Introduction to Game Theory

Part 2: Dynamic Games

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Agenda

1. Introduction to Dynamic Games
   - Game Form and the **Tree** (the rules)
   - Exogenous personal features (tastes)
2. Beliefs: 1^{st}-order and 2^{nd}-order
3. Best replies
4. Predictions: Iterated Elimination of Never-Best Replies

Key References for Game Theory

- BATTIGALLI, P. (2025): *Mathematical Language and Game Theory.* Typescript, Bocconi University. [Downloadable from Pierpaolo Battigalli's webpage]
- BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2025): *Game Theory: Analysis of Strategic Thinking.* Typescript, Bocconi University. [Downloadable from Pierpaolo Battigalli's webpage]

Key References for Psychological Game Theory (PGT)

- BATTIGALLI, P., AND M. DUFWENBERG (2022): "Belief-Dependent Motivations and Psychological Game Theory," *Journal of Economic Literature*, 60, 833–882.
- BATTIGALLI, P., AND M. DUFWENBERG (2009): "Dynamic Psychological Games," *Journal of Economic Theory*, 144, 1–35.
- BATTIGALLI, P., R. CORRAO, AND M. DUFWENBERG (2019): "Incorporating Belief-Dependent Motivation in Games," *Journal of Economic Behavior & Organization*, 167, 185–218.

Game theory

Game theory is the formal analysis of *interactive decision making*

- interactive situations among *n* individuals called **players**
- some (or all) take **actions**, which affect the **outcome/consequences** for everybody

  We will focus on monetary outcomes only!

We differentiate two classes of games:

- **Static Games:** Active players move *simultaneously once and for all*
- **Dynamic Games:** moves are *sequential* (some of them may be simultaneous)

  As the game unfolds, new information leads to belief-updating & revision.

Game Form vs. Game

It is crucial to distinguish:

- the description of the "*rules of the game*" (in experiments, controlled by the experimenter), called **game form**, from
- the description of the *exogenous personal features* of the participating individuals, such as their "tastes" (e.g., preferences over lotteries of outcomes).

Game = (Game form) + (description of players' exogenous features)

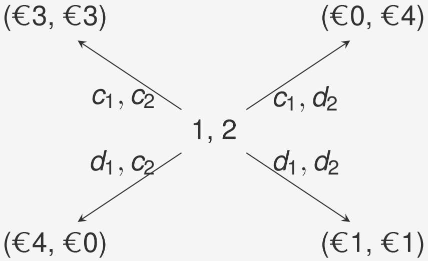
Example of static **game form**: Prisoners' Dilemma (PD)

P1 and P2 choose simultaneously between actions c and d. Outcomes are monetary payoffs in €. Static **game forms** admit both:

Payoff matrix:

	c_2	d_2
c_1	€3, €3	€0, €4
d_1	€4, €0	€1, €1

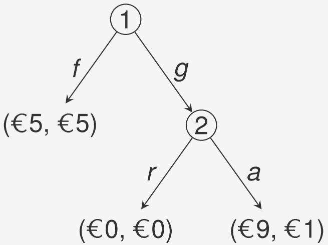
Game tree form:



Example of dynamic **game form**: Ultimatum mini-game

There are €10 to split. P1 can implement the *fair* allocation (€5, €5) (action f) or a *greedy* offer of only €1 to Bob (action g). If P1 makes the greedy offer, P2 can *reject* (r) or *accept* (a).

UmG (not a game!)



Could also have **chance moves**, and **imperfect information**

- **Root (initial history):** $\varnothing$.
- **Decision (non-terminal) history** h**:**
  - *Active players* at h: $\iota(h)$
  - *Available actions* for i at h: $A_i(h)$
  - *Decision histories* where i is active: H_i
  - All decision histories: H
- **Terminal histories** $z \in Z$
- **Material payoff function:** $\pi_i : Z \to Y_i$

Chance, Information, and the Game

- **Chance probability function:** $p_0 = (p_0(\cdot \mid h))_{h \in H_0}$, with $p_0(\cdot \mid h) \in \Delta(A_0(h))$ specifies the (objective) probabilities of chance moves.
- For simplicity, we assume here that active players **perfectly observe earlier choices**, and consequently we do not represent the non-terminal information of inactive players (not essential for what we study).
- The **terminal information** of each $i \in I$ is given by a partition $\mathcal{P}_i$ of Z.
  - For each $z \in Z$, $P_i(z)$ is the cell containing z.
- *Traditional GT*: addition of a profile of utility functions $(v_i \colon \prod_{j \in I} Y_j \to \mathbb{R})_{i \in I}$ we obtain a **game**. I.e. by defining $u_i = v_i \circ \pi \colon Z \to \mathbb{R}$.

