Reasoning about Bounded Reasoning

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Game Theory: Strategic Reasoning and Bounded Reasoning

- Bounded Reasoning in Games:
  - level-k model

    (Nagel, 1995, Stahl II and Wilson, 1994, 1995)

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    Explicit model of reasoning

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    Rationalizability

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Main question:

What is the relation of these approaches?

Is the former consistent with an explicit model of reasoning?

Illustrative Example

Player 1 \ 2	l	c	r
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  - $R^3 = \{U, M\} \times \{l\}$
  - $R^4 = \{(U, l)\} = R^\infty$
  - Behavioral implications of RCBR

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- More generally, $L^k[p] \subseteq R^k$
- level-k is consistent with assumptions about rationality and higher-order reasoning about rationality

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- We study ***complete*** information games
  - A player is uncertain only about action choices of others

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- In models like level-k or CH, there is ***additional uncertainty*** about the level of co-players, at least potentially (and also about the reasoning about this)
- We analyze the complete information game by transforming it into an ***incomplete*** information game
  - Allows to unify level-k, CH, and a robust (belief-free) generalization within one framework

Agenda

1. Model set-up and background knowledge
2. Downward Rationalizability
3. Level-k Rationalizability

1 – The Set-Up

The Primitive Object: A Complete Information Game

- A finite two-player game with complete information $G = \langle I, (A_i, \pi_i)_{i \in I} \rangle$
  - Set of players $I = \{1, 2\}$ and finite set of actions A_i for each player i.
  - $\pi_i : A_i \times A_{-i} \to \mathbb{R}$ is player i's payoff.
  - Two players and finiteness mostly for convenience and simplification.

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- $R^\infty := R_1^\infty \times R_2^\infty := \bigcap_{n\geq 0} R_1^n \times \bigcap_{n\geq 0} R_2^n$ denote the ***rationalizable*** action profiles.
  - Behavioral implications of *rationality and common belief in rationality* (RCBR)

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- Fixing an anchor p, the level-k solution is given inductivity as follows:

  Step $1.$ For every $i \in I$, $L_i^1[p] := r_i(p_{-i})$.

  Step $k+1.$ For every $i \in I$ and $a_i \in A_i$, $a_i \in L_i^{k+1}[p]$ if and only if there exists $\nu^i \in \Delta(A_{-i})$ such that:

  (i) $a_i \in \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \nu^i(a_{-i}) \pi_i(a_i, a_{-i})$, and

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- In contrast to most of behavioral GT literature, we *do not assume a tie-breaker.*

  (*cf.*, Brandenburger, Friedenberg, and Kneeland, 2020)

The Derived Object: An Incomplete Information Game

- $G = \langle I, (A_i, \pi_i)_{i \in I} \rangle$ induces an incomplete information game $\hat{G} = \langle I, (A_i, \Theta_i, u_i)_{i \in I} \rangle$
  - Players and actions as before.
  - ***Payoff types***: refer to as ***level***-k ***types*** with $\Theta_i := \{\theta_{i,0}, \theta_{i,1}, ...\} = \{\theta_{i,k} : k \in \mathbb{N}_0\}$

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  - (Private values) ***Payoffs***:

$$u_i(\theta, a) = \begin{cases} 0 & \text{if } \theta_i = \theta_{i,0}, \\ \pi_i(a) & \text{otherwise.} \end{cases}$$

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- Note: Not a Bayesian Game (or incomplete information á la Fudenberg and Tirole, '91)

Reasoning in the incomplete information game

- We want to model *reasoning* explicitly in this game of incomplete information
- In particular, we want to impose (some) restrictions on what each level-k type "thinks" and considers possible
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- But, not only do we want to impose these restrictions, but they should be ***transparent*** to the players themselves too
  - Informally, these restrictions should be *common knowledge* among the players
  - More formally, the *restrictions are true and there is common correct belief* in these restrictions among the players

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- We formalize this by relying on Δ-rationalizability

  due to Battigalli (1999) and Battigalli and Siniscalchi (2003).

