CSE332: Data Structures & Parallelism

Lecture 2: Algorithm Analysis

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Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
- Asymptotic Analysis
- Big-Oh Definition

What do we care about?

- Correctness:
  - Does the algorithm do what is intended.

- Performance:
  - Speed time complexity
  - Memory space complexity

- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.

Q: How should we compare two algorithms?

A: How should we compare two algorithms?

- Uh, why NOT just run the program and time it??
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when n doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!

Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
  - **How to count different code constructs**
  - Best Case vs. Worst Case, and more
  - Ignoring Constant Factors
- Asymptotic Analysis
- Big-Oh Definition

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Loops	Num iterations * time for loop body
Conditionals	Time of condition plus time of slower branch
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Examples

```
b = b + 5
c = b / a
b = c + 100

for (i = 0; i < n; i++) {
    sum++;
}

if (j < 5) {
  sum++;
} else {
  for (i = 0; i < n; i++) {
    sum++;
  }
}
```

Another Example

```
int coolFunction(int n, int sum) {
   int i, j;
   for (i = 0; i < n; i++) {
      for (j = 0; j < n; j++) {
        sum++;
      }
   }
   print "This program is great!"
   for (i = n; i > 1; i = i / 2) {
        sum++;
   }
   return sum
}
```

Using Summations for Loops

```
for (i = 0; i < n; i++) {
    sum++;
}
```

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Complexity cases

We'll start by focusing on two cases:

- **Worst-case complexity**: max # steps algorithm takes on "most challenging" input of size N

- **Best-case complexity**: min # steps algorithm takes on "easiest" input of size N

Example

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    ???
}
```

Linear search – Best Case & Worst Case

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case:

Worst case:

Linear search – Running Times

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

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boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6 “ish” steps = *O*(1)

Worst case: 5 “ish” * (arr.length)

= *O*(arr.length)

Remember a faster search algorithm?

Complexity cases, again

Adding two more cases:

- **Worst-case complexity**: max # steps algorithm takes on "most challenging" input of size N

- **Best-case complexity**: min # steps algorithm takes on "easiest" input of size N

- **Average-case complexity**: **avg** # steps algorithm takes on *random* inputs of size N

- **Amortized complexity**: **max** total # steps algorithm takes on **M** "most challenging" *consecutive* inputs of size N, *divided by **M*** (i.e., divide the max total by **M**).

Example – Growing an array

```java
public void add(T value){
    if (data.length == size)
        resize();
    data[size] = value;
    size++;
}
private void resize(){
    T[] oldData = data;
    data = (T[]) new Object[data.length * 2];
    for (int i = 0; i < oldData.length; i++)
        data[i] = oldData[i];
}
```

- What is the <u>**worst case**</u> running time of add?

Growing an array: Amortized Analysis

```
public void add(T value){
    if (data.length == size)
        resize();
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private void resize(){
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```

- Amortized Analysis Idea:
  - Suppose we have a program that in total does m adds.
  - How much time was spent "**on average" across all m**?

