About Controlling Fee Rates and P&L for Lightning Network Routing Nodes

Work in Progress: This peace of work should be interpreted as thoughts by the author only. It probably still contains minor and possibly even major errors at this stage.

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1. Summary

An approach for a more quantitative fee modelling for Lightning Network Routing Nodes is presented, which considers inbound and outbound fee rates as well. A key assumption of the approach is that a competitive fee setting is good for the network and that a node can incentivize inflows and outflows with its fee settings. If a node uses its fees as opportunity costs for the calculation of the fee limits, the fee settings influence the rebalancing payments. The incentivization can only work if the remote side has healthy fee settings, at least within a certain range. If a node uses high fee walls regardless of the liquidity state of the channel and there is no usage by the sender, the question arises whether the channel should remain open.

The basic dynamic of the proposed model is simple: lower outbound fees and raise inbound fees if liquidity is local, and vice versa if liquidity is remote. However, we are not modelling the fee rates directly; instead, we model two components, called margin and spread, and derive the fee rates from

these components. Moreover, we introduce a five-parameter model for the spread using exponential moving averages and a two-parameter model for the margin using mean reversion. Ultimately, the model enables us to parameterize the fee rate so that it is slower immediately after a depletion compared to a longer-lasting depletion.

The automatic determination of optimal parameterization would be the subject of future research. However, the model can already be used manually, provided the parameter dynamics are understood.

About the structure of the document

First, in Chapter 2, we show how to introduce negative forward fees that are compatible with #bLIP18 using inbound fees. We demonstrate when these are arbitrage-free, meaning that an attacker cannot gain money by routing through a node due to negative fees. Together with the findings from Chapter 3, we conclude that the existence of a positive margin is crucial for this.

In Chapter 3, we explain how to model margin and spread rates, and in a second step, how to compose fee rates from these. This also allows a better explanation of the actual P&L, as we can view forwards as liquidity trades: buying liquidity on the incoming, selling liquidity on the outgoing channel, and earning a margin. We mathematically decompose forward fees and rebalancing costs in two ways: one separates the margin of the source channel, the other separates the margin of the target channel. This enables node operators to better explain their profit and losses at the channel level.

In Chapters 4 and 5, we present an approach to model spread and margin rates. The spread rates adjust using a kind of PID Controller, increasing the spread when the channel is depleted. The controller for spread rates uses an exponential moving average for the integral and an exponential decay for the integral. This expresses the controller as a continuous function of current depletion and five parameters. The margin rate is instead controlled by a mean reverting controller, which smoothly converges to a given mean reversion level at a specified speed.

We assume that all parameters for the controllers are externally provided. But for automated node management, we also need processes for parameter calibration. Some rough ideas for further automation are presented in Section 6. The overall goal is to build a model that maximizes the node's returns.

1.1. Comparison of Dynamic Lndg Auto-Fees

We do not aim to provide a complete overview of existing fee tools. However, we can categorize existing algorithms into three main categories:

1. **Static**: The fee rates are constant until the node operator changes them.
2. **Semi-dynamic**: The fee rates are a function of the current state of a channel or all channels with one peer, e.g., based on current local balances. The same state at different times implies the same fee rates. An example of this is balance-dependent configurations like charge-lnd.
3. **Dynamic**: Dynamic algorithms are not only a function of the current channel state, but also consider historical trends. The key difference from semi-dynamic is that the same state at a different time can lead to different fee rates.

One example of a dynamic fee algorithm is the Auto-Fees implemented in Lndg, which we'd like to compare with our proposed spread rate controller. We focus on where Lndg decreases fees if local liquidity exceeds a given `ExcessLimit` (Code) and discuss two key differences.

1. - The check of the `ExcessLimit` serves only as a trigger point for the algorithm, but the size of the resulting fee change does not scale with the size of the excess. It also does not scale with other used metrics, such as revenues or the amount routed. Moreover, the historical excesses and recent evolutions do not influence the result.
   - Our spread rate controller, on the other hand, has a target for liquidity. The resulting change in the spread rate scales with the size of the difference to the target. This means that slight

excesses only cause slight adjustments to the spread rate. The exponential moving average (EWMA) also yields different results depending on whether the target has been exceeded for a short or long period. We also factor in recent trends: if the liquidity is already moving in the desired direction, future adjustments will be smaller.

2.
   - In Lndg, the change scales linearly with the number of calls to the algorithm, not with time. Lndg uses `UpdateHours`, which sets the time that must pass since the last fee rate change before it can change again. Over a longer period, `UpdateHours` influences the result, as a lower value leads to more frequent algorithm calls.
   - The spread rate controller, in contrast, evolves continuously over time and produces real-number results. The decision to broadcast a result via gossip is handled at a higher level. At this level, we consider the last update and a minimum delta rate. The continuous model also allows for better mathematical analysis. Additionally, parameters do not need to be adjusted based on the call frequency, as the continuous model will yield the same value at any time t.

2. About Arbitrage Freeness with Negative Forward Fees

We introduce the concept of negative forward fees, where inbound fees can overpay outbound fees. In the review process of #6703, there was a discussion about allowing overall negative forward fees. At that point, it seemed better to prevent node operators from losing money by not allowing this feature.

Here, we introduce the concept of **arbitrage freeness**, which ensures that an attacker cannot extract money from a node by sending sats back and forth due to bad relationships between inbound and outbound fees.

Let c be a channel policy, $r_{c,i}$ and $r_{c,o}$ be the inbound and outbound fee rates (in decimals, e.g., 1000ppm = 0.001), and $b_{c,i}$ and $b_{c,o}$ be the base fees in msat. All information is represented as a vector $c = (r_{c,o}, b_{c,o}, r_{c,i}, b_{c,i})^T$ and $c \in C$, where $C = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}$ is the policy space.

The function

$$f_{c,o}(x) := (1 + r_{c,o}) \cdot x + b_{c,o}$$

calculates the amount to receive from an amount sent using the outbound fees of channel c. The function

$$f_{c,i}(x) := (1 + r_{c,i}) \cdot x + b_{c,i}$$

calculates the amount to send from an amount received using the inbound fees of the channel c. This function only applies to negative forward fees, which are currently floored by the next outbound fee. The function becomes:

$$\tilde{f}_{c,i}(x) := x + \max(r_{c,i} \cdot x + b_{c,i}, -k) = \max((1 + r_{c,i}) \cdot x + b_{c,i}, x - k)$$

with the previous outbound fee k.

With negative forward fees, it is possible for $f_{c,i}(x) < 0$ for small amounts of x if the inbound base fee $b_{c,i}$ is negative. Economically, this would mean that the routing node pays the sender. We assume the function:

$$\bar{f}_{c,i}(x) = \max((1 + r_{c,i}) \cdot x + b_{c,i}, 0)$$

for future proofs. Additionally, we assume there are no upper limits for the amount x, i.e., no constraints by channel capacity or max HTLC amounts.

