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ON TWO CLASSICAL RAMSEY NUMBERS OF THE FORM $R(3, n)$*

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Abstract. New lower bounds are given for the classical Ramsey numbers $R(3, 10)$ and $R(3, 12)$. Both constructions were made using a variant of the Metropolis Algorithm and were built on smaller cyclic constructions.

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Two new lower bounds for Ramsey numbers of the form $R(3, n)$ are proved below. Both proofs are completed by extending cyclic constructions for $R(3, n-1)$.

In general our notation follows that of Harary [2]. We use $R(s, t)$ to denote the classical Ramsey number of K_s versus K_t, defined to be the smallest integer n such that in any 2-coloring of the edges of K_n there is a monochromatic copy of K_s in color 1 or a monochromatic copy of K_t in color 2. A coloring of a complete graph is called an (s, t)-coloring if there are no monochromatic copies of K_s in color 1 or of K_t in color 2. A graph of order n is called a *cyclic* $n(a_1, \cdots, a_k)$ graph if its vertices can be labeled with the integers from 0 to $n-1$ so that two vertices are adjacent if and only if their difference is a_i, for some i, $1 \leqq i \leqq k$.

The underlying algorithm we used to make these constructions is a procedure that has been called *simulated annealing* [3], and is based on an algorithm devised by Metropolis et al. [4] for application to statistical mechanics. We offer a brief description. Let f be an integer-valued function of integer variables $x_1, \cdots, x_n$, and suppose we wish to find the minimum value of f. At each step of the algorithm we have a *current vector* $(x_1, \cdots, x_n)$ that may initially be chosen at random. We consider a small random change in one of the variables x_i, yielding a new vector $(x_1, \cdots, x_i', \cdots, x_n)$. The values $y = f(x_1, \cdots, x_i, \cdots, x_n)$ and $y' = f(x_1, \cdots, x_i', \cdots, x_n)$ are compared, and $\Delta Y = y' - y$ is computed. If $\Delta Y \leqq 0$, then the new vector is accepted as the current vector, otherwise the new vector is accepted with probability $\exp(-\Delta Y/k_B T)$, where k_B is the analogue of the Boltzmann constant and T is an analogue of temperature. In the course of running the algorithm we usually begin with a relatively large value for T (i.e., a high temperature), and gradually lower the value of T (i.e., allow the system to cool).

The first problem that arises when applying this procedure to Ramsey numbers is that of determining f. This issue has been discussed in some detail in [1]. New issues arise when dealing with far off-diagonal cases. Specifically, with $R(3, t)$, we must decide how much weight to give to a K_3 in color 1, as opposed to a K_t in color 2. In the context of the Metropolis Algorithm, the random change in the current vector corresponds to recoloring one edge in a given 2-coloring. Suppose that coloring a given edge in color 1 yields m_1 monochromatic K_3's in color 1, while coloring it with color 2 yields m_2 monochromatic K_t's in color 2. Let $\rho = m_2/m_1$. The question that must be answered is: For what values of ρ do we prefer color 1 and for what values do we prefer color 2? Let ρ_0

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be the value of ρ for which we are indifferent. In other words, for values of $\rho > \rho_0$ we choose color 1, for values of $\rho < \rho_0$ we choose color 2, and when $\rho = \rho_0$ we make a random choice. In practice we have found that choosing

$$\rho_0 = \binom{t}{2} \Big/ 3$$

is a good choice, so that the weights are inversely proportional to the number of edges in the graphs we are trying to avoid.

The table of Ramsey numbers given in [5] seems to be the most recently published. The values listed there for numbers of the form $R(3, n)$ are as follows:

$$\begin{array}{ll}
R(3,3)=6, & R(3,4)=9, \\
R(3,5)=14, & R(3,6)=18, \\
R(3,7)=23, & 28 \leqq R(3,8) \leqq 29, \\
R(3,9)=35, & 39 \leqq R(3,10) \leqq 44, \\
46 \leqq R(3,11) \leqq 54, & 49 \leqq R(3,12) \leqq 63.
\end{array}$$

We improve the lower bounds for $R(3, 10)$ and $R(3, 12)$ by one.

THEOREM 1. $R(3, 10) \geqq 40$.

Proof. Begin with the cyclic $(3, 9)$-coloring of K_{35} given by having the edges of the cyclic graph 35(1, 7, 11, 16, 19, 24, 28, 34) colored in color 1, and the edge of the complement colored in color 2. To this graph we add four vertices labeled a, b, c, and d. The edges joining a to c and b to d are colored in color 2. The remaining edges among these four vertices are colored in color 1. In addition, the four new vertices are joined in color 1 to those of the original 35 as listed below:

a:	2	15	19	27	32		
b:	11	17	25	29			
c:	8	16	26	28	34		
d:	1	4	10	18	22	24	30.

The remaining edges are in color 2.

THEOREM 2. $R(3, 12) \geqq 50$.

Proof. The construction proceeds just as in Theorem 1. We begin with the cyclic coloring derived from the graph 45(3, 10, 11, 12, 16, 29, 33, 34, 35, 42). Again we add four vertices, a, b, c, and d, with a adjacent to c in color 2 and b adjacent to d in color 2. All other edges among these four vertices are in color 1. The color 1 edges joining the new vertices to the original 45 are given below:

a:	1	22	24	31	39			
b:	6	7	14	28	33	37		
c:	10	12	27	34	35	36	40	
d:	2	3	4	17	21	25	30	43.

We note that evidence seems to be accumulating for the conjecture that it is the exception, rather than the rule, for Ramsey colorings to be cyclic.

REFERENCES

[1] G. Exoo, *Constructing Ramsey graphs with a computer*, Congr. Numer., 59 (1987), pp. 31–36.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[3] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, *Optimization by simulated annealing*, Science, 220 (1983), pp. 671–680.

[4] N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, *Equation of state calculations for fast computing machines*, J. Chem. Phys., 6 (1953), p. 1087.

[5] S. P. Radziszowski and D. L. Kreher, *Search algorithm for Ramsey graphs by union of group orbits*, J. Graph Theory, 12 (1988), pp. 59–72.