Growing an array: Amortized Analysis

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}
```

Every time we resize, we earn `data.length` more adds guaranteed to not resize!

- Amortized Analysis Idea:
  - Suppose we have a program that in total does m adds.
  - How much time was spent **"on average" across all m**?
- Let c be the initial size of `data`
  - The first c adds take: $c + c = 2c$
  - The next $2c$ adds: $2c + 2c = 4c$
  - The next $4c$ adds: $4c + 4c = 8c$
  - Average time per operation: $\frac{14c}{7c} = 2$

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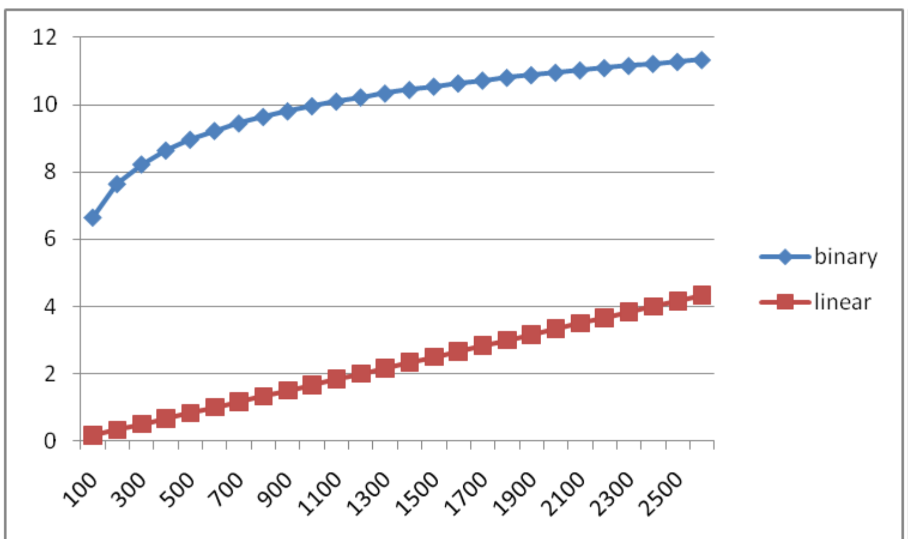
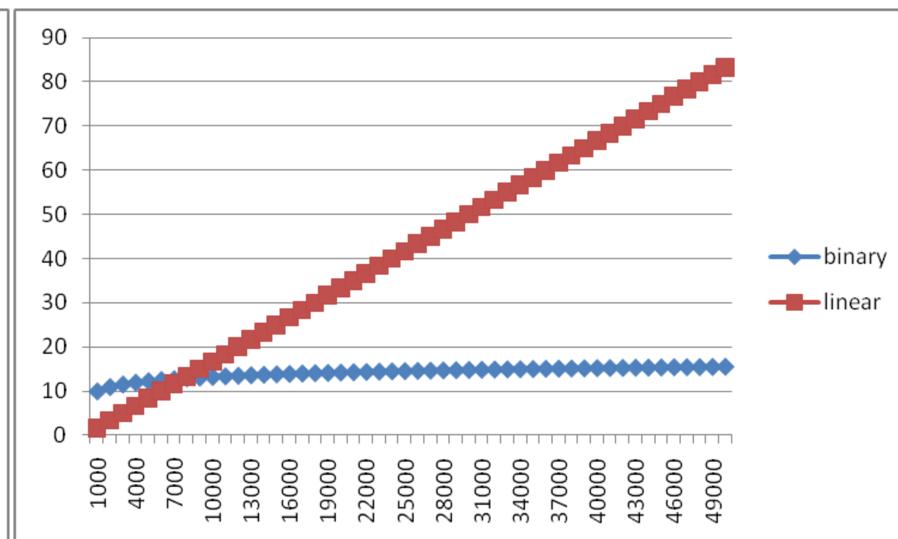
Ignoring constant factors

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which will actually be faster?
  - Depending on **constant factors** and **size of n**, in a particular situation, linear search could be faster….
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each *n*
- And could depend on size of *n* – what if n is small?
- **But** there exists some n_0 such that for all $n > n_0$ binary search "wins"
- Let's look at a couple plots to get some intuition…

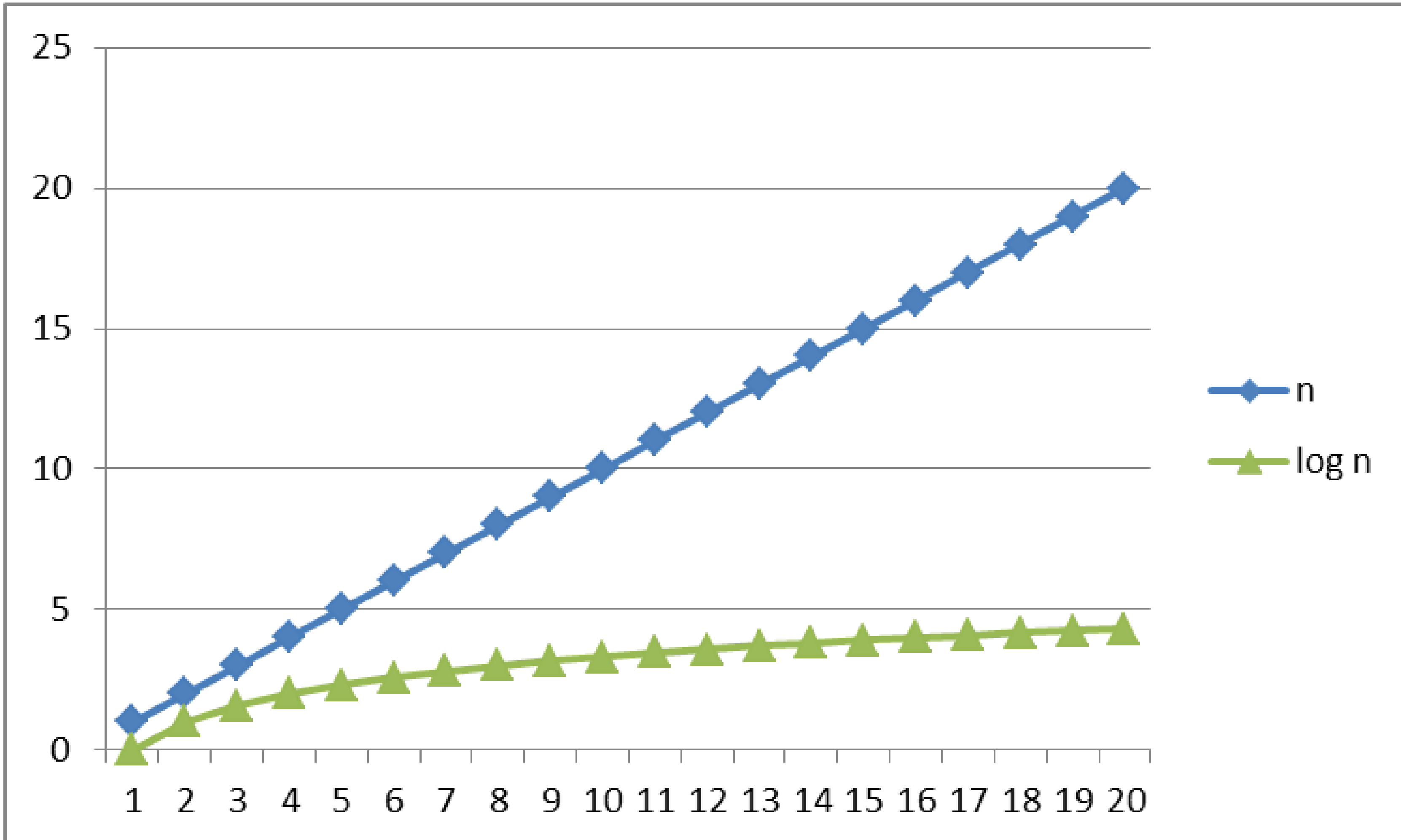
Linear Search vs. Binary Search