Example: a reporting **game form** (Implemented in the lab to study deception)

Initial die roll (where face-6 of the die counts as 0). One active (real) player privately observes realization x and reports a number y: **she can lie!** A passive player (constant payoffs) observes (only) the report. The payoff of the active player is equal to her report.

$I_0 = I \cup \{0\} = \{0, 1, 2\}$,

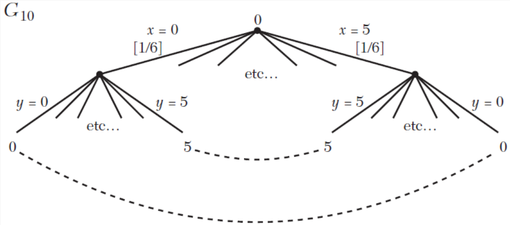
$H = \{\varnothing\} \cup \{0, \ldots, 5\}$ $Z = \{0, \ldots, 5\} \times \{0, \ldots, 5\}$

$\iota(\varnothing) = \{0\}$, $A_0(\emptyset) = \{0, \ldots, 5\}$

$\forall x \in A_0(\varnothing), \iota(x) = \{1\},\ A_1(x) = \{0, \ldots, 5\}, p_0(x|\varnothing) = \frac{1}{6}$

$\forall (x, y) \in Z$,

$\mathcal{P}_1(x, y) = \{(x, y)\}, \quad \mathcal{P}_2(x, y) = \{0, \ldots, 5\} \times \{y\}$



$\forall (x, y) \in Z,\ \pi_1(x, y) = y, \pi_2(x, y) = 0$

First-order Beliefs

For a (real) player i, given each $h \in H \cup \mathcal{P}_i$, i has **conditional** 1^{st}-order belief

$$\alpha_i(\cdot|h) \in \Delta(Z)$$

about the play.

System $\alpha_i = (\alpha_i(\cdot \mid h))_{h \in H \cup \mathcal{P}_i}$ describe i's initial belief about paths, how i would update or revise her beliefs, and include her terminal beliefs (this makes the analysis "dynamic" even if the game has only one stage).

Need to make some "consistency" assumptions about α_i.

First-order Beliefs: Maintained Assumptions

- For each $h \in H$ and $z \in Z$, $\alpha_i(z|h) > 0$ only if z follows h (denoted as $h \prec z$)
- For each $z \in Z$, $\alpha_i(z'|P_i(z)) > 0$ only if $z' \in P_i(z)$ (i believes what she observes)
- α_i is consistent with p_0
- **Chain rule:** α_i satisfies the rules of conditional probabilities when applicable,

  that is, if player i did not assign prob. 0 to what she later observed
- **Self vs others independence:** i's beliefs about simultaneous or past and unobserved actions of other players do not depend on i's chosen action:

  Implies that conditional beliefs about the *continuation* can be obtained by multiplication of i's plan $\alpha_{i,i}(\cdot|h) \in \Delta(A_i(h))$ and conjecture $\alpha_{i,-i}(\cdot|h) \in \Delta(A_{-i}(h))$

Δ_i^1 denotes the space of i's 1$^{\text{st}}$-order beliefs satisfying these assumptions

Second-order Beliefs

2nd-order belief matter if players care about others' 1st-order beliefs: $u_i : Z \times \left(\prod_{j \in I} \Delta_j^1\right) \to \mathbb{R}$.

For (real) player i, given each $h \in H \cup \mathcal{P}_i$, i has **conditional** 2nd-order belief

$$\beta_i(\cdot|h) \in \Delta\left(Z \times \left(\prod_{j \neq i} \Delta_j^1\right)\right).$$

Second-order belief systems $\beta_i = (\beta_i(\cdot \mid h))_{h \in H \cup \mathcal{P}_i}$ describe i's initial and conditional beliefs about paths of play and the 1st-order beliefs of others. We assume they satisfy properties similar to those of 1st-order belief systems.

From second-order to first-order

From $\beta_i = (\beta_i(\cdot \mid h))_{h \in H \cup \mathcal{P}_i}$ we derive a corresponding 1$^{\text{st}}$-order belief system $\alpha_i = (\alpha_i(\cdot|h))_{h \in H \cup \mathcal{P}_i}$ by *marginalization*:

$$\alpha_i(z \mid h) = \beta_i\left(\{z\} \times \left(\prod_{j \neq i} \Delta_j^1 \right) \middle| \, h \right)$$

Our assumptions on β_i imply that the derived α_i satisfies the properties of 1$^{\text{st}}$-order beliefs, that is $\alpha_i \in \Delta_i^1$.

