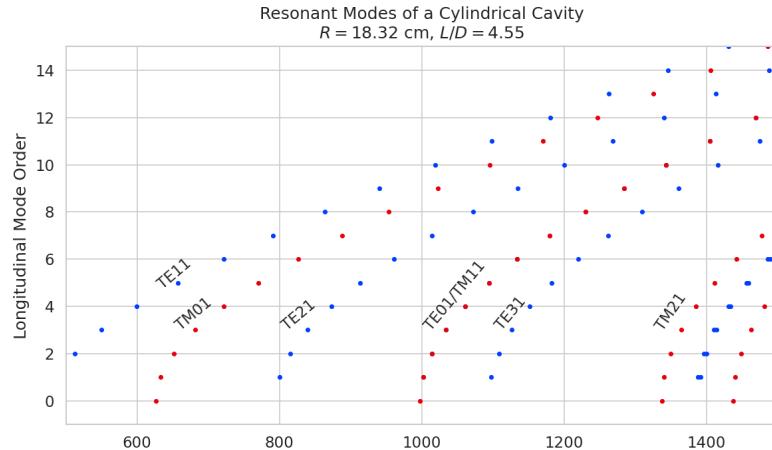
4134 systematics include unpredicted mode hybridization or changes in the mode shapes from
4135 imperfections in the cavity construction, which would prevent reconstruction of the
4136 electron's starting kinetic energies with adequate resolution. For this reason, it is ideal
4137 to operate with magnetic fields that give cyclotron frequencies near the fundamental
4138 frequency of the cavity, where the mode structure is relatively simple (see Figure 6.8).
4139 In this frequency region it is possible to perform CRES by coupling to only a single
4140 resonant mode, however, it is currently an open question if a single mode measurement
4141 will provide enough information about an individual electron's position to reconstruct
4142 the full event. Regardless, developing a solid understanding of the CRES phenomenology
4143 when an electron is coupling to a single mode will be a necessary step towards a future
4144 multi-mode cavity experiment.

4145 Considerations for Resonant Mode Selection

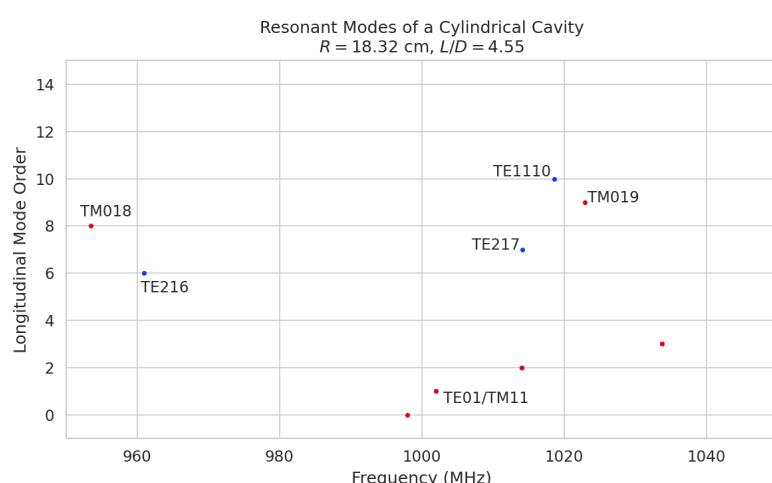
4146 A single-mode cavity experiment begs the question, which resonant mode is best for
4147 CRES measurements? There is an immediate bias towards low order TE_{nm} and TM_{nm}
4148 modes due to the multi-mode considerations discussed above. Additionally, there is a
4149 preference towards modes with longitudinal index $\ell = 1$ with a single antinode along the
4150 vertical axis of the cylindrical cavity. The reason for this is that there is a phase change
4151 in the electric fields between antinodes that leads to modulation effects that destroy the
4152 carrier frequency signal information.

4153 A second consideration for mode selection is the volumetric efficiency of the mode.
4154 Volumetric efficiency can be thought of as an integral over the volume of the cavity
4155 weighted by the relative amplitude of the mode. From the perspective of simply maximiz-
4156 ing the volume useable for CRES measurements this integral would be as close to unity
4157 as possible. However, there is a requirement to reconstruct the position of the electrons
4158 inside the cavity volume so that the local magnetic fields can be used to convert the
4159 measured cyclotron frequency to a kinetic energy. With a single mode this necessarily
4160 requires a variable transverse mode amplitude, which lowers the volumetric efficiency, so
4161 that position of the electron in the cavity can be estimated from the average amplitude
4162 of the CRES signal. Longitudinal indices of $\ell = 1$ have an advantage in volumetric
4163 efficiency over higher order ℓ modes, since there are only two longitudinal nodes, one at
4164 each end of the cavity. Therefore, the average coupling strength of trapped electrons as
4165 they oscillate axially is higher for $\ell = 1$ modes.

4166 The longitudinal variation in the mode strength is ultimately critical for achieving the
4167 energy resolution required for neutrino mass measurements. Correcting for the change in



(a)



(b)

Figure 6.8: Examples of the resonant mode frequencies of a cylindrical cavity. This cavity has a radius of 18.32 cm and a length to diameter ratio of 4.55.

4168 the average magnetic fields experienced by electrons with different pitch angles requires
 4169 that information on the axial motion of the electron be encoded into the CRES signal.
 4170 The longitudinal variation in the mode amplitude leads to amplitude modulation of the
 4171 CRES signal with a frequency proportional to the electron's pitch angle.

4172 An additional factor for mode selection is the intrinsic or unloaded Q of the mode. In
 4173 terms of SNR it is advantageous to use a mode with a very high Q_0 , which is then highly
 4174 overcoupled to achieve the necessary bandwidth to cover the tritium endpoint spectrum.
 4175 This scheme leads to a decoupling of the physical cavity temperature from the effective
 4176 noise temperature after the amplifier, which allows us to achieve adequate SNR without

4177 the requirement of cooling the entire cavity to single Kelvin temperatures.

4178 An example of a resonant mode that exhibits these traits is the TE₀₁₁ mode. At present
4179 the TE₀₁₁ mode is the preferred resonance for a single-mode cavity CRES experiment
4180 by the Project 8 collaboration. TE₀₁₁ is a low order mode located in a region relatively
4181 far from other cavity modes. Furthermore, the separation of the TE₀₁₁ mode can be
4182 improved by various mode-filtering techniques discussed in Section 6.4.2 below. TE₀₁₁
4183 consists of a single longitudinal antinode that can provide pitch angle information in the
4184 form of amplitude modulation, and has an electric field with a radial profile given by the
4185 J'_0 Bessel function allowing for radial position estimation. Lastly, the TE₀₁₁ mode has a
4186 relatively high intrinsic Q compared to nearby modes, which helps with SNR. Unloaded
4187 Q's greater than 80000 are achievable for a 1 GHz TE₀₁₁ resonance using a copper walled
4188 cavity.