- Plotting linear f(N) = N and binary search f(N) = log N
  - Let's even give linear search a boost (N/600)
- For small values of N, linear search might run faster
- But eventually, binary search will win

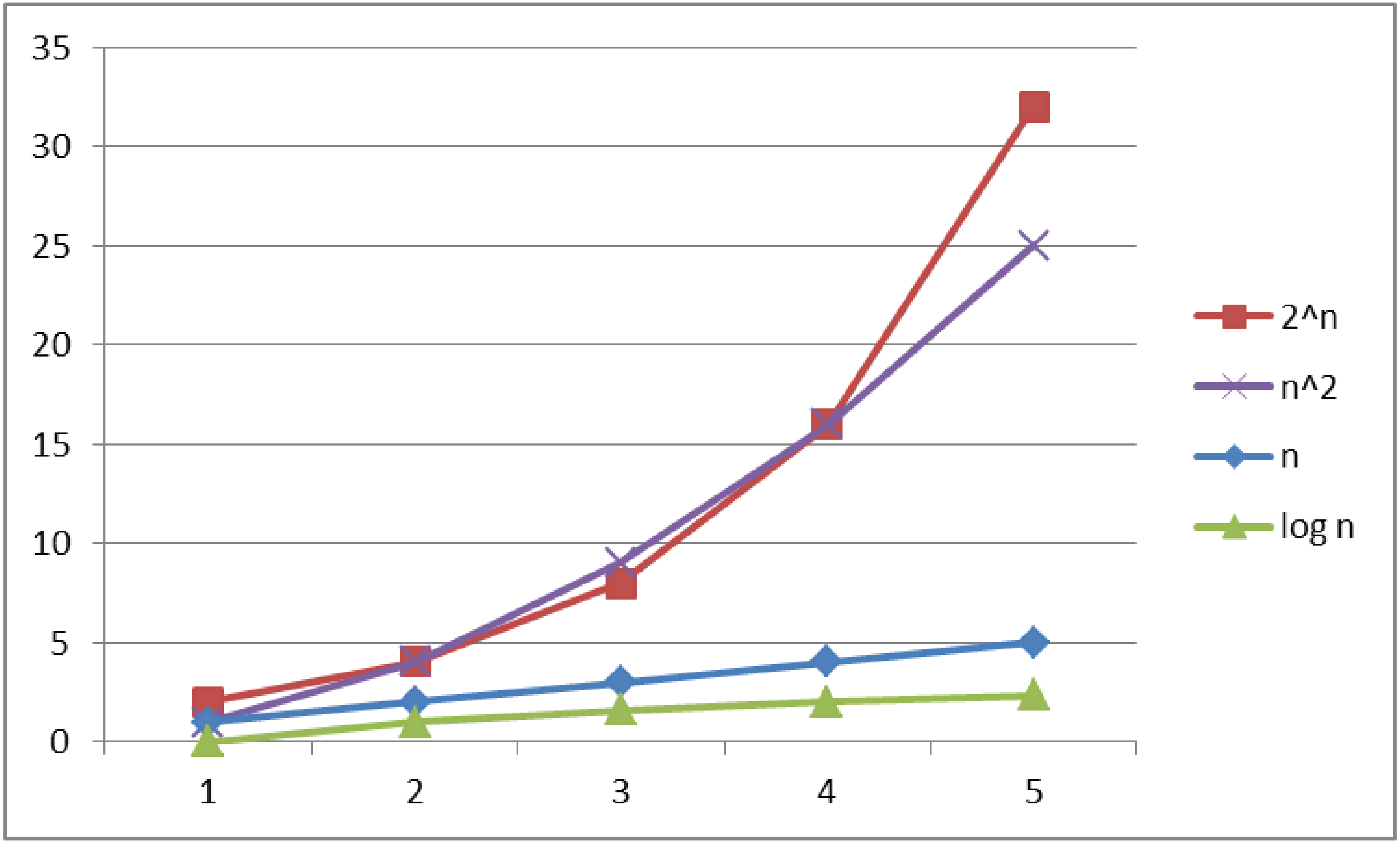




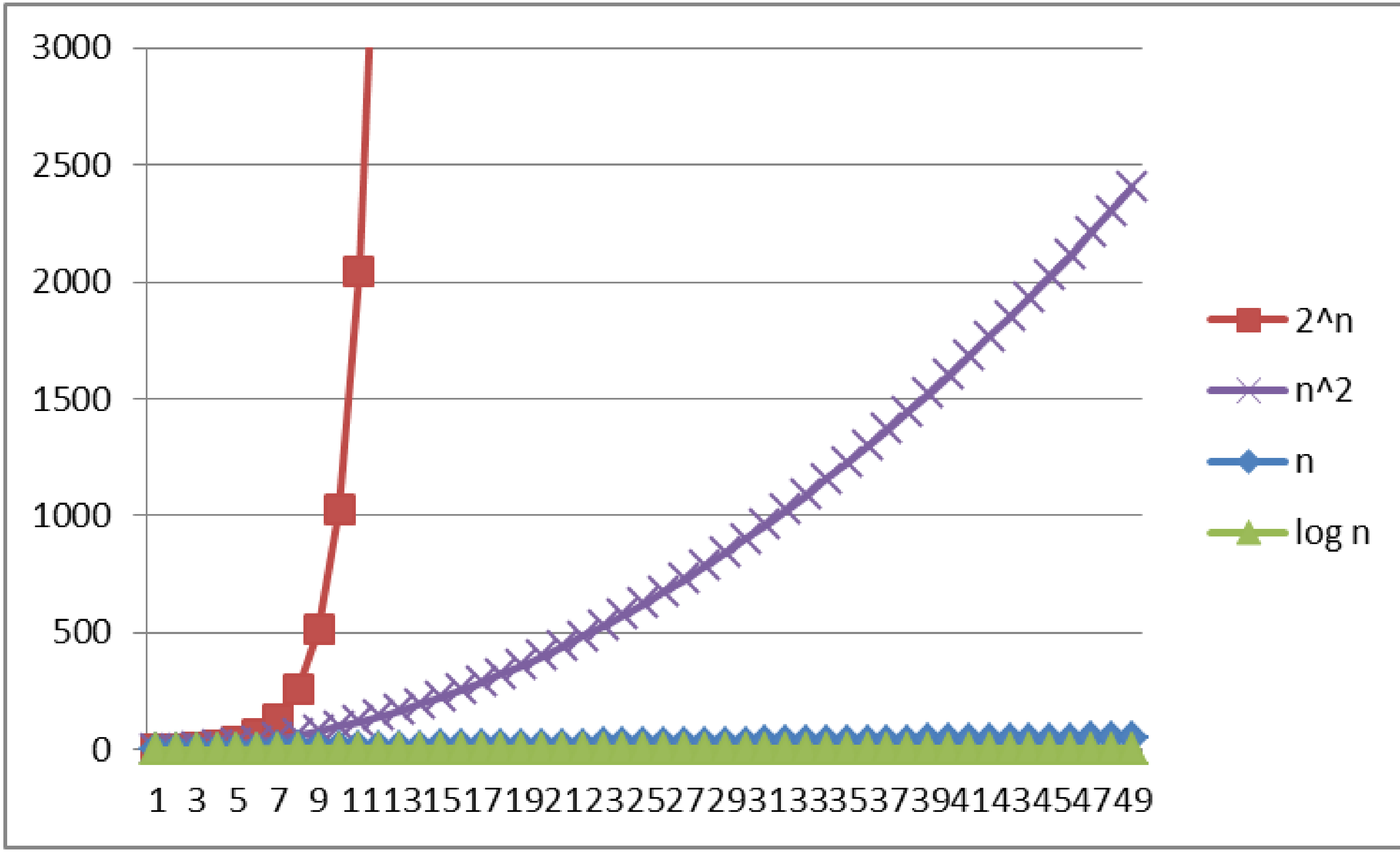
Logarithms and Exponents



Logarithms and Exponents



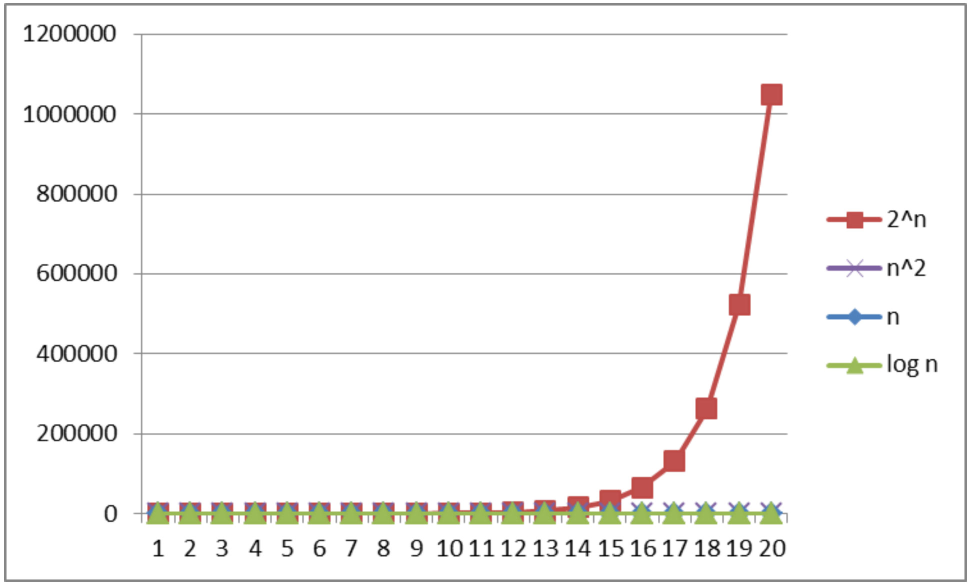
Logarithms and Exponents



Review: Logarithms and Exponents

- Since so much is binary in CS, **`log`** almost always means $\mathtt{log_2}$
- Definition: $\mathtt{log_2\ x = y}$ if $\mathtt{x = 2^y}$
- So, $\mathtt{log_2}$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

See Excel file for plot data – play with it!



Aside: Log base doesn't matter (much)

"Any base *B* log is equivalent to base 2 log within a constant factor"

- **And we are about to stop worrying about constant factors**!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- Say, to convert from base B to base A:

$$\log_B x = (\log_A x) / (\log_A B)$$

Review: Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So $\log(N^k) = k \log N$
- **`log(A/B) = log A – log B`**
- $x = \log_2 2^x$