For any representation of f, we see that $f_{c,i}(x+\Delta) \geq f_{c,i}(x)$ for $x, \Delta \geq 0$ because

$$f_{c,i}(x+\Delta) = f_{c,i}(x) + \Delta(1 + r_{c,i})$$

and the last summand is positive if $r_{c,i} \geq -1$, i.e., at least -100%.

2.1 Arbitrage Freeness

To introduce **arbitrage freeness**, we analyze when an attacker cannot profit from sending sats between nodes. We assume the attacker controls all nodes in a channel's parties, enabling them to send sats without additional hop costs. The attacker chooses an arbitrary sequence of channels with policies $c_1, \ldots, c_n$ where $c_1 = c_n$. They send an amount x_1 from c_1 to c_2, receiving x_2 in c_2. They continue sending and receiving through the channels, and we want to ensure that $x_1 \geq x_n$.

A channel policy c is **weak arbitrage free** if an attacker cannot profit from a self-payment within the same channel, i.e., sending x_1 and receiving x_2 with $x_2 > x_1$ is not possible.

Lemma: A channel policy c is weak arbitrage free if and only if:

$$f_{c,io}(x) := (f_{c,i} \circ f_{c,o})(x) \geq x$$

for all x. This condition is satisfied if both conditions hold:

- $r_{c,o} + r_{c,i} + r_{c,o} \cdot r_{c_i} \geq 0$
- $b_{c,o} + b_{c,i} + b_{c,o} \cdot r_{c,i} \geq 0$

Proof: We set $c = c_1 = c_2$, meaning an attacker makes a circular payment in the same channel, sending x_1 and receiving x_2. Hence, the condition $f_{c,io}(x_2) \geq x_2$ is both necessary and sufficient for each x_2.

Further, we have:

$$f_{c,io}(x) = x + (r_{c,o} + r_{c,i} + r_{c,o} \cdot r_{c,i}) \cdot x + b_{c,o} + b_{c,i} + b_{c,o} \cdot r_{c,i}$$

If both inequalities hold, the function's value is $\geq x$ for all x. Moreover, using $\tilde{f}_i$ or $\bar{f}_i$ instead of f_i only increases the function's value. $\square$

At this point, we cannot say whether an attacker could gain money when combining channel policy c with other policies. Thus, we introduce **strong arbitrage freeness (SAF)**, a property of a subset $\tilde{C} \subseteq C$. We say $\tilde{C}$ is SAF if for any sequence $c_1, \ldots, c_n$ with $c_i \in \tilde{C}$ and any amount x:

$$((f_{c_1,i} \circ f_{c_2,o}) \circ \cdots \circ (f_{c_{n-2},i} \circ f_{c_{n-1},o}) \circ (f_{c_{n-1},i} \circ f_{c_1,o}))(x) \geq x$$

In other words, the attacker cannot profit by sending an arbitrary amount through an arbitrary sequence of channels.

If a set is SAF, every subset is SAF as well, since every policy sequence of the subset is also a sequence of the larger set. This leads us to the question of whether we can define a maximum set C_A that is SAF, meaning no other set $\tilde{C}$ is SAF and $C_A \subset \tilde{C}$.

For the next lemma, we define another composition $f_{c,oi} := f_{c,o} \circ f_{c,i}$.

Lemma: The set $C_A := \{c \in C \mid \forall x \geq 0,\ f_{c,oi}(x) \geq x \ \wedge \ f_{c,io}(x) \geq x\}$ is the maximum SAF set.

Proof: The first part shows that every channel policy $c \in \tilde{C}$ of an arbitrary SAF set is also in C_A. The second part shows that the constructed set C_A is still SAF.

Part 1:
The first step is to recognize that the zero channel policy $c_0 = (0,0,0,0)^T$ can be used to construct a SAF set: $C_0 = \{c_0\}$. By definition, we have $f_{c_0,io}(x) = f_{c_0,oi}(x) = x$, so every composition leads to the identity function, which satisfies the SAF condition for each x.

Next, if we have a SAF set $\tilde{C}$ that does not include c_0, then $\tilde{C} \cup \{c_0\}$ is also SAF. This can be shown by taking any arbitrary policy sequence from $\tilde{C}$, and if we enrich the sequence with c_0, the SAF definition still holds because the identity function does not alter the fees.

Finally, let $c \in \tilde{C}$ be a policy from any arbitrary SAF set. By the definition of SAF, c alone must satisfy $f_{c,io}(x) \geq x$ for all x. Moreover, the set $C_c = \{c_0, c\}$ is SAF, as previously mentioned. From the SAF definition, we also know the set is safe against an attacker sending an amount into the channel with policy c_0, then sending it back to the channel with policy c. Because:

$$((f_{c_0,i} \circ f_{c,o}) \circ (f_{c,i} \circ f_{c_0,o}))(x) = (f_{c,o} \circ f_{c,i})(x) = f_{c,oi}(x)$$

we have demonstrated that $f_{c,oi}(x) \geq x$ for all x, and hence $c \in C_A$.

Part 2:
Let $c_1, \ldots, c_n$ be a sequence where $c_i \in C_A$. We need to show that for all x:

$$x_1 = ((f_{c_1,i} \circ f_{c_2,o}) \circ \cdots \circ (f_{c_{n-2},i} \circ f_{c_{n-1},o}) \circ (f_{c_{n-1},i} \circ f_{c_1,o}))(x) \geq x$$

holds.

Choosing an arbitrary amount $x_n \geq 0$ and using the fact that composition is associative, we have:

$$x_1 = (f_{c_1,i} \circ (f_{c_2,o} \circ f_{c_2,i}) \circ \cdots \circ (f_{c_{n-1},o} \circ f_{c_{n-1},i}) \circ f_{c_n,o})(x_n)$$

By our definitions, this becomes:

$$x_1 = (f_{c_1,i} \circ (f_{c_2,oi} \circ \cdots \circ f_{c_{n-1},oi}) \circ f_{c_1,o})(x_n)$$

If we assume that $f_{c,oi}(x) \geq x$ for each channel c and every possible amount x, then the inequality holds for any composition of such functions in sequence. As a consequence, there exists a $\Delta \geq 0$ such that:

$$(f_{c_2,oi} \circ \cdots \circ f_{c_{n-1},oi})(f_{c_1,o}(x_n)) = f_{c_1,o}(x_n) + \Delta$$

As already demonstrated, for any $x, \Delta \geq 0$, we have $f_{c,i}(x + \Delta) \geq f_{c,i}(x)$, and thus:

$$x_1 = f_{c_1,i}(f_{c_1,o}(x_n) + \Delta) \geq f_{c_1,i}(f_{c_1,o}(x_n)) = f_{c_1,io}(x_n)$$

Because of the C_A definition, we have $f_{c,io}(x) \geq x$ for each channel c and any amount x. Therefore, $x_1 \geq f_{c_1,io}(x_n) \geq x_n$, proving that C_A is SAF. □

Corollary:

The set of channel policies where the parameters satisfy the following conditions is denoted by C_R:

1. $r_{c,o} + r_{c,i} + r_{c,o} \cdot r_{c_i} \geq 0$
2. $b_{c,o} + b_{c,i} + b_{c_o} \cdot r_{c,i} \geq 0$
3. $b_{c,o} + b_{c,i} + b_{c,i} \cdot r_{c,o} \geq 0$

C_R is SAF.