4189 **6.3.3 Trade-offs Between the Antenna and Cavity Approaches**

4190 The choice between cavities and antennas for large-scale CRES measurements is not
4191 without trade-offs. Both the antenna array and cavity approaches are relatively immature
4192 techniques, at present there are no known obstacles that would prevent either approach
4193 from being used for a large scale neutrino mass experiment. The preference for cavities
4194 is largely driven by important practical considerations that could make a cavity based
4195 experiment significantly cheaper than an antenna experiment of similar size and scope.
4196 However, the switch to cavities also introduces new challenges less relevant to the
4197 antenna array, which must be solved in order for Project 8 to achieve its neutrino mass
4198 measurement goals.

4199 One of the major relative drawbacks of the antenna array approach is the size and
4200 complexity of the data-acquisition system. A large-scale antenna array experiment
4201 requires $O(100)$ antennas independently digitized at rates of $O(10)$ to $O(100)$ MHz. Since
4202 there is insufficient information in a single antenna channel to detect or reconstruct the
4203 CRES signal, the entire array output must be processed during the signal reconstruction.
4204 Because data storage becomes an issue with these data volumes, there is a real-time
4205 signal reconstruction requirement that allows one to detect CRES signals buried in the
4206 thermal noise. As we discuss in Section 4.4, the computational cost of these real-time
4207 detection algorithms are potentially quite large for even a small scale antenna array
4208 experiment. However, the operating principle of a cavity experiment allows the CRES
4209 signal to be detected using only a single read-out channel digitized at rates of $O(10)$ MHz,
4210 which reduces the cost of the data acquisition system by many orders of magnitude.

4211 From an engineering perspective, the simple geometry and thin-walls of a cylindrical
4212 cavity are simpler to interface with the cryogenic and magnetic subsystems needed for a
4213 CRES experiment. Whereas, the antenna array requires careful design and engineering
4214 to accommodate the antenna array and receiver electronics in proximity to the trapping
4215 magnets. Additionally, due to near-field interference effects, the antenna array is unable
4216 to reconstruct CRES events within the reactive near-field distance of the antennas.
4217 Because atom trapping requirements require magnetic fields which correspond to cyclotron
4218 frequencies for endpoint electrons less than 1 GHz, the required stand-off distance leads to
4219 a significant loss in useable experiment volume, necessitating larger and more expensive
4220 magnets.

4221 Another advantage to the cavity approach is the relatively compact sideband structure,
4222 which is a result of the low modulation index for cavity CRES signals. The axial motion
4223 in an antenna array experiment leads to frequency modulation and sidebands. The shape
4224 of the sideband structure is determined by the modulation index, $h = \frac{\Delta f}{f_a}$, where Δf
4225 is the size of the frequency deviation and f_a is the axial frequency. The large electron
4226 traps required for a cubic-meter-scale experiment leads to high modulation indices, which
4227 causes the signal spectrum to be made up of numerous low power sidebands that make
4228 reconstruction and detection challenging. This behavior was observed in simulations
4229 of the FSCD in which carrier power decreased with pitch angle due to the increase in
4230 modulation index (see Figure 4.31). For cavities, however, the modulation index remains
4231 near $h = 1$ even for very long magnetic traps due to the high phase velocity in cavities
4232 relative to the axial velocity of the electron. This results in an almost ideal spectrum
4233 shape that has a strong carrier frequency with a few sidebands whose relative amplitudes
4234 encode pitch angle information.

4235 A downside of the cavity approach is the apparent difficulty of estimating the position
4236 of the electron using only the coupling of the electron to a single mode. The amplitude of
4237 the TE₀₁₁ mode is completely independent of the azimuthal coordinate, therefore, position
4238 reconstruction using the TE₀₁₁ mode is only able to estimate the radial position of the
4239 electron. This position degeneracy may lead to magnetic field uniformity requirements
4240 that are too challenging to meet due to mechanical uncertainties in cavity and magnet
4241 construction, as well as uncertainties caused by nuisance external magnetic fields such
4242 as the Earth's field and magnetic fields from building materials. A multi-mode cavity
4243 experiment may provide a way to extract more precise information on the position of
4244 the electron by analyzing the coupling of the electron to several modes that overlap in
4245 different ways.

4246 6.4 Single-mode Resonant Cavity Design and Simulations

4247 The single-mode cylindrical cavities envisioned for the Phase III and IV experiments must
4248 be carefully engineered in order to measure the neutrino mass with the desired sensitivity.
4249 In this section I summarize some simulation studies performed to analyze early design
4250 concepts for a single-mode cavity. The primary tool for these investigations was Ansys
4251 HFSS, which was also used for the development of the SYNCA antenna described in
4252 Section 5.3.

4253 6.4.1 Open Cylindrical Cavities with Coaxial Terminations

4254 Design Concept

4255 A basic cavity design question relevant to Project 8’s ultimate goal of an atomic tritium
4256 CRES experiment is how to build a cavity that can be efficiently filled with atomic
4257 tritium. To keep the rate of atom loss from recombination on surfaces it is ideal if the
4258 ends of the cylindrical cavity are as open as possible so that tritium atoms can flow
4259 inside unimpeded. Additionally, one of the primary calibration techniques planned for
4260 future CRES experiments involves CRES measurements using electrons injected from
4261 an electron gun source, which also requires an opening at the cavity end. Cylindrical
4262 cavities with open ends can be manufactured, however, the intrinsic Q-factors of these
4263 cavities are orders of magnitude less than their sealed counterparts, which reduces the
4264 signal-to-noise ratio when that cavity is used for CRES measurement.

4265 Cylindrical cavities with mostly open ends that also exhibit Q values for the $TE_{01\ell}$
4266 modes similar to sealed cavities can be built by using coaxial endcaps to terminate the
4267 cavity. Cavities of this type have been manufactured for specialized applications related
4268 to the measurements of the dielectric constants of liquefied gasses (see Figure 6.9) [65, 66].
4269 This cavity design leaves the ends of the cavity wide open, but retains high Q-values for
4270 the $TE_{01\ell}$ modes due to the coaxial endcap, which are designed to perfectly reflect the
4271 electric fields of $TE_{01\ell}$ modes. Coupling to the $TE_{01\ell}$ mode is achieved via an aperture
4272 located at the center of the cavity wall.

4273 A cavity similar to Figure 6.9 is a candidate design for the future CRES experiments
4274 by Project 8, since it appears to elegantly solve many practical issues that arise when
4275 combining cavity CRES and atomic tritium. The coaxial endcaps leave significant regions
4276 of the cavity ends completely open, which allows for the entrance of atomic tritium as
4277 well as the pumping away of molecular tritium that has recombined on the cavity walls.