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows fast
  - Ex:
    $$\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$$
- **`(log x)(log x)`** is written $\log^2 x$
  - It is greater than **`log x`** for all $x > 2$

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Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$

Big-Oh relates functions

We use O on a function f(n) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(n)

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same **asymptotic behavior**

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17)$ **=** $O(n^2)$

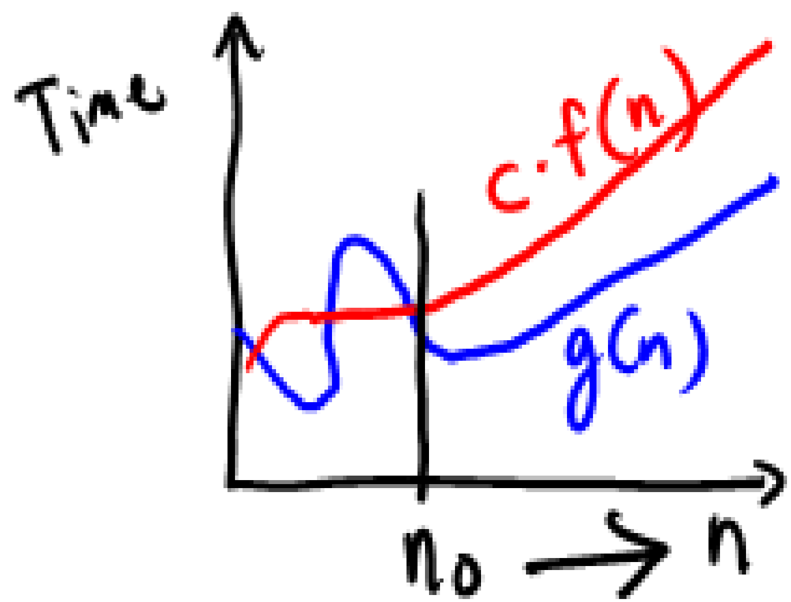
But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh

Definition: $g(n)$ is in $O(\ f(n)\ )$ iff there exist positive constants c and n_0 such that

$g(n) \leq c\ f(n)$ for all $n \geq n_0$

Note: $n_0 \geq 1$ (and a natural number) and $c > 0$



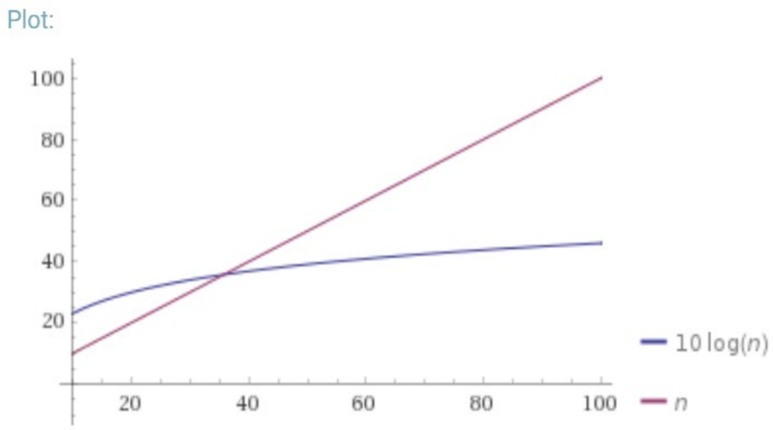
Why n_0? Why c?

Definition: $\mathbf{g}(n)$ is in O($\mathbf{f}(n)$) iff there exist positive constants c and n_0 such that

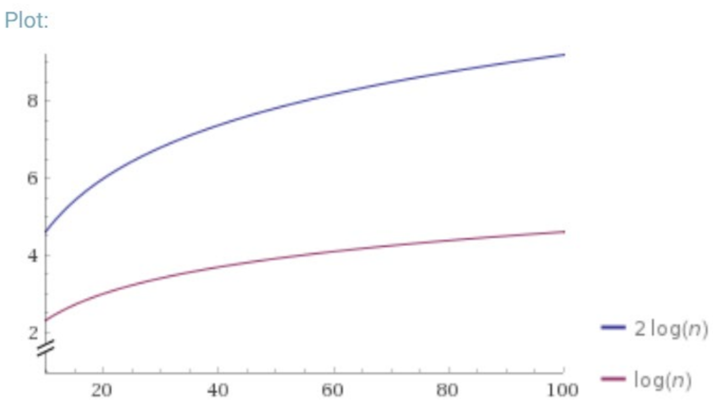
$\mathbf{g}(n) \leq c\, \mathbf{f}(n)$ for all $n \geq n_0$

Note: $n_0 \geq 1$ (and a natural number) and $c > 0$

Why n_0?



