Homework 1: Counting

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CSE 312(12973)/Section B

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1. Syllabus (5 points)

Ans: I have read the syllabus, and agree to follow the collaboration policies.

2. Softball (15 points)

Twelve people (7 students and 5 faculty) on CSE's softball team show up for a game

(a) How many ways are there to choose which 4 players will be infielders (for this part, you should just account for who is playing and who is not)?

(b) How many ways are there to assign the 4 infield positions by selecting players from the 12 people who show up (for this part, account for both who plays and which of the 4 different positions they are playing. There are four different infield positions)?

(c) How many ways are there to choose which 4 players will be infielders if at least one of these players must be a faculty member (for this part, we select only who are infielders, not the positions, like part (a))?

Ans:

(a) Choose 4 infielders from 12 people order doesn't matter. Use Formula for Combinations: $\binom{12}{4} = \frac{12!}{(12-4)! \cdot 4!} = 495$

(b) Choose 4 infielders from 12 people order does matter. Use Formula for Permutation: $P(12, 4) = \frac{12!}{(12-4)!} = 11880$

(c) We can use the total ways choose 4 infielders from 12 people minus ways that all 4 infielders are students. The remain ways will be at least one faculty is the infielder. $\binom{12}{4} - \binom{7}{4} = \frac{12!}{(12-4)! \cdot 4!} - \frac{7!}{(7-4)! \cdot 4!} = 495 - 35 = 460$

3. Getting from here to there (20 points)

   (a) How many paths are there from point (0,0) to (125,150) if every step increments one coordinate and leaves the other unchanged?

   (b) How many paths are there from point (0,0) to (125,150) if every step increments one coordinate and leaves the other unchanged and you want the path to go through (70,80)?

   (c) How many paths are there from point (0,0) to (125,150) if every step increments one coordinate and leaves the other unchanged and the path cannot go through (30, 40) or (70,80)? (Hint: Try inclusion-exclusion.)

   (d) How many paths are there from point (0,0,0) to (20,40,300) if every step increments one coordinate and leaves the other two unchanged?

Ans:

(a) We're going to take total $125 + 150 = 275$ steps. Choose which SET of 125 of the steps will be up. So the answer would be $\binom{275}{125}$

(b) We are divide the travel to two part, first part is from (0,0) to (70,80) and the second one is from (70,80) to (125,150).
We're going to take total $70 + 80 = 150$ steps for first half. Chose 70 steps will be up, so $\binom{150}{70}$.
We're going to take total $275 - 150 = 125$ steps for first half $125 - 70 = 55$ steps will be up, so $\binom{125}{75}$.
Use Product Rule, the total ways would be $\binom{150}{70} \cdot \binom{125}{75}$

(c) Inclusion-exclusion: paths cannot go through (30, 40) or (70,80) = Total paths - (paths go through (30, 40) + paths go through (70, 80) - paths go through (30, 40) and paths cannot go through (70, 80))
Paths from (0,0) to (125,150) is $\binom{275}{125}$.
Paths from (0,0) to (125,150) go through the (70,80) is $\binom{150}{70} \cdot \binom{125}{75}$.
We can divide the travel from (0,0) to (125,150) to two part, first part is from (0,0) to (30,40) and the second one is from (30,40) to (125,150) (similar way in (b)) ...
So we got paths from (0,0) to (125,150) go through the (30,40) is $\binom{70}{30} \cdot \binom{205}{95}$.
We can divide the travel from (0,0) to (125,150) to three part, first part is from (0,0) to (30,40), the second one is from (30,40) to (70,80), and the third one is from (70,80) to (125,150) (similar way in (b))...
So we got paths from (0,0) to (125,150) go through the (30,40) and (70,80) is $\binom{70}{30} \cdot \binom{80}{40} \cdot \binom{125}{55}$
paths cannot go through (30, 40) or (70,80) =
$\binom{275}{125} - (\binom{150}{70} \cdot \binom{125}{75} + \binom{70}{30} \cdot \binom{205}{95} - \binom{70}{30} \cdot \binom{80}{40} \cdot \binom{125}{55})$

(d) We're going to take total $20 + 40 + 300 = 360$ steps.
Step 1: 20 of 360 steps will be go x-direction, so $\binom{360}{20}$
Step 2: 40 of remain 340 steps will be go y-direction, so $\binom{340}{40}$
Product Rule, total paths $= \binom{360}{20} \cdot \binom{340}{40}$

4. 5-suit-deck with 6-card-hands (10 points)

Suppose you are playing poker with a non-standard deck of cards. The deck has 5 suits, each of which contains 12 values (so the deck has 60 cards total).
How many 6-card hands are there, where you have at least one card from each suit?
A hand of cards is an unordered set.

Ans:
There are always two cards in one suit and one card in each of the other four suits.

Step 1: Chose the two cards with same suit
We have 5 choice for suit.

Step 2: chose 2 cards from the suit We chose 2 cards from 12 values, so there are $\binom{12}{2} = 66$ possible situation.

Step 3: Choose remain suits cards
Choose 1 card from each of the 4 suits: $\binom{12}{1}^4 = 12^4$

By Product Rule, total number of valid hands is: $5 \cdot \binom{12}{2} \cdot 12^4 = 6842880$

5. Sitting around (15 points)

Archer (A), Bilbo (B), Cersei (C), Dante (D), Eowyn (E), Frodo (F), and Gollem (G) are sitting in a row of nine seats (Note: there are only seven people). Archer and Bilbo are exes, so they cannot sit next to each other. Cersei and Dante are dating, so they must sit next to each other. Eowyn, Frodo, and Gollem are best friends, so they also want to sit next to each other, but Frodo must be in the middle of Eowyn and Gollem (with no spaces between the three). Our goal is to figure out how many ways they can sit in a row. Build up to the answer by answering the following questions: In how many ways can they sit in a row?