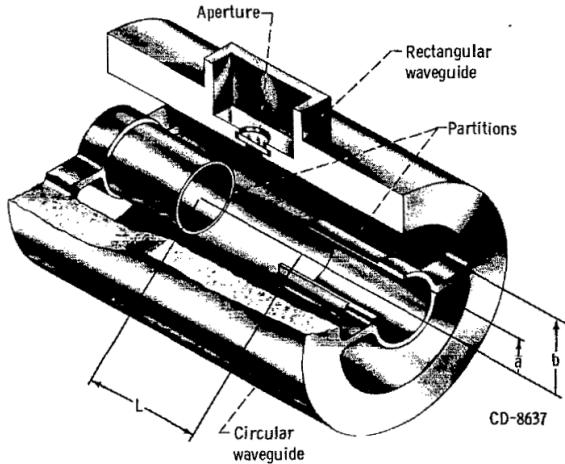


Figure 6.9: An image of an open cavity with coaxial terminations used for dielectric constant measurements. Figure from [66].

4278 These open ends are achieved while preserving the high Q-values of the $\text{TE}_{01\ell}$ modes,
 4279 which is important for extracting as much signal power from the electron as possible. In
 4280 subsequent sections we shall analyze this cavity design in more detail, primarily by using
 4281 HFSS simulations to analyze the resonant mode structure of this cavity geometry.

4282 Coaxial Terminator Constraints

4283 The reason that coaxial endcaps can be used to achieve high Q-values for the $\text{TE}_{01\ell}$
 4284 modes is that the electric fields for these modes are purely azimuthally polarized (see
 4285 Equations 6.12 and 6.13). Therefore, the boundary conditions that require the electric
 4286 field to go to zero at the cavity ends can be supplied using a coaxial partition of the
 4287 correct radius (see Figure 6.10). Because the cylindrical shape enforced by the partition
 4288 does not match the boundary conditions of other cavity modes, these terminations also
 4289 significantly suppress the Q-factors of non- $\text{TE}_{01\ell}$ modes, which is potentially beneficial
 4290 for a single-mode cavity CRES experiment.

4291 The correct radius of the cylindrical partition is derived by setting up the boundary
 4292 value problem in Figure 6.10, and analyzing the reflection and transmission coefficients
 4293 for waves incident on the coaxial terminators. The basic problem is to identify the radius
 4294 a where the reflection coefficient for the $\text{TE}_{01\ell}$ modes becomes equal to 1. One can show
 4295 that if the coaxial partitions are made sufficiently long relative to the wavelength of the
 4296 TE_{01} modes than perfect reflection can be achieved. This derivation is quite lengthy
 4297 and complex and is presented in full in [65]. Here, we shall simply explain the resulting

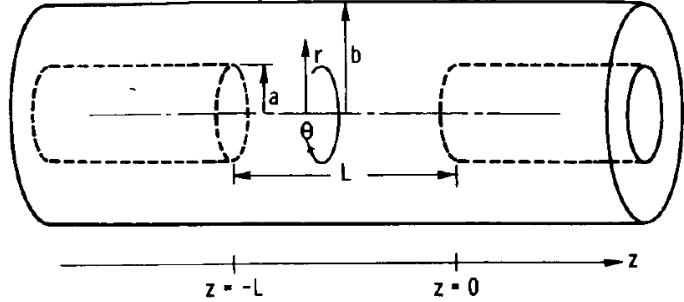


Figure 6.10: The simplified geometry of an open cavity with coaxial terminations. Figure from [65].

4298 conditions on the partition radius for perfect reflection.

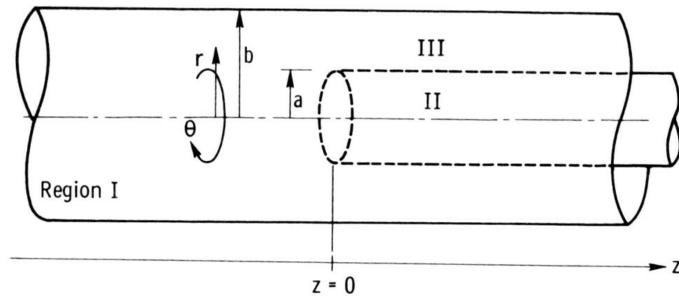


Figure 6.11: Electric field regions for the open cavity boundary value problem. Figure from [65].

4299 The open cavity boundary value problem is solved by expressing the forms of the
 4300 electric fields in the different regions of the cavity and requiring that the electric fields are
 4301 continuous. There are effectively three distinct regions in the open cavity corresponding
 4302 to the central cavity volume, the inner coaxial volume, and the outer coaxial volume (see
 4303 Figure 6.11).

4304 In Region I, the boundary conditions are those of a cylindrical waveguide, and we
 4305 require that E_ϕ for the TE_{0m} modes go to zero at the cavity wall ($r = b$). This requires
 4306 that $J'_{0m}(k_{c0m} b) = 0$. We aim to solve for the radius a in the specific situation where the
 4307 TE_{01} mode can propagate but all other TE_{0m} modes are below the cutoff frequency for
 4308 the circular waveguide. This is equivalent to requiring

$$3.832 < k_{c0m} b < 7.016, \quad (6.41)$$

4309 where the numbers 3.832 and 7.016 correspond to the first and second zeros of the Bessel

4310 function (see Table 6.1).

4311 In Region II the boundary conditions are those of a cylindrical waveguide, but with
4312 a smaller radius. The condition that $E_\phi = 0$ at the cylindrical partition radius is that
4313 $J'_{0m}(k_{c0m}a) = 0$. To ensure perfect reflection, we want all modes in Region 1 of the cavity
4314 to be below the cutoff frequency of the circular waveguide formed by the inner volume of
4315 the coaxial terminator. Therefore, we consider the solutions where

$$k_{c0m}a < 3.832. \quad (6.42)$$

4316 Finally, in Region III the boundary condition are those of a coaxial waveguide. We
4317 need to guarantee that $E_\phi = 0$ at both $r = b$ and $r = a$, which involves finding the
4318 eigenvalues of the following equation

$$J'_0(k_{c0m}a)Y'_0(k_{c0m}b) - J'_0(k_{c0m}b)Y'_0(k_{c0m}a) = 0, \quad (6.43)$$

4319 where Y'_0 the zeroth-order derivatives of the Bessel function of the second kind. The
4320 solutions to this equation depend on the value of the ratio b/a . The approximate solution
4321 is given by

$$\delta_n a \simeq \frac{n\pi}{b/a - 1}, \quad (6.44)$$

4322 where δ_n are eigenvalues of Equation 6.43. Similar to Region II, we are interested in
4323 solutions for which the TE₀₁ modes of Region I are below the cutoff of Region III.
4324 Therefore, we require that

$$k_{c0m} < \delta_1. \quad (6.45)$$

4325 In general, one has some freedom in specifying the value of b/a . A value typically used
4326 in practice is $b/a = 2.082$, which corresponds to positioning the radius of the cylindrical
4327 partition at the maxima of the TE₀₁ electrical fields.

4328 Using the constraints from the three field regions one can develop a coaxial terminator
4329 that acts as a virtual perfectly conducting surface for the TE₀₁ modes. The only required
4330 inputs are the desired frequency of the TE₀₁₁ mode and a choice for the value of b/a .

