Binary Classifiers as Dilations

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Binary Diagnostic Tests

- Medical diagnostic tests:
  - HIV test, caries, cancer screening, diabetes, pulmonary embolism...
  - Antigen tests: Covid, influenza, RSV, streptococcus, ...
  - Antibody/ELISA tests

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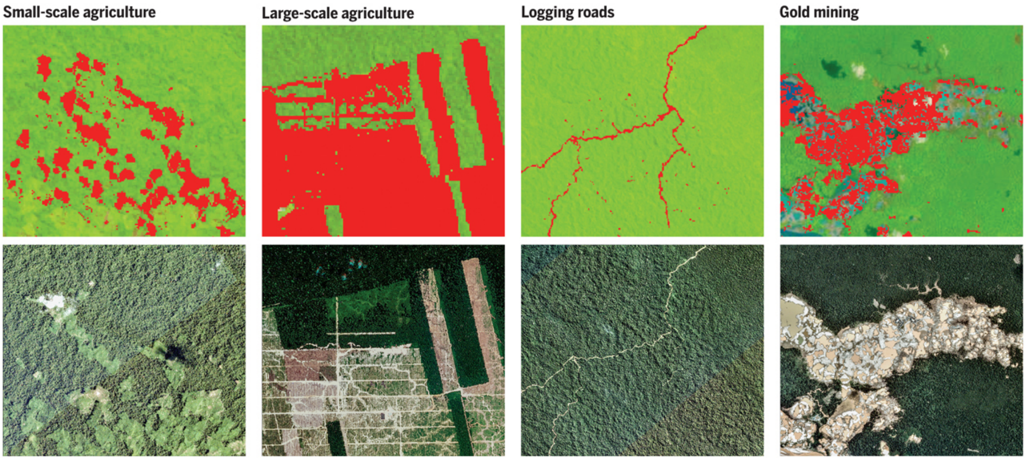
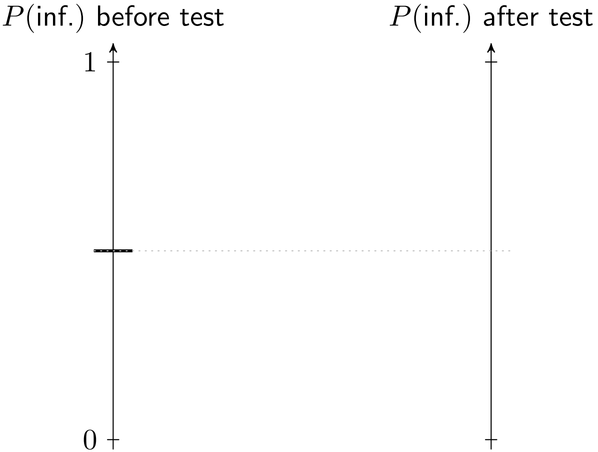
  - Antibody/ELISA tests
- Binary expert testing, binary classification, ...

Land Cover Change



Finer, Matt, et al. "Combating deforestation: From satellite to intervention." *Science* (2018).

Usual Binary Classifiers



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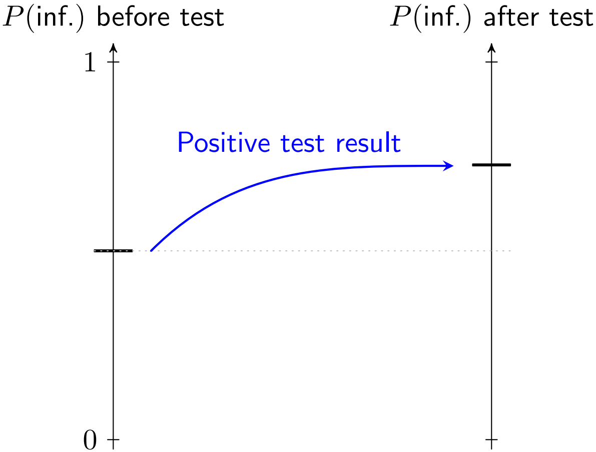

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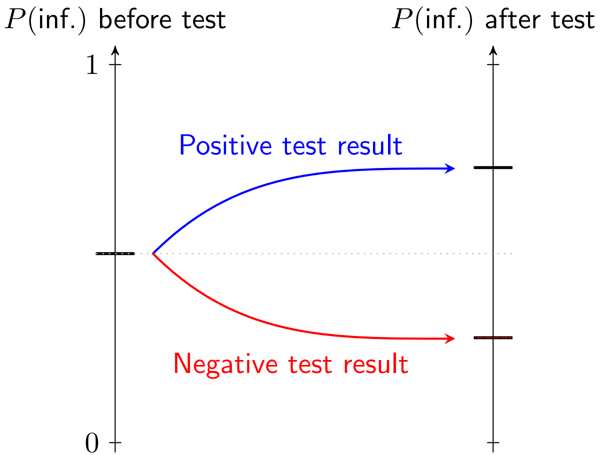

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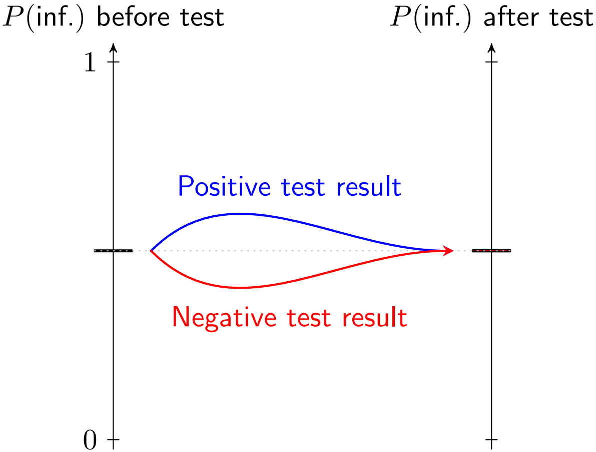

