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Van der Corput sequences, Kakutani transforms and one-dimensional numerical integration

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Abstract

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It is shown that many functions can be empirically integrated with the Van der Corput sequences with an $O(\sqrt{\ln n}/n)$ or $O(1/n)$ rate according to their smoothness (the first rate derives from a new Koksma–Hlawka-like inequality using the diaphony). Then we show that, as for the discrepancy estimates, these properties still hold when considering an extension of these sequences generated by ergodic transforms.

Keywords: Numerical integration; discrepancy; diaphony; ergodic theory.

0. Introduction

It is well known (see [14]) that the integral $\int_0^1 f(x)\,\mathrm{d}x$ of any Riemann integrable function (that is bounded, $\mathrm{d}x$-a.s. continuous) defined on [0, 1] can be computed using some [0, 1]-valued sequences $(\xi_n)_{n \geqslant 0}$ by the formula

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_k) \to \int_0^1 f(x)\,\mathrm{d}x.$$

This is the case, typically, of almost every path $(U_n(\omega))_{n \geqslant 0}$ of a sequence $(U_n)_{n \geqslant 0}$ of i.i.d. uniformly distributed random variables, with a rate of convergence $O(\sqrt{\ln(\ln n)/n})$ (law of the iterated logarithm). But it also works with some deterministic sequences, called *uniformly distributed sequences* mod 1, sometimes with a much better rate of convergence than in the "random" case: the $(\{n\alpha\})_{n \geqslant 0}$, $\alpha \notin \mathbb{Q}$, sequences, the Van der Corput sequences, defined below, are among them (see [14], $\{\cdot\}$ denotes the fractional part).

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Uniformly distributed sequences (denoted u.d. from now on) are characterized by the following equivalent properties:

(i) $\forall f \in \mathscr{R}_{\mathrm{d}x}([0,1]) := \{f:[0,1]\to\mathbb{R}$, bounded, $\mathrm{d}x$-a.s. continuous$\}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_k) \;\to\; \int_0^1 f(x)\,\mathrm{d}x;$$

(ii) $$D_n^*(\xi) := \sup_{x\in[0,1]}\left|\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_{[0,x]}(\xi_k) - x\right| \;\to\; 0,$$

where $\mathbf{1}_A$ is the indicator function of the Borel subset A: $\mathbf{1}_A(x)=1$ if $x\in A$, $\mathbf{1}_A(x)=0$ elsewhere. $D_n^*(\xi)$ is called the *discrepancy* (at the origin) of $\xi_1,\ldots,\xi_n$. The better asymptotical rate of convergence down to 0 of the discrepancy $D_n^*(\xi)$ of an infinite sequence is $O(\ln n/n)$. The Van der Corput sequences achieve, for example, such a rate (see [3,14]). If a function f has finite variation over [0, 1], then the Koksma–Hlawka inequality holds:

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_k) \to \int_0^1 f(x)\,\mathrm{d}x\right| \leqslant \int_{]0,1]} |\mathrm{d}f|\, D_n^*(\xi).$$

Actually, for the $\{n\alpha\}$ ones (see [14]) for instance, this rate can be substantially improved when restricting to certain classes of (periodic) functions.

The aim of this paper is to show, using various methods, that similar results hold for the Van der Corput (denoted VdC from now on) sequences and their ergodic extensions (see [11]). Let us recall now their definition: let $p\geqslant 2$ be a fixed integer; the nth term ξ_n of the the p-adic VdC sequence displays as

$$\xi_n = \frac{a_0}{p} + \cdots + \frac{a_r}{p^{r+1}},$$

with $n = a_0 + \cdots + a_r p^r$ the p-adic expansion of n. $v(n) = \max\{k\geqslant 0$ such that $p^k \mid n\}$ will denote the p-adic valuation of n.

The VdC sequences had been already generalized in many arithmetic directions (multi-adic, scrambled, ...). The ergodic way consists in introducing a (uniquely) ergodic transform T of the unit interval that generates the VdC sequence by $\xi_k = T^k(x)$ for a certain x.