4331 **6.4.2 Mode Filtering**

4332 The general case of an electron coupling to a resonant cavity is complicated. This is
4333 because cavities contain an infinite number of resonant modes, which for higher order
4334 modes, have couplings to the electron with a complex spatial dependence. The danger is

4335 that improper modeling of the electron's coupling to the cavity can lead to systematic
4336 errors in the CRES measurements that prevent a high-resolution measurement of the
4337 electron's kinetic energy. This in part drives the preference for a single-mode cavity
4338 experiment that uses only the electron's coupling to the TE₀₁₁ mode to perform CRES,
4339 assuming that sufficient information on the electron's position can be obtained with a
4340 single mode.

4341 The TE₀₁₁ mode is in a region where there are relatively few other modes to which
4342 the electron could couple(see Figure 6.8). However, one can see that the frequency of
4343 the TE₀₁₁ is perfectly degenerate with the TM₁₁₁ mode, which means that electrons will
4344 inevitably couple to both modes if they have the correct cyclotron frequency.

4345 The magnitude of the impact of the electron coupling to both TE₀₁₁ and TM₁₁₁ is
4346 currently unknown. To first order an electron coupling to more both modes will lose more
4347 energy overtime, which can be measured by observing the frequency chirp rate of the
4348 signal. This effect may be small enough to be negligible or simple enough to model that
4349 the cavity can be treated as an effective single-mode cavity. Alternatively, the one could
4350 consider devising a coupling scheme that is sensitive to both the TE₀₁₁ and the TM₁₁₁
4351 modes. By measuring the coupling of the electron to both modes more information on
4352 the position of the electron could be obtained, which could improve the position and
4353 energy resolution of the CRES measurements.

4354 A different approach is the mode filtering approach, which seeks to obtain a single
4355 TE₀₁₁ mode cavity using perturbations to the cavity walls that selectively impede the
4356 TM modes, while leaving the TE modes mostly unperturbed. The type of perturbations
4357 required can be determined by visualizing the surface currents induced in the cavity
4358 walls by each type of mode (see Figure 6.12). By definition, all TM have electric fields
4359 directed along the vertical axis of the cylindrical cavity, which means that perturbations
4360 that impede currents in this direction will modify TM resonances. On the other hand,
4361 the TE₀₁ modes induce azimuthal currents in the cavity walls, therefore, it is possible to
4362 break the degeneracy between TE₀₁ and TM₁₁ using a cavity perturbation that impedes
4363 axial currents, but does not affect the flow of azimuthal currents.

4364 Figure 6.12 shows two cavity design concepts that achieve this selective current
4365 perturbation. The resistive approach inserts a series of thin dielectric rings into the walls
4366 of the cavity that introduces a resistive and capacitive impedance to the longitudinal
4367 currents, while leaving azimuthal current paths intact. Cavities of this type with high
4368 TE₀₁ Q's have also been constructed by tightly wrapping a thin, dielectric coated wire
4369 around a mold to form the cavity wall. An alternative method is to introduce an inductive

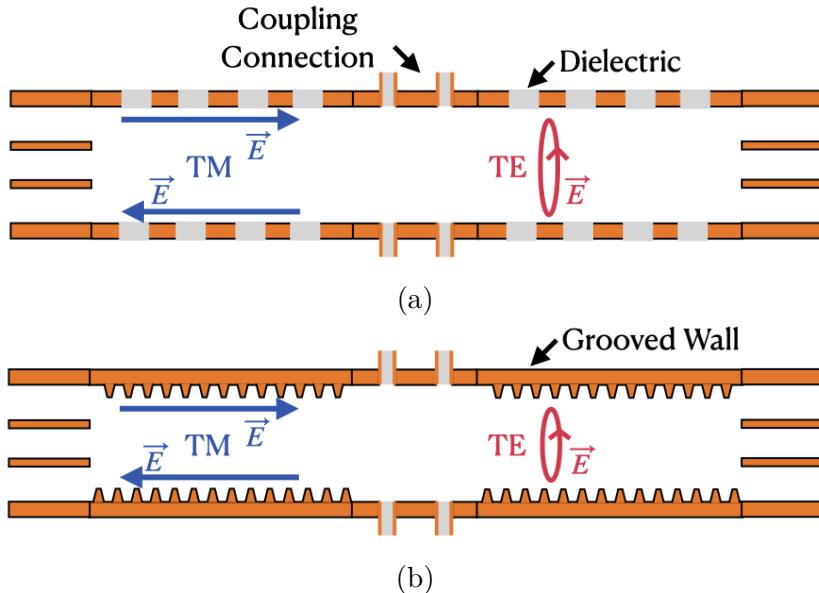


Figure 6.12: Two mode filtering concepts to break the degeneracy of TE_{01} and TM_{11} modes. The resistive approach uses dielectric materials to impede currents that travel vertically along the cavity while leaving azimuthal currents unperturbed. An alternative approach is to impede the currents using grooves cut into the cavity wall, which achieve the same effect with an inductive impedance.

4370 impedance by cutting grooves or a thread pattern on the inside wall of the cavity. For
 4371 reasons of manufacturability and compatibility with tritium the grooved cavity approach
 4372 is the preferred method for mode-filtered cavity construction by Project 8.

4373 6.4.3 Simulations of Open, Mode-filtered Cavities

4374 A candidate design for a single TE_{011} mode CRES experiment is a cavity that utilizes
 4375 the coaxial terminations combined with a mode-filtering wall. The first step towards
 4376 validating that a cavity that combines these two design features will operate as expected
 4377 is a thorough simulation effort for which finite element method (FEM) simulation software
 4378 is invaluable. The primary tool for electromagnetic FEM calculations inside Project 8 is
 4379 Ansys HFSS, which has a robust and well-established eigenmode solver that can identify
 4380 the resonant frequencies and associated Q-factors for given structure.

4381 Four variations of a cavity design with a ~ 1 GHz TE_{011} resonance were implemented
 4382 in HFSS (see Figure 6.13). The four designs include a standard cylindrical cavity, an
 4383 open cavity with smooth walls, an open cavity with resistive walls, and an open cavity
 4384 with grooved walls. The relevant design parameters are summarized in Table 6.3. All

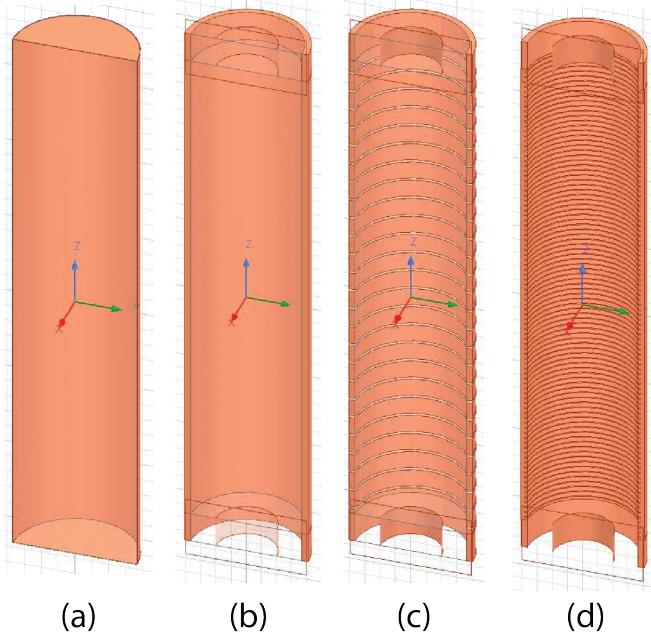


Figure 6.13: Four cavity design variations. (a) is a standard sealed cylindrical cavity, (b) is an open cavity with smooth walls, (c) is an open cavity with resistive walls, and (d) is an open cavity with grooved walls. The main cavity and coaxial terminator parameter are identical for all four cavities.