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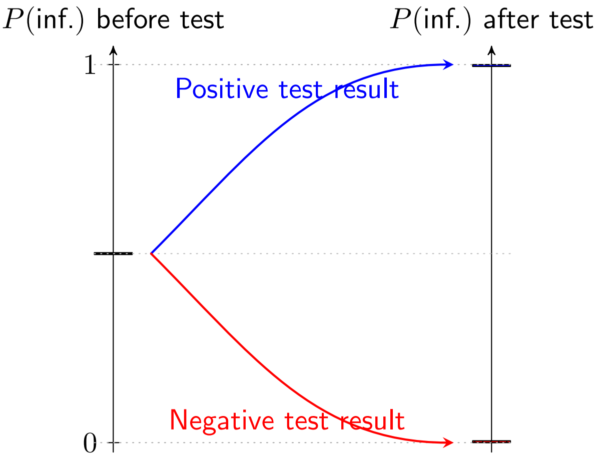

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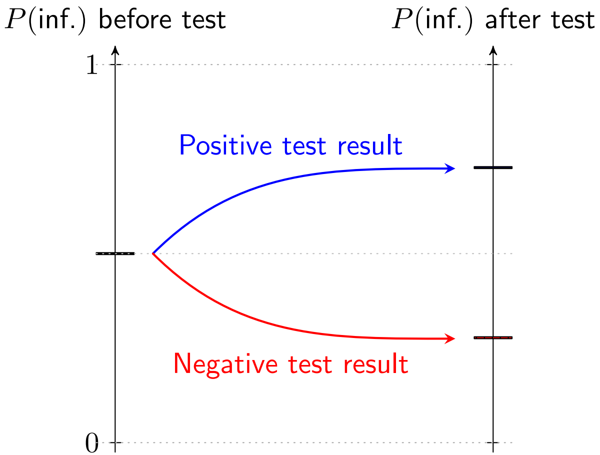

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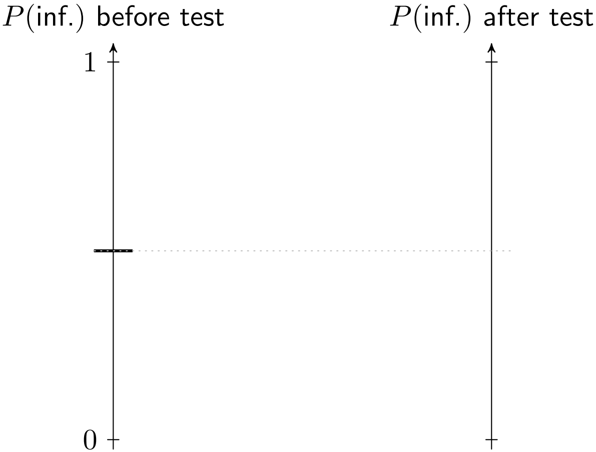
- always (weakly) informative à la Blackwell

Tests with ambiguous information



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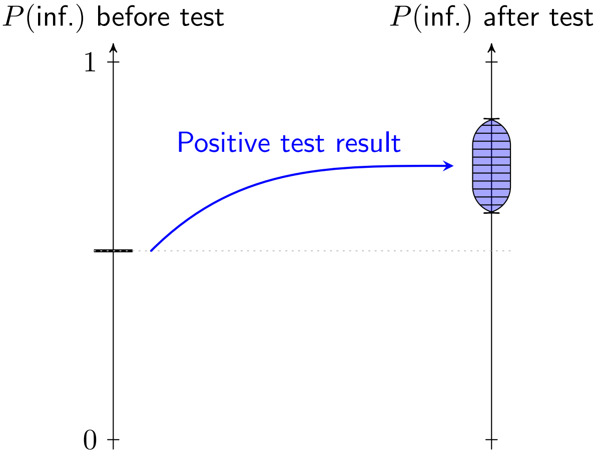

After test result: set of posteriors

aka *ambiguity/Knightian uncertainty*

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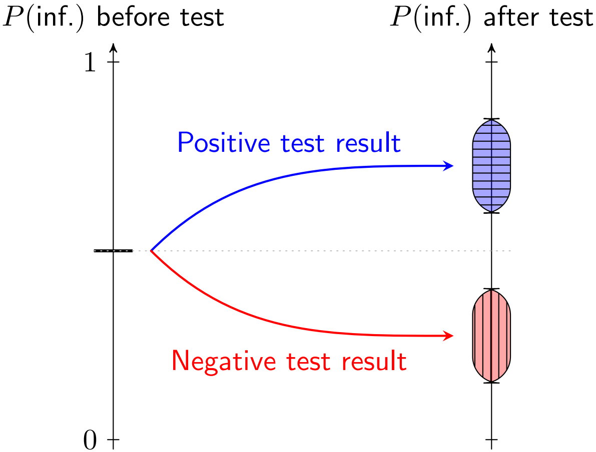

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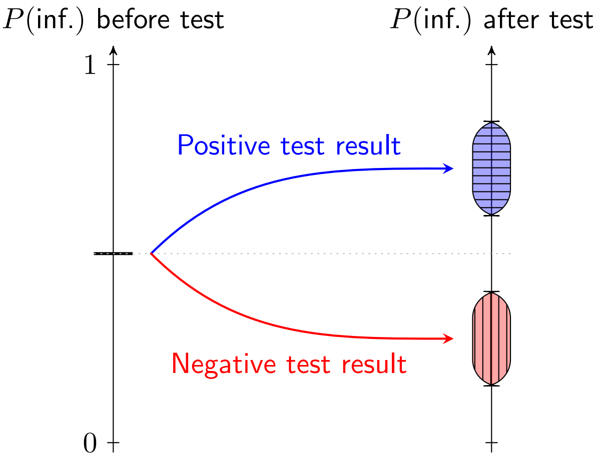

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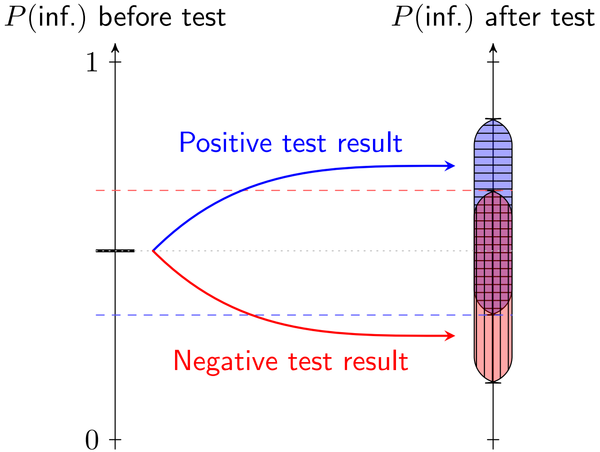

