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Ramsey Numbers for Circulant Colorings

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Abstract. A 2-coloring of the diagonals (and sides) of a convex n-gon is called a circulant 2-coloring of K_n if all diagonals of the same length have the same color. For graphs G and H, the circulant Ramsey number $RC_1(G,H)$ is the largest number such that for every $n < RC_1$ a circulant 2-coloring without G in the first and H in the second color exists. The circulant Ramsey number $RC_2(G,H)$ is the smallest number such that for $n \geq RC_2$ every circulant 2-coloring of K_n contains G of the first or H of the second color. The numbers RC_1 and RC_2 are determined for paths and cycles, for small complete graphs and for graphs with up to five vertices. As a by-product the new lower bounds $R(K_4,K_{10}) \geq 92$ and $R(K_5,K_8) \geq 101$ for the classical Ramsey numbers are determined instead of 80 and 95, respectively.

1. Introduction

For given graphs G and H the classical Ramsey number $R = R(G,H)$ is the smallest number R of vertices of a complete graph K_R such that every 2-coloring of the edges of K_R contains G in the first color or H in the second color. For the known exact values of $R(G,H)$ see [5]. Many lower bounds for $R(G,H)$ are determined by the so-called circulant colorings, that means, the edges are represented by the diagonals of an R-gon and all diagonals having the same length, that is, having the same distance between their endpoints are colored equally. This may be a reason to ask for all values of n such that every circulant 2-coloring of the edges of K_n contains G or H in the first or second color, respectively. It is different from the classical case that for circulant colorings the property for K_n does not imply this property for K_{n+1}, too. So KALBFLEISCH [4] observed already that every circulant 2-coloring of K_{16} contains a monochromatic K_4, however, there exists the well-known circulant 2-coloring of K_{17} proving $R(K_4) = R(K_4,K_4) \geq 18$. Therefore we will ask for two circulant Ramsey numbers. On the one hand, we will ask for the largest number $RC_1(G,H)$ such that for every $n < RC_1(G,H)$ a critical circulant 2-coloring of K_n exists, that is, a 2-coloring without G or H in the first or second color, respectively. On the other hand, we will ask for the smallest number $RC_2(G,H)$ such that for every $n \geq RC_2(G,H)$ no critical circulant 2-coloring of

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K_n does exist. Clearly, it holds

$$RC_1(G,H) \leq RC_2(G,H) \leq R(G,H).$$

We will discuss these circulant Ramsey numbers for paths P_s and cycles C_s, small complete graphs K_s, and all connected graphs with up to five vertices.

2. Paths and Cycles

For path P_s and cycles C_s the classical Ramsey numbers are $R(P_s,P_s) = R(P_s) = s + \lfloor \frac{s}{2} \rfloor - 1$ [2] and $R(C_s) = 6$ if $s = 3,4$, $R(C_s) = 3s/2 - 1$ if $s \equiv 0 \pmod 2$, $s \geq 6$, and $R(C_s) = 2s - 1$ if $s \equiv 1 \pmod 2$, $s \geq 5$ [1,6,7]. The corresponding circulant Ramsey numbers are given in the following theorem.

Theorem 1. For paths P_s and cycles C_s we have

$$RC_1(P_s) = RC_2(P_s) = RC_1(C_s) = s$$

and

$$RC_2(C_s) = \begin{cases} 6 & \text{if } s = 3 \text{ or } s = 4, \\ s & \text{if } s \equiv 0 \pmod 2,\ s \geq 6, \\ 2s-1 & \text{if } s \equiv 1 \pmod 2,\ s \geq 5. \end{cases}$$

Proof. The first three values are trivial since the diagonals of length 1 of an s-gon form a monochromatic graph C_s and the diagonals of length 1 of an n-gon contain a monochromatic P_s for $n \geq s$.

For $s = 3,4$ the well-known coloring of a 5-gon with the sides of the first color (green) and the diagonals of length 2 of the second color (red) shows $RC_2(C_s) > 5$. The upper bound $RC_2(C_s) \leq R(C_s) = 6$ for $s = 3,4$ follows from the classical Ramsey number.

For $s \equiv 0 \pmod 2$, $s \geq 6$, the lower bound s is trivial. For the upper bound we consider an n-gon with $n \geq s$ and distinguish three cases.

Diagonals of lengths 1 and 2 are of one color: We choose two paths $P_{s/2}$ with $(s-2)/2$ diagonals of length 2, both clockwise and starting at two vertices of distance 1. Since both pairs of endpoints are connected by diagonals of length 1 we obtain a monochromatic C_s (see Figure 1 (a)).

Diagonals of lengths 1 and 3 are of one color: Clockwise we choose two paths $P_{s/2}$ starting at consecutive vertices of the n-gon. The first path uses alternatingly diagonals of lengths 3 and 1 and ends after $s/4$ or $(s-2)/4$ diagonals of length 3 or 1 for $s \equiv 0$ or $s \equiv 2 \pmod 4$, respectively. The second path uses alternatingly diagonals of lengths 1 and 3 and ends after $s/4$ or $(s-2)/4$ diagonals of length 1 or 3 for $s \equiv 2$ or $s \equiv 0 \pmod 4$, respectively. Connecting the pairs of endpoints by diagonals of length 1 closes a monochromatic C_s (see Figure 1 (b)).

Diagonals of lengths 2 and 3 are of one color: Clockwise we choose two paths $P_{s/2}$ of the n-gon starting at vertices of distance 3 and using diagonals of length 2 only. Connecting both pairs of endpoints by diagonals of length 3 we obtain a monochromatic C_s if $n \geq s+2$ (see Figure 1 (c)). For $n = s$ the diagonals of length 1 form a monochromatic C_s so that $n = s+1$ remains. In this case we modify the two just used paths $P_{s/2}$ by substituting the second diagonal of the first and the first diagonal of the second path by a diagonal of length 3, respectively (see Figure 1 (d)).

For $s \equiv 1 \pmod 2$, $s \geq 5$, the classical Ramsey number determines $2s-1$ as an upper bound. The lower bound is established by the following circulant 2-coloring of K_{2s-2}. If the vertices are labeled clockwise by 1 to $2s-2$ then all vertices of different parity are connected in green and the remaining diagonals are red. Then the green subgraph is bipartite and thus it does not contain an odd cycle C_s. The red subgraph consists of components of order $s-1$ so that a red C_s does not exist. □

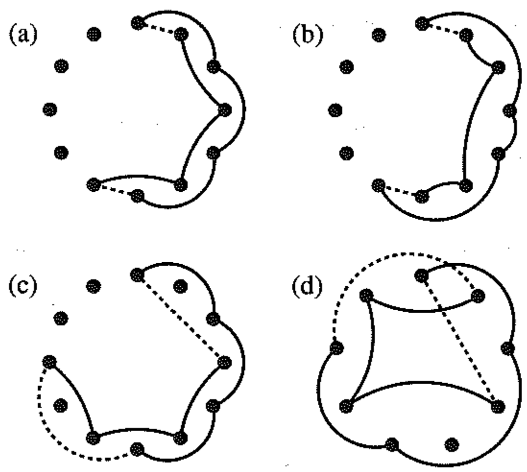


Figure 1. Monochromatic cycles with diagonals of two lengths 1, 2, or 3.

It may be remarked that for $s \equiv 1 \pmod 2$ critical circulant 2-colorings of K_n, that are colorings without a monochromatic C_s, can be constructed for n even with $s+1 \leq n \leq 2s-4$ corresponding to the above construction for $n = 2s-2$. For n odd with $n \geq s+2$, any circulant 2-coloring contains a monochromatic C_s.

3. Complete Graphs

For pairs of small complete graphs we start with a lower bound.

Lemma 1. For $3 \leq s \leq t$ and $t \geq 4$ we have

$$RC_1(K_s, K_t) \geq (s-1)t.$$

Proof. Critical circulant 2-colorings (green/red) of the K_n have to be constructed for all $n < (s-1)t$. For $n < t$ a red K_n is critical trivially.

For $t \leq n \leq 2s-1$ all diagonals of length 1 are colored in green and the remaining diagonals in red. Every set of s vertices determines at least one diagonal of length 1 and one diagonal of larger length. Since $s \leq t$ no green K_s and no red K_t does exist.

For $2s \leq n \leq (s-1)t-1$ the diagonals of lengths $1, 2, \ldots, s-2$ are green and the remaining diagonals are red. Any set of s vertices contains one pair having a distance of at least $s-1$. The corresponding diagonal is red and thus a green K_s does not occur. If a red K_t exists then consecutive vertices of the t-gon must have distances of at least $s-1$ in the n-gon, however $t(s-1) > n$. □

For small s and t we determine exact values and bounds of $RC_1(K_s, K_t)$ and $RC_2(K_s, K_t)$, some of them with the help of a computer.

Theorem 2. All known circulant Ramsey numbers $RC_1(K_s, K_t)$ and $RC_2(K_s, K_t)$, compared with the classical Ramsey numbers $R(K_s, K_t)$, most of them as in [5], are given in Tables 1 to 4.

$t =$	3	4	5	6	7	8	9	10	11	12
$RC_1(K_3, K_t)$	3	9	14	15	22	25	34	37	46	49
$RC_2(K_3, K_t)$	6	9	14	17	22	27	36	39	46	49
$R(K_3, K_t)$	6	9	14	18	23	28	36	40\43	46\51	52\59

Table 1.

$t =$	4	5	6	7	8	9	10	11
$RC_1(K_4, K_t)$	14	23	32	43	52	69	76	96
$RC_2(K_4, K_t)$	18	25	34	47	52	69	92	96
$R(K_4, K_t)$	18	25	35\41	49\61	56\84	69\115	92\149	96\191

Table 2.

$t =$	5	6	7	8
$RC_1(K_5, K_t)$	39	54	76	95
$RC_2(K_5, K_t)$	42	57	80	101
$R(K_5, K_t)$	43\49	58\87	80\143	101\216

Table 3.

$t =$	6
$RC_1(K_6, K_t)$	69
$RC_2(K_6, K_t)$	102
$R(K_6, K_t)$	102\165

Table 4.

Proof. For Table 1 the case $t = 3$ is covered by Theorem 1. For $t = 4$ we use $R(K_3, K_4) = 9$, Lemma 1, and the well-known circulant 2-coloring of the 8-gon with green edges of lengths 1 and 4. If $t = 5$ then $R(K_3, K_5) = 14$, Lemma

1, and the 2-colorings of a 10-, 11-, 12-, and 13-gon with green edges of pairs of lengths $(1,4)$, $(1,3)$, $(2,3)$, and $(1,5)$, respectively, determine $RC_1(K_3,K_5) = RC_2(K_3,K_5) = 14$.

For Table 1, $t = 6$, the 2-coloring of K_{14} with green edges of lengths 1 and 4 proves $RC_1(K_3,K_6) \geq 15$. To prove $RC_1 \leq 15$ we assume the existence of a circulant 2-coloring of K_{15} without a green K_3 and a red K_6.

If a green edge of length d exists with $(d,15) = 1$ then choosing every dth vertex, the 2-coloring of the 15-gon is mapped into an isomorphic one with edges of length 1 in green. Then edges of lengths 2, 5, and 7 have to be in red. The K_6 in Figure 2 forces green for the length 3. This implies red edges of lengths 4 and 6 yielding the red K_6 in Figure 3, a contradiction.

It remains that all edges of length d with $(d,15) = 1$ are red ($d = 1,2,4,7$). Moreover, diagonals of length 5 are red. Also diagonals of lengths 3 or 6 are red and the red K_6 in Figure 2 or 3, respectively, is a contradiction.

The 2-coloring of K_{16} with green edges of lengths 1, 3, 8 proves $RC_2(K_3,K_6) \geq 17$.

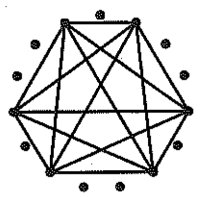


Figure 2. K_6 using edges of lengths 2, 3, 5, and 7 only.

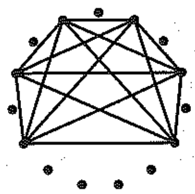


Figure 3. K_6 using edges of lengths 2, 4, 5, 6, and 7 only.

Again choosing every dth vertex of a 17-gon we may assume diagonals of length 1 in green in a circulant 2-coloring of K_{17} without a green K_3 and red K_6. Then edges of lengths 2 and 8 have to be in red. To avoid green triangles, at most one green length occurs in the pairs of lengths $(3,4)$, $(4,5)$, $(5,6)$, $(6,7)$, $(3,7)$, $(3,6)$, and $(5,7)$. Consider the three triples of lengths $(3,5,6)$, $(3,5,7)$, and $(4,6,7)$. For at least one of them all diagonals are in red since otherwise one of the pairs is covered by green edges. Since the three triples determine the red 6-gons in Figures 4, 5, and 6, respectively, a contradiction is guaranteed.

The remaining values in Table 1 for $7 \leq t \leq 12$ are determined with the help of a computer. The circulant colorings used for the lower bounds are listed in Table 5 if they are not mentioned earlier.

For Table 2, $t = 4$, the circulant 2-colorings of K_{12} and K_{13} with green edges of lengths 1, 2, 6 together with Lemma 1 prove $RC_1(K_4) \geq 14$.

To prove $RC_1(K_4) \leq 14$ we assume the existence of a circulant 2-coloring of K_{14} without a monochromatic K_4. The seven 4-gons in Figure 7 are determined by triples of diagonal lengths. By the assumption no triple is allowed to be monochro-

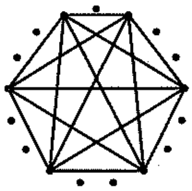


Figure 4. K_6 using edges of lengths 2, 3, 5, 6, and 8 only.

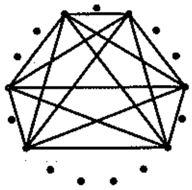


Figure 5. K_6 using edges of lengths 2, 3, 5, 7, and 8 only.

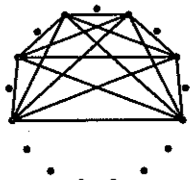


Figure 6. K_6 using edges of lengths 2, 4, 6, 7, and 8 only.

n	t	lengths
15	7	3
17	7	1,4
18	7	2,3
19	7	1,3,7
20	7	1,5,8
21	7	1,3,8
22	8	1,5,8
23	8	1,3,8
24	8	1,6,10
26	8	1,3,8,13
25	9	1,3,8

n	t	lengths
27	9	1,3,11
28	9	1,3,8
29	9	1,5,13
30	9	1,3,7,12
31	9	1,3,5,12
32	9	1,4,6,15
33	9	1,6,10,15
35	9	1,7,11,16
34	10	1,3,8,14
36	10	1,3,8,13,18
38	10	1,3,5,12,19

n	t	lengths
37	11	1,3,7,12
39	11	1,3,9,14
40	11	1,3,5,12,19
41	11	1,3,5,7,16
42	11	1,3,9,15,20
43	11	1,4,6,9,20
44	11	1,4,10,16,21
45	11	1,3,5,12,19
46	12	1,3,8,14,20
47	12	1,3,5,12,19
48	12	1,3,8,14,18,24

Table 5. Green edges in critical circulant colorings of K_n for K_3 versus K_t.

matic. The seven triples form a Steiner triple system of order 7, that is, each length occurs in exactly three triples and each pair of lengths occurs in exactly one triple. In one color, say green, there are at most three lengths. Two of them occur in five

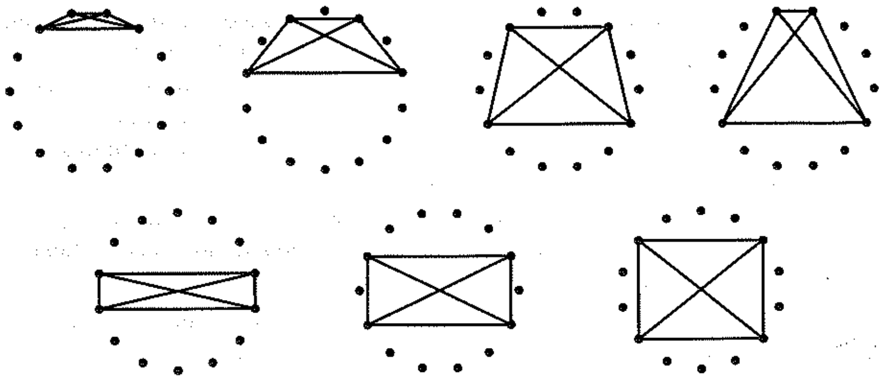


Figure 7. Subgraphs K_4 of K_{14} using edges of lengths $(1,2,3)$, $(2,4,6)$, $(3,5,6)$, $(1,4,5)$, $(1,6,7)$, $(2,5,7)$, and $(3,4,7)$, respectively.

triples. The triple containing both of these lengths cannot contain the third green length. Thus this third length occurs in two of the five triples and in a sixth triple. Then the seventh triple remains with red lengths, a contradiction.

n	t	lengths
15	5	1,2,6
16	5	2,4
18	5	1,3,4
19	5	1,7,8
20	5	1,2,4,9
21	5	1,3,8,9
22	5	1,2,8,9
24	5	1,2,4,8,9
23	6	1,2,4,8
25	6	1,2,4,5,12
26	6	1,2,4,9,10
27	6	1,2,4,8,9
28	6	1,2,4,8,9
29	6	1,2,4,5,7,14
30	6	1,2,4,5,10,12
31	6	1,2,4,5,8,15
33	6	1,2,4,10,11,15
32	7	1,2,4,8,9
34	7	2,4,8,16
35	7	1,2,4,5,10,12
36	7	1,2,4,8,9,16
37	7	1,2,4,5,7,14,15
38	7	1,2,4,11,15,16
39	7	1,2,4,5,7,14,15
40	7	1,2,4,5,10,11,18
41	7	1,2,4,10,12,16,17
42	7	1,7,10,12,14,15,18
44	7	1,2,4,10,11,13,17,18
45	7	1,2,5,6,8,12,20,21
46	7	1,2,4,12,13,17,19,20
43	8	1,2,4,5,7,11,21
47	8	1,2,4,5,7,11,13,21
48	8	1,2,4,5,14,16,21,24
49	8	1,2,4,5,13,14,21,24
50	8	1,2,4,5,8,13,15,16,23
51	8	1,2,4,10,13,17,18,24
52	9	1,2,4,5,13,14,24
53	9	1,2,7,16,17,24,25
54	9	1,2,4,5,10,12,17
55	9	1,2,9,10,17,23,25,26
56	9	1,2,11,12,17,18,26
57	9	1,3,4,6,14,17,21,24
58	9	1,2,10,11,17,18,26,27
59	9	1,2,7,8,18,19,27,28
60	9	2,3,5,6,21,22,24,25
61	9	1,2,4,5,7,10,17,18,20,28
62	9	1,2,4,12,13,17,20,27,29
63	9	1,2,6,7,14,17,24,26,30
64	9	1,2,4,5,12,13,18,19,28,30
65	9	1,2,4,5,12,14,18,20,29,30
66	9	1,2,4,12,15,19,20,28,29,31
67	9	1,2,4,5,12,13,15,19,21,30,31
68	9	1,2,7,8,16,18,19,27,28,32
69	10	1,2,4,9,11,12,14,15,30,32
70	10	1,2,4,12,20,21,25,27,28,35
71	10	1,2,4,7,11,13,22,27,29,30
72	10	1,2,4,5,7,14,18,20,27,28,35
73	10	1,2,4,7,9,10,18,20,24,32,33
74	10	1,2,4,8,17,18,20,21,23,27,28,32
75	10	1,2,4,9,10,18,20,21,23,29,33,35
77	10	1,2,4,5,8,10,20,21,28,31,32,35
79	10	1,2,4,5,8,13,17,23,24,26,31,33,37
91	10	1,2,4,7,8,14,16,17,23,27,28,32,34,37,45
76	11	1,2,4,5,7,8,16,21,29,34
78	11	1,2,4,14,16,24,25,27,32,34
80	11	1,2,4,5,7,10,18,20,23,31,36
81	11	1,2,4,5,14,18,20,28,30,35,36
82	11	1,2,4,5,12,14,18,20,29,30,35
83	11	1,2,6,9,14,15,17,36,37,39,40
84	11	1,2,6,7,9,10,22,23,25,37,40
85	11	1,2,6,7,9,10,15,29,30,32,33
86	11	1,2,4,5,12,13,15,20,21,30,32,36
87	11	1,2,4,7,13,15,16,26,29,33,34,36,41
88	11	1,2,4,7,11,12,21,22,29,32,40,42
89	11	1,2,4,9,10,16,20,21,26,28,29,33,43
90	11	1,2,4,12,15,23,24,28,30,34,35,39,41,42
92	11	1,2,4,14,15,21,23,24,32,35,39,40,42
93	11	1,2,4,8,9,16,18,21,23,29,32,35,36,42,46
94	11	1,2,4,8,14,17,21,22,32,33,35,41,42,46
95	11	1,2,4,5,8,13,16,17,25,27,32,34,35,41,45

Table 6. Green edges in critical circulant colorings of K_n for K_4 versus K_t.

n	t	lengths
23	5	1,3,4,7
25	5	1,2,3,5,8
26	5	1,2,4,5,6
27	5	1,3,4,6,7
28	5	1,2,10,11,12
29	5	1,2,3,8,10,11
30	5	1,4,5,6,7,8
31	5	1,2,3,11,12,13
32	5	1,2,3,10,12,13,14
33	5	1,2,3,5,12,13,15
34	5	1,2,6,7,8,15,16
35	5	1,2,3,5,12,13,14,16
36	5	1,2,4,5,12,14,15,16
37	5	1,2,6,8,9,11,12,17
38	5	1,2,6,10,11,12,15,17,18
40	5	1,2,4,5,7,12,16,17,18
41	5	1,2,3,5,7,10,13,15,16,17
39	6	1,2,3,5,6,7,13,16
42	6	1,2,3,5,8,13,18,20,21
43	6	1,2,3,5,6,7,9,14,20
44	6	1,2,3,5,6,9,15,16,18
45	6	1,2,3,5,6,7,10,15,16,21
46	6	1,2,3,5,6,11,13,14,15,20
47	6	1,2,3,5,6,7,9,15,16,21

n	t	lengths
48	6	1,2,3,5,6,11,13,14,20,23,24
49	6	1,2,3,5,6,7,9,11,16,17,23
50	6	1,2,3,6,8,9,11,21,22,24,25
51	6	1,2,3,5,7,9,10,16,17,19,21,22
52	6	1,2,3,5,8,9,12,13,15,18,20,21
53	6	1,2,3,5,6,7,9,15,16,21,23,26
55	6	1,2,6,10,11,13,14,16,19,20,21,25
56	6	1,2,3,5,6,9,14,16,18,19,24,25,28
54	7	1,2,6,7,13,14,15,17,24
57	7	1,2,3,6,7,13,14,16,21,27
58	7	1,5,6,12,14,17,22,26,27,28
59	7	1,2,3,5,6,7,13,18,24,25,27
60	7	1,2,6,7,9,10,11,15,18,24,25
61	7	1,2,3,7,8,15,16,17,19,24,27
62	7	1,2,3,5,7,15,16,18,23,28,29,31
63	7	1,2,3,5,8,10,12,14,15,16,18,24
64	7	1,2,3,5,6,7,10,16,17,19,20,30,31
65	7	1,2,3,8,10,11,12,14,20,24,30,31
66	7	1,2,3,5,6,17,18,19,24,25,27,28,29
67	7	1,2,3,8,10,11,12,14,20,24,31,32
68	7	1,4,8,9,13,15,16,19,21,25,32,33,34
69	7	1,2,3,8,9,10,12,13,14,20,24,32,33
70	7	1,2,3,6,7,8,10,12,15,16,19,27,31,33
71	7	1,2,3,5,7,10,13,16,17,24,29,30,31,33
72	7	1,4,5,6,7,8,11,13,17,19,30,32,35,36

Table 7. Green edges in critical circulant colorings of K_n for K_5 versus K_t.

It may be remarked, that by similar arguments one can prove that every circulant 2-coloring of K_{15} and K_{16} contains a monochromatic K_4. To prove $RC_2(K_4) = 18$ we use the well-known circulant coloring of K_{17} with green edges of lengths 1, 2, 4, 8 proving $R(K_4) = 18$.

The remaining values in Table 2 for $5 \leq t \leq 11$ and all values of Tables 3 and 4 are determined by computer. The corresponding circulant colorings for the lower bounds are listed in Tables 6 to 9. Those critical colorings of K_n for K_t being critical for K_{t-1} are not repeated. $\square$

It may be remarked that the circulant colorings proving $RC_2(K_4, K_{10}) \geq 92$ and $RC_2(K_5, K_8) \geq 101$ improve the lower bounds $R(K_4, K_{10}) \geq 80$ and $R(K_5, K_8) \geq 95$ (see [5]) by 12 and 6, respectively.

4. Small Graphs

For all connected graphs G with up to five vertices the values of $RC_1(G)$ and $RC_2(G)$ are determined.

n	t	lengths
73	7	1,2,3,5,8,11,16,17,18,28,29,31,32,33,35
74	7	1,2,3,8,10,11,12,14,15,16,23,27,33,35,37
75	7	1,2,3,5,8,9,10,17,19,20,28,30,33,34,36
79	7	1,2,3,5,8,10,18,20,21,22,30,31,34,35,36,38
76	8	1,2,3,5,7,9,11,12,19,20,22,27,34,37
77	8	1,2,3,5,6,7,9,14,20,21,25,27,30,33
78	8	1,2,3,6,17,19,20,21,23,24,29,30,31,36
80	8	1,2,3,6,8,13,17,19,20,23,25,28,33,36,37
81	8	1,2,3,5,6,7,9,11,13,14,24,25,26,30,32,35
82	8	1,2,3,5,7,9,10,17,19,21,22,28,29,30,32,35
83	8	1,2,3,5,7,8,9,11,19,20,24,28,33,35,37,40
84	8	1,2,3,5,6,7,9,11,16,21,25,26,27,33,36,40,42
85	8	1,3,4,9,10,11,12,16,19,25,30,31,34,35,36,39
86	8	1,2,4,7,10,11,12,15,17,23,26,31,32,34,36,41,43
87	8	1,2,3,5,7,8,9,14,22,23,25,32,33,35,36,37,42
88	8	1,2,3,5,6,7,9,11,15,16,20,22,25,28,34,35,41,44
89	8	1,2,3,8,9,10,12,14,15,17,19,20,21,23,25,31,38,42
90	8	1,2,3,5,7,9,10,17,18,24,25,27,29,32,38,39,40,42,45
91	8	1,2,3,5,8,13,18,20,21,22,24,27,29,30,33,34,36,41,43,44
92	8	1,2,3,5,6,7,9,11,13,14,19,24,25,27,29,36,41,42,44,46
93	8	1,2,3,5,9,12,15,16,22,26,27,30,32,34,35,36,38,41,43,45
94	8	1,2,3,5,7,9,11,12,18,19,21,23,26,33,34,36,38,39,40,46,47
100	8	1,2,3,6,7,8,13,15,19,22,25,26,29,31,33,36,37,40,42,43,47,48,50

Table 8. Green edges in critical circulant colorings of K_n for K_5 versus K_t.

Theorem 3. The circulant Ramsey numbers $RC_1(G)$ and $RC_2(G)$ compared with the classical Ramsey number $R(G)$ from [3] are as listed in Table 10.

Proof. For the graphs G_1, G_2, and G_3 the classical Ramsey number $R = 6$ and the 2-colorings of K_4 and K_5 with diagonals of length 1 in green prove $RC_1 = RC_2 = 6$.

For G_4 to G_{16} we obtain $RC_1 \geq 7$ from the 2-colorings of 4-, 5-, and 6-gons with diagonals of length 2 in green containing no monochromatic G_4, G_5, or G_{15}. Any 2-coloring of K_7 contains a monochromatic $K_7 - C_7$. This proves $RC_1 \leq 7$ since $K_7 - C_7$ contains G_{14} and G_{16} being supergraphs of the remaining graphs.

For G_4 and G_5 it follows $RC_2 = 7$ by the classical Ramsey number. We prove the non-existence of critical circulant colorings of K_8 and K_9 to obtain $RC_2(G_6) \leq 7$. To avoid a monochromatic G_6, diagonals of lengths 1 and 2 so as of lengths 2 and 4 have to be of different colors in K_8 and in K_9. Then lengths 1 and 4 in K_9 determine a monochromatic G_6. In K_8 lengths 2 and 3 have to be of different color. Then lengths 1, 3, and 4 are of the same color implying a monochromatic G_6 in K_8.

n	lengths
54	1,2,3,4,6,7,8,9,11,17,20,23
57	1,2,3,4,6,7,8,9,12,16,24,25
58	1,2,3,4,6,7,8,9,11,18,20,22,24
59	1,2,3,4,6,7,8,9,14,17,21,25,28
60	1,2,4,5,7,10,11,12,14,16,17,24,28
61	1,2,3,4,6,7,8,18,19,20,21,27,28
62	1,2,3,4,6,7,8,9,14,16,20,23,26,27
63	1,2,3,4,6,7,8,9,12,13,20,21,24,25,27
64	1,2,3,4,6,7,8,9,13,16,20,22,24,26
65	1,2,3,4,6,7,8,9,11,16,17,20,24,28,30
66	1,2,3,4,7,9,11,12,13,14,16,20,27,30,31,33
67	1,2,3,4,6,7,8,9,12,13,18,20,24,26,32
68	1,2,3,4,7,8,10,12,14,15,17,18,19,27,28,31,33
101	1,4,5,6,9,13,14,16,17,19,20,21,22,23,24,25,30,31,33,36,37,43,45,47,49

Table 9. Green edges in critical circulant colorings of K_n for K_6 versus K_6.

For G_7 to G_{14} an 8-gon with diagonals of lengths 1 and 3 in green proves $RC_2 \geq 9$ since the green subgraph does not contain a triangle and the red subgraph consists of two disjoint K_4s. The coloring of K_8 with edges of lengths 1 and 2 in green proves $RC_2(G_{15}) \geq 9$. By the classical Ramsey number it follows $RC_2 \leq 9$ for G_7 to G_{11}. Again any 2-coloring of K_9 with diagonals of pairs of lengths $(1,2)$, $(1,4)$, and $(2,4)$ of the same color contains a monochromatic G_{14} and thus G_{12} and G_{13}, too. To avoid a monochromatic G_{15} in a circulant coloring of K_9, exactly one green length occurs in pairs of lengths $(3,1)$, $(3,2)$, and $(3,4)$. Thus edges of lengths 1, 2, and 4 are of the same color determining a monochromatic G_{15}. Then by $R = 10$ we have $RC_2 \leq 9$ for G_{12} to G_{15}.

For G_{16} the critical coloring of K_9 with diagonals of lengths 1 and 2 in green together with $R = 10$ proves $RC_2 = 10$.

For G_{17} and G_{18} we obtain $RC_1 \geq 11$ by the 2-colorings used for G_{15} and by 2-colorings of K_7, K_9, and K_{10} where edges of lengths 1 are in green, edges of length 2 in green for K_9 and K_{10}, and all other edges in red. There are five different lengths of diagonals in K_{11}. Three of them are of one color, say green. We may assume diagonals of length 1 to be in green since every dth vertex of an 11-gon can be chosen if a diagonal of length d is in green. For each of the six possibilities to select two further green diagonal lengths a green G_{17} and G_{18} is guaranteed (see Figures 8 and 9). This proves $RC_1 \leq 11$. The classical Ramsey numbers $R = 14$ and $R = 15$ and 2-colorings of K_{13} and K_{14} where the green edges are of lengths 1, 3, and 4 prove $RC_2 = 14$ and $RC_2 = 15$ for G_{17} and G_{18}, respectively.

For G_{19} and G_{20} the proofs correspond to the proofs for K_4. For G_{21} we obtain $RC_1 \geq 19$ using the additional colorings of K_{15} and K_{16} where the green edges are

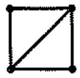
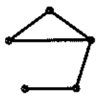
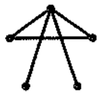
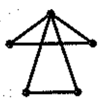
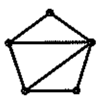
G_1  $RC_1 = 6$ $RC_2 = 6$ $R = 6$	G_2  $RC_1 = 6$ $RC_2 = 6$ $R = 6$	G_3  $RC_1 = 6$ $RC_2 = 6$ $R = 6$
G_4  $RC_1 = 7$ $RC_2 = 7$ $R = 7$	G_5  $RC_1 = 7$ $RC_2 = 7$ $R = 7$	G_6  $RC_1 = 7$ $RC_2 = 7$ $R = 10$
G_7  $RC_1 = 7$ $RC_2 = 9$ $R = 9$	G_8  $RC_1 = 7$ $RC_2 = 9$ $R = 9$	G_9  $RC_1 = 7$ $RC_2 = 9$ $R = 9$
G_{10}  $RC_1 = 7$ $RC_2 = 9$ $R = 9$	G_{11}  $RC_1 = 7$ $RC_2 = 9$ $R = 9$	G_{12}  $RC_1 = 7$ $RC_2 = 9$ $R = 10$
G_{13}  $RC_1 = 7$ $RC_2 = 9$ $R = 10$	G_{14}  $RC_1 = 7$ $RC_2 = 9$ $R = 10$	G_{15}  $RC_1 = 7$ $RC_2 = 9$ $R = 10$
G_{16}  $RC_1 = 7$ $RC_2 = 10$ $R = 10$	G_{17}  $RC_1 = 11$ $RC_2 = 14$ $R = 14$	G_{18}  $RC_1 = 11$ $RC_2 = 15$ $R = 15$
G_{19}  $RC_1 = 14$ $RC_2 = 18$ $R = 18$	G_{20}  $RC_1 = 14$ $RC_2 = 18$ $R = 18$	G_{21}  $RC_1 = 19$ $RC_2 = 19$ $R = 22$

Table 10.

of lengths 1, 3, 4, and 5 and of K_{18} where the green edges are of lengths 1, 6, 7, and 8. For $RC_2 \leq 19$ we have used a computer. □

Finally, we remark that for $G = H = K_n$ we so far have failed to determine $RC_1(K_7)$ and $RC_2(K_7)$. In general, the orders of magnitude of $RC_1(K_n)$ and $RC_2(K_n)$ remain open.

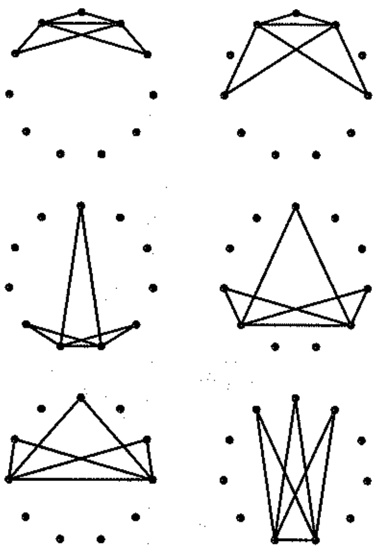


Figure 8. G_{17} using edges of lengths $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,3,4)$, $(1,3,5)$, and $(1,4,5)$, respectively.

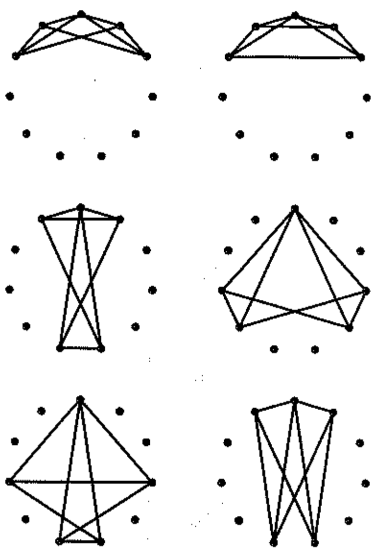


Figure 9. G_{18} using edges of lengths $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,3,4)$, $(1,3,5)$, and $(1,4,5)$, respectively.

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