4385 cavities were simulated using copper walls and filled with a vacuum dielectric. The
 4386 identities of the resonant modes found by HFSS were validated by visual inspection of
 4387 the electric and magnetic field patterns and by comparison to analytical calculations of
 4388 the mode frequencies.

Table 6.3: A table of cavity design parameters used for HFSS simulations.

| Name | Qty. | Unit | Description |
|----------------------------|--------|------|--------------------------------------|
| D_{cav} | 326.4 | mm | Cavity diameter |
| L_{cav} | 1668.0 | mm | Cavity length |
| D_{term} | 200.2 | mm | Inner diameter of coaxial terminator |
| L_{term} | 100.0 | mm | Terminator length |
| l_{die} | 8.3 | mm | Dielectric spacer thickness |
| Δl_{die} | 66.7 | mm | Distance between dielectric spacers |
| l_{groove} | 3.0 | mm | Groove height |
| d_{groove} | 9.0 | mm | Groove depth |
| Δl_{groove} | 18.3 | mm | Distance between grooves |

4389 The results of the HFSS simulations validate our predictions of the resonant behavior
 4390 of an open, mode-filtered cavity developed in the preceding sections (see Figure 6.14) One

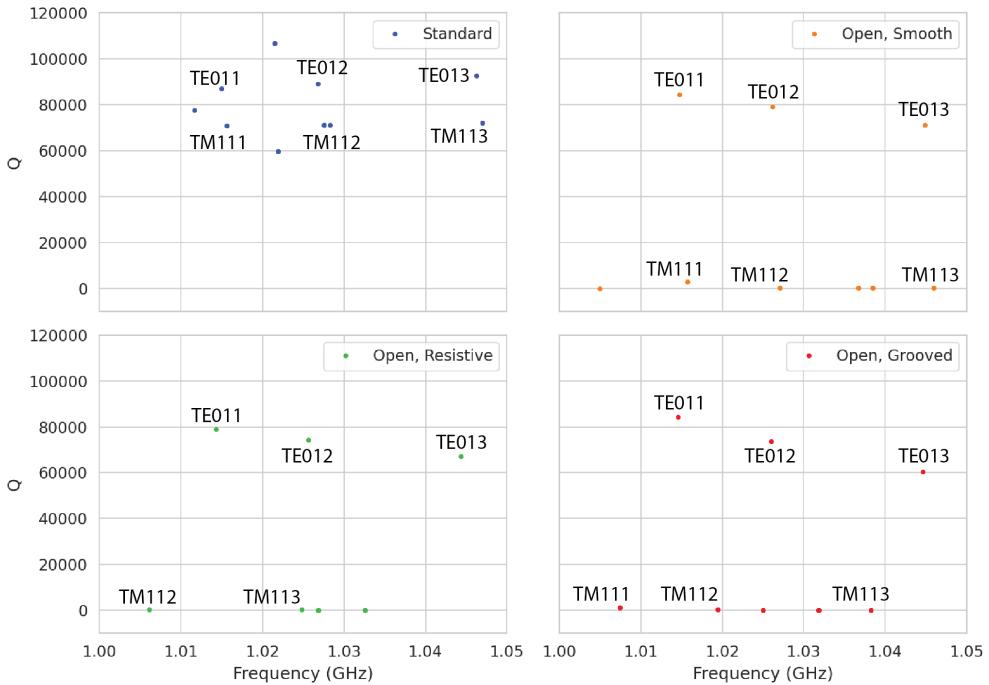


Figure 6.14: The frequencies and Q-factors of the resonant modes identified by HFSS for the cavity variations shown in Figure 6.13. The fully-sealed cavity with smooth walls has several high-Q modes near the TE_{011} resonance. Introducing the open-termination preserves the Q-factors of the $TE_{01\ell}$ modes and suppresses the Q-factors of the modes whose boundary conditions do not match the cylindrical partition. Both the resistive and grooved wall perturbations shift the resonant frequencies of the TM modes away from the TE_{011} mode. By properly tuning the geometry of the grooves or the resistive spacers several MHz of frequency separation can be achieved.

can see that for a standard cavity the TE_{01} and the TM_{11} are degenerate in frequency with relatively high Q-factors. The open-ended cavity preserves the high Q-factors of the TE_{01} modes, while the other modes, since their boundary conditions do not match the coaxial geometry, have their Q-factors suppressed. One can see that the effect of the resistive and inductive mode-filtering schemes is to effectively shift the resonant frequencies of the TM_{11} modes below those of the associated TE_{01} modes, which breaks the degeneracy. Optimization of the dielectric spacer or groove parameters can ensure that the TE_{011} mode is isolated from other modes by $O(10)$ MHz, which provides sufficient bandwidth for a measurement of the tritium spectrum endpoint.

Further optimization of the cavity design requires a more detailed cavity simulation that includes the cavity coupling mechanism as well as other geometry modifications required for integration into the magnetic and tritium gas subsystems. Perhaps more

4403 important is the development of the capability to simulate the interaction of electrons
4404 with the cavity so that simulated CRES signals can be generated using cavities designed
4405 for CRES measurements. Simulated CRES signals can then be used to estimate the
4406 neutrino mass sensitivity of the experiment, which allows for the optimization of the cavity
4407 design towards the configuration that provides the best measurement of the neutrino
4408 mass.

4409 **6.5 Single-mode Resonant Cavity Measurements**

4410 Measurement test stands play an important role in the research and development process
4411 that cannot be replaced by simulations. For example, constructing a prototype CRES
4412 cavity forces one to consider important practical issues such as manufacturability and
4413 machine tolerances that may require modifications to the design. Furthermore, by
4414 comparing laboratory measurements of a real cavity to simulations, one can quantify
4415 the impact of imperfections and real-life measurement systematics, which allows for
4416 more accurate sensitivity estimates of the experiment. Lastly, the development of these
4417 prototypes helps to build the necessary experience and expertise within the collaboration
4418 required for more complicated experiments to succeed.

4419 In this spirit a prototype cavity was constructed to demonstrate the open, mode-
4420 filtered cavity concept explored in the previous sections. The primary goal of the
4421 measurements was to validate that an open, mode-filtered cavity suppressed the TM_{11}
4422 modes as predicted by HFSS simulations.