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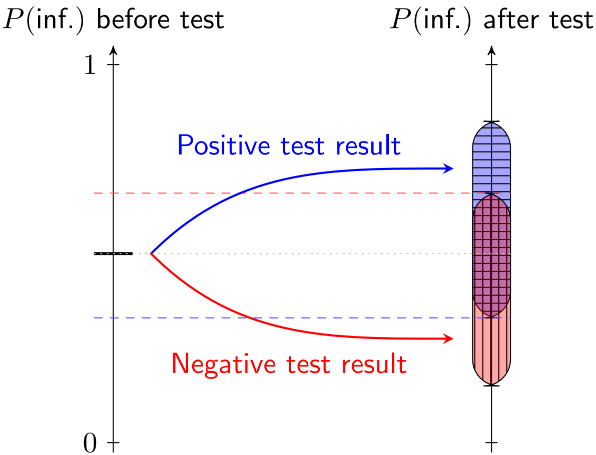

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- Here: **Robust approach** to allow for all possible correlations
  - Possible interpretation: Ambiguity or group of disagreeing experts

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Policy maker might want know when new test is a dilation

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3. Statistical inference for "$H0$: new test is a dilation" based on subvector inference
4. Empirical application: Covid LFT tests ✓, but other procedures ?, ML prediction

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- Since then mainly a theoretical curiosity
  - Some interesting applications to mechanism design/Bayesian persuasion

    Bose and Renou (2014), Beauchêne, Li, and Li (2019), Cheng (2021), Pahlke (2022)
  - Recently, also experimental evidence: in particular, Shishkin and Ortoleva (2023)

    Experiments on ambiguous information: Kellner, Le Quement, and Riener (2022), Kops and Pasichnichenko (2022), Liang (2022), Epstein and Halevy (2023)

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- This paper is the first illustration of this curious property "in the field"
  - But see Manski (2018)

Outline

1. Model
2. Illustrations & Applications
3. Statistical Inference (work in progress)
4. Extension and Conclusions

1 – Model

New test: Accuracy

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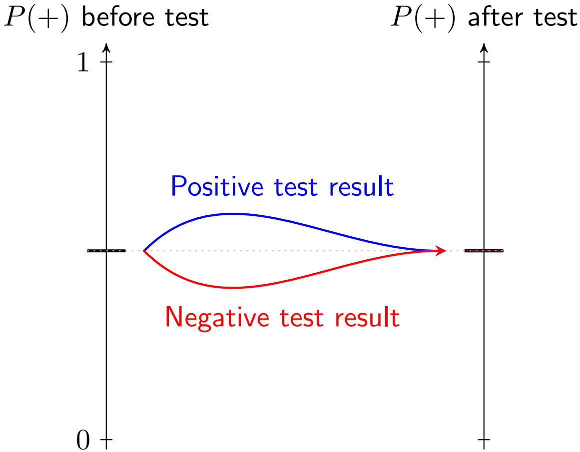
Assumption (Non-triviality)

The population satisfies $\mathbb{P}(+) \in (0, 1)$.

No Information



Observation

The new test provides no information (in Blackwell sense) if and only if

$$\theta_0 + \theta_1 = 1.$$

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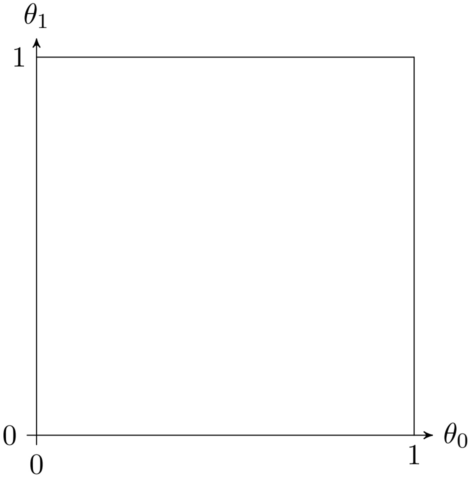

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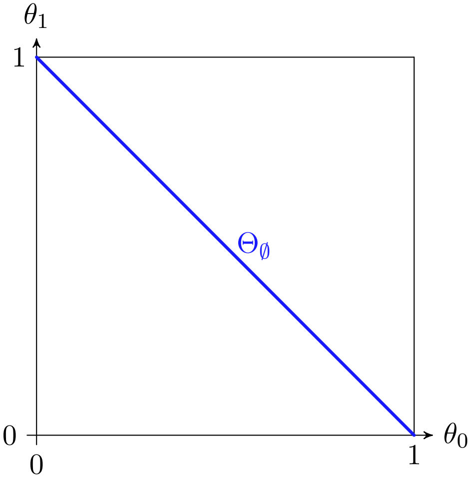

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$$\Theta_\emptyset := \Big\{(\theta_0, \theta_1) : \theta_0 + \theta_1 = 1\Big\}$$

Gold Standards

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Data: $\mathbb{P}(r, t)$

Need: $\mathbb{P}(t|\pm)$ (or $\mathbb{P}(r, t, \pm)$)

Multiplicity and Ambiguous Information

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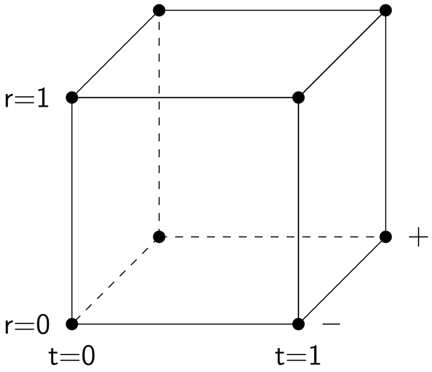

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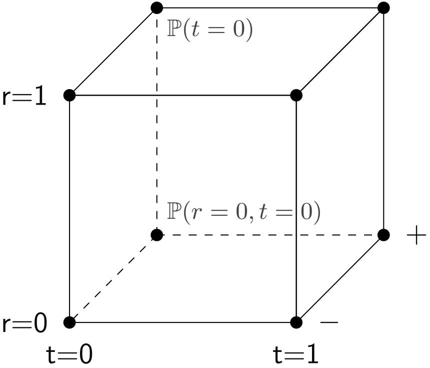