Proof:
Regardless of whether we use $\tilde{f}_i$, $\bar{f}_i$, or f_i, we can state that:

$$f_{c,io}(x) \geq x + (r_{c,o} + r_{c,i} + r_{c_o} \cdot r_{c,i}) \cdot x + b_{c,o} + b_{c,i} + b_{c_o} \cdot r_{c,i}$$

$$f_{c,oi}(x) \geq x + (r_{c,o} + r_{c,i} + r_{c_o} \cdot r_{c,i}) \cdot x + b_{c,o} + b_{c,i} + b_{c,i} \cdot r_{c,o}$$

Given these assumptions, both right-hand sides are ≥ 0, which implies SAF.

Remark:
We can also say that $C_R \subseteq C_A$, e.g., if we only work with policies that have no base fees and do not floor the fees at outbound fees.

3. Modelling Outbound and Inbound Fee Rates with Margin and Liquidity Spread

3.1. The Margin Spread Model

Assume for a specific channel c, there are two given variables: a non-negative margin rate m_c and a liquidity spread rate s_c, both of which can be real numbers. We aim to set the outbound and inbound fee rates based on this information.

If $m_c + s_c \geq 0$, we set:

$$\begin{aligned} r_{c,o} &= \frac{1}{1 - s_c} \cdot (m_c + s_c) \\ r_{c,i} &= -s_c \end{aligned}$$

A continuous extension of these functions for the case $m_c + s_c < 0$ is:

$$\begin{aligned} r_{c,o} &= 0 \\ r_{c,i} &= m_c \end{aligned}$$

We can consolidate these two cases into one by introducing $\tilde{s_c} := s_c$ if $s_c + m_c \geq 0$ and $\tilde{s_c} := -m_c$ if $s_c + m_c < 0$, and then set:

$$\begin{aligned} r_{c,o} &= \frac{1}{1 - \tilde{s_c}} \cdot (\tilde{s_c} + m_c) \\ r_{c,i} &= -\tilde{s_c} \end{aligned}$$

This makes subsequent proofs simpler, as we only need to consider one case.

3.2. Rationale for Margin and Spread

3.2.1. Margin in Circular Case Why do we call these two variables margin and spread? Let's consider a circular payment with channel policy c. Assuming there are no restrictions for the incoming fees, the net earnings from such a payment can be calculated as:

$$E(x) = f_{c,oi}(x) - x = ((1 + r_{c,o}) \cdot (1 + r_{c,i}) - 1) \cdot x + B(x)$$

where $B(x)$ is a residual driven by base fees, which we will ignore for now. Using:

$$(1 + r_{c,o}) \cdot (1 + r_{c,i}) - 1 = \left(1 + \frac{1}{1 - \tilde{s_c}} \cdot (m_c + \tilde{s_c})\right) \cdot (1 - \tilde{s_c}) - 1 = m_c$$

we find that the node earns a margin amount proportional to the margin rate m_c. Ignoring $B(x)$, the spread does not influence earnings.

Additionally:

$$m_c = (1 + r_{c,o}) \cdot (1 + r_{c,i}) - 1 = r_{c,o} + r_{c,i} + r_{c,o} \cdot r_{c_i}$$

Thus, we can replace the first inequality for checking whether $c \in C_R$ by the margin rate.

Corollary: For a channel policy c, we have $c \in C_R$ if $m_c \geq 0$ and the second and third inequalities are satisfied.

3.2.2. Margin in Two-Party Case Now consider a case where two channels with policies c_1 and c_2 are involved. First, we have forwards from c_1 to c_2 and then back in the opposite direction. Assuming there are no restrictions for incoming fees, the earnings are given by:

$$E(x) = (f_{c_1,i} \circ f_{c_2,o} \circ f_{c_2,i} \circ f_{c_1,o})(x) - x = ((1 + r_{c_1,i}) \cdot (1 + r_{c_2,o}) \cdot (1 + r_{c_2,i}) \cdot (1 + r_{c_1,o}) - 1) \cdot x + B(x)$$

After some algebra, we find:

$$E(x) = ((1 + m_{c_1}) \cdot (1 + m_{c_2}) - 1) \cdot x + B(x)$$

This shows a compounding effect of the margin rates. The spread still has no influence on the earnings if we ignore $B(x)$.

3.2.3. Role of the Spread As we've seen, the spread does not directly affect earnings, but this is only partially true. The spread essentially sets the price for liquidity, either through rebalancing or incoming forwards. A lower spread makes it harder to incentivize inbound liquidity, as the inbound fee rate is lower. Additionally, if outbound fees are used as opportunity costs for rebalancing, a lower spread implies a lower outbound fee and thus a reduced rebalancing budget.

Conversely, a higher spread makes it easier to attract inbound liquidity, but it also raises the outbound fees, which can limit outgoing forwards on the channel.

We conclude that the spread rate acts more as a price for liquidity, influencing the forwarding volume of a channel. This, in turn, affects the margin a node can earn.

The volatility of the spread is also important. If spread volatility is high— meaning that the spread increases quickly when liquidity is remote and decreases quickly when liquidity is local—there's a risk of buying liquidity at prices that make it unprofitable to sell. This can negatively impact total profits due to an impairment effect driven by the spread rates. Volatility, in itself, is not necessarily bad, as it could lead to higher volumes on the node. The key question is finding the optimal balance between volatility and total profits.

3.3. Practical Setting

This model will later help us understand the profit and loss (P&L) structure of a routing node. If we can model the margin and liquidity spread, we can set fee rates immediately. However, in practice, we cannot set fee rates as real numbers, and we may not broadcast each changed parts-per-million (ppm) rate to the network. Thus, we introduce residuals $\epsilon_{c,o}$ and $\epsilon_{c,i}$, which are the result of rounding and other practical constraints. Additionally, in practice, the factor $\frac{1}{1-\tilde{s_c}} \sim 1$. We will set this factor to 1 and include the difference in $\epsilon_{c,o}$.

Summarizing:

$$\begin{aligned} r_{c,o} &= \tilde{s_c} + m_c + \epsilon_{c,o} \\ r_{c,i} &= -\tilde{s_c} + \epsilon_{c,i} \end{aligned}$$

3.4. Explain the P&L of a Routing Node

3.4.1 P&L Explain for Forwards We denote the fee a sender wants to pay us with f for a forward incoming in channel A with Alice and outgoing in channel B with Bob. The function:

$$f(x) = x \cdot ((1 + r_{B,o}) \cdot (1 + r_{A,i}) - 1) + b_{B,o} \cdot (1 + r_{A,i}) + b_{A,i}$$

calculates the theoretical fee if there is no floor at 0, based on the outgoing amount x. Our goal is to build two views: one that separates the margin of channel A and one that extracts the margin of channel B.