4423 **6.5.1 Cavities and Setup**

4424 Two rudimentary, cavities were constructed using segments of copper pipe available from
4425 McMaster-Carr (see Figure 6.15). The design consists of copper pipes of two diameters.
4426 The larger diameter pipe forms the main cavity wall and the smaller diameter pipe is
4427 used to create a coaxial termination. The diameter of the outer pipe was chosen to
4428 produce a TE_{011} resonance of approximately 6 GHz, while the diameter of the smaller
4429 pipe was selected based on the open termination criteria introduced in Section 6.4.1. The
4430 approximate diameters and lengths of the copper pipe are summarized in Table 6.4.

4431 Coupling to the cavity was achieved using a hand-formable segment of coaxial cable
4432 stripped at one end to form a loop antenna. This was inserted into a small hole located
4433 at the center of the main cavity wall. The coaxial terminators were supported inside the

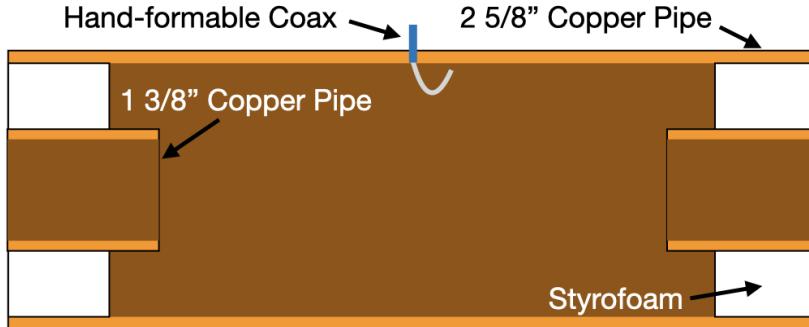


Figure 6.15: A cartoon depicting the design of the open-ended cavity prototype designed to operate at approximately 6 GHz. The main cavity wall was composed of a single copper pipe. A mode-filtered version of this cavity was constructed by

⁴⁴³⁴ main cavity by carving a spacer from polystyrene foam (styrofoam) so that they could
⁴⁴³⁵ be easily inserted into the cavity and repositioned. The dielectric constant of styrofoam
⁴⁴³⁶ is quite close to air at microwave frequencies so this is expected to have minimal impact
⁴⁴³⁷ on the resonant properties of the cavity.

Table 6.4: A table of parameters describing the cavity prototypes. Certain values such as the cavity length and the distance between dielectric spacers are approximate due to variation in the machining of the copper. In particular, the filtered cavity was constructed from conducting copper segments that varied in size from 1.50" to 1.85".

| Name | Qty. | Unit | Description |
|-------------------------|--------------------------|------|--------------------------------------|
| D_{cav} | 2.625 | in | Cavity diameter |
| L_{cav} | ≈ 13 | in | Cavity length |
| D_{term} | 1.375 | in | Inner diameter of coaxial terminator |
| L_{term} | 1.575 | in | Terminator length |
| l_{die} | 0.75 | in | Dielectric spacer thickness |
| Δl_{die} | ≈ 1.50 to 1.85 | in | Distance between dielectric spacers |

⁴⁴³⁸ The actual length of the cavity is given by the distance between the inner edges of the
⁴⁴³⁹ coaxial terminations. The length of the outer section of pipe that forms the main wall of
⁴⁴⁴⁰ the cavity is approximately 16" in length which leads to a cavity length of $\approx 13"$ when
⁴⁴⁴¹ both terminators are inserted in the cavity. Because the terminators were not rigidly
⁴⁴⁴² mounted this distance is only approximate, however, the uncertain length of the cavity
⁴⁴⁴³ will not prevent us from validating the open cavity design.

⁴⁴⁴⁴ Along with the smooth-walled open cavity a resistively mode-filtered cavity was
⁴⁴⁴⁵ constructed by creating dielectric spacers out of segments of clear PVC pipe (see Figure
⁴⁴⁴⁶ 6.16). The spacers were machined such that the conductive segments of the cavity would

4447 be separated by 0.75" when the cavity was fully assembled. Due to variations in the
 4448 lengths of the copper segments that make up the cavity wall the distance between spacers
 4449 has significant variation with average value of about 1.7". Eight total spacers were used
 4450 to build the cavity, which when assembled was approximately 16" in total length similar
 to the non-filtered cavity.