Actually, a continuous mapping into a compact probability space $(X, \mathscr{X}, \lambda)$ is uniquely ergodic (see [17]) iff one of the following equivalences holds:

(i) $\{\mu$, probability on $(X,\mathscr{X})$ such that $T\mu = \mu\} = \{\lambda\}$;

(ii) $\forall x\in X$, $(T^k x)_{k\geqslant 0}$ is *uniformly distributed*, that is, $\forall f\in\mathscr{R}_\lambda(X) := \{f: X\to\mathbb{R}$, bounded, λ-a.s. continuous$\}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \;\to\; \int_X f\,\mathrm{d}\lambda;$$

(iii) $$\forall f\in\mathscr{C}(X,\mathbb{R})\ \sup_{x\in[0,1]}\left|\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) - \int_X f\,\mathrm{d}\lambda\right| \;\to\; 0.$$

Thus we get for free an infinity of uniformly distributed sequences as the initial value x runs over [0, 1]. This approach (see [7,11]) is developed in Section 3 with the introduction of the generalized Von Neumann–Kakutani transforms, the first and the second being dedicated to the study of the computation of exponential sums of the sole VdC sequences and its straightforward applications.

1. Computation of the exponential sums of the Van der Corput sequences

The exact computation of the exponential sums of the Van der Corput sequence was first achieved for all p by Xiao [20] (when $p=2$, Proïnov and Atanassov probably used it in [1] as an intermediary step to compute the diaphony of the 2-adic Van der Corput sequence). The following just yields a very simplified approach of this computation.

Theorem 1.1. $\forall n \geqslant 1$, $\forall h \in \mathbb{Z}^*$,

$$\begin{aligned} S_n(h) &:= \sum_{k=0}^{n-1} \mathrm{e}^{2\mathrm{i}\pi h \xi_k} \\ &= \mathrm{e}^{2\mathrm{i}\pi h \xi_n} \left(\sum_{j=0}^{v(h)-1} a_j p^j + \frac{1-\exp\left(-2\mathrm{i}\pi h a_{v(h)}/p^{v(h)+1}\right)}{\exp\left(2\mathrm{i}\pi h/p^{v(h)+1}\right)-1} p^{v(h)} \right). \end{aligned} \tag{1}$$

Proof. Let $k = k_0 + \cdots + k_r p^r$ be the p-adic expansion of $k \in \{0, \ldots, n-1\}$;

$$k<n \quad \Leftrightarrow \begin{cases} 0 \leqslant k_r \leqslant a_r - 1, \quad k_j \in \{0,\ldots,p-1\},\ 0 \leqslant j \leqslant r-1, \\ \text{or} \\ k_r = a_r, \quad 0 \leqslant k_{r-1} \leqslant a_{r-1}-1, \quad k_j \in \{0,\ldots,p-1\},\ 0 \leqslant j \leqslant r-1, \\ \text{or} \\ \cdots \\ \text{or} \\ k_j = a_j,\quad l+1 \leqslant j \leqslant r, \quad 0 \leqslant k_l \leqslant a_l - 1, \\ \quad k_j \in \{0,\ldots,p-1\},\ 0 \leqslant j \leqslant l-1, \\ \text{or} \\ \cdots \\ \text{or} \\ k_j = a_j, \quad 1 \leqslant j \leqslant r, \quad 1 \leqslant k_0 \leqslant a_0 - 1. \end{cases}$$

Hence

$$\begin{aligned} S_n(h) &= \sum_{l=0}^{r} \exp\left(2\mathrm{i}\pi h \left(\frac{a_r}{p^{r+1}} + \cdots + \frac{a_{l+1}}{p^{l+2}}\right)\right) \\ &\quad \times \sum_{k_l=0}^{a_l-1} \sum_{k_{l-1}=0}^{p-1} \cdots \sum_{k_0=