2 – Downward Rationalizability

Baseline Assumptions and Downward Rationalizability

Allow for all conjectures for a k-type that satisfy:

(K1) $\mu^i(\theta_{-i,0}) > 0 \implies \mu^i(a_{-i}|\theta_{-i,0}) = p_{-i}(a_{-i})$ (imposes the anchor)

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- ***robust***: belief-free (*cf.*, robust mechanism design) and "broadly downward looking"
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- Notation: $R^n_{i,p}$ for the resulting Δ-rationalizability: ***Downward Rationalizability***

Downward Rationalizability: Example

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$p = (\delta_D, \delta_r)$

n/k	R^n	$L^k[p]$
1	$A_1 \times \{l, c\}$	$\{(D, c)\}$
2	$\{U, M\} \times \{l, c\}$	$\{(M, c)\}$
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$R_1^n(\theta_{i,k}),\ R_2^n(\theta_{i,k})$	$n \rightarrow$			
$k \downarrow$	1	2	3	$\geq$ 4
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2	$A_1,\{l,c\}$	$\{M,D\},\{c\}$		
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3	$\{U, M\} \times \{l\}$	$\{(M, l)\}$
4	$\{(U, l)\}$	$\{(U, l)\}$

$R_1^n(\theta_{i,k}),\ R_2^n(\theta_{i,k})$	$n \rightarrow$			
$k \downarrow$	1	2	3	$\geq$ 4
1	$\{D\}, \{c\}$	$\{D\}, \{c\}$	$\{D\}, \{c\}$	$\{D\}, \{c\}$
2	$A_1, \{l, c\}$	$\{M, D\}, \{c\}$	$\{M, D\}, \{c\}$	$\{M, D\}, \{c\}$
3	$A_1, \{l, c\}$			
$\geq$ 4	$A_1, \{l, c\}$			

Downward Rationalizability: Example

Player 1 \ 2	l	c	r
U	$3,2$	$2,1$	$1,0$
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D	$1,1$	$1,2$	$3,0$

$p = (\delta_D, \delta_r)$

n/k	R^n	$L^k[p]$
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3	$\{U,M\} \times \{l\}$	$\{(M,l)\}$
4	$\{(U,l)\}$	$\{(U,l)\}$

$R_1^n(\theta_{i,k}),\ R_2^n(\theta_{i,k})$	$n \rightarrow$			
$k \downarrow$	1	2	3	≥ 4
1	$\{D\},\{c\}$	$\{D\},\{c\}$	$\{D\},\{c\}$	$\{D\},\{c\}$
2	$A_1,\{l,c\}$	$\{M,D\},\{c\}$	$\{M,D\},\{c\}$	$\{M,D\},\{c\}$
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Downward Rationalizability: Properties

$R_1^n(\theta_{i,k}),\ R_2^n(\theta_{i,k})$ $k \downarrow$	$n \rightarrow$ 1	2	3	≥ 4
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3	$A_1, \{l, c\}$	$A_1, \{l, c\}$	$\{M, D\}, \{l, c\}$	$\{M, D\}, \{l, c\}$
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Proposition (Bounded Reasoning)

For every $i \in I$, every $k \in \mathbb{N}_0$, and every $t \geq k$, $R^t_{i,p}(\theta_{i,k}) = R^k_{i,p}(\theta_{i,k})$.

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Proposition (Increasing Monotone: higher level, more predictions)

For every $i \in I$, *every* $n \in \mathbb{N}_0 \cup \{\infty\}$, *and every* $k \in \mathbb{N}$ $R_{i,p}^n(\theta_{i,k}) \subseteq R_{i,p}^n(\theta_{i,k+1})$,

Downward Rationalizability: Characterization

Even high level and high reasoning, do not get consistency with R^n, but robust characterization across all anchors is possible.

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Theorem

For every $i \in I$, $R_i^1 = \bigcup_{p \in \Delta(A_1) \times \Delta(A_2)} \bigcap_{k \in \mathbb{N}} R_{i,p}^{\infty}(\theta_{i,k}) = \bigcup_{p \in \Delta(A_1) \times \Delta(A_2)} \bigcup_{k \in \mathbb{N}} R_{i,p}^{\infty}(\theta_{i,k})$.

- Every undominated action is consistent with some anchor and ***all*** type levels.
- Conversely, every downward rational. action for some anchor and level is undominated.

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- Every undominated action is consistent with some anchor and ***all*** type levels.
- Conversely, every downward rational. action for some anchor and level is undominated.
- ***Design Insight***: Robustness to (as-if) "bounded reasoning" requires mechanism that implements in undominated actions. ***Identification*** of levels requires strong assumption.