Forwards: Separation of the Source Margin We can express the identity as:

$$\begin{aligned} x \cdot ((1 + r_{B,o}) \cdot (1 + r_{A,i}) - 1) &= x \cdot (r_{B,o} + r_{A,i} + r_{B,o} \cdot r_{A,i}) \\ &= x \cdot r_{B,o} + (x + f) \cdot r_{A,i} + x \cdot r_{B,o} \cdot r_{A,i} - f \cdot r_{A,i}. \end{aligned}$$

Considering the following expression:

$$\begin{aligned} r_{A,i} = -\tilde{s_A} + \epsilon_{A,i} &= -(m_A + \tilde{s_A} + \epsilon_{A,o}) + m_A + (\epsilon_{A,i} + \epsilon_{A,o}) \\ &= -r_{A,o} + m_A + (\epsilon_{A,i} + \epsilon_{A,o}) \end{aligned}$$

we can decompose the total fee into seven components:

$$f = \sum_i f_i$$

where:

$$
\begin{aligned}
f_1 &= f - f(x)^+ \\
f_2 &= f(x)^+ - f(x) \\
f_3 &= b_o \cdot (1 + r_i) + b_i \\
f_4 &= (x + f) \cdot m_A \\
f_5 &= -(x + f) \cdot r_{A,o} \\
f_6 &= x \cdot r_{B,o} \\
f_7 &= x \cdot r_{B,o} \cdot r_{A,i} - f \cdot r_{A,i} + (x + f) \cdot (\epsilon_{A,i} + \epsilon_{A,o})
\end{aligned}
$$

The components are explained as follows:

1. f_1: Represents an overpayment between the actual fee and the theoretical fee based on bLIP18 formulas, which considers a floor at 0 (current lnd implementation). This can happen if the payer is unaware of lower fees or cannot use inbound discounts.
2. f_2: Unexpected margin, typically occurring when $(1+r_{B,o}) \cdot (1+r_{A,i}) < 1$, meaning the outbound fee is lower than the inbound discount. This can occur when forwarding from a "sink" node to a "source" node.
3. f_3: Base margin, representing the net amount from base fees.
4. f_4: The margin using Alice's margin rate, which scales with the received incoming amount.
5. f_5: Represents the purchasing cost of the fee potential. Liquidity of amount $(x + f)$ has been received in Alice's channel. Selling it could earn $(x + f) \cdot r_{A,o}$.
6. f_6: The fee received for selling liquidity to Bob.
7. f_7: Residual P&L consisting of various second-order terms.

The concept of 'fee potential,' introduced by DerEwige, is the product of the fee rate and its potential earnings. We can also define spread potentials for spread rates and margin potentials for margin rates.

The main idea here is that routing is like liquidity trading. f_5 and f_6 do not directly cause P&L—P&L arises from the other components. P&L only changes if the fee rates are adjusted between the purchasing and selling of liquidity.

Forwards: Separation of the Target Margin We can also build a decomposition that separates the margin of Bob's channel:

$$
\begin{aligned}
r_{B,o} &= (\tilde{s_B} + \epsilon_{B,o}) + m_B \\
r_{A,i} &= -(\tilde{s_A} + \epsilon_{A,o}) + (\epsilon_{A,i} + \epsilon_{A,o})
\end{aligned}
$$

Decomposing f gives:

$$
f = \sum_i f_i
$$

where:

$$
\begin{aligned}
f_1 &= f - f(x)^+ \\
f_2 &= f(x)^+ - f(x) \\
f_3 &= b_o \cdot (1 + r_i) + b_i \\
f_4 &= x \cdot m_B \\
f_5 &= -(x + f) \cdot (\tilde{s_A} + \epsilon_{A,o}) \\
f_6 &= x \cdot (\tilde{s_B} + \epsilon_{B,o}) \\
f_7 &= x \cdot r_{B,o} \cdot r_{A,i} - f \cdot r_{A,i} + (x + f) \cdot (\epsilon_{A,i} + \epsilon_{A,o})
\end{aligned}
$$

In this decomposition:

- f_4 is Bob's margin, scaling with the outgoing amount x.
- f_5 and f_6 focus on liquidity trade related to spread potentials (and the epsilons).

Now, we have two decompositions: one that extracts the margin of the incoming channel and one for the outgoing channel. Any convex combination (e.g., 50% of each) is a valid decomposition.

3.4.2. P&L Explain for Rebalancing Costs In the context of rebalancing costs, we want to decompose the rebalancing costs into two views.

Suppose we are rebalancing from our source channel A (with Alice) to channel B (with Bob), and the rebalancing costs amount to $c < 0$ in sats. In future cases of negative forward fees, we also want to account for cases where $c > 0$, e.g., when rebalancing from a sink to a source. Effectively, rebalancing for a routing node is akin to liquidity trading, better described as a liquidity arbitrage trade: selling liquidity in channel A and buying liquidity in channel B.

Rebalancing: Separation of the Benefit in the Source Margin We decompose the rebalancing cost c as follows:

$$
\begin{aligned}
c &= \sum_i c_i \\
c_1 &= (x + c) \cdot r_{A,o} \\
c_2 &= -x \cdot r_{B,o} \\
c_3 &= m_B \cdot (x + f_B) - m_A \cdot (x + f_A) \\
c_4 &= c - c_1 - c_2 - c_3
\end{aligned}
$$

1. c_1: Represents the virtual amount received from selling the fee potential in Alice's channel.
2. c_2: Represents the virtual amount paid for buying the fee potential in Bob's channel.
3. c_3: The margin advantage of forwarding from Alice to Bob compared to the reverse direction (from Bob to Alice). Here, f_A and f_B are the hypothetical fees for forwards to calculate the correct incoming amounts in both channels. If A and B have nearly identical fee rates such that $c_1 + c_2 = 0$, rebalancing is only beneficial if the margin from Alice to Bob is higher than from Bob to Alice, i.e., $c > c_3$.
4. c_4: The difference between c_1 and c_3 compared to the actual paid fee c. It is essentially the rebalancing margin. Rebalancing is only beneficial if $c_4 > 0$.

Rebalancing: Separation of the Benefit in the Target Margin The next step is to build a decomposition that separates the margin benefit of the target channel.

$$
\begin{aligned}
c &= \sum_i c_i \\
c_1 &= (x+c)\cdot(\tilde{s_A}+\epsilon_{A,o}) = (x+c)\cdot r_{A,o} - (x+c)\cdot m_A \\
c_2 &= -x\cdot(\tilde{s_B}+\epsilon_{B,o}) = -x\cdot r_{B,o} + x\cdot m_B \\
c_3 &= m_A\cdot(x+f_A) - m_B\cdot(x+f_B) \\
c_4 &= c - c_1 - c_2 - c_3
\end{aligned}
$$

In this decomposition, the components are interpreted similarly to the previous one, but here we focus only on the spread potentials (and epsilons), rather than the entire fee potentials.