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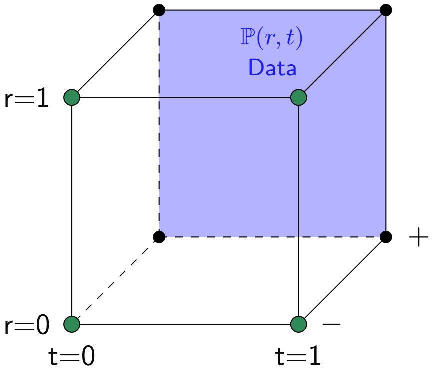

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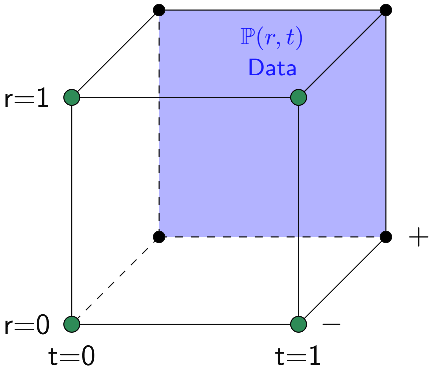

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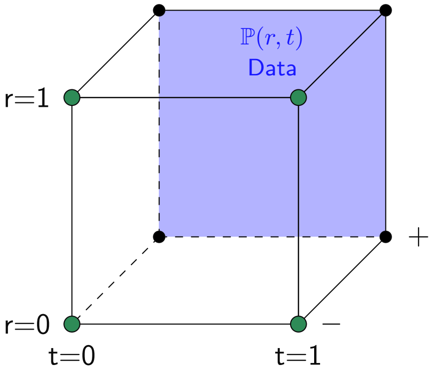
- Multiple $\theta \in: \Theta$ possible
- One *ambiguous* experiment or

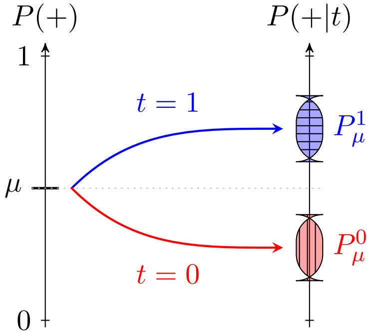
  a *set* of Blackwell experiments



Dilating test

For $\mu \in \Delta(\pm)$, define

$$P_\mu^1 := \left\{ \frac{\mu\theta_1}{\mu\theta_1 + (1-\mu)(1-\theta_0)} : \theta \in \Theta \right\} \text{ and } P_\mu^0 := \left\{ \frac{\mu(1-\theta_1)}{\mu(1-\theta_1) + (1-\mu)\theta_0} : \theta \in \Theta \right\}$$



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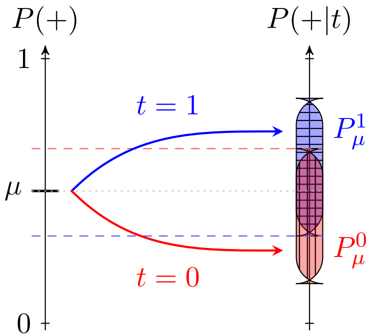

Definition

Fix a prior $\mu \in \Delta(\pm)$. The new test is a *μ-dilation* if $\{\mu\} \subsetneq P^0_\mu \cap P^1_\mu$.

It's a *dilation* if it's a μ-dilation for all $\mu \in \Delta(\pm)$.

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Reference test's sensitivity $\mathbb{P}(r = 1|+) =: s_1$ *and specificity* $\mathbb{P}(r = 0|-) =: s_0$ *are known.*

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Assumption (Informative reference test)

The reference test satisfies $s_1 + s_0 > 1$.

Characterization of the ambiguous experiment

Proposition

Fix data $\mathbb{P}(t, r)$. Under the assumptions the following hold:

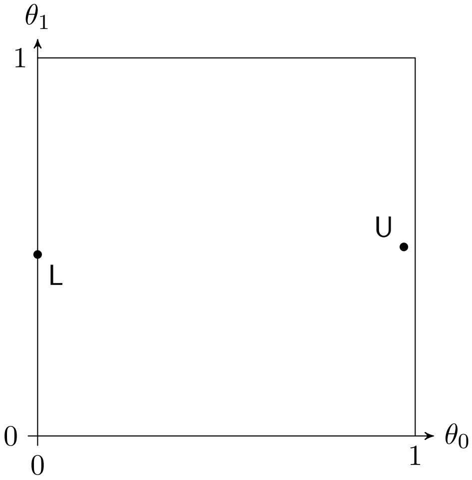
$$\Theta(s) := \Big\{(\theta_0, \theta_1) : \theta_1 \in [L_s, U_s], \mathbb{P}_s(-)\theta_0 - \mathbb{P}_s(+)\theta_1 = \mathbb{P}(t=0)\Big\}.$$

$$\mathbb{P}_s(+) = \frac{\mathbb{P}(r=1) + s_0 - 1}{s_1 + s_0 - 1} \qquad \mathbb{P}_s(-) = 1 - \mathbb{P}_s(+)$$

$$L_s = \frac{\big[\mathbb{P}(t=1, r=0) - s_0\mathbb{P}_s(-)\big]^+ + \big[\mathbb{P}(t=1, r=1) - (1-s_0)\mathbb{P}_s(-)\big]^+}{\mathbb{P}_s(+)}$$

$$U_s = \frac{\min\{\mathbb{P}(t=1, r=0), (1-s_1)\mathbb{P}_s(+)\} + \min\{\mathbb{P}(t=1, r=1), s_1\mathbb{P}_s(+)\}}{\mathbb{P}_s(+)}$$

Visualizing the characterization



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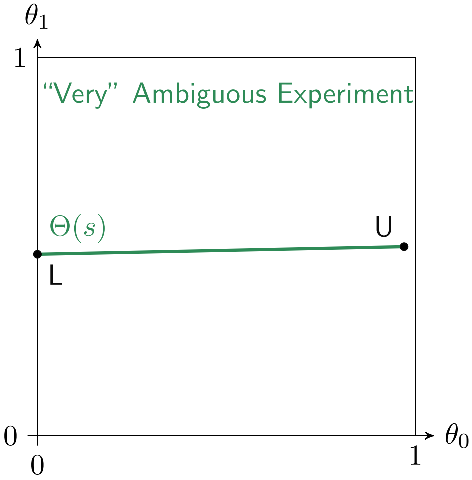