Why c?



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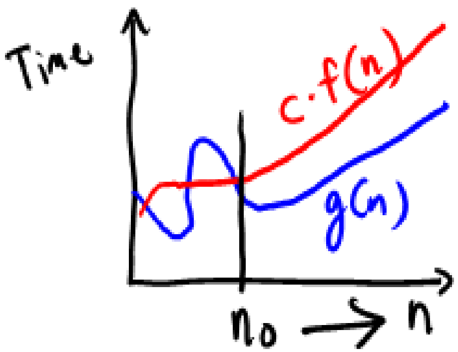

Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

g(*n*) ≤ ***c*** **f(*n*)** for all ***n*** ≥ n_0

Note: n_0 ≥ 1 (and a natural number) and *c* > 0

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms".

Example: Let **g(*n*)** = 3*n* + 4 and **f(*n*)** = *n*

c = 4 and n_0 = 5 is one possibility

This is "less than or equal to"

- So 3*n* + 4 is also $O(n^5)$ and $O(2^n)$ etc.

*What's with the **c**?*

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:

  g(n) = 7n+5

  f(n) = n
- These have the same asymptotic behavior (linear),
  so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger

- There is **no** positive $\mathbf{n_0}$ such that **g(n)** ≤ **f(n)** for all n ≥ $\mathbf{n_0}$
- The '**c**' in the definition allows for that:

  $\mathbf{g}(n) \le c\, \mathbf{f}(n)$ for all $n \ge n_0$
- To show **g(n)** is in O(**f(n)**), have **c** = 12, $\mathbf{n_0}$ = 1

An Example

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $4n^2 + 3n + 4$ and **f(*n*)** = n^3

Examples

True or false?

1. $4+3n$ is $O(n)$
2. $n+2\log n$ is $O(\log n)$
3. $\log n+2$ is $O(1)$