Adjustments of the Potentials We've seen that the decomposition of earnings and costs implicitly includes liquidity trades at the current fee or spread rates. However, these rates must be the current rates at the time of trade, and they can change between buying liquidity through rebalancing and selling it through a forward. If a node lowers its fees in between, this creates a negative P&L component that is not reflected in the above decomposition. To handle this, we can calculate additional value adjustments through a process that revalues the fee potentials at regular intervals.

1. For time T_{n-1}, we know the potential for fees, spreads, or margins.
2. At time T_n, we determine the current potentials by multiplying the current balance by the current fees, spreads, or margins.
3. We calculate the net balance of liquidity bought and sold for fees, spreads, or margins between these times.
4. We compare the sum of 1. and 3. with the value at time 2. If the sum is lower, this results in a negative value adjustment (a loss). If it is higher, it results in a gain.

3.5. Margin Spread Model with #bLIP19 Inbound Fees

There is an alternative proposal for inbound fees with #bLIP19. The main idea is that a node communicates the inbound fee to the peer, and the peer increments its outbound fee accordingly.

> MUST ensure it's next `channel_update` message for the corresponding channel has its `fee_proportional_millionths` and `fee_base_msat` fields incremented by the `inbound_forwarding_fee_proportional_millionths` and `inbound_forwarding_fee_base_msat` fields, respectively. Fields which fall below zero MUST be set to zero.

The zero floor would make it impossible to use the proposed margin spread model with such inbound fees because the inbound discounts would be constrained by the outbound fees of the peer.

Therefore, the proposed margin spread model cannot be fully applied with #bLIP19 inbound fees.

4. Modelling the Spread Rate with Exponential Moving Averages

4.1. Basics

The model for the spread rate is based on the idea of PID (Proportional-Integral- Derivative) controllers. PID controllers use a measured process variable, compare it to a target value, and calculate an error function, denoted as $e(t)$. They use a linear function to adjust the control variable based on the error, its integral over time, and its derivative, aiming to minimize the error in subsequent iterations.

We are not using classical integrals and derivatives. Instead, an exponentially weighted moving average with a smoothing parameter (α_i) defines the implicit length of the error history. Additionally, an exponential decay with a parameter (α_d) is applied to the error delta as a derivative component. You can also specify a drift for adjustments that scale over time but are not influenced by the error.

To calculate the error, the remote balance is compared to a target value, typically derived from the average liquidity ratio of all channels. The difference between the observed remote balance and the target is mapped to an error (e) in the range [-0.5, 0.5] using linear interpolation. When the remote balance equals the target, e is set to 0.

4.2. Modelling the Dynamic of the Spread Rate

Let T be the current time, and T_0 represent the oldest observed historic timestamp.

Our approach is based on control theory, i.e., we have a system with a time- dependent input function, our error function $e(t)$. The system returns a time- dependent output $y(t)$, which is the marginal increase in the spread rate. Thus, we can model the spread rate by the following differential equation:

$$ds(t) = y(t)dt$$

This equation leads to:

$$s(T_n) = s(T_{n-1}) + \int_{T_{n-1}}^{T_n} y(t)dt$$

The system output $y(t)$ itself is a linear combination of several components:

$$y(t) = K_p \cdot e(t) + K_i \cdot E_{\alpha_i}(t) + K_d \cdot D_{\alpha_d}(t)$$

The functions E_α and D_α are defined later. Our goal is that they exhibit the following properties: while the first summand scales with the current value of the error function, the second summand should scale with an exponential moving average $E_{\alpha_i}(t)$ over a longer time horizon. α_i serves as a smoothing parameter to control the implicit length of the history. The third summand should scale with recent changes in the error function. However, we don't want to apply the changes immediately. Instead, we want to smooth the changes with an exponential decay function $D_{\alpha_d}(t)$, parametrized by a smoothing parameter α_d for the implicit decay period.

Overall, the model has five parameters that need to be calibrated. For the next sections, we assume the parameters are externally given.

Our goal is now to find a recursive representation of the following integral, which allows us to update the controller incrementally:

$$\int_{T_{n-1}}^{T_n} y(t)dt = K_p \int_{T_{n-1}}^{T_n} e(t)dt + K_i \int_{T_{n-1}}^{T_n} E_{\alpha_i}(t)dt + K_d \int_{T_{n-1}}^{T_n} D_{\alpha_d}(t)dt$$

Now, let's delve into the different components. We will assume that the error function $e(t)$ is piecewise linear over $t \in]T_{n-1}, T_n]$, i.e.,

$$e(t) = \beta_1 \cdot (t - T_{n-1}) + \beta_0, \ \beta_1 = \frac{e(T_n) - e(T_{n-1})}{T_n - T_{n-1}}, \ \beta_0 = e(T_{n-1})$$

4.2.1. Proportional Part This part is relatively straightforward:

$$\begin{aligned}
K_p \int_{T_{n-1}}^{T_n} e(t)dt &= K_p \cdot (T_n - T_{n-1}) \cdot \Big(\frac{m}{2} \cdot (T_n - T_{n-1}) + \beta_0\Big) \\
&= K_p \cdot (T_n - T_{n-1}) \cdot \frac{e(T_n) + e(T_{n-1})}{2}
\end{aligned}$$

The outcome of the controller scales with the average error during the time period.

4.2.2. Exponential Decay of a Function Before we define the concrete integral and derivative parts, some general remarks about function decay: given an integrable input function $x(t)$ (which will later be the error $e(t)$ or its derivative) and a parameter α, we define a weight function $W_{x,\alpha}(t)$ by

$$W_{x,\alpha}(t) := \int_{T_o}^{t} x(\tau) \cdot \alpha \cdot \exp\left(\alpha(\tau - t)\right) d\tau$$

and with $h_\alpha(t) = \alpha \exp(-\alpha t)$, we have

$$W_{x,\alpha}(t) := \int_{T_o}^{t} x(\tau) \cdot h_\alpha(t - \tau) d\tau$$

which is a convolution integral over the local area $[T_0; t]$. This fact is not relevant now but might be interesting for future analysis of the controller.