We use Δ_i^2 to denote the space of i's 2$^{\text{nd}}$-order beliefs.

Best Replies

- **Best replies depend on information.** Whenever active, a rational player chooses actions that are best replies to his conditional beliefs, which include how he planned to continue afterward: rational planning, or one-step optimality, or intrapersonal equilibrium.
- Formally, at each history $h \in H_i$, let

$$u_i(a_i, \beta_i) := \mathbb{E}_{\beta_i(\cdot|h)}(u_i \mid h, a_i)$$

denote the subjective expected utility of taking action a_i at h.

- **Rational planning:** Given β_i,

$$(\forall h \in H_i, a_i \in A_i) \quad \alpha_{i,i}(a_i \mid h) > 0 \implies a_i \in \arg\max_{a_i' \in A_i(h)} u_i(a_i', \beta_i)$$

where α_i is derived from β_i as above

Dynamic Consistency

If u_i satisfies own-plan independence (that is, u_i does not depend on $\alpha_{i,i}$), then preferences over continuation plans given updated beliefs satisfy **dynamic consistency**.

This implies, by a "folding-back" argument, that one-step optimality is equivalent to re-optimization over continuation plans starting from every h (**One-Deviation Principle**). We omit the details.

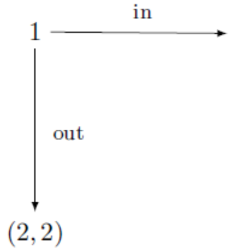
Exercise 1: Battle of the Sexes with an Outside Option

Suppose each player is selfish and risk neutral.

Find the rational plan $\alpha_{1,1}$ of player 1 when

1. $\alpha_{1,2}(B_2 \mid \text{In}) = \frac{1}{2}$
2. $\alpha_{1,2}(B_2 \mid \text{In}) = \frac{3}{4}$
3. $\alpha_{1,2}(B_2 \mid \text{In}) = \frac{1}{4}$
4. $\alpha_{1,2}(B_2 \mid \text{In}) = \frac{2}{3}$

Why don't we need to specify the beliefs of player 1 at the initial history $\varnothing$?



$1\backslash 2$	B_2	S_2
B_1	$3, 1$	$0, 0$
S_1	$0, 0$	$1, 3$

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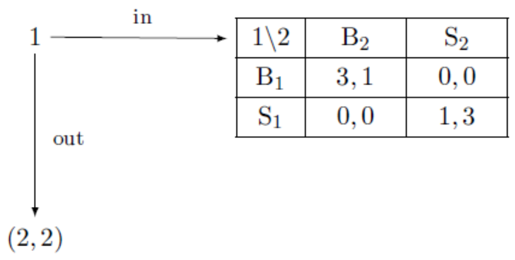

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Exercise 1: Battle of the Sexes with an Outside Option

(i) Given $\alpha_{1,2}(B_2 \mid \text{In}) = \frac{1}{2}$, at history (In), player 1's expected payoff of using B_1 is

$$\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 0 = 1.5$$

while the expected payoff of using S_1 is

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = 0.5$$

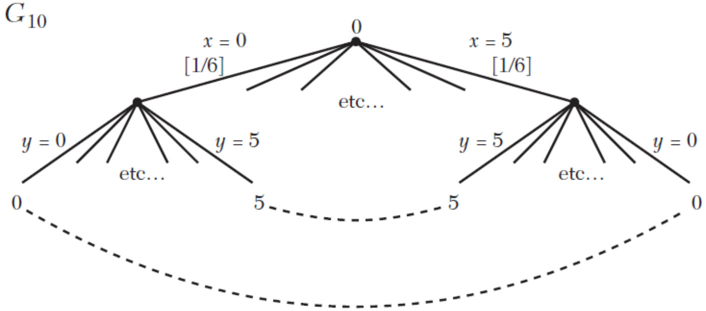
Hence $\alpha_{1,1}(\cdot \mid \text{In}) = \delta_{B_1}$, and consequently $\alpha_{1,1}(\cdot \mid \varnothing) = \delta_{\text{out}}$.