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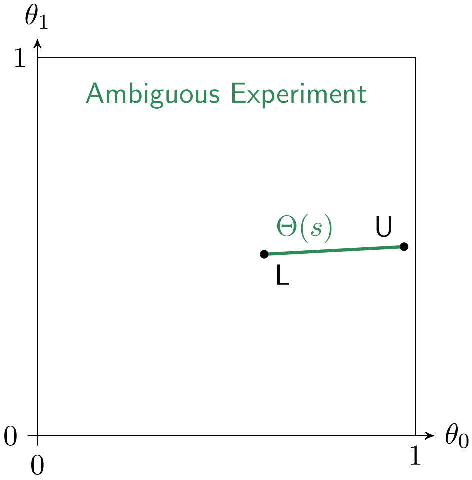

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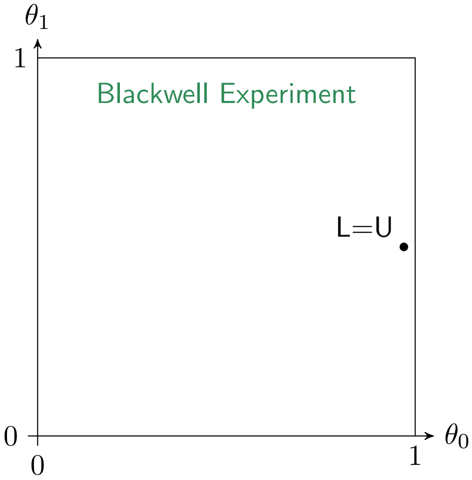
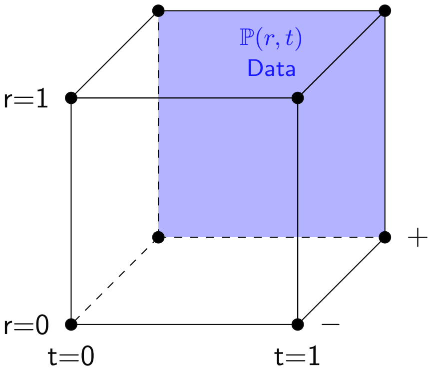


Proof



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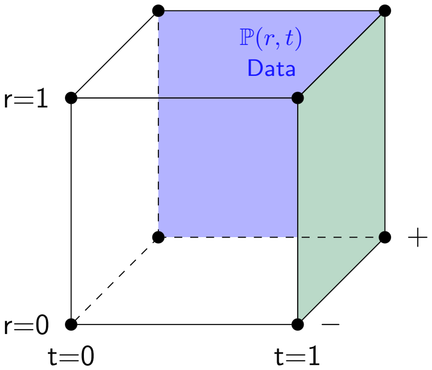
1. Informative reference $\implies \mathbb{P}(r, \pm)$



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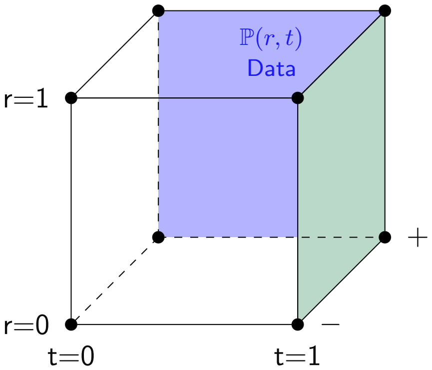

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3. A Copula problem:
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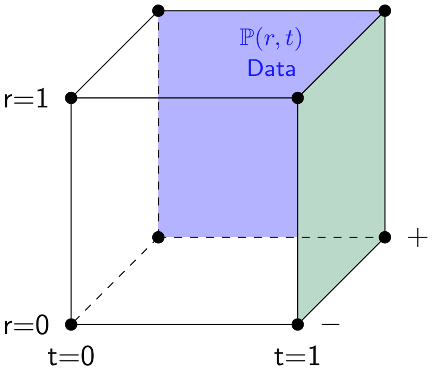

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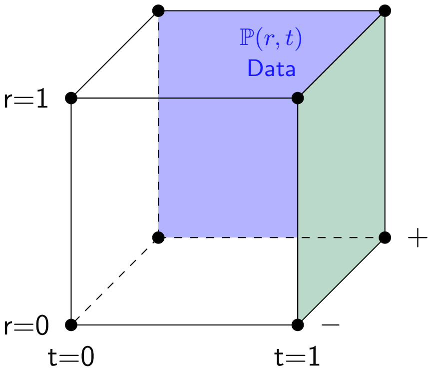
4. U/L from most/least correlated distributions



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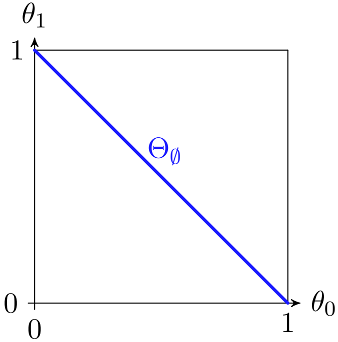
Lemma 4: Together with other assumptions, sufficient for proper ambiguity, *i.e.* $|\Theta| > 1$

Characterizing Dilations

Theorem

Under the mentioned assumptions,

t *is a dilation if and only if there exists* $\theta \in \Theta$ *such that* $\theta_1 + \theta_0 = 1$.



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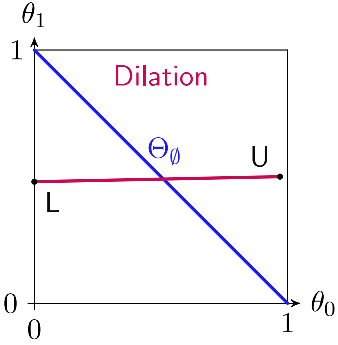

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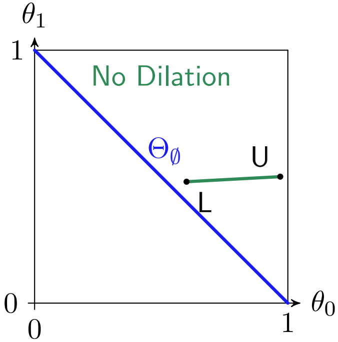

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Corollary: Dilation if and only if $\mathbb{P}(t = 1) = \theta_1$ for some $\theta \in \Theta$.