Now, given $T_{n-1} > T_0$, we want to express $W_{h,\alpha}(t)$ for $t > T_{n-1}$ recursively using $W_{h,\alpha}(T_{n-1})$. It becomes:

$$\begin{aligned}
W_{x,\alpha}(t) &= \int_{T_{n-1}}^{t} x(\tau) \cdot \alpha \cdot \exp\left(\alpha(\tau - t)\right) d\tau + \exp(\alpha(T_{n-1} - t)) \int_{T_o}^{T_{n-1}} x(\tau) \cdot \alpha \cdot \exp\left(\alpha(\tau - T_{n-1})\right) d\tau \\
&= \int_{T_{n-1}}^{t} x(\tau) \cdot \alpha \cdot \exp\left(\alpha(\tau - t)\right) d\tau + \exp(\alpha(T_{n-1} - t)) \cdot W_{x,\alpha}(T_{n-1})
\end{aligned}$$

4.2.3. Integral Part To calculate the integral part, we apply our piecewise linear error function $e(t)$ to W_x as $x(t)$.

Our weight function will lead to an exponentially weighted moving average, which we define as

$$E_\alpha(t) := W_{e,\alpha}(t)$$

Given the assumption for $e(t)$, we need to solve

$$E_\alpha(t) = \int_{T_{n-1}}^{t} \alpha \exp\left(\alpha(\tau - t)\right) \cdot (\beta_1 \cdot \tau + \beta_0) \cdot d\tau + \exp\left(\alpha(T_{n-1} - t)\right) \cdot E_\alpha(T_{n-1})$$

The solution is

$$E_\alpha(t) = e(t) - \frac{\beta_1}{\alpha} + \exp\left(\alpha(T_{n-1} - t)\right) \cdot \left(E_\alpha(T_{n-1}) + \frac{\beta_1}{\alpha} - \beta_0\right)$$

Now, we can define the integral part of the controller as:

$$
\begin{aligned}
&K_i \int_{T_{n-1}}^{T_n} E_\alpha(t)dt \\
=&K_i \left((T_n - T_{n-1}) \cdot \left(\frac{e(T_n) + e(T_{n-1})}{2} - \frac{\beta_1}{\alpha} \right) + \frac{1}{\alpha}(1 - \exp(\alpha(T_{n-1} - T_n))) \cdot \left(E_\alpha(T_{n-1}) + \frac{\beta_1}{\alpha} - \beta_0 \right) \right)
\end{aligned}
$$

This allows us to update the controller recursively, using only the knowledge of $E_\alpha(T_{n-1})$.

4.2.4. Derivative Part For the derivative part, we use the partial derivative $\frac{\partial e}{\partial \tau}$ as $x(t)$. Thus, we solve

$$D_\alpha(t) := W_{\frac{\partial e}{\partial \tau},\alpha}(t) = \int_{T_{n-1}}^{t} \alpha \exp\left(\alpha(\tau - t)\right) \cdot \beta_1 \cdot d\tau + \exp\left(\alpha(T_{n-1} - t)\right) \cdot D_\alpha(T_{n-1})$$

The solution is

$$D_\alpha(t) = \beta_1 + \exp\left(\alpha(T_{n-1} - t)\right) \cdot \left(D_\alpha(T_{n-1}) - \beta_1\right)$$

Now, we can define the derivative part of the controller as:

$$
\begin{aligned}
&K_d \int_{T_{n-1}}^{T_n} D_\alpha(t)dt \\
=&K_d \left(e(T_n) - e(T_{n-1}) + \frac{1}{\alpha}(1 - \exp(\alpha(T_{n-1} - T_n))) \cdot (D_\alpha(T_{n-1}) - \beta_1) \right)
\end{aligned}
$$

This also allows us to update the controller recursively.

4.3. Behavior after a Simple Impulse

To show the dynamics of the spread controller, we want to investigate its long- term behavior after a simple impulse using all information provided. At $T_0 = 0$, we set $s(T_0) = 0$ and $e(T_0) = e_1$. For $t \geq T_1 = 1$, we observe $e(t) = e_1$. We want to determine an analytical function for the spread $s(T)$ for $T \geq T_1$ based on our derived formulas. For $t \in]0; 1]$, we have $\beta_1 = e_1 - e_0$ and $\beta_0 = e_0$. For $t > 1$, we have $\beta_1 = 0$ and $\beta_0 = e_0$. The other five model parameters are chosen arbitrarily.

First, we want to show how $E_\alpha(t)$ evolves if there is no change in the error from T_1. For $t > T_1$, we get:

$$E_\alpha(t) = e(t) - \frac{\beta_1}{\alpha} + \exp\left(\alpha(T_1 - t)\right) \cdot \left(E_\alpha(T_1) - e_1\right)$$

Because the second summand goes to zero for $t \to \infty$, $E_\alpha(t)$ converges to e_1. Hence, the contribution of the integral part for time intervals with large values of T_{n-1} and T_n is nearly $K_i \cdot (T_n - T_{n-1}) \cdot e_1$.

Applying the same thoughts to the derivative part, we see that $D_\alpha(t)$ converges to $\beta_1 = 0$, and hence the long-term contribution of this part also converges to 0. However, it is more interesting to show that the overall contribution of this part converges to $K_d \cdot (e_1 - e_0)$. We set $D_\alpha(T_0) = 0$ because there was no impulse before T_0. We get:

$$D_\alpha(T_1) = e_1 - e_0 - \beta_1 \cdot \exp(-\alpha) = \beta_1 \cdot (1 - \exp(-\alpha))$$

$$\int_{T_0}^{T_1} D_\alpha(t)dt = e_1 - e_0 - \frac{\beta_1}{\alpha} \cdot (1 - \exp(-\alpha))$$

For arbitrary $T > T_1$:

$$\begin{aligned}\int_{T_1}^{T} D_\alpha(t)dt &= \frac{1}{\alpha} \cdot (1 - \exp(-\alpha(T - T_1))) \cdot D_\alpha(T_1) \\ &= \frac{1}{\alpha} \cdot (1 - \exp(-\alpha(T - 1))) \cdot \beta_1 \cdot (1 - \exp(-\alpha))\end{aligned}$$

Adding the two integrals leads to:

$$\begin{aligned}\int_{T_0}^{T} D_\alpha(t)dt &= e_1 - e_0 - \frac{\beta_1}{\alpha} \cdot (1 - \exp(-\alpha)) + \frac{1}{\alpha} \cdot (1 - \exp(-\alpha(T - 1))) \cdot \beta_1 \cdot (1 - \exp(-\alpha)) \\ &= e_1 - e_0 - \frac{\beta_1}{\alpha} \cdot (1 - \exp(-\alpha)) \cdot (-1 + (1 - \exp(-\alpha(T - 1)))) \\ &= e_1 - e_0 + \frac{\beta_1}{\alpha} \cdot (1 - \exp(-\alpha)) \cdot \exp(-\alpha(T - 1))\end{aligned}$$

Since $\exp(-\alpha(T-1))$ converges to zero for large T, the integral converges to $e_1 - e_0$, and thus the overall contribution of the derivative part converges to $K_d \cdot (e_1 - e_0)$.

Moreover, the total spread rate adjustment converges to $K_\infty = e_1 \cdot (K_p + K_i)$, and since the error function is bounded, the spread rate adjustments are bounded to values in $[-0.5 \cdot (K_p + K_i), 0.5 \cdot (K_p + K_i)]$.