(ii) $\alpha_{1,1}(\cdot \mid \varnothing) = \delta_{\text{In}}, \quad \alpha_{1,1}(\cdot \mid \text{In}) = \delta_{B_1}$

(iii) $\alpha_{1,1}(\cdot \mid \varnothing) = \delta_{\text{out}}, \quad \alpha_{1,1}(\cdot \mid \text{In}) \in \Delta(\{B_1, S_1\})$

(iv) $\alpha_{1,1}(\cdot \mid \varnothing) \in \Delta(\{\text{In}, \text{out}\}), \quad \alpha_{1,1}(\cdot \mid \text{In}) = \delta_{B_1}$

Exercise 2: Reporting game



Assume the (parametric) utility function to be: $u_1(x, y, \alpha_2) = y - \theta_1 \sum_{x'=0}^{5} \alpha_2(x' \mid y)[y - x']^+$

Suppose that player 1 is certain that report $y = 0$ would be believed, and any $y > 0$ would not, with all lower numbers deemed equally likely.

Find the rational plan $\alpha_{1,1}$ for (*i*) $\theta_1 = 1$, (*ii*) $\theta_1 = 2$, and (*iii*) $\theta_1 = 5/3$.

Solution: Rational Plans in the Reporting Game

(i) When $\theta_1 = 1$, by reporting $y = 1$ player 1's payoff is $1 - \frac{1}{1}(1) = 0$.

By reporting $y = 2$ player 1's payoff is $2 - \frac{1}{2}(2 + 1) = 0.5$, and so on.

By reporting $y > 0$ player 1's payoff is $y - \frac{1}{y}\sum_{y'=1}^{y} y' = y - \frac{(y+1)y}{2y} = \frac{y-1}{2}$.

Hence, $\alpha_{1,1}(\cdot \mid x) = \delta_5$.

(ii) When $\theta_1 = 2$, by reporting $y > 0$, player 1's payoff is $y - 2 \cdot \frac{1}{y}(y + \cdots + 1) = -1$.

Hence, $\alpha_{1,1}(\cdot \mid x) = \delta_0$.

(iii) When $\theta_1 = \frac{5}{3}$, by reporting $y > 0$, player 1's payoff is $y - \frac{5}{3} \cdot \frac{1}{y}(y + \cdots + 1) = \frac{y-5}{6}$.

Hence, $\alpha_{1,1}(\cdot \mid x) \in \Delta(\{0, 5\})$.

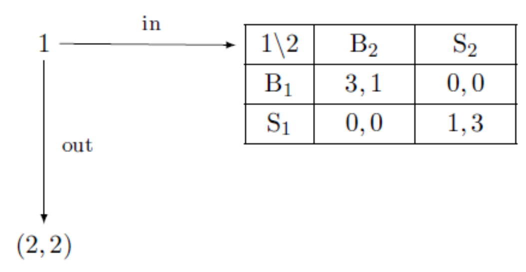
Prediction

- The dominant paradigm in traditional GT is to derive predictions using refinements (strengthenings) of the **Nash equilibrium** concept, such as “subgame perfect equilibrium”, or “sequential equilibrium”.
  - Such methodology is of little use for predicting subjects’ behavior in experimental settings.
- One can define notions of “**iterated deletion of never (sequential) best replies**”. Sometimes 2–3 rounds of deletion are useful to derive meaningful predictions in experiments.

Example 1: Prediction in the Battle of the Sexes with an Outside Option

Suppose each player is selfish and risk neutral. Furthermore, assume (higher-order) belief in rationality whenever possible.

1. Delete plan (In, S_1) (because $2 > 1$)
2. Delete plan S_2: if 2 maintains the assumption that 1 is rational, then In "signals" that 1 will continue with B_1
3. Delete plan Out: if 1 reasons as above about 2, then his unique best reply is to play In with the plan to continue with B_1



$1\backslash 2$	B_2	S_2
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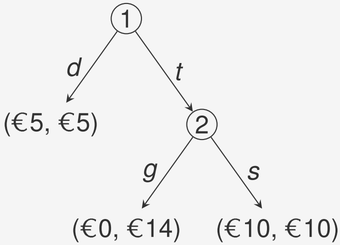
Example 2: Prediction in the Trust mini-Game



Parametric psychological utility functions of *guilt aversion*:

$$u_1(z, \alpha_2) = \pi_1(z)$$

$$u_2(z, \alpha_1) = \pi_2(z) - [\mathbb{E}_{\alpha_1}[\pi_1] - \pi_1(z)]^+$$

1. P1 trusts P2 only if $\alpha_{1,2}(s|t) \geq \frac{1}{2}$, so delete all (t, α_1) with $\alpha_{1,2}(s|t) < \frac{1}{2}$
2. P2 maintains (when possible) the assumption that P1 is rational:

$$\beta_2 \left(Z \times \left\{ \alpha_1 \middle| \alpha_{1,2}(s|t) \geq \frac{1}{2} \right\} \middle| t \right) = 1,$$

   so delete g, because $14 - 5 = 9 < 10$.
3. P1 understands this and trusts P2, delete d