Applications of algebraic effect theories

Žiga Lukšič
Mentor: doc. dr. Matija Pretnar

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Reasoning about programs

$$f_1 = \texttt{fun}\ x \mapsto x + x$$
$$f_2 = \texttt{fun}\ x \mapsto 2 * x$$

We know mathematical properties of operations + and $*$ so we can argue that the functions are equal (whatever that means).

Harder with effectful operations, such as *print*.

$$print\_twice = \texttt{fun}\ s \mapsto print\ s;\ print\ s$$
$$print\_double = \texttt{fun}\ s \mapsto print\ s\,\hat{}\,s$$

“Implementing print”

We give meaning to operations with handlers. An example:

$$
\begin{aligned}
&\mathit{collect\_prints} \;=\; \texttt{handler}\,\{\\
&\quad \mid \mathit{print}(s;\, k) \mapsto\\
&\qquad\quad \texttt{do}\ (v, s') \leftarrow k\,()\ \texttt{in}\ (v, s \mathbin{\hat{}} s')\\
&\quad \mid \texttt{ret}\ x \mapsto (x, \text{""})\\
&\}
\end{aligned}
$$

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$$k = \texttt{fun}\ ()\mapsto \texttt{with}\ \mathit{collect\_prints}\ \texttt{handle}\ (\mathit{print}\ \text{"}\mathit{test}\text{"})$$

and proceeds with the evaluation

$$\begin{aligned}&\texttt{do}\ (v, s') \leftarrow k\ ()\ \texttt{in}\\&(v, \text{"}\mathit{test}\text{"}\ \hat{}\ s')\end{aligned}$$

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We want reasoning tools that can separate handlers from the rest of our code!

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We can use equations such as

$$\textit{print } s_1;\ \textit{print } s_2 \sim \textit{print } s_1 \hat{}\ s_2$$

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Can we achieve that without restricting ourselves to a global effect theory?

Idea

Include the desired equations in types.

$$\underline{C} = A!\Sigma/\mathcal{E}$$

Operations of type $\underline{C}$ either return a value of type A or call an operation from Σ in the effect theory $\mathcal{E}$.

Equations in $\mathcal{E}$ tell us what computations we deem equal.

Term Syntax

$$
\begin{array}{rcll}
\text{values } v & ::= & x & \text{variable} \\
& \mid & (\,) & \text{unit constant} \\
& \mid & \texttt{true} \mid \texttt{false} & \text{boolean constants} \\
& \mid & \texttt{fun}\ x \mapsto c & \text{function} \\
& \mid & \texttt{handler}\,(\texttt{ret}\ x \mapsto c_r; h) & \text{handler} \\
\\
\text{computations } c & ::= & \texttt{if}\ v\ \texttt{then}\ c_1\ \texttt{else}\ c_2 & \text{conditional} \\
& \mid & v_1\ v_2 & \text{application} \\
& \mid & \texttt{ret}\ v & \text{returned value} \\
& \mid & op(v;\ y.c) & \text{operation call} \\
& \mid & \texttt{do}\ x \leftarrow c_1\ \texttt{in}\ c_2 & \text{sequencing} \\
& \mid & \texttt{with}\ v\ \texttt{handle}\ c & \text{handling} \\
\\
\text{operation clauses } h & ::= & \emptyset \mid h \cup \{op(x;\ k) \mapsto c_{op}\} &
\end{array}
$$

Type Syntax

$$
\begin{array}{rcll}
\text{(value) type } A, B & ::= & \texttt{unit} & \text{unit type} \\
 & | & \texttt{bool} & \text{boolean type} \\
 & | & A \to \underline{C} & \text{function type} \\
 & | & \underline{C} \Rightarrow \underline{D} & \text{handler type} \\
\\
\text{computation type } \underline{C}, \underline{D} & ::= & A!\Sigma/\mathcal{E} & \\
\\
\text{signature } \Sigma & ::= & \emptyset \mid \Sigma \cup \{op : A \to B\} &
\end{array}
$$

Type Syntax (additions)

$$
\begin{array}{rcl}
\text{value context } \Gamma & ::= & \varepsilon \mid \Gamma, x : A \\[1em]
\text{template context } \mathtt{Z} & ::= & \varepsilon \mid \mathtt{Z}, z : A \to * \\[1em]
\text{template } T & ::= & z\ v \\
& \mid & \mathtt{if}\ v\ \mathtt{then}\ T_1\ \mathtt{else}\ T_2 \\
& \mid & op(v;\ y.T) \\[1em]
\text{(effect) theory } \mathcal{E} & ::= & \emptyset \mid \mathcal{E} \cup \{\Gamma; \mathtt{Z} \vdash T_1 \sim T_2\}
\end{array}
$$

The *any type* $*$ used in template types can be instantiated to any computation type so that we can reuse templates.

Full equation

$$\Gamma; \mathtt{Z} = (x : \mathtt{string}, y : \mathtt{string}); (z : \mathtt{unit} \to *)$$

$$\Gamma; \mathtt{Z} \vdash \mathit{print}(x; \_.\mathit{print}(y; \_.z\ ())) \sim \mathit{print}(x\,\hat{}\,y; \_.z\ ())$$

We can use different kinds of logics

We can use any logic that implements some kind of respects relation (though there are requirements for denotational semantics to make sense).

The simplest logic we can use is the free logic, in which

$$\frac{\Gamma \vdash h : \Sigma \Rightarrow \underline{D}}{\Gamma \vdash h : \Sigma \Rightarrow \underline{D} \text{ respects } \emptyset}$$

and corresponds to the conventional approach where we ignore equations.