4.4. Example for a Spread Rate Controller

We want to show how the spread rate controller evolves in the following scenario: The funds in our channel are fully local and haven't moved for a while. Thus, $D_\alpha(t) = 0$ and $e(t) = E_\alpha(t) = 0.5$. Suddenly, an impulse depletes the channel immediately, moving the error to -0.5. For this scenario analysis, we use the following parameters:

$$K_p = 40,\ \alpha_i = 0.1,\ K_i = 80,\ \alpha_d = 0.9,\ K_d = 30$$

As a result, we observe the following total adjustments of the spread rate. We interpret the time scale as days, but depending on the implementation, it could also be hours, weeks, or another time unit. The first 3 to 4 days show no significant increase in the spread rate. By day 9, the spread rate increases by about 100 ppm. Another 100 ppm increase occurs by day 12. By day 21, an additional 400 ppm increase is observed. On average, it takes roughly 2 days to increase by 100 ppm, which is consistent with our observation that the spread rate adjustments converge to $K_\infty = 0.5 \cdot (K_p + K_i) = 60$.

For a better understanding of the dynamics, we examine how the different components of the controller evolve, applying parameters for one or two parts at a time while setting others to zero. The proportional part leads to a daily increase of 20 ppm. The integral part initially decreases the spread rate due to a negative moving average $E_\alpha(t)$. As E rises, the decrease becomes less until it turns positive. Combining the proportional and integral parts yields a slight initial drop in the spread rate, reaching a minimum of around -30 ppm on day 3. However, decreasing the rate after channel depletion makes little sense.

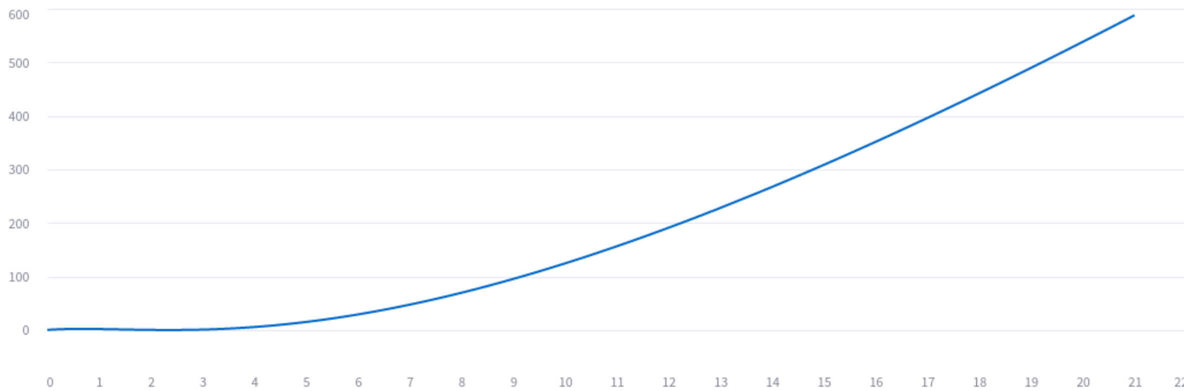
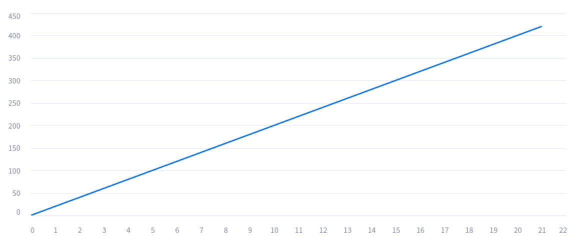


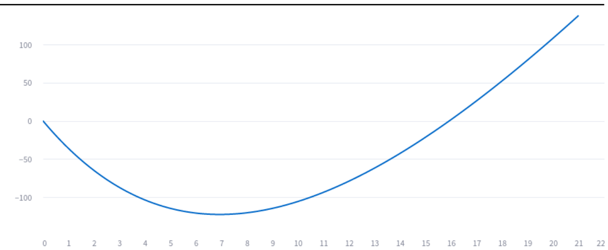
Figure 1: Aggregated Adjustments of the Spread Rate Controller

The derivative part corrects this, contributing nearly 30 ppm on day 3, with minimal contribution afterward.

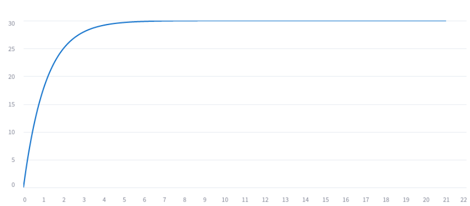
Proportional $K_p = 40$



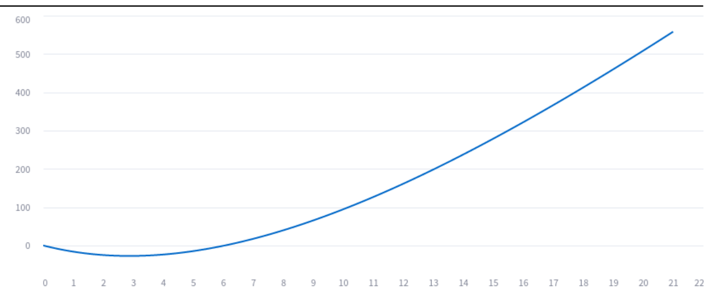
Integral $\alpha_i = 0.1,\ K_i = 80$



Derivative $\alpha_d = 0.9,\ K_d = 30$



Proportional & Integral $K_p = 40,\ \alpha_i = 0.1,\ K_i = 80$



4.4.1. Scaling the Example Interestingly, we can construct other functions for spread adjustments using the given parameters, resulting in curves of a similar shape. We want to create a function that remains almost flat for around 15 days and converges to an adjustment rate of about 96 ppm per day.

The given parameters remain flat for around 3 days. If we interpret one time unit as "per 5 days," then by dividing by 5, we get a new parameter set that results in a flat curve until $t = 15$. The only parameter we keep the same is K_d, as the 30 ppm only depends on the rate of change, not the timing.

$$K_p = 8,\ \alpha_i = 0.02,\ K_i = 16,\ \alpha_d = 0.18,\ K_d = 30$$

This controller converges to 12 ppm. Scaling the K values by a factor of 8 produces a controller converging to $K_\infty = 96$.

$$K_p = 64,\ \alpha_i = 0.02,\ K_i = 128,\ \alpha_d = 0.18,\ K_d = 240$$

This is the result of the controller. Notably, there is a local maximum of around 15 ppm within the first 15 days. Before that, it was about 2 ppm and practically irrelevant. Using mathematical optimization techniques, it should be possible to smooth the curve further.

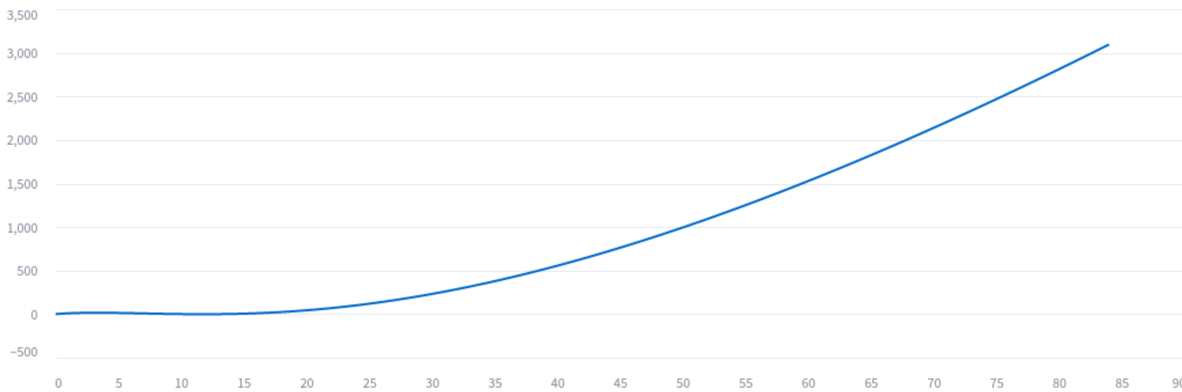


Figure 2: Scaled Spread Rate Controller

4.5. Spread Rate Controller as a Linear Time-Invariant System

Due to the structure of the convolution integral, my research points to linear time-invariant systems (LTI). Our spread rate controller appears to be LTI, as the output (the change of the controller) is linear depending on the input function (the error function), since the integrals are linear functions, and the derivative is linear as well. The output is also time-invariant, meaning if you apply the same error function with some time delay, the change of the spread controller will return the same output with the exact time delay.

> The fundamental result in LTI system theory is that any LTI system can be characterized entirely by a single function called the system's impulse response.

This is also important for calibrating the model. For example, if one calibrates the parameters to achieve suitable spread rate changes for a given error function, this calibration applies to all other possible error functions as well.

4.6. Outlook: Scenario-Based Model Calibration of the Parameters

Calibration involves finding the optimal parameters for the model. One idea is to use a scenario-based calibration, where the user or another algorithm suggests a dynamic for the controller in a depletion scenario, like the one we analyzed.

Only rough thoughts and ideas from here without deeper analysis

The dynamic could be described by three parameters: the adjustment rate K_∞ to which the controller converges, a time T_c up to which the spread adjustments are minimal, and an adjustment rate $K_{2 \cdot T_c}$, representing the marginal adjustment rate at time $2 \cdot T_c$. This redefinition simplifies the dynamic but needs to be converted into the actual five parameters of the controller. One way to find a parameter set with minimal adjustments at the beginning is to solve the following optimization problem with two constraints:

$$\text{argmin} \int_0^{T_c} \left(s(t) - s(0)\right)^2 dt$$

With knowledge of the model's scaling properties, the problem can be solved first with the parameters $K_\infty = 1$, $T_c = 1$, and $K_2 = \frac{K_{2 \cdot T_c}}{K_\infty}$. Due to the non-polynomial nature of α's effect on the objective function, solving this directly is complex.

An alternative could be an alternating minimization:

1. Fix α_d and α_i.
2. Allow K_p, K_i, and K_d to vary. Now, minimize a quadratic function $f(x) = x^T Ax$ with two linear constraints. This optimization problem can be solved using a system of linear equations. If A is not positive semi-definite, replace A with $\frac{1}{2}(A^T + A)$, which is p.s.d. by construction.
3. Determine the partial derivatives of step 2 with respect to α_d and α_i and use a gradient method to update the alphas. Repeat from step 1 until an end condition is met.

A possible way to adjust the three parameters K_∞, T_c, and $K_{2 \cdot T_c}$ is described in the next section.

5. Modelling the Margin Rate with a Mean Reverting Controller

The margin $m(t)$ is controlled by the following differential equation:

$$dm(t) = \alpha \cdot (K_m - m(t))dt$$

K_m is called the mean reversion level. If K_m equals $m(t)$, then $dm(t) = 0$, and no further adjustments of the margin are needed. $\alpha > 0$ is called the mean reversion speed and determines how quickly $m(t)$ reverts to the mean reversion level K_m.

The solution of the differential equation for $t > T_{n-1}$ with a given initial value $m(T_{n-1})$ is:

$$m(t) = K_m \cdot (1 - \exp\left(\alpha(T_{n-1} - t)\right)) + m(T_{n-1}) \cdot \exp\left(\alpha(T_{n-1} - t)\right)$$

or equivalently:

$$m(t) = m(T_{n-1}) + (K_m - m(T_{n-1})) \cdot (1 - \exp\left(\alpha(T_{n-1} - t)\right))$$

A way to calibrate K_m is explained in the next section.

6. Overall Model Design and Calibration

There are many possibilities to build an overall model with the introduced building blocks. Here is one example the author will focus on in the next months, with the long-term goal of having a fully automated model that optimizes the node's risk-return ratio. To accomplish this, we plan to implement various processes that each optimize different objective functions. The aim is for the interaction of these processes to lead to the desired outcome.

1. There is one margin controller for all channel parties and, for each channel party, a separate spread controller with manual adjustment of all parameters. This is the current state of the prototype.

2. For each spread controller, we have to adjust K_∞ and $K_{2 \cdot T_c}$. A simple approach would be setting the K's proportional to the margin, with fixed factors initially. The factors should be high enough so that changing T_c significantly affects the adjustment speed of the controllers.

3. For the margin controller, we can keep α constant at some preconfigured value, e.g., 1% to 3% per day. The critical task is calibrating the level K_m. We want to adjust the margin to maximize the node's profit by considering dependencies with flow and potential rebalancing margins. This requires the proposed P&L decomposition of each transaction, which helps us understand the interaction between forwarding, rebalancing margins, and adjustments on different potentials.

4. For setting T_c of each spread controller, it may be helpful to analyze the P&L at the channel level. If T_c is set too high, the channel may remain depleted for too long, possibly generating margins but lacking potential adjustments. If T_c is too low, excessive negative P&L could occur, with the spread controller overreacting due to the short T_c.

5. Each channel party needs a target. By default, the ratio of remote liquidity to capacity is used. However, for some parties, it might make sense to use different values, e.g., if the party lowers their fee rates late.

6. Using spread rate and margin controllers for risk-averse calculation of maximum rebalancing costs could be beneficial. For example, lower cost limits might be applied to almost full channels, as they are more likely to experience negative value adjustments in the future. Similarly, source channels could have reduced cost limits if near capacity to anticipate future negative adjustments. An advanced cost calculation could use a statistical simulation of rates, e.g., via a Monte Carlo model.

7. Automating the factors from step 2 could be possible, perhaps fine-tuning the P&L volatility to achieve a stable P&L over time.