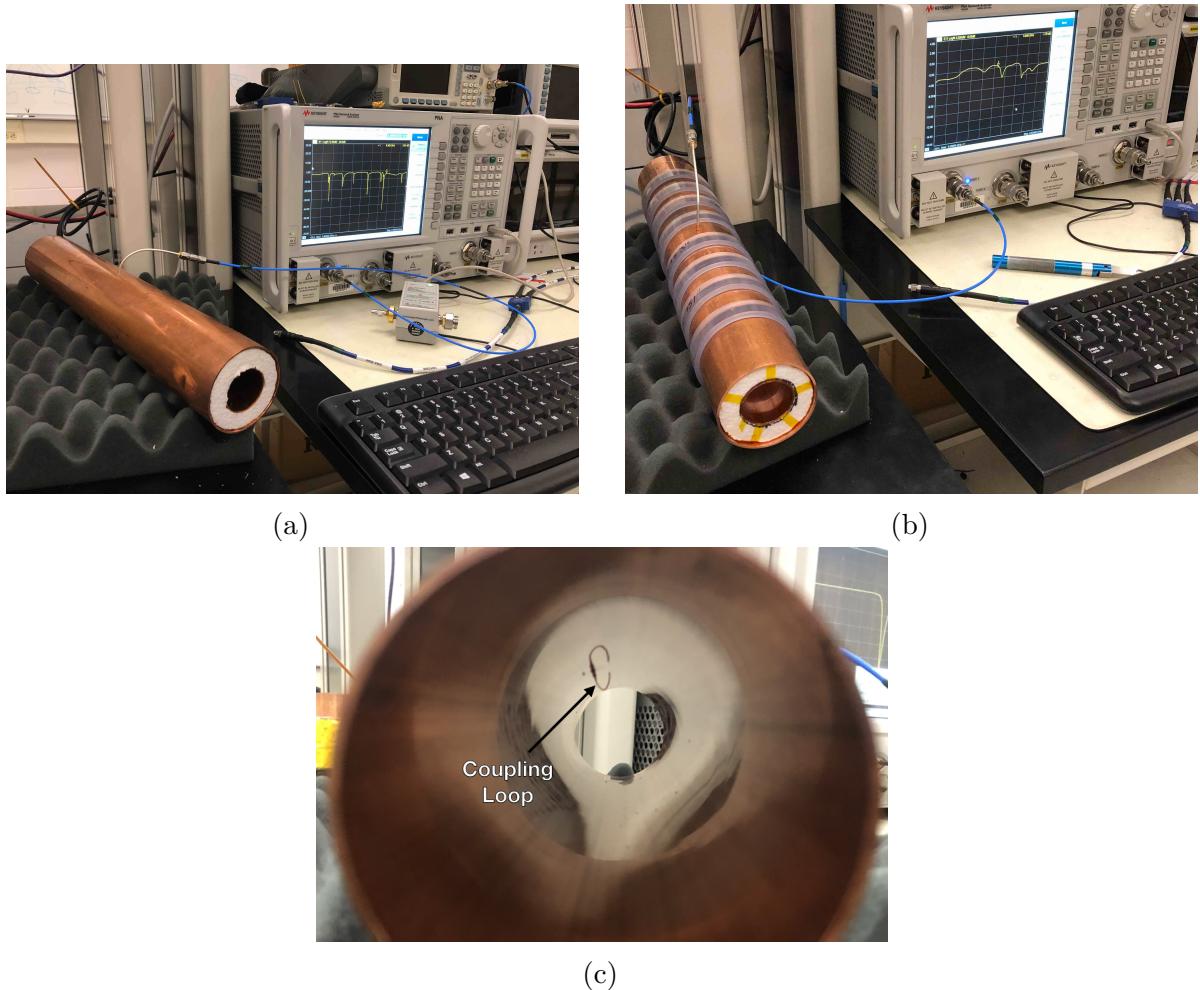


Figure 6.16: Images depicting the measurement of the filtered and non-filtered open cavities using the VNA. The coupling loop in the figure is shown in the TE orientation.

4451 Measurements of both cavities were performed using a VNA connected to the cavity
 4452 coupling probe (see Figure 6.16). By measuring the return loss over a range of frequencies
 4453 one can measure the frequencies and relative Q-factors of the resonant modes in the
 4454 cavity. Due to the opposite polarity of the electric fields for the TE and TM modes,
 4455 the loop coupling probe must be rotated 90° to change the polarity of the loop antenna.
 4456 When the antenna is oriented such that the loop opening faces the ends of the cavity, it

4458 couples primarily to the TE modes which have magnetic fields directed along the long
 4459 axis of the cavity (see Figure 6.16). If the coupling loop is turned by 90° from where
 4460 it is shown in the image then it will couple to the TM modes which have azimuthally
 4461 directed magnetic fields. In this way both the TE and TM resonances can be measured
 4462 independently.

4463 **6.5.2 Results and Discussion**

4464 The primary analysis method for the prototype cavities involved simply visualizing the
 4465 return loss measured by the VNA and comparing between the filtered and non-filtered
 4466 cavities. Since the resonances measured by the VNA are not labeled, there is some
 4467 uncertainty about the true identities of the modes measured by the VNA. To help with
 4468 this we performed a simulation of the simplest possible cavity that could be created from
 4469 the prototype components, which is a fully open cavity created by simply removing the
 4470 coaxial inserts from the non-filtered cavity configuration. The fully open cavity with the
 4471 as-built dimensions was simulated in HFSS to get estimates on the positions of the TE₀₁₁
 4472 and TM₁₁₁ modes (see Figure 6.17).

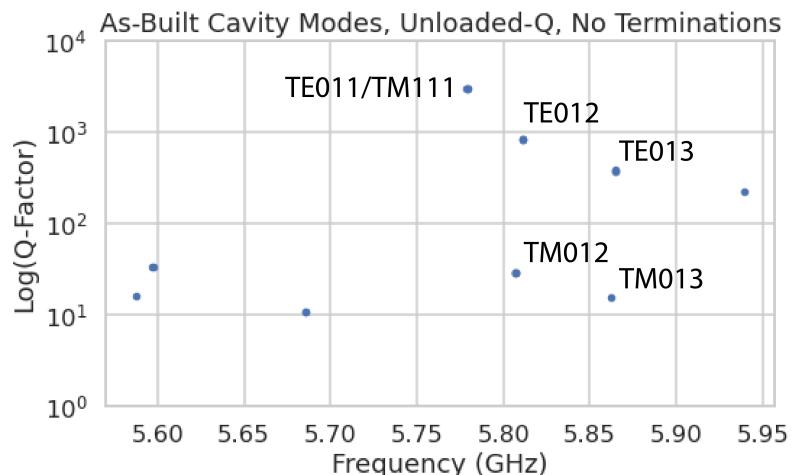


Figure 6.17: HFSS simulation results for a the as-built cavity with the coaxial terminators removed. The TE₀₁₁/TM₁₁₁ frequency is approximately 5.78 GHz.

4473 Simulation of the fully open cavity shows that the TE₀₁₁/TM₁₁₁ modes have a
 4474 frequency of approximately 5.78 GHz in the fully open cavity. If the frequency of this
 4475 mode is compared to the measurments of the fitered and non-filtered cavities with the
 4476 terminators removed we can easily identify the TE₀₁₁ mode at approximately 5.75 GHz

4477 (see Figure 6.18).

4478 For the non-filtered cavity one sees that the TE_{011} mode is degenerate in frequency
 4479 with what appears to be a doublet of TM modes located at the TM_{111} frequency position.
 4480 This doublet is actually the TM_{111} mode, which has two polarizations with opposite
 4481 polarizations. Because the pipe used to construct the cavity is not perfectly round, the
 4482 frequency degeneracy between the two polarizations is broken producing the doublet
 peak. In the case of the filtered cavity with no terminators there is an isolated TE