(a) How many ways are there to place the 7 people into the 9 chairs if EFG must sit together in that order and CD must sit together in that order (This is not unlike the rearrangements of DOGGY that we discussed/will discuss in lecture (where the empty seats are like the two Gs))?

(b) How many ways are there to place the 7 people into the 9 chairs if EFG must sit together but E and G can swap positions and CD must sit together in either order?

(c) How many ways are there to place the 7 people into the 9 chairs if EFG must sit together (but E and G can swap positions), CD must sit together in either order and AB must sit together in either order?

(d) How many ways are there to place the 7 people into the 9 chairs if EFG must sit together (but E and G can swap positions), CD must sit together in either order and A and B must not sit next to each other?

Ans: We can seen the dating group as G1(Group1) and friend group as G2(Group2).

(a) Step 1: how we chose the 7 chairs from 9
we have A B two individuals, G1 G2 two groups, and two empty seats. Actually, we chose 2 empty seats from these six parts so $\binom{6}{2} = 15$
Step 2: How to sit the 7 chairs
If there is no restriction that AB cannot be together, total ways of seats for 4 parts A B G1 G2 is 4!.
By Product Rule, total way is $\binom{6}{2} \cdot 4! = 360$

By hint, we can directly seen the two empty seats are like the two Gs. We can also directly use $\frac{6!}{2!} = 360$

(b) Step 1: From (a) we got if G1 and G2 in that order the total ways should be 360.
Step 2: E and G can swap places providing 2 additional arrangements.
Step 3: C and D can swap places providing 2 more additional arrangements.
Product Rule: Total ways $= 360 \cdot 2 \cdot 2 = 1440$

(c) Step 1: how we chose the 7 chairs from 9
we have AB(BA), G1, G2 three groups, and two empty seats. Actually, we chose 2 empty seats from these five parts so $\binom{5}{2} = 10$
Step 2: How to sit the 7 chairs
Total ways of seats for 3 parts AB(BA) G1 G2 is $3! = 6$.
AB, EG and CD can swap the place so total ways to sit is $2 \cdot 2 \cdot 2 \cdot 3! = 48$
By Product Rule, total ways for AB G1 G2 must sit together but can swap position is $\binom{5}{2} \cdot 2 \cdot 2 \cdot 2 \cdot 3! = 480$

By hint, we can also directly seen the two empty seats are like the two Gs. We can directly use $\frac{5!}{2!} \cdot 4 \cdot 2 = 480$

(d) If we subtract the total arrangements without restrictions AB(we got in (b)) with arrangements where A and B are together(we got in (c)), we will get Total arrangements where A and B are NOT together: $\frac{6!}{2!} \cdot 4 - \frac{5!}{2!} \cdot 8 = 1440 - 480 = 960$

For this problem I didn't see the hint at first that's why I used two way to solve the problem.

6. Binomial Theorem applications (15 points)

For part (a) of this question (as with many others for 312), you could find the numerical answer in a few seconds by asking wolfram alpha (in this case, by asking it to expand the polynomial). We have learning goals associated with this problem that mean we want you to practice solving this problem by hand even though you could easily answer it with computational power.

Remember that you must give an explanation of an answer such that another student would understand the principles that go into solving the problem, and such that they could find the answer with a simple calculator (that doesn't have an "expand a polynomial" operation)

You may find it beneficial to verify your answer using WolframAlpha, but you may not use the "show steps" option on WolframAlpha or any similar tool.

(a) What is the coefficient of x^5y^{10} in the expansion of $(2x - y^2)^{10}$?

(b) Use the binomial theorem to prove that

$$\sum_{i=0}^{100} \binom{100}{i} (-4)^{100-i} = 3^{100}$$

Ans:

(a) Here are ten $(2x-y^2)$ multiplied together. For each $(2x-y^2)$, we can select 2x or $(-y^2)$for operation. According to the requirements of the question x^5y^{10}, we chose 2x 5 times and $(-y^2)$ 5 times. There are a total of $\binom{10}{5} = 252$ times that you can select this combination. The combination result should be $\binom{10}{5} \cdot (2x)^5 \cdot (-y^2)^5 = 255 \cdot 32 \cdot (-1)x^5y^{10} = -8160x^5y^{10}$, so the coefficient is -8160.

(b) Binomial Theorem: $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} (x)^i (y)^{n-i}$
For the problem we have n = 100, x = 1, and y = -4.
Substitute values into the binomial expansion:

$$(1 - 4)^{100} = \sum_{i=0}^{100} \binom{100}{i} (1)^i (-4)^{100-i}$$

$$(-3)^{100} = \sum_{i=0}^{100} \binom{100}{i} (-4)^{100-i}$$

Use properties of exponents: $(-3)^{100} = 3^{100}$ (because raising -3 to an even power eliminates the negative sign), we have

$$\sum_{i=0}^{100} \binom{100}{i} (-4)^{100-i} = 3^{100}$$

7. Coding and Reflection (15 points)

Ans: Among the various topics, I find the reshape function most interesting because it allows for dynamic reorganization of data into different dimensions without altering its values. This is particularly powerful for tasks involving multidimensional datasets. On the other hand, indexing with arrays is a bit tricky. The difference between slicing (:) and using specific indices can be confusing at first, especially when dealing with multidimensional arrays. Understanding how NumPy interprets these indexing commands requires careful attention, particularly for beginners transitioning from basic Python lists.

8. Feedback (1 points)

- I spent approximately 8 hours on this assignment.
- I spent the most time on Problem 5: Sitting around
- When communicating with classmates, I found that there are different solutions to the same problem. Sometimes when you are sure that you have found all the methods, you will find more or less. This may be the trouble of this kind of problem.