4. n^{50} is $O(1.1^n)$

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we cannot count operations very accurately

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic
$O(n)$	linear
$O(n\ \texttt{log}\ n)$	"n **`log`** *n*"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant > 1)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

More Asymptotic Notation

- **Upper bound**: O(**f(n)**) is the set of all functions asymptotically less than or equal to **f(n)**
  - g(n) is in O(**f(n)**) if there exist constants c and n_0 such that

    g(n) $\leq$ c **f(n)** for all $n \geq n_0$

- **Lower bound**: Ω(**f(n)**) is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(n) is in Ω(**f(n)**) if there exist constants c and n_0 such that

    g(n) $\geq$ c **f(n)** for all $n \geq n_0$

- **Tight bound**: θ(**f(n)**) is the set of all functions asymptotically equal to **f(n)**
  - Intersection of O(**f(n)**) and Ω(**f(n)**) (can use *different* c values)

Summary of Complexity cases

Problem size **N**

- **Worst-case complexity**: **max** # steps algorithm takes on "most challenging" input of size **N**
- **Best-case complexity: min** # steps algorithm takes on "easiest" input of size **N**

- **Average-case complexity**: **avg** # steps algorithm takes on *random* inputs of size **N**
- **Amortized complexity**: **max** total # steps algorithm takes on **M** "most challenging" *consecutive* inputs of size **N**, divided by **M** (i.e., divide the max total by **M**).

Regarding use of terms

A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is *also* $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
- Somewhat incomplete; instead say it is $\theta(n)$
- That means that it is not, for example $O(\texttt{log}\ n)$

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is $\omega(\texttt{log}\ n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm

- Example: True statements about binary-search algorithm
  - Common: θ(`log` n) running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is Ω(`log log` n) in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is Ω(`log` n) in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time
- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
- Sometimes an *amortized guarantee*

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for ***large n*** and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful