Computer Algebra in the study of Automorphism Groups of Smooth Hypersurfaces

Zigang Zhu

Joint work with Song Yang and Xun Yu

Tianjin University

November 11, 2023

Overview

Smooth hypersurfaces $X_{n,d}$ in $\mathbb{P}^{n+1}$ and $\mathrm{Aut}(X_{n,d})$

We work over the complex numbers $\mathbb{C}$.
$X := X_{n,d} \subset \mathbb{P}^{n+1}$: smooth hypersurface of degree d defined by F.

$\mathrm{Lin}(X_{n,d}) := \{g \in \mathrm{Aut}(\mathbb{P}^{n+1}) | \, g(X_{n,d}) = X_{n,d}\}$ algebraic group with finitely many components.

Theorem (Matsumura–Monsky, 1963)

If $n \geq 2, d \geq 3$ and $(n, d) \neq (2, 4)$, then $\mathrm{Aut}(X_{n,d}) = \mathrm{Lin}(X_{n,d})$, and it is a finite group.

Theorem (Chang, 1978)

If $d \geq 4$, then $\mathrm{Aut}(X_{1,d}) = \mathrm{Lin}(X_{1,d})$.

Classification of $G \subseteq \mathrm{Aut}(X_{n,d})$

We say a finite group G is an (n, d)-*group* if G is isomorphic to a subgroup of $\mathrm{Aut}(X)$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d.

For fixed $n \geq 1, d \geq 3$ and $(n, d) \neq (1, 3), (2, 4)$, classification of all possible (n, d)-groups is equivalent to the following *projective* linear algebra problem in the classical invariant theory:

Problem

Find all finite subgroups $G \subset \mathrm{PGL}(n+2, \mathbb{C})$ such that there is a smooth homogeneous polynomial $F = F(x_1, ..., x_{n+2})$ of degree d such that for each $g \in G$ there is $A = (a_{ij}) \in \mathrm{GL}(n+2, \mathbb{C})$ such that $[A] = g$ and $F = A(F) := F(\sum_{i=1}^{n+2} a_{1i}x_i, \cdots, \sum_{i=1}^{n+2} a_{(n+2)i}x_i)$.

Classification of $G \subseteq \mathrm{Aut}(X_{5,3})$

Theorem (Yang–Yu–Z, 2023)

A finite group G can act faithfully on a smooth cubic fivefold if and only if G is isomorphic to a subgroup of one of the following 20 groups:

No.	group	order	No.	group	order
1	$C_3^6 \rtimes S_7$	3674160	11	$C_{63} \rtimes C_6$	378
2	$((C_3^2 \rtimes C_3) \rtimes C_4) \times (C_3^3 \rtimes S_4)$	69984	12	$C_3.M_{10}$	2160
3	$C_8 \times (C_3^3 \rtimes S_3)$	1296	13	$S_7 \times C_3$	15120
4	$S_5 \times (C_3^3 \rtimes S_3)$	19440	14	$C_3 \times ((C_8 \times C_2) \rtimes C_2)$	96
5	$C_{48} \times S_3$	288	15	$C_3 \times (\mathrm{PSL}(3,2) \rtimes C_2)$	1008
6	$\mathrm{PSL}(2,11) \times (\mathrm{C}_3^2 \rtimes \mathrm{C}_2)$	11880	16	$C_3.A_7$	7560
7	$((C_3^2 \rtimes C_3) \rtimes C_4)^2 \rtimes C_2$	23328	17	$C_3 \times \mathrm{GL}(2,3)$	144
8	$((C_3^2 \rtimes C_3) \rtimes C_4) \times C_8$	864	18	$((C_3^2 \rtimes C_3) \rtimes Q_8) \rtimes C_3$	648
9	$S_5 \times ((C_3^2 \rtimes C_3) \rtimes C_4)$	12960	19	C_{64}	64
10	C_{96}	96	20	$C_{43} \rtimes C_7$	301

Remark

- $|G|$ *divides* $2^6 \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 43$.
- 5 *non-abelian simple* G: A_5, $\mathrm{PSL}(3,2)$, A_6, $\mathrm{PSL}(2,11)$, A_7.

Classification of $G \subseteq \mathrm{Aut}(X_{5,3})$

Table: 20 examples of smooth cubics

G	Order	X
$C_3^6 \rtimes S_7$	3674160	$x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3=0$
$((C_3^2 \rtimes C_3) \rtimes C_4) \times (C_3^3 \rtimes S_4)$	69984	$x_1^3+x_2^3+x_3^3+3(\sqrt{3}-1)x_1x_2x_3+x_4^3+x_5^3+x_6^3+x_7^3=0$
$C_8 \times (C_3^3 \rtimes S_3)$	1296	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^3+x_5^3+x_6^3+x_7^3=0$
$S_5 \times (C_3^3 \rtimes S_3)$	19440	$\{x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3+x_8^3=x_1+x_2+x_3+x_4+x_5=0\} \subseteq \mathbb{P}^7$
$C_{48} \times S_3$	288	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^2x_5+x_5^3+x_6^3+x_7^3=0$
$\mathrm{PSL}(2,11) \times (\mathrm{C}_3^2 \rtimes \mathrm{C}_2)$	11880	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^2x_5+x_5^2x_1+x_6^3+x_7^3=0$
$((C_3^2 \rtimes C_3) \rtimes C_4)^2 \rtimes C_2$	23328	$x_1^3+x_2^3+x_3^3+3(\sqrt{3}-1)x_1x_2x_3+x_4^3+x_5^3+x_6^3+3(\sqrt{3}-1)x_4x_5x_6+x_7^3=0$
$((C_3^2 \rtimes C_3) \rtimes C_4) \times C_8$	864	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^3+x_5^3+x_6^3+x_7^3+3(\sqrt{3}-1)x_5x_6x_7=0$
$S_5 \times ((C_3^2 \rtimes C_3) \rtimes C_4)$	12960	$\{x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3+x_8^3+3(\sqrt{3}-1)x_6x_7x_8=x_1+x_2+x_3+x_4+x_5=0\} \subseteq \mathbb{P}^7$
C_{96}	96	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^2x_5+x_5^2x_6+x_6^3+x_7^3=0$
$C_{63} \rtimes C_6$	378	$x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^2x_5+x_5^2x_6+x_6^2x_1+x_7^3=0$
$C_3.M_{10}$	2160	$(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3)+1/5(-3\xi_{24}^7-3\xi_{24}^5+3\xi_6-3\xi_8+6\xi_{24}-3)\cdot(x_1x_2x_3+x_1x_2x_4+(\xi_6-1)x_1x_2x_5+x_1x_2x_6+(\xi_6-1)x_1x_3x_4+x_1x_3x_5+x_1x_3x_6+(\xi_6-1)x_1x_4x_5-\xi_6x_1x_4x_6-\xi_6x_1x_5x_6+(\xi_6-1)x_2x_3x_4+(\xi_6-1)x_2x_3x_5-\xi_6x_2x_3x_6+x_2x_4x_5+x_2x_4x_6-\xi_6x_2x_5x_6+x_3x_4x_5-\xi_6x_3x_4x_6+x_3x_5x_6+x_4x_5x_6)=0$

Classification of $G \subseteq \mathrm{Aut}(X_{5,3})$

Table: 20 examples of smooth cubics (continued)

G	Order	X
$S_7 \times C_3$	15120	$\{x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3+x_8^3 = x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 0\} \subseteq \mathbb{P}^7$
$C_3 \times ((C_8 \times C_2) \rtimes C_2)$	96	$x_1^2x_2 + x_2^2x_5 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_2x_4x_6 + x_6^3 + x_7^3 = 0$
$C_3 \times (\mathrm{PSL}(3,2) \rtimes C_2)$	1008	$x_1^3 + 8x_2^3 + 8(-5+4\sqrt{2})x_2x_3^2 + 2\xi_4(-11+6\sqrt{2})x_3(x_4^2+x_5^2) - 4\xi_4x_2((-5+4\sqrt{2})x_4x_5 + 2(-3+\sqrt{2})x_6x_7) + (1+\xi_4)(-12+11\sqrt{2})(x_5x_6^2 - x_4x_7^2) = 0$
$C_3.A_7$	7560	$x_1^3+x_2^3+x_3^3+\frac{12}{5}x_1x_2x_3+x_1x_4^2+x_2x_5^2+x_3x_6^2+\frac{4\sqrt{15}}{9}x_4x_5x_6+x_7^3 = 0$
$C_3 \times \mathrm{GL}(2,3)$	144	$x_1^3+x_2^3+x_2x_5^2-\frac{2}{3}x_2x_5x_7+x_2x_7^2-\frac{2\xi_6}{3}x_2x_5x_6-\frac{2\xi_6}{3}x_2x_6x_7+(-1+\xi_6)x_2x_6^2+x_3^2x_5-x_3x_4x_5+x_4^2x_5+x_4^2x_7+3\xi_6x_3x_4x_6+(-1+\xi_{24}-\xi_8-\xi_{24}^5)x_3^2x_7+(-1-2\xi_{24}+2\xi_8+2\xi_{24}^5)x_3x_4x_7+(\xi_{24}-\xi_6-\xi_{24}^7)x_4^2x_6+(-\xi_{24}-\xi_6+\xi_{24}^7)x_3^2x_6+x_5^3-x_6^3-x_5^2x_7-x_5x_7^2+x_7^3-\xi_6x_5^2x_6+2\xi_6x_5x_6x_7-\xi_6x_6x_7^2+(1-\xi_6)x_5x_6^2+(1-\xi_6)x_6^2x_7 = 0$
C_{64}	64	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_7 + x_7^3 = 0$
$C_{43} \rtimes C_7$	301	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_7 + x_7^2x_1 = 0$

Table: 20 examples of smooth cubics (continued)

G	Order	X
$((C_3^2 \rtimes C_3) \rtimes Q_8) \rtimes C_3$	648	$x_1^3+x_2^3+(\frac{3}{2}\xi_4-\xi_6+\frac{1}{2})x_2^2x_3+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-1)x_2x_3^2+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_3^3+(\xi_4-2\xi_6-\xi_{12}+2)x_2^2x_4+(2\xi_{12}-1)x_2x_3x_4+(\frac{1}{2}\xi_4+\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12})x_3^2x_4+(\xi_4-2\xi_6+1)x_2x_4^2+(-\frac{3}{2}\xi_4+\xi_6+\xi_{12}-\frac{1}{2})x_3x_4^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-\frac{1}{2})x_4^3+(\xi_4+\xi_6-1)x_2^2x_5+(-\xi_4-\xi_6+\xi_{12}-1)x_2x_3x_5+(-\frac{1}{2}\xi_4-\frac{1}{2})x_3^2x_5+(2\xi_{12})x_2x_4x_5+(\xi_6-\xi_{12}-1)x_3x_4x_5+(-\frac{3}{2}\xi_4+\frac{1}{2}\xi_6+\frac{3}{2}\xi_{12}-1)x_4^2x_5+(-\xi_6-\xi_{12})x_2x_5^2+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_3x_5^2+(\frac{1}{2}\xi_4+\xi_6-\xi_{12}-\frac{1}{2})x_4x_5^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}+\frac{1}{2})x_5^3+(\xi_{12}-2)x_2^2x_6+(-\xi_4+2\xi_6-1)x_2x_3x_6+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_3^2x_6+(-2\xi_4+2\xi_6+2\xi_{12}-2)x_2x_4x_6+(\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_4^2x_6+(-2\xi_4)x_2x_5x_6+(\xi_6-\xi_{12})x_3x_5x_6+(\xi_4-\xi_6-\xi_{12})x_4x_5x_6+(-\frac{1}{2}\xi_4+\xi_6+\xi_{12}-\frac{1}{2})x_5^2x_6+(\xi_6-\xi_{12}+1)x_2x_6^2+(\xi_4-\frac{3}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_3x_6^2+(\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_4x_6^2+(\frac{3}{2}\xi_4-\frac{1}{2}\xi_6-\frac{3}{2}\xi_{12}+1)x_5x_6^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-\frac{1}{2})x_6^3+(\frac{1}{2}\xi_4-\frac{3}{2}\xi_6+\frac{1}{2}\xi_{12})x_2^2x_7+(-2\xi_4+\xi_6+\xi_{12}-1)x_2x_3x_7+(\frac{1}{2}\xi_4-2\xi_6-\xi_{12}+\frac{5}{2})x_3^2x_7+(\xi_6+\xi_{12}-2)x_2x_4x_7+(\xi_6-\xi_{12}-1)x_3x_4x_7+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_4^2x_7+(-2\xi_4+2\xi_{12})x_2x_5x_7+(-\xi_4+\xi_6-\xi_{12})x_3x_5x_7+(-\xi_4-1)x_4x_5x_7+(\frac{3}{2}\xi_6-\frac{1}{2}\xi_{12}-\frac{1}{2})x_5^2x_7+(2\xi_6-\xi_{12})x_2x_6x_7+(\xi_4-2\xi_6+1)x_3x_6x_7+(-\xi_{12}+1)x_4x_6x_7+(2\xi_4-\xi_6-2\xi_{12}+1)x_5x_6x_7+(-\frac{1}{2}\xi_4-\xi_6+\xi_{12}-\frac{1}{2})x_6^2x_7+(-\frac{1}{2}\xi_4+\xi_6-\frac{1}{2})x_2x_7^2+(\frac{1}{2}\xi_4-\frac{5}{2}\xi_6+\frac{1}{2}\xi_{12}+2)x_3x_7^2+(\frac{1}{2}\xi_4-\xi_{12}+\frac{1}{2})x_4x_7^2+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_5x_7^2+(-\frac{1}{2}\xi_4-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_6x_7^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}+\frac{1}{2})x_7^3=0$

Classification of $G \subseteq \mathrm{Aut}(X_{4,3})$

Theorem (Yang–Yu–Z, 2023)

A finite group G can act faithfully on a smooth cubic fourfold if and only if G is isomorphic to a subgroup of one of the following 15 groups:

No.	group	order	No.	group	order
1	$C_3^5 \rtimes S_6$	174960	9	$C_{21} \rtimes C_6$	126
2	$((C_3 \times (C_3^3 \rtimes C_3)) \rtimes C_3) \rtimes (C_4 \times C_2)$	5832	10	M_{10}	720
3	$C_8 \times (C_3^2 \rtimes C_2)$	144	11	S_7	5040
4	$S_5 \times (C_3^2 \rtimes C_2)$	2160	12	$(C_8 \times C_2) \rtimes C_2$	32
5	C_{48}	48	13	$\mathrm{PSL}(3,2) \rtimes C_2$	336
6	$\mathrm{PSL}(2,11) \times \mathrm{C}_3$	1980	14	$\mathrm{GL}(2,3)$	48
7	$((C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_3) \rtimes (C_4^2 \rtimes C_2)$	7776	15	$(C_3^2 \rtimes Q_8) \rtimes C_3$	216
8	C_{32}	32			

Remark

- $|G|$ *divides* $2^5 \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$.
- 5 *non-abelian simple* G: A_5, $\mathrm{PSL}(3,2)$, A_6, $\mathrm{PSL}(2,11)$, A_7.

Classification of $G \subseteq \mathrm{Aut}(X_{4,3})$

Table: 15 examples of smooth cubics

G	Order	X
$C_3^5 \rtimes S_6$	174960	$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0$
$((C_3 \times (C_3^3 \rtimes C_3)) \rtimes C_3) \rtimes (C_4 \times C_2)$	5832	$x_1^3 + x_2^3 + x_3^3 + 3(\sqrt{3} - 1)x_1x_2x_3 + x_4^3 + x_5^3 + x_6^3 = 0$
$C_8 \times (C_3^2 \rtimes C_2)$	144	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^3 + x_5^3 + x_6^3 = 0$
$S_5 \times (C_3^2 \rtimes C_2)$	2160	$\{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 = x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \subseteq \mathbb{P}^6$
C_{48}	48	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^3 + x_6^3 = 0$
$\mathrm{PSL}(2,11) \times \mathrm{C}_3$	1980	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 + x_6^3 = 0$
$((C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_3) \rtimes (C_4^2 \rtimes C_2)$	7776	$x_1^3 + x_2^3 + x_3^3 + 3(\sqrt{3} - 1)x_1x_2x_3 + x_4^3 + x_5^3 + x_6^3 + 3(\sqrt{3} - 1)x_4x_5x_6 = 0$
C_{32}	32	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^3 = 0$
$C_{21} \rtimes C_6$	126	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_1 = 0$
M_{10}	720	$(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3)+1/5(-3\xi_{24}^7-3\xi_{24}^5+3\xi_6-3\xi_8+6\xi_{24}-3)\cdot (x_1x_2x_3+x_1x_2x_4+(\xi_6-1)x_1x_2x_5+x_1x_2x_6+(\xi_6-1)x_1x_3x_4+x_1x_3x_5+x_1x_3x_6+(\xi_6-1)x_1x_4x_5-\xi_6x_1x_4x_6-\xi_6x_1x_5x_6+(\xi_6-1)x_2x_3x_4+(\xi_6-1)x_2x_3x_5-\xi_6x_2x_3x_6+x_2x_4x_5+x_2x_4x_6-\xi_6x_2x_5x_6+x_3x_4x_5-\xi_6x_3x_4x_6+x_3x_5x_6+x_4x_5x_6) = 0$ (Höhn–Mason, 2019)

Classification of $G \subseteq \mathrm{Aut}(X_{4,3})$

Table: 15 examples of smooth cubics (continued)

G	Order	X
S_7	5040	$\{x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3 = x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 0\} \subseteq \mathbb{P}^6$
$(C_8 \times C_2) \rtimes C_2$	32	$x_1^2x_2 + x_2^2x_5 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_2x_4x_6 + x_6^3 = 0$
$\mathrm{PSL}(3,2) \rtimes C_2$	336	$8x_1^3+8(-5+4\sqrt{2})x_1x_2^2+2\xi_4(-11+6\sqrt{2})x_2(x_3^2+x_4^2)-4\xi_4x_1((-5+4\sqrt{2})x_3x_4+2(-3+\sqrt{2})x_5x_6)+(1+\xi_4)(-12+11\sqrt{2})(x_4x_5^2-x_3x_6^2) = 0$
$\mathrm{GL}(2,3)$	48	$x_1^3 + x_1x_4^2 - \frac{2}{3}x_1x_4x_6 + x_1x_6^2 - \frac{2\xi_6}{3}x_1x_4x_5 - \frac{2\xi_6}{3}x_1x_5x_6 + (-1+\xi_6)x_1x_5^2 + x_2^2x_4 - x_2x_3x_4 + x_3^2x_4 + x_3^2x_6 + 3\xi_6x_2x_3x_5 + (-1+\xi_{24}-\xi_8-\xi_{24}^5)x_2^2x_6 + (-1-2\xi_{24}+2\xi_8+2\xi_{24}^5)x_2x_3x_6 + (\xi_{24}-\xi_6-\xi_{24}^7)x_3^2x_5 + (-\xi_{24}-\xi_6+\xi_{24}^7)x_2^2x_5 + x_4^3 - x_5^3 - x_4^2x_6 - x_4x_6^2 + x_6^3 - \xi_6x_4^2x_5 + 2\xi_6x_4x_5x_6 - \xi_6x_5x_6^2 + (1-\xi_6)x_4x_5^2 + (1-\xi_6)x_5^2x_6 = 0$

Classification of $G \subseteq \mathrm{Aut}(X_{4,3})$

Table: 15 examples of smooth cubics (continued)

G	Order	X
$((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$	216	$x_1^3+(\frac{3}{2}\xi_4-\xi_6+\frac{1}{2})x_1^2x_2+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-1)x_1x_2^2+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_2^3+(\xi_4-2\xi_6-\xi_{12}+2)x_1^2x_3+(2\xi_{12}-1)x_1x_2x_3+(\frac{1}{2}\xi_4+\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12})x_2^2x_3+(\xi_4-2\xi_6+1)x_1x_3^2+(-\frac{3}{2}\xi_4+\xi_6+\xi_{12}-\frac{1}{2})x_2x_3^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-\frac{1}{2})x_3^3+(\xi_4+\xi_6-1)x_1^2x_4+(-\xi_4-\xi_6+\xi_{12}-1)x_1x_2x_4+(-\frac{1}{2}\xi_4-\frac{1}{2})x_2^2x_4+(2\xi_{12})x_1x_3x_4+(\xi_6-\xi_{12}-1)x_2x_3x_4+(-\frac{3}{2}\xi_4+\frac{1}{2}\xi_6+\frac{3}{2}\xi_{12}-1)x_3^2x_4+(-\xi_6-\xi_{12})x_1x_4^2+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_2x_4^2+(\frac{1}{2}\xi_4+\xi_6-\xi_{12}-\frac{1}{2})x_3x_4^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}+\frac{1}{2})x_4^3+(\xi_{12}-2)x_1^2x_5+(-\xi_4+2\xi_6-1)x_1x_2x_5+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_2^2x_5+(-2\xi_4+2\xi_6+2\xi_{12}-2)x_1x_3x_5+(\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_3^2x_5+(-2\xi_4)x_1x_4x_5+(\xi_6-\xi_{12})x_2x_4x_5+(\xi_4-\xi_6-\xi_{12})x_3x_4x_5+(-\frac{1}{2}\xi_4+\xi_6+\xi_{12}-\frac{1}{2})x_4^2x_5+(\xi_6-\xi_{12}+1)x_1x_5^2+(\xi_4-\frac{3}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_2x_5^2+(\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_3x_5^2+(\frac{3}{2}\xi_4-\frac{1}{2}\xi_6-\frac{3}{2}\xi_{12}+1)x_4x_5^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}-\frac{1}{2})x_5^3+(\frac{1}{2}\xi_4-\frac{3}{2}\xi_6+\frac{1}{2}\xi_{12})x_1^2x_6+(-2\xi_4+\xi_6+\xi_{12}-1)x_1x_2x_6+(\frac{1}{2}\xi_4-2\xi_6-\xi_{12}+\frac{5}{2})x_2^2x_6+(\xi_6+\xi_{12}-2)x_1x_3x_6+(\xi_6-\xi_{12}-1)x_2x_3x_6+(-\frac{1}{2}\xi_4+\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_3^2x_6+(-2\xi_4+2\xi_{12})x_1x_4x_6+(-\xi_4+\xi_6-\xi_{12})x_2x_4x_6+(-\xi_4-1)x_3x_4x_6+(\frac{3}{2}\xi_6-\frac{1}{2}\xi_{12}-\frac{1}{2})x_4^2x_6+(2\xi_6-\xi_{12})x_1x_5x_6+(\xi_4-2\xi_6+1)x_2x_5x_6+(-\xi_{12}+1)x_3x_5x_6+(2\xi_4-\xi_6-2\xi_{12}+1)x_4x_5x_6+(-\frac{1}{2}\xi_4-\xi_6+\xi_{12}-\frac{1}{2})x_5^2x_6+(-\frac{1}{2}\xi_4+\xi_6-\frac{1}{2})x_1x_6^2+(\frac{1}{2}\xi_4-\frac{5}{2}\xi_6+\frac{1}{2}\xi_{12}+2)x_2x_6^2+(\frac{1}{2}\xi_4-\xi_{12}+\frac{1}{2})x_3x_6^2+(-\frac{1}{2}\xi_6-\frac{1}{2}\xi_{12}+\frac{1}{2})x_4x_6^2+(-\frac{1}{2}\xi_4-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12})x_5x_6^2+(-\frac{1}{2}\xi_6+\frac{1}{2}\xi_{12}+\frac{1}{2})x_6^3=0$

Remark

Previous known results on classification of $G \subseteq \mathrm{Aut}(X_{n,d})$ for (n,d) with $n \geq 2$, $d \geq 3$, $(n,d) \neq (2,4)$:

- classification of $\mathrm{Aut}(X)$ for cubic surfaces is known (Segre 1942, Hosoh 1997, Dolgachev 2012).
- González-Aguilera–Liendo 2011 determined all possible prime factors of $|\mathrm{Aut}(X_{n,d})|$ for arbitrary (n,d);
- Fu 2016 classified cyclic $G \subseteq \mathrm{Aut}^s(X_{4,3})$ of primary orders.
- Oguiso–Yu 2019 classified all possible groups of automorphisms for $X_{3,5} \subset \mathbb{P}^4$.
- Wei–Yu 2020 classified all possible groups of automorphisms for $X_{3,3} \subset \mathbb{P}^4$.
- Laza–Zheng 2022 classified $\mathrm{Aut}^s(X_{4,3})$ and proved that the Fermat cubic fourfold has the largest possible order for $|\mathrm{Aut}(X_{4,3})|$.
- Zheng 2022 determined abelian G satisfying some technical conditions; in particular, he determined all cyclic $G \subseteq \mathrm{Aut}(X_{n,d})$.
- González-Aguilera–Liendo–Montero 2023 determined Sylow p-subgroups $\mathrm{Aut}(X_{n,d})$ for p satisfying certain conditions.
- ...

Classification of $G \subseteq \mathrm{Aut}(X_{3,5})$ and $G \subseteq \mathrm{Aut}(X_{3,3})$

Theorem (Oguiso–Yu, 2019)

For a finite group G, the following two conditions are equivalent to each other:
(i) G has a faithful action on a smooth quintic threefold, and
(ii) G is isomorphic to a subgroup of one of the 22 groups below:

$C_5^4 \rtimes S_5$, $C_4 \times (C_5^3 \rtimes S_3)$, $(C_5^2 \times C_4^2) \rtimes C_2$, $C_{16} \times (C_5^2 \rtimes C_2)$, $S_3 \times (C_5^3 \rtimes S_3)$, $C_5 \times C_{16} \times C_4$, $C_{64} \times C_5$, $C_5^2 \times C_4 \times S_3$, $(C_5^2 \rtimes C_2) \times (C_{13} \rtimes C_3)$, $C_{16} \times (C_5 \times S_3)$, C_{256}, $C_4 \times C_5 \times (C_{13} \rtimes C_3)$, $C_5 \times (C_{51} \rtimes C_4)$, $(C_5^2 \times C_3^2) \rtimes D_8$, $C_{205} \rtimes C_5$, $C_5 \times S_3 \times (C_{13} \rtimes C_3)$, $C_5 \times ((\mathrm{SL}(2,3).C_2) \rtimes C_2)$, $\mathrm{SL}(2,3) \rtimes C_4$, $C_5 \times (C_3 \rtimes Q_8)$, $C_5 \times D_{24}$, $C_5 \times S_5$, $C_{32} \times C_2$.

Theorem (Wei–Yu, 2020)

For a finite group G, the following two conditions are equivalent to each other:
(i) G has a faithful action on a smooth cubic threefold, and
(ii) G is isomorphic to a subgroup of one of the 6 groups below:

$C_3^4 \rtimes S_5$, $((C_3^2 \rtimes C_3) \rtimes C_4) \times S_3$, C_{24}, C_{16}, $\mathrm{PSL}(2,11)$, $S_5 \times C_3$.

Oguiso–Yu's method for $(n, d) = (3, 5), (3, 3)$

How to use smoothness?

Lemma (Oguiso-Yu)

Let $F = F(x_1, ..., x_{n+2})$ be an irred. homogeneous poly. of degree $d \geq 3$. Let a and b be two nonnegative integers, and $2a + b \leq n + 1$. Then $X := \{F = 0\} \subseteq \mathbb{P}^{n+1}$ is not smooth if $\exists$ $a + b$ distinct variables $x_{i_1}, \ldots, x_{i_{a+b}}$ such that $F \in (x_{i_1}, \ldots, x_{i_a}) + (x_{i_{a+1}}, \ldots, x_{i_{a+b}})^2$.

Proposition (Oguiso-Yu)

Let $X := \{F = 0\} \subseteq \mathbb{P}^4$ of degree $d \geq 3$. Then X is singular if one of the following three conditions is true:
(1) $\exists$ $1 \leq i \leq 5$, such that for all $1 \leq j \leq 5, x_i^{d-1} x_j \notin F$;
(2) $\exists$ $1 \leq p, q \leq 5, p \neq q$, such that $F \in (x_p, x_q)$;
(3) $\exists$ x_i, x_j, x_k, such that $F \in (x_i) + (x_j, x_k)^2$.

This "combinatorial" test can be handled by computer effectively.

F-liftability

reduce to classification of finite subgroups in $\mathrm{GL}(n+2,\mathbb{C})$ instead of $\mathrm{PGL}(n+2,\mathbb{C})$ via F-liftability

Definition (Oguiso-Yu)

$G \subset \mathrm{PGL}(n,\mathbb{C})$ finite subgroup, $F \in \mathbb{C}[x_1, x_2, ..., x_n]$ homogeneous. We say G *is F-liftable* if :

i) G admits a lifting $\widetilde{G} \subset \mathrm{GL}(n,\mathbb{C})$ (i.e., $\widetilde{G} \cong G$ under natural projection $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$);

ii) $A(F) = F$, for all A in $\widetilde{G}$.

In this case, we say $\widetilde{G}$ is an F-lifting of G.

F-liftability

Example

- Let $F = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3$, $C_3^2 \cong G = \langle [A_1], [A_2] \rangle \subset \mathrm{PGL}(6, \mathbb{C})$, where $A_1 = \mathrm{diag}(\xi_3, 1, 1, 1, 1, 1)$, $A_2 = \mathrm{diag}(1, \xi_3, 1, 1, 1, 1)$. Then G is F-liftable.
- Let $F = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3$, $C_3^2 \cong G = \langle [A_3], [A_4] \rangle \subset \mathrm{PGL}(6, \mathbb{C})$, where
$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_4 = \mathrm{diag}(1, \xi_3, \xi_3^2, 1, \xi_3, \xi_3^2).$$ Then G is not F-liftable ($A_3 A_4 = \xi_3 A_4 A_3$, and G has no lifting in $\mathrm{GL}(6, \mathbb{C})$).
- The action $[A_4]$ on $X : F = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_6 + x_6^2 x_1 = 0$ is not F-liftable (note that $A_4(F) = \xi_3 F$).

F-liftability

Theorem (Oguiso-Yu)

Let G be a finite subgroup of $\mathrm{PGL}(n, \mathbb{C})$. Let homogeneous $F \in \mathbb{C}[x_1, ..., x_n]$ of degree p, where p is a prime number. Suppose F is G-invariant (i.e., for any $g \in G$, $\exists A \in \mathrm{GL}(n, \mathbb{C})$, such that $A(F) = F$ and $[A] = g$). Let G_p be a Sylow p-subgroup. Then G is F-liftable if the following two conditions are satisfied:
(1) G_p is F-liftable; and
(2) either G_p has no element of order p^2 or G has no normal subgroup of index p.

It turns out that there exists no automorphism of smooth quintic threefolds of order 25.

Theorem (Oguiso-Yu)

Let X be a smooth quintic threefold defined by F, and $G \subseteq \mathrm{Aut}(X)$. Then G is F-liftable if and only if a Sylow-5 subgroup G_5 is F-liftable.

F-liftability

Theorem (Wei-Yu, 2020)

Let $X_F \subset \mathbb{P}^4$ be a smooth cubic threefold defined by F, and $G \subseteq \mathrm{Aut}(X_F)$. Then G is F-liftable.

Recent generalization:

Theorem (González-Aguilera–Liendo–Montero, 2023)

Let $n \geq 1, d \geq 3, (n, d) \neq (1, 3), (2, 4)$. Then the automorphism group of every smooth hypersurface X_F of dimension n and degree d in $\mathbb{P}^{n+1}$ is F-liftable if and only if d and $n+2$ are relatively prime.

The differential method

Let $F = F(x_1, x_2, ..., x_n)$ homogeneous poly. of degree d. For $1 \le i \le d$, we have the natural i-th order differential map naturally defined by F:

$$D_i^F : \mathcal{D}_i(x_1, ..., x_n) \to \mathbb{C}[x_1, ..., x_n] \ ,$$

where $\mathcal{D}_i(x_1, ..., x_n)$ is the vector space of i-th order differential operators. E.g., $D_1^F(\frac{\partial}{\partial x_j}) = \frac{\partial F}{\partial x_j}$, $D_2^F(\frac{\partial^2}{\partial x_i \partial x_j}) = \frac{\partial^2 F}{\partial x_i \partial x_j}$.

Theorem (Oguiso-Yu)

Let $G = G(y_1, ..., y_n)$ be a homogeneous polynoimal in variables $y_1, ..., y_n$. If $F(x_1, ..., x_n) = G(y_1, ..., y_n)$ under an invertible linear change of coordinates, in other words, there exists an invertible matrix $L = (l_{ij})_{1 \le i,j \le n}$, such that

$$F(x_1, ..., x_n) = G(\sum_{i=1}^{n} l_{1i} x_i, ..., \sum_{i=1}^{n} l_{ni} x_i), \text{ then } \mathrm{Rank}(D_i^F) = \mathrm{Rank}(D_i^G), \text{ for all } 1 \le i \le d.$$

New ingredients for cubic fivefolds and fourfolds

Let $F = F(x_1, x_2, \ldots, x_m)$ be a homogeneous polynomial of degree d.

Definition (partitionability and characteristic sets)

(1) If there exists $A \in \mathrm{GL}(m, \mathbb{C})$ and positive integers $a_1, \ldots, a_t$ such that

$$A(F) = H_1(x_1, \ldots, x_{a_1}) + \cdots + H_t(x_{a_1+a_2+\cdots+a_{t-1}+1}, \ldots, x_{a_1+a_2+\cdots+a_t}),$$

where $a_1 + \cdots + a_t \leq m$ and $t \geq 2$, then we say F is *partitionable* or F has an $(a_1, a_2, \ldots, a_t)$*-type partition*. If, moreover, all H_i ($i = 1, ..., t$) are unpartitionable, we say F has a *maximal* $(a_1, a_2, \ldots, a_t)$-type partition.
(2) Let r be a positive integer. We define

$$S_r^F := \{(l_1, \ldots, l_m) \in \mathbb{C}^m \mid \mathrm{rk}(D_1^{l_1 \frac{\partial F}{\partial x_1} + l_2 \frac{\partial F}{\partial x_2} + \cdots + l_m \frac{\partial F}{\partial x_m}}) = r\}.$$

We call S_r^F the *r-th characteristic set* of F.

Partitionability and characteristic sets

Lemma (Yang–Yu–Z)

Let $G = G(y_1, y_2, \ldots, y_m)$ be a homogeneous polynomial of degree d, and let $F(x_1, \ldots, x_m) = G(\sum_{i=1}^m a_{1i}x_i, \ldots, \sum_{i=1}^m a_{mi}x_i)$, where $A = (a_{ij})_{1\le i,j\le m} \in \mathrm{GL}(m, \mathbb{C})$. Consider the following linear transformation:

$$P : \mathbb{C}^m \longrightarrow \mathbb{C}^m,\ (l_1, \ldots, l_m) \longmapsto (\sum_{j=1}^m a_{1j}l_j, \ldots, \sum_{j=1}^m a_{mj}l_j).$$

Then $P(S_r^F) = S_r^G$. In particular, if $A(F) = F$, then $P(S_r^F) = S_r^F$.

Lemma (Yang-Yu-Z)

Let $F = F(x_1, x_2, \ldots, x_m)$ be a smooth cubic form. Then F has a $(1, m-1)$-type partition if and only if the first characteristic set $S_1^F \neq \emptyset$.

Partitionability and characteristic sets

Example

(1) Let $F = x_1^3 + x_2^2x_3 + x_3^2x_2$. Then F has a $(1,2)$-type partition given by $F = H_1 + H_2$, where $H_1 = x_1^3$ and $H_2 = x_2^2x_3 + x_3^2x_2$. This partition is not maximal. In fact, H_2 has a $(1,1)$-type partition since $A(H_2) = x_2^3 + x_3^3$, where $A = \begin{pmatrix} -1 & -1 \\ \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} \end{pmatrix}$. Thus, F has a maximal $(1,1,1)$-type partition.

(2) Let $F := H + K$, where $H = x_1^3 + x_2^3 + x_3^3 + x_4^3$ and $K = x_5^3 + x_6^3 + x_7^3 + 3(\sqrt{3}-1)x_5x_6x_7$. Then $S_1^K = \emptyset$, and F has a maximal $(1,1,1,1,3)$-type partition.

Control $\mathrm{Aut}(X_F)$ via partitionability and characteristic sets

We define $G_F := \{A \in \mathrm{GL}(m, \mathbb{C}) \mid A(F) = F\}$.

Proposition (Yang–Yu–Z)

Let $F = F(x_1, \ldots, x_m)$ be a smooth cubic form with $m \geq 4$. Suppose $F = H(x_1, \ldots, x_k) + K(x_{k+1}, \ldots, x_m)$, where $3 \leq k \leq m-1$, $S_1^H = \emptyset$, and $K = x_{k+1}^3 + \ldots + x_m^3$. Then $G_F \subset \{\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \mathrm{GL}(k, \mathbb{C}), C \in \mathrm{GL}(m-k, \mathbb{C})\}$.

Proposition (Yang–Yu–Z)

Let $F = F(x_1, \ldots, x_7)$ be a smooth cubic form such that $F(x_1, \ldots, x_7) = H(x_1, \ldots, x_3) + K(x_4, \ldots, x_7)$, where H and K are unpartitionable. Then $G_F \subset \{\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \mathrm{GL}(3, \mathbb{C}), C \in \mathrm{GL}(4, \mathbb{C})\}$.

Control $\mathrm{Aut}(X_F)$ via partitionability and characteristic sets

Proposition (Yang–Yu–Z)

Let $F = F(x_1, \ldots, x_7)$ be a smooth cubic form with $F = x_1^3 + H(x_2, x_3, x_4) + K(x_5, x_6, x_7)$, where H and K are unpartitionable. If $A \in \mathrm{GL}(7, \mathbb{C})$ satisfies $A(F) = F$, then either $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$ or $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix}$, where $B, C \in \mathrm{GL}(3, \mathbb{C})$.

A relation among partitionability, F-liftability, $\mathrm{Aut}(\mathrm{X_F})$

Theorem (Yang–Yu–Z)

Let $F = F(x_1, \dots, x_m)$ be a smooth form of degree d with $F = H(x_1, \dots, x_k) + K(x_{k+1}, \dots, x_m)$, where $m \geq 5$, $d \geq 3$, $4 \leq k \leq m-1$, $(k, d) \neq (4, 4)$. Suppose the following statements hold:

(1) $\mathrm{Aut}(X_H)$ *admits an H-lifting* $\widetilde{\mathrm{Aut}(X_H)}$;

(2) $G_F \subset \{\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \mathrm{GL}(k, \mathbb{C}), C \in \mathrm{GL}(m-k, \mathbb{C})\}$.

Then $\mathrm{Aut}(X_F)$ *is isomorphic to* $\mathrm{Aut}(X_H) \times G_K$ *and* $\mathrm{Aut}(X_F)$ *has an F-lifting* $\widetilde{G} := \{\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \widetilde{\mathrm{Aut}(X_H)}, C \in G_K\}$.

Special $(5,3)$-representations

Definition

Let $A, B \in \mathrm{GL}(n+2,\mathbb{C})$. Let $\rho : G \to \mathrm{GL}(n+2,\mathbb{C})$ be a faithful linear representation of a finite group G.

(1) We say ρ is an *(n,d)-representation* of G if there exists a smooth form $F = F(x_1, \dots, x_{n+2})$ of degree d such that $\rho(G)$ is an F-lifting of a subgroup of $\mathrm{Aut}(X_F)$.

(2) We say ρ *is an almost (n,d)-representation* of G if for every proper abelian subgroup $H < G$, the restriction $\rho|H$ is an (n,d)-representation of H.

(3) Let $n = 5$. We say ρ is a *special* representation of G if there exists no $A \in \rho(G)$ such that A is similar to either $\xi_3^a \cdot \mathrm{diag}(\xi_3, \xi_3, 1, 1, 1, 1, 1)$ or $\xi_3^a \cdot \mathrm{diag}(\xi_3, \xi_3, \xi_3, 1, 1, 1, 1)$ for some $a \in \{0, 1, 2\}$.

d-equivalent

Definition

Let $\rho_i : G \to \mathrm{GL}(n+2, \mathbb{C})$ be two faithful linear representations of a finite group G with characters χ_i $(i = 1, 2)$. Let d be a positive integer. We say ρ_1 and ρ_2 are *d-equivalent* if the groups $\langle \rho_1(G), \xi_d I_{n+2} \rangle$ and $\langle \rho_2(G), \xi_d I_{n+2} \rangle$ are conjugate in $\mathrm{GL}(n+2, \mathbb{C})$. We say χ_1 and χ_2 are d-equivalent if ρ_1 and ρ_2 are d-equivalent.

The motivation of the definition of d-equivalence is the following

Lemma

Let $\rho_i : G \to \mathrm{GL}(n+2, \mathbb{C})$ be d-equivalent faithful linear representations of a finite group G $(i = 1, 2)$. If ρ_1 is an (n, d)-representation, so is ρ_2.

Classification of $(5,3)$-groups $G \subseteq \mathrm{Aut}(X_F)$

By González-Aguilera–Liendo–Montero's result, $\mathrm{Aut}(X_F)$ of any smooth cubic fivefold X_F has an F-lifting. From this, we conclude that a finite group G can act faithfully on a smooth cubic fivefold X_F defined by F admitting neither $(2,5)$ nor $(3,4)$-type partition only if G has a special $(5,3)$-representation. Thus, there are two cases:

(1) F of the form

$$F = H(x_1, ..., x_a) + K(x_{a+1}, ..., x_7),\ 2 \leq a \leq 3.$$

In this case, by results on partitionability, we get all possible subgroups G of $\mathrm{Aut}(X_F)$ from previously known classifications for cubic surfaces and threefolds in PC free way.

(2) F not of the above form (up to linear change of coordinates). Then $|\mathrm{Aut}(X_F)| \leq 90720 = 2^5 \cdot 3^4 \cdot 5 \cdot 7$ and G must have a special $(5,3)$-representation. We use a (computer-aided) strategy to rule out groups which have no special (almost) $(5,3)$-representations.

$(4,3)$-groups and C_3-covering $(5,3)$-groups

Unlike cubic fivefolds, the automorphism groups of cubic fourfolds have no F-lifting in general, which is a key obstruction in our classification of such groups. To deal with this issue, we introduce the notion of C_d-covering group.

Definition

Let G and $\widehat{G}$ be two finite groups. Let d be a positive integer. We say $\widehat{G}$ is a *C_d-covering group* of G if the centre $Z(\widehat{G})$ of $\widehat{G}$ contains a subgroup N such that $N \cong C_d$ and $\widehat{G}/N \cong G$.

Lemma (Yang–Yu–Z)

Every $(4,3)$-group has at least one C_3-covering $(5,3)$-group.

This gives strong constraints on $(4,3)$-groups.

Partitionability of defining polynomials of cubic fourfolds and fivefolds are closely related.

Lemma (Yang–Yu–Z)

Let $F = F(x_1, ..., x_m)$ be a smooth cubic form with $m \geq 3$. We define $\widehat{F} := F + x_{m+1}^d$. Then F is partitionable if and only if $\widehat{F}$ has an (a_1, a_2)-type partition with $a_1 \geq 2$, $a_2 \geq 2$.

Based on the relations between automorphism groups of cubic fourfolds and fivefolds, we classify all $(4,3)$-groups by taking advantages of our results and strategies for $(5,3)$-groups.

Symplectic automorphism groups of cubic fourfolds

The *symplectic automorphism group* $\mathrm{Aut}^s(X_F)$ of a smooth cubic fourfold X_F consists of the symplectic automorphisms f of X_F (i.e., the induced action on $H^{3,1}(X_F) \cong \mathbb{C}$ is trivial). If F and matrix generators of $\mathrm{Aut}(X_F)$ are explicitly given, one can directly compute $\mathrm{Aut}^s(X_F)$.

Lemma (Fu, 2016)

Let X be a smooth cubic fourfold defined by $F(x_1, \ldots, x_6)$. Let $f = [A]$ be an element in $\mathrm{Aut}(X)$, $A \in \mathrm{GL}(6, \mathbb{C})$, with $\mathrm{ord}(f) = \mathrm{ord}(A)$ and $A(F) = \lambda F$ with $\lambda \in \mathrm{C}$, then f is symplectic if and only if $\det(A) = \lambda^2$.

Example

Consider $X_5' \subset \mathbb{P}^5$ defined by $F_5' = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^3 + x_6^3$. Then $\mathrm{Aut}(X_5') \cong C_{48}$ is generated by $A_{X_5'} := \mathrm{diag}(\xi_{16}, \xi_8^7, \xi_4, -1, 1, \xi_3)$. Since $A_{X_5'}(F_5') = F_5'$ and $\det(A_{X_5'})$ is a 48-th primitive root of unity, by the lemma, we have that $\mathrm{Aut}^s(X_5')$ is trivial.

Symplectic automorphism groups of cubic fourfolds

The symplectic automorphism groups $\mathrm{Aut}^s(X_i')$ $(i = 1, 2, ..., 15)$ can be computed similarly and the result is summarized in the following table. This is consistent with Laza–Zheng's result.

i	$\mathrm{Aut}(X_i')$	$\lvert\mathrm{Aut}(X_i')\rvert$	$\mathrm{Aut}^s(X_i')$	$\lvert\mathrm{Aut}^s(X_i')\rvert$	maximal
1	$C_3^5 \rtimes S_6$	174960	$C_3^4 \rtimes A_6$	29160	✓
2	$((C_3 \times (C_3^3 \rtimes C_3)) \rtimes C_3) \rtimes (C_4 \times C_2)$	5832	$(C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_2$	486	
3	$C_8 \times (C_3^2 \rtimes C_2)$	144	S_3	6	
4	$S_5 \times (C_3^2 \rtimes C_2)$	2160	$A_5 \rtimes S_3$	360	✓
5	C_{48}	48	trivial	1	
6	$\mathrm{PSL}(2, 11) \times \mathrm{C}_3$	1980	$\mathrm{PSL}(2, 11)$	660	✓
7	$((C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_3) \rtimes (C_4^2 \rtimes C_2)$	7776	$((C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_3) \rtimes Q_8$	1944	✓
8	C_{32}	32	trivial	1	
9	$C_{21} \rtimes C_6$	126	$C_7 \rtimes C_3$	21	
10	M_{10}	720	M_{10}	720	✓
11	S_7	5040	A_7	2520	✓
12	$(C_8 \times C_2) \rtimes C_2$	32	QD_{16}	16	
13	$\mathrm{PSL}(3, 2) \rtimes C_2$	336	$\mathrm{PSL}(3, 2)$	168	
14	$\mathrm{GL}(2, 3)$	48	$\mathrm{GL}(2, 3)$	48	✓
15	$((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$	216	$(C_3 \times C_3) \rtimes Q_8$	72	

Table: Symplectic automorphism groups of cubic fourfolds

Equations of two cubic fourfolds with maximal $\mathrm{Aut}^s(X)$

Based on the global Torelli theorem for cubic fourfolds, Laza–Zheng proved that A_7 (resp. M_{10}) can act faithfully and symplecticly on exactly two smooth cubic fourfolds $X^1(A_7)$ and $X^2(A_7)$ (resp. $X^1(M_{10})$ and $X^2(M_{10})$). It is known that $X^1(A_7) \cong X'_{11}$ and $X^1(M_{10}) \cong X'_{10}$. However, explicit defining equations for $X^2(A_7)$ and $X^2(M_{10})$ seem unknown. As a by-product of our classification of $(4,3)$-groups and $(5,3)$-groups, we solve this problem.

Theorem (Yang–Yu–Z)

Let $F_{A_7} = x_1^3 + x_2^3 + x_3^3 + \frac{12}{5}x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + x_3x_6^2 + \frac{4\sqrt{15}}{9}x_4x_5x_6$. Then the smooth cubic fourfold $X_{F_{A_7}}$ satisfies

$$\mathrm{Aut}^s(X_{F_{A_7}}) = \mathrm{Aut}(X_{F_{A_7}}) \cong A_7.$$

In particular, $X_{F_{A_7}}$ is isomorphic to $X^2(A_7)$.

Theorem (Yang–Yu–Z)

Let $F_{M_{10}} = x_1^3 + 1/1815(1036\xi_{24}^7 - 5800\xi_4 - 1576\xi_{24}^5 + 2016\xi_6 + 4180\xi_8 + 3632\xi_{12} - 2644\xi_{24} - 3939)x_1x_2^2 + 1/605(1028\xi_{24}^7 - 864\xi_4 - 1468\xi_{24}^5 - 1448\xi_6 + 1280\xi_{24}^3 + 3072\xi_{12} + 152\xi_{24} - 2270)x_1x_3x_4 + 1/3993(25574\xi_{24}^7 + 9032\xi_4 - 20826\xi_{24}^5 - 18220\xi_6 - 13744\xi_{24}^3 + 13592\xi_{12} + 22080\xi_{24} - 1231)x_2x_3^2 + 1/3993(41818\xi_{24}^7 + 64576\xi_4 - 1314\xi_{24}^5 - 79580\xi_6 - 60500\xi_{24}^3 - 20552\xi_{12} + 70716\xi_{24} + 43177)x_2x_4^2 + 1/19965(-16944\xi_{24}^7 - 50216\xi_4 - 100168\xi_{24}^5 + 192272\xi_6 - 55224\xi_{24}^3 + 153712\xi_{12} - 145288\xi_{24} - 22288)x_2x_5x_6 + 1/6655(-20096\xi_{24}^7 + 5560\xi_4 + 22156\xi_{24}^5 + 6216\xi_6 - 452\xi_{24}^3 - 17296\xi_{12} - 11268\xi_{24} + 13556)x_3x_5^2 + 1/6655(89336\xi_{24}^7 + 48240\xi_4 - 8948\xi_{24}^5 - 88392\xi_6 - 106476\xi_{24}^3 + 29824\xi_{12} + 50884\xi_{24} + 70396)x_4x_6^2.$
Then the smooth cubic fourfold $X_{F_{M_{10}}}$ *is isomorphic to* $X^2(M_{10})$.

Remark

The automorphism groups $\mathrm{Aut}(X_{F_{A_7}})$ *and* $\mathrm{Aut}(X_{F_{M_{10}}})$ *have no* F_{A_7}*-lifting and* $F_{M_{10}}$*-lifting respectively.*

Remark

We can compute all $(5, 3)$-representations of any $(5, 3)$-groups. In particular, for any $(4, 3)$-group G, we can determine whether G can act faithfully and symplectically on cubic fourfolds via computing $(5, 3)$-representations of C_3-covering $(5, 3)$-groups of G.

In this way, without using the global Torelli theorem for cubic fourfolds, we can give a new proof of Laza–Zheng's classification of symplectic automorphism groups of cubic fourfolds.

GAP software

GAP is a system for computational discrete algebra. Website: www.gap-system.org

GAP provides:
(1) a programming language,
(2) a library of thousands of functions implementing algebraic algorithms,
(3) large data libraries of algebraic objects.

GAP is used in research and teaching for studying groups and their representations, rings, vector spaces, algebras, combinatorial structures, and more.

Started at Lehrstuhl D für Mathematik, RWTH Aachen in 1986.

GAP installation

Webpage for downloading and installing GAP: https://www.gap-system.org/Download/

The basic steps of a GAP installation:
(1) verify required tools
(2) download the archive and compile the core system
(3) build up packages

An alternative method for Mac users: Homebrew (a package manager for macOS and Linux)

Basic functions of GAP

[Demonstrate in the terminal]

Example: sub-test using GAP

Let $\mathcal{G}_m := \{G | \, G$ acts faithfully on some $X_{n,d}$, and $|G| = m\}$.
We determine $\mathcal{G}_m$ inductively w. r. t. m. Suppose $\mathcal{G}_1$,..., $\mathcal{G}_{m-1}$ are determined, then we determine $\mathcal{G}_m$ via the following steps:

(1) rule out all finite groups G of order m s. t. $\exists H \subsetneq G$ and H is not in $\mathcal{G}_{|H|}$;

(2) if G already realizable by known examples, add G to $\mathcal{G}_m$;

(3) for undecided G after steps (1) and (2), check case by case.

For quintic 3-fold and cubic 3-fold cases, this “sub-test” turns out to be effective.

Example: sub-test using GAP

Code logic for orders ≤ 2000:

(1) Create a list (Lcubic2000) containing all groups in $\mathcal{G}_1, \ldots, \mathcal{G}_{m-1}$.

(2) Generate a list (L) of all solvable or nonsolvable groups of order m.

(3) List all groups after completing step (1).

(4) List all groups after completing step (2).

For orders greater than 2000: use the "GrpConst" package or do argument.

[Demonstrate in the terminal]

Wolfram Mathematica

Mathematica is a powerful symbolic computation system.

It provides

(1) rich library of built-in functions and packages,

(2) interactive interface,

(3) large community and support.

Installation:
purchasing a license from the official website and following the installation instructions for your operating system. (see https://www.wolfram.com/mathematica/)

Mathematica can be started in the terminal by executing
"/Applications/Mathematica.app/Contents/MacOS/MathKernel"

Basic symbols and functions

[Demonstrate in the Mathematica notebook]

Detailed descriptions of Mathematica functions can be found in https://reference.wolfram.com/

Example: smoothness test

We use the following critetia in our computer program.

Lemma (González-Aguilera–Liendo–Montero, 2020)

Let $F = F(x_1, \ldots, x_{n+2})$, $n \geq 2$, be a homogeneous polynomial of degree 3. If there exist three mutually disjoint collections of variables V_1, V_2, V_3 such that $\bigcup_i V_i = \{x_1, \ldots, x_{n+2}\}$, $|V_1| > |V_2|$ and for every monomial $\mathfrak{m} \in F$, $\mathfrak{m}$ can be expressed in one of the following forms:

(i) $\mathfrak{m} = x_p x_q x_r$, $x_p, x_q \in V_1$ *and* $x_r \in V_2$;

(ii) $\mathfrak{m} = x_p x_q x_r$, $x_p \in V_1$ *and* $x_q, x_r \in V_2 \cup V_3$; *or*

(iii) $\mathfrak{m} = x_p x_q x_r$, $x_p, x_q, x_r \in V_2 \cup V_3$;

then F is not smooth.

Example: smoothness test

Code logic:

(1) for a given n, find all possible collections of variables V_1, V_2, V_3,

(2) give the list of monomials in the polynomial F,

(3) Check each collection one by one to see if it meets the condition. If there exists such a collection, return 'not smooth'(2); otherwise, return 'smooth'(1).

[Demonstrate in the Mathematica notebook]

SageMath

SageMath is a comprehensive open-source mathematics software system that combines mathematical tools for symbolic and numerical computations, data visualization, and more.

(1) nearly 100 open-source packages,
(2) cost-free and open-source,
(3) using Python, allowing users to write custom code and algorithms.

Installation:
Download the suitable installer from the official website and follow the provided instructions. (see https://www.sagemath.org/)

Linux and Mac users can also use Homebrew to install SageMath.

Example: basis of invariant polynomials

[Demonstrate in the terminal]

Interactions among GAP, Sage and Mathematica

We use the “sed” command to facilitate interaction between software.

$$\text{the output of software A} \xrightarrow{\text{“sed”}} \text{the input of software B}$$

The “sed” is a command-line tool for text processing and transformation, enabling functions like search, replace, filter, and edit text data.

For example, the usage of the “sed” command is referenced in McMullen’s Salem number/Coxeter group/K3 surface package. (see dx.doi.org/doi:10.7910/DVN/29211)

An example of using “sed”

Matrices obtained via GAP calculations:

```
#I  DeclareGlobalFunction: too many arguments in /opt/homebrew/Cellar/gap/4.12.2/libexec/pkg/repndecomp/lib/block_diagonalize.gd:18
#I  DeclareGlobalFunction: too many arguments in /opt/homebrew/Cellar/gap/4.12.2/libexec/pkg/repndecomp/lib/block_diagonalize.gd:29
[ [ [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, -1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, E(4) ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, -1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, 1 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, -1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, 1 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, E(3)^2, 0, 0, 0, 0 ], [ 0, 0, 0, E(3), 0, 0, 0 ],
          [ 0, 0, 0, 0, E(3)^2, 0, 0 ], [ 0, 0, 0, 0, 0, E(3), 0 ],
          [ 0, 0, 0, 0, 0, 0, E(3)^2 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, -1 ] ] ] ]
```

Figure: before using “sed”

An example of using “sed”

The format obtained after using the “sed” command, which can be input into SageMath:

```
UCF = UniversalCyclotomicField();L=[ [ [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, -1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, UCF.gen(4) ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, -1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, 1 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, -1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, 1 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, UCF.gen(3)^2, 0, 0, 0, 0 ], [ 0, 0, 0, UCF.gen(3), 0, 0, 0 ],
          [ 0, 0, 0, 0, UCF.gen(3)^2, 0, 0 ], [ 0, 0, 0, 0, 0, UCF.gen(3), 0 ],
          [ 0, 0, 0, 0, 0, 0, UCF.gen(3)^2 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0 ],
          [ 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0 ],
          [ 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0 ],
          [ 0, 0, 0, 0, 0, 0, -1 ] ] ] ]
F=CyclotomicField(12)
MS=MatrixSpace(F,7,7)
matlist=[MatrixGroup([MS(p) for p in i]) for i in L]
[[L[j],matlist[j].order(),matlist[j].invariants_of_degree(3)]for j in \
[0..len(matlist)-1]]
```

Figure: after using “sed”

A strategy for (5,3)-group classification

Let m be a positive integer. Suppose that all $(5,3)$-groups of orders $m' < m$ satisfying $m' \mid m$ have been found. We classify non-abelian $(5,3)$-groups of order m as follows.
Step 1: We compute the (finite) set $\mathcal{B}_m$ of non-abelian groups G of order m satisfying the following conditions:

(1) All proper subgroups of G are $(5,3)$-groups;

(2) G_{X_i} has no subgroup isomorphic to G for all $1 \le i \le 20$;

(3) G contains none of the 19 abelian groups.

If $\mathcal{B}_m = \emptyset$, then we are done. Otherwise, we do case-by-case check for groups in $\mathcal{B}_m$. For each $G \in \mathcal{B}_m$, go to **Step 2**.
Step 2: Compute the (finite) set $\mathcal{R}_G$ of the 3-equivalence classes of the special almost $(5,3)$-characters of G. If $\mathcal{R}_G = \emptyset$, then G is ruled out. Otherwise, go to **Step 3**.
Step 3: For each $\chi \in \mathcal{R}_G$, (i) we compute a representation ρ affording χ; (ii) we compute the cubic forms F invariant by all matrices in $\rho(G)$; (iii) we prove that such forms F are not smooth.

A strategy for (5,3)-group classification

We take $m = 96$ as an example.

[Demonstrate in the terminal]

A strategy for (4,3)-group classification

Let m be a positive integer. Suppose that all $(4,3)$-groups of orders $m' < m$ satisfying $m' \mid m$ have been found. We classify $(4,3)$-groups of order m as follows.

Step 1: Compute the set $\mathcal{B}'_m$ of groups G of order m satisfying the following conditions:

(1) All proper subgroups of G are $(4,3)$-groups;

(2) $\mathrm{Aut}(X'_i)$ has no subgroup isomorphic to G for all $1 \le i \le 15$.

If $\mathcal{B}'_m = \emptyset$, then we are done. Otherwise, we do case-by-case check for groups in $\mathcal{B}'_m$. For each $G \in \mathcal{B}'_m$, go to **Step 2**.

Step 2: Compute the (finite) set $\mathcal{C}_G$ defined as follows: if $3 \mid m$,

$$\mathcal{C}_G := \{\widehat{G} \mid \widehat{G} \text{ is a } C_3\text{-covering } (5,3)\text{-group of } G\};$$

if $3 \nmid m$, $\mathcal{C}_G := \{\widehat{G} \mid \widehat{G} \cong C_3 \times G \text{ and } \widehat{G} \text{ is a } (5,3)\text{-group}\}$. If $\mathcal{C}_G = \emptyset$, then G is ruled out. Otherwise, go to **Step 3**.

Step 3: For each $\widehat{G} \in \mathcal{C}_G$, we prove that either (i) $\widehat{G}$ contains one of the 19 groups by computing abelian subgroups of $\widehat{G}$ or (ii) $\widehat{G}$ has no special $(5,3)$-representation ρ with $\rho(\widehat{G})$ containing $\mathrm{diag}(\xi_3, ..., \xi_3, 1)$ by computing $\mathcal{R}_{\widehat{G}}$ (and applying **Step 3** of strategy for $(5,3)$-groups to each $\chi \in \mathcal{R}_{\widehat{G}}$ if $\mathcal{R}_{\widehat{G}} \neq \emptyset$). Then G is ruled out.

A strategy for (4,3)-group classification

We take $m = 16$ as an example.

[Demonstrate in the terminal]

Find the 2nd representation of A_7

[Demonstrate in the terminal]

THANK YOU !