That is, there is at most one intersection point with $\Theta_\emptyset$.

2 – Illustrative Applications

Application 1: Covid Antigen test (LFT) from Leber et al. (2021)

new \ ref.	$r = 0$	$r = 1$	Sum
$t = 0$	179	38	217
$t = 1$	22	788	819
Sum	201	826	1027

- "Patients with ... flu-like symptoms attending a general practice network in an Austrian district" – Leber et al. (2021, p.1)
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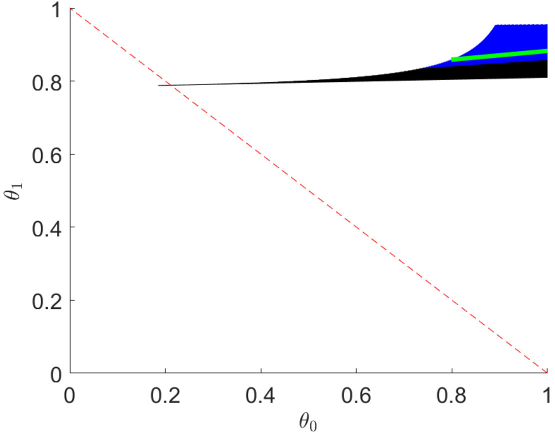

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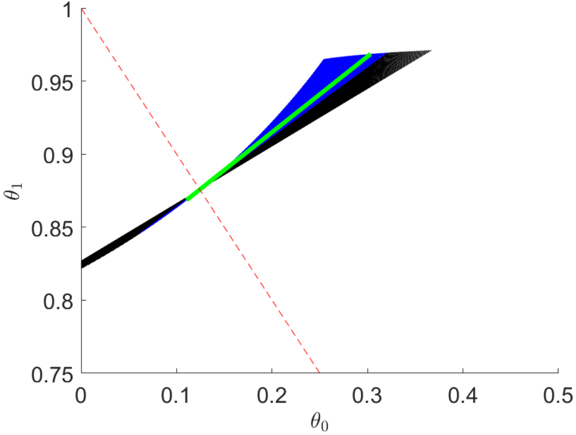

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3 – Statistical Inference

Statistical Test(s) for Dilations

Hypothesis: $H_0 =$ new test is a dilation ($\iff \exists(\theta_1, \theta_0) \in \Theta : \theta_1 + \theta_0 = 1$)

First, somewhat naive, attempt based on Goodman (1965)

new \ ref.	$r = 0$	$r = 1$
$t = 0$	p_{00}	p_{01}
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- Data $\sim$ multinomial distribution
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- Get CI for $\theta_0 + \theta_1$, reject if 1 not in it

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- Issues:
  - Conservative
  - Low power
  - Uniformity problems

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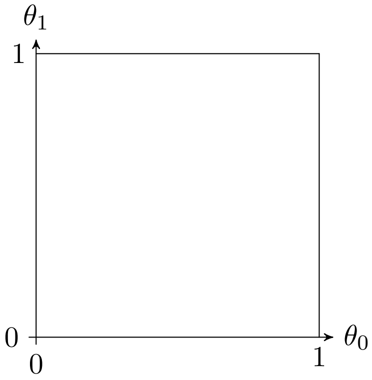
Key insight from Obradović (2024): Θ can be expressed via **moment inequalities**



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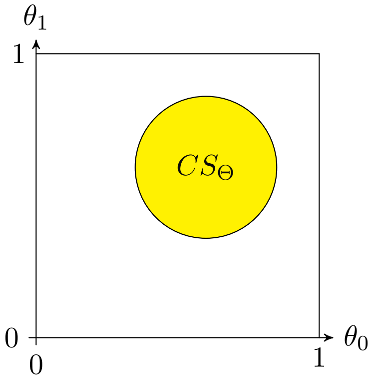

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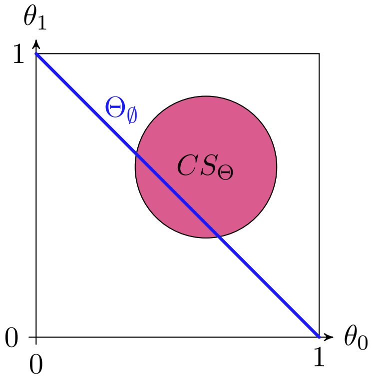

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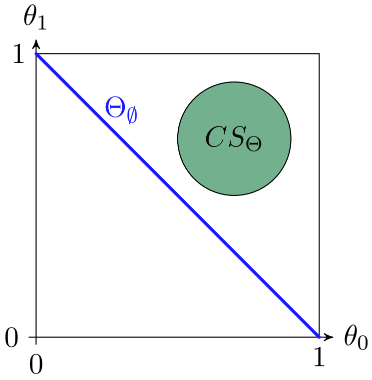

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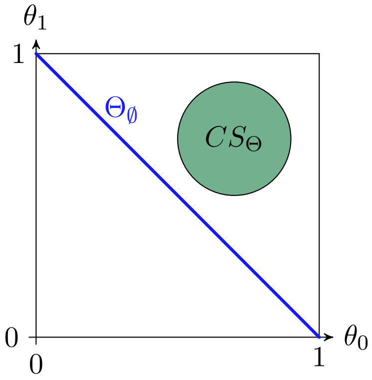

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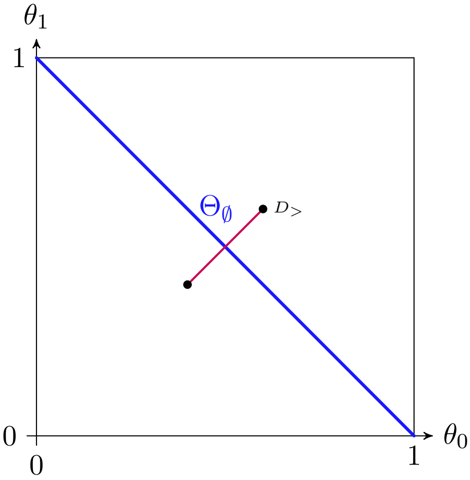
   - ... but only need to look across $\Theta_\emptyset$
2. Subvector inference:
   - based on Bugni, Canay, and Shi (2017)
   - better coverage and power