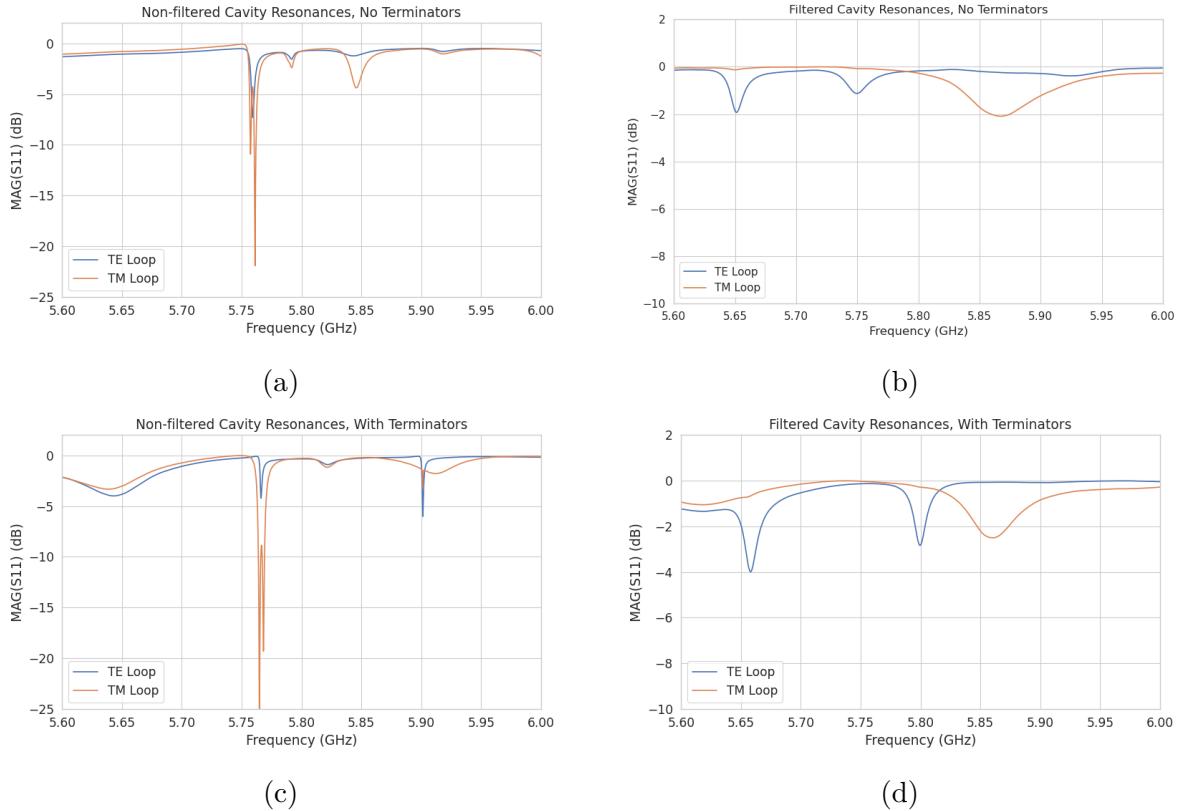


Figure 6.18: Measurements of the filtered and non-filtered prototype cavities acquired with the VNA.

4483
 4484 resonance at 5.75 GHz that appears to be the TE_{011} , however, there is no apparent TM_{111}
 4485 doublet at the same frequency. This is what one would expect if the mode-filtering was
 4486 effective at suppressing the TM modes. There is a notable difference in the Q of the
 4487 TE_{011} resonance for the non-filtered and filtered cavities indicated by the relative widths
 4488 of the resonances. This is likely caused by the large width of the dielectric spacers that
 4489 are partially impeding the TE modes. When the terminators are inserted into the cavity
 4490 one sees that Q-factors of the modes improves as expected, by noticing the narrowing of
 4491 the peaks compared to the no terminator plots.

⁴⁴⁹² In conclusion, one see from these cavity measurements that, in principle, mode-
⁴⁴⁹³ filtering can be used to separate the TE₀₁₁ resonance from the degenerate TM₁₁₁ mode in
⁴⁴⁹⁴ combination with the an open cavity design. The ideal next step would be to construct a
⁴⁴⁹⁵ open, mode-filtered cavity that could be used to perform CRES measurements. In order
⁴⁴⁹⁶ to study the coupling of an electron to the isolated TE₀₁₁ mode.

4497 Chapter 7 | 4498 Conclusion and Future Prospects

4499 In this dissertation we have discussed research and development efforts towards the
4500 development of a scalable CRES measurement technology that can be used to build a
4501 CRES experiment at cubic-meter scales with sensitivity to neutrino masses of 40 meV.
4502 The primary contributions of my dissertation are the development and analysis of signal
4503 reconstruction algorithms for an antenna array based CRES experiment [67], which leads
4504 to estimates of the neutrino mass sensitivity; the development of a synthetic cyclotron
4505 radiation antenna (SYNCA) [29], which allowed for laboratory validation of antenna
4506 array CRES simulation models [6]; and the development of an open-ended cavity design
4507 compatible with atomic tritium for a cavity based CRES experiment. A measurable
4508 impact of this work is the transition of the Project 8 collaboration’s experimental plan
4509 from an antenna array based approach to a cavity based approach, where my work played
4510 a key role in demonstrating the significantly higher cost and complexity of the antenna
4511 array experiment.

4512 The transition from antenna arrays to cavities requires a new set of demonstrator
4513 experiments to make incremental progress towards a 40 meV measurement of the neutrino
4514 mass. At the time of writing, the near-term plan of Project 8 is to design and construct a
4515 small-scale cavity CRES experiment utilizing the 1 T magnet installed in the UW-Seattle.
4516 This cavity is designed to have a TE011 resonance with a frequency of about 26 GHz with
4517 a length-to-diameter ratio that mimics the larger cavities intended for the pilot-scale and
4518 Phase IV experiments. The goal of this experiment is to demonstrate cavity CRES as
4519 well as validate models of CRES systematics using electrons from ^{83m}Kr and an electron
4520 gun. Though the primary goal is demonstration, near-term physics measurements are
4521 available in the form of high-resolution measurements of the ^{83m}Kr conversion spectrum
4522 of interest to the KATRIN collaboration.

4523 Furthermore, Project 8 is currently constructing a low-frequency CRES setup located
4524 at Yale University to better understand the principles of cavity based CRES at lower

4525 magnetic fields. The Low, UHF Cavity Krypton Experiment at Yale (LUCKEY) is
4526 a 1.5 GHz cavity CRES experiment the will use conversion electrons from ^{83m}Kr to
4527 perform CRES measurements at the lowest frequencies ever attempted with the technique.
4528 LUCKEY will validate frequency scaling models developed by Project 8 and will pave
4529 the way for the future Low-Frequency Apparatus (LFA), which will be a larger, 1 GHz
4530 cavity CRES experiment that includes a molecular tritium source. The target for the
4531 LFA is a measurement of the neutrino mass with a sensitivity of approximately 0.2 eV,
4532 which will build towards the atomic pilot-scale CRES experiment.

4533 In parallel to the development of cavity CRES is the development of the atomic
4534 tritium source. Recent demonstrations of the production of atomic hydrogen are excellent
4535 steps towards the atomic tritium production needed for the pilot-scale experiment. One
4536 area of future study includes the development of a more detailed understanding of the
4537 efficiency of atomic hydrogen production. Near-term plans include the development of a
4538 magnetic, evaporatively cooled beamline, as well as the prototyping of a Halbach array
4539 atoms trap. Nearly all of the components of the atomic tritium system will require
4540 demonstration before the complete system can be built. The long-term goal of the
4541 atomic tritium work is to construct a full atomic tritium prototype that demonstrates
4542 the production, cooling, trapping, and recycling of tritium at the rates needed for the
4543 pilot-scale experiment.

4544 More broadly, the long-term goal of the Project 8 collaboration is to fully develop
4545 both the atomic tritium and cavity CRES technologies so that both can be combined in
4546 a pilot-scale CRES experiment. It is envisioned that this process will take approximately
4547 10 years for both atomic tritium and cavity CRES. After these developments comes
4548 the pilot-scale experiment which will be the first CRES experiment that simultaneously
4549 demonstrates all the required technologies for Phase IV. Scaling to Phase IV with cavity
4550 CRES will require the construction of multiple copies (approximately 10) of the pilot-scale
4551 experiment to obtain sufficient statistics for 40 meV sensitivity.

4552 Development of the CRES experimental technique by Project 8 has led to new
4553 experiments utilizing the CRES technique for basic physics research, such as the ^6He -
4554 CRES collaboration [68], and has also found applications as a new approach to x-ray
4555 spectroscopy [69]. Recently, a new experimental effort called CRESDA has begun in
4556 the UK to develop new quantum technologies applied to CRES measurements for the
4557 neutrino mass [70]. This flourishing of new experimental efforts based on the CRES
4558 technique is likely to continue as Project 8 continues to develop the technique towards
4559 its neutrino mass measurement goal.

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