Downward Rationalizability: Proof

Step 1: $R_i^1 \subseteq \bigcup_{p \in \Delta(A_1) \times \Delta(A_2)} \bigcap_{k \in \mathbb{N}} R_{i,p}^{\infty}(\theta_{i,k})$

Step 2: $\bigcup_{p \in \Delta(A_1) \times \Delta(A_2)} \bigcup_{k \in \mathbb{N}} R_{i,p}^{\infty}(\theta_{i,k}) \subseteq R_i^1$

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- Take undominated action a_i, by Pearce's Lemma, it is justified by ν^i, then set $p_{-i} = v_i$.
- $a_i \in R_{i,p}^1(\theta_{i,1})$ by construction, $a_i \in R_{i,p}^{\infty}(\theta_{i,1})$ by "bounded reasoning", and
- $a_i \in R_{i,p}^{\infty}(\theta_{i,k})$ for every $k \geq 1$ because "increasing monotone"

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- ▷ Take $a_i \in R_{i,p}^{\infty}(\theta_{i,k})$ for some $k \in \mathbb{N}$ and p. Clearly, it is a best-reply to some conjecture, and therefore undominated by Pearce's Lemma.

3 – Level-k Rationalizability

Towards level-k

Recall for a k-type: (K2) $\operatorname{supp} \operatorname{marg}_{\Theta_{-i}} \mu^i \subseteq \{\theta_{-i,0}, \theta_{-i,1}, \ldots, \theta_{-i,k-1}\}$.

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Level-k, but not CH, imposes something stronger:

- Put $\mu^i \in \Delta^i_k$ if the conjecture satisfies:

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(K1) $\mu^i(\theta_{-i,0}) > 0 \implies \mu^i(a_{-i}|\theta_{-i,0}) = p_{-i}(a_{-i})$ (imposes the anchor)

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- Notation: $L^n_{i,p}$ for the resulting Δ-rationalizability: ***L(evel-k)-Rationalizability***

- Same interpretation as before: transparent and unbounded reasoning

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- Notation: $L^n_{i,p}$ for the resulting Δ-rationalizability: ***L(evel-k)-Rationalizability***
- Same interpretation as before: transparent and unbounded reasoning
- Observation: (KL) $\implies$ (K2) so that $L^n_{i,p} \subseteq R^n_{i,p}$, i.e. a *refinement*

L-Rationalizability: Example

Player 1 \ 2	l	c	r
U	$3,2$	$2,1$	$1,0$
M	$2,2$	$3,1$	$2,0$
D	$1,1$	$1,2$	$3,0$

$p = (\delta_D, \delta_r)$

n/k	R^n	$L^k[p]$
1	$A_1 \times \{l, c\}$	$\{(D, c)\}$
2	$\{U, M\} \times \{l, c\}$	$\{(M, c)\}$
3	$\{U, M\} \times \{l\}$	$\{(M, l)\}$
4	$\{(U, l)\}$	$\{(U, l)\}$

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$L_1^n(\theta_{i,k}),\ L_2^n(\theta_{i,k})$	$n \rightarrow$			
$k \downarrow$	1	2	3	≥ 4
1	$\{D\}, \{c\}$			
2				
3				
≥ 4				

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3				
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$\geq$ 4	$A_1, \{l, c\}$	$\{U, M\}, \{l, c\}$	$\{U, M\}, \{l\}$	$\{U\}, \{l\}$

L-Rationalizability: Characterization

Now, consistency with rationality and higher-order reasoning about rationality in G:

Proposition (Consistency)

For every $n \in \mathbb{N}_0$, $k \geq n$, *and every* $i \in I$, $L^n_{i,p}(\theta_{i,k}) \subseteq R^n_i$.

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Theorem (Foundation of the level-k model)

For every $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, *and every* $i \in I$,

$$L^n_{i,p}(\theta_{i,k}) = \begin{cases} R^n_i, & \text{if } n < k, \\ L^k_i[p], & \text{if } n \geq k. \end{cases}$$

L-Rationalizability: An Observational Challenge

"*In our view, people stop at low levels mainly because they believe others will not go higher, not due to cognitive limitations[.]*" – Crawford, Costa-Gomes, and Iriberri (2013, JEL)

$k \downarrow / n \rightarrow$	1	2	3	$\geq$ 4
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2	$A_1, \{l, c\}$	$\{M\}, \{c\}$	$\{M\}, \{c\}$	$\{M\}, \{c\}$
3	$A_1, \{l, c\}$	$\{U, M\}, \{l, c\}$	$\{M\}, \{l\}$	$\{M\}, \{l\}$

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- ▷ In practice, we might not observe level of reasoning (n) or level-k type (or neither)
  - ▷ If $k > 1$ known, but any n possible: get again only undominated actions.

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- In practice, we might not observe level of reasoning (n) or level-k type (or neither)
    - If $k>1$ known, but any n possible: get again only undominated actions.
    - If n known, but any k is possible? Get (potentially) more than R^n.

That's it!

Thanks!

Comments and questions much appreciated! As said, very much work in progress.

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