Simulations



- 5 specifications with $s = (90\%, 90\%)$:
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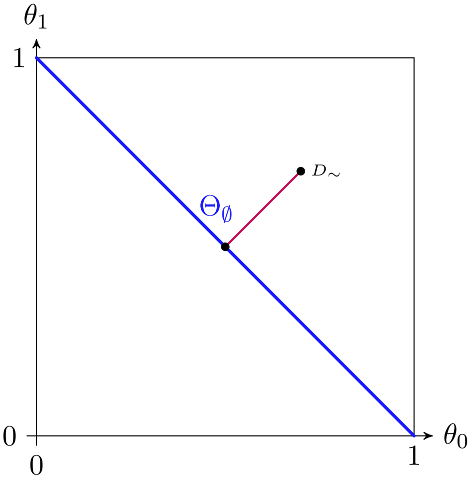

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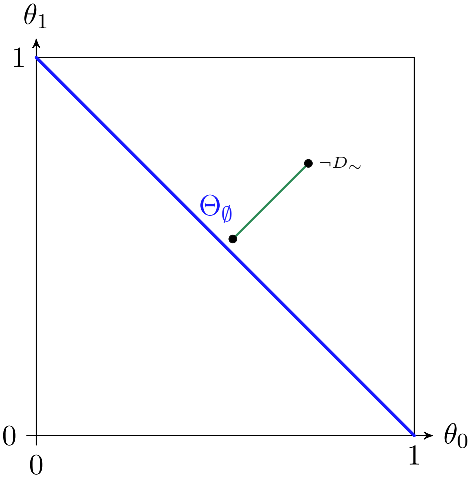

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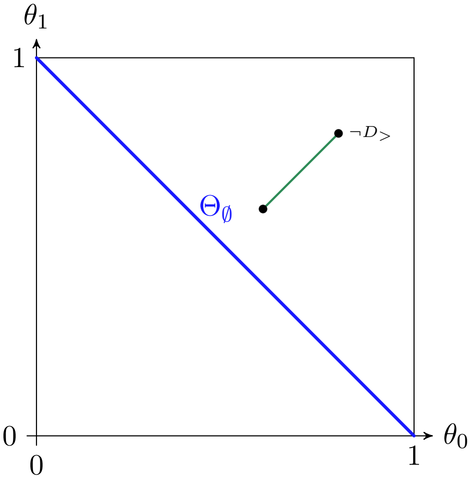

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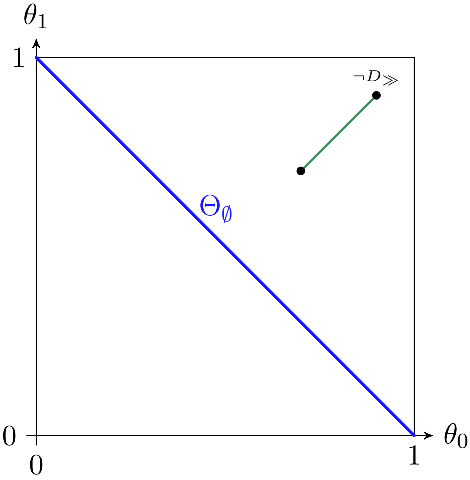

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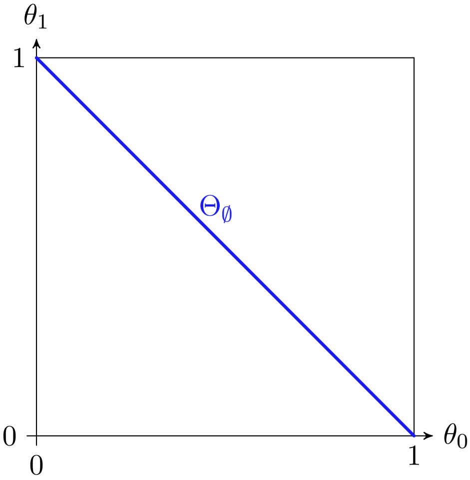

- 5 specifications with $s = (90\%, 90\%)$:
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- Sample Size:
  - $N \in \{50, 100, 500\}$
- Simulations replications: 1000

Simulations - Rejection Probabilities ($\alpha = 5\%$)

		$D_>$	$D_\sim$	$\neg D_\sim$	$\neg D_>$	$\neg D_\gg$
	G	0.9%	0.6%			
$N = 50$	RSW	0%	0%			
	BCS	0%	0.8%			
	G	0%	0%			
$N = 100$	RSW	0%	0%			
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$N = 500$	RSW	0%	0%			
	BCS	0%	1.3%			

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		$D_{>}$	$D_{\sim}$	$\neg D_{\sim}$	$\neg D_{>}$	$\neg D_{\gg}$
	G	0.9%	0.6%	0.2%	0.7%	4.3%
$N = 50$	RSW	0%	0%	0%	0.1%	4.3%
	BCS	0%	0.8%	3.9%	22%	76%
	G	0%	0%	0%	1%	30%
$N = 100$	RSW	0%	0%	0%	0.1%	19%
	BCS	0%	1%	3.1%	43%	100%
	G	0%	0%	0%	50%	100%
$N = 500$	RSW	0%	0%	0%	28%	100%
	BCS	0%	1.3%	9.2%	99%	100%

Moment (in)equalities

Fix $P = \mathbb{P}(r, t)$. Obradović (2024) shows that $\Theta(s)$ is can be rewritten as

$$\tilde{\Theta}_P(s) = \left\{\theta \in [0,1]^2 : \mathbb{E}_P[m_j(r,t;\theta,s)] \leq 0, j = 1, \ldots, 6 \text{ and } \mathbb{E}_P[m_j(r,t;\theta,s)] = 0\right\},$$

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- **Moment Inequality Model**

We are interested in testing the hypothesis that H_0: $\theta_1 + \theta_0 = 1$, i.e. based on a linear function of the parameters.

- **Subvector Inference**
- Bugni, Canay, and Shi (2017) propose bootstrap-based inference method

Assumptions

We consider a (relatively large) class of distributions $\mathcal{P} \subset \Delta^3 \subset \mathbb{R}^4$.

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Then, strengthen identification assumptions to avoid (well-known) uniformity problems:

- Without: For any n, there might exist $P \in \mathcal{P}$ rendering confidence regions invalid.
- Often better finite sample performance if uniformity holds.
- Especially important for partially identified models.

Stronger Assumptions

Want the probability of infection to be strictly (and uniformly) bounded away from boundary:

Assumption

There exists $\varepsilon_y \in \left(0, \frac{1}{2}\right)$ *such that for all* $P \in \mathcal{P}$*, we have* $\mathbb{P}_s(+) \in [\varepsilon_y, 1 - \varepsilon_y]$.

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Strengthen "Everything goes" assumption similarly:

Assumption

There exists an $\varepsilon_d \in \left(0, \frac{1}{4}\right)$ *s.t. for every* $P \in \mathcal{P}$ *and every* $(t, r) \in \{0, 1\}^2$, $P(t, r) \geq \varepsilon_d$ *holds.*

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- We show Bugni et al (2017, Thm 4.1) applies, so our test asympt. and unif. controls size:

$$\limsup_{n\to\infty} \sup_{(\theta,P):H_0 \text{ is true}} \mathbb{P}(\text{Type 1 error}) \leq \alpha.$$

4 – Applications

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  Critical value $\hat{c}_n^{95\%} = 1.11$

  Cannot reject $H0$ of it being a dilation

  (p-value $> 99\%$)

Deep Neural Net to Predict Loan Approval (Abakarim et al. '18)

- Predict "good" ($t = 1$) vs. "bad" risk ($t = 0$)

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  Reject $H0$: $T_n = 35.96$ vs. $\hat{c}_n^{95\%} = 5.25$

  Also reject with $s = (0.95, 0.95)$

5 – Extensions and Conclusion

Extensions: Partial Knowledge about Reference Test

- What if s is not known exactly?

Extensions: Partial Knowledge about Reference Test

- What if s is not known exactly?
- If *set* $\mathcal{S}$ is known and satisfies some regularity conditions:
  - Identification generalization

Extensions: Partial Knowledge about Reference Test

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  - Identification generalization
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- What if s is not known exactly?
- If *set* $\mathcal{S}$ is known and satisfies some regularity conditions:
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  - However, we were not able (yet) to prove size control

Extensions: No Knowledge about Reference Test

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- Given data $\rightarrow$ find $s = (s_0, s_1)$ for dilation
- Thinking of $\Theta(s)$ as correspondence: $\mathcal{D} := \Theta^-(\Theta_\emptyset)$
- Again, a moment inequality approach is applicable

Illustration 1: Independent tests

new \ ref.	$r = 0$	$r = 1$	Sum
$t = 0$	25%	25%	50%
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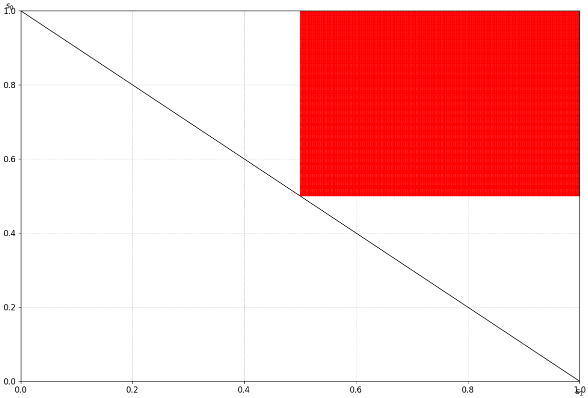

Illustration 2: Weakly correlated tests

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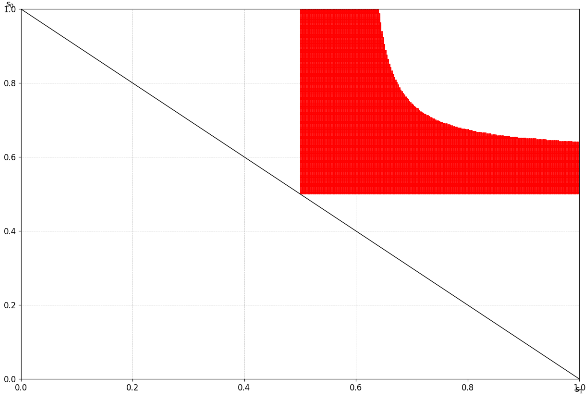

Illustration 3: Highly correlated tests

new \ ref.	$r = 0$	$r = 1$	Sum
$t = 0$	49%	1%	50%
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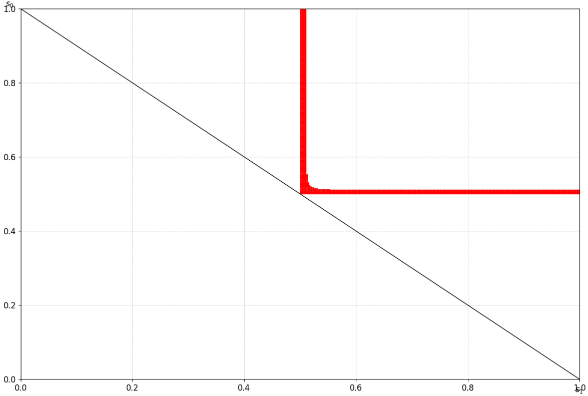

Policy implication: Regulation of medical tests

new \ ref.	$r = 0$	$r = 1$	Sum
$t = 0$	388	120	508
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- **BUT:** even if perfect specificity of reference

  if s_1 close to 62%

  we have a dilation

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- Sequential testing? I.e. take both tests
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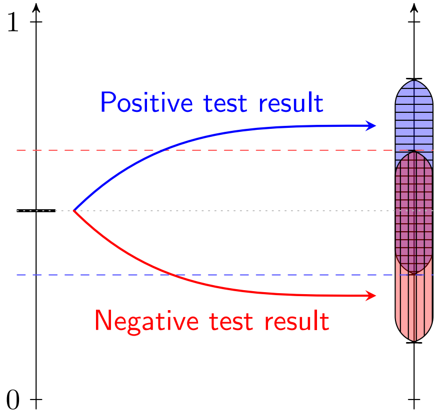
- Theory of ambiguous information

Summary

Do we encounter dilations in (existing?) classifiers?



Thanks! – Gabriel Ziegler, U Edinburgh, ziegler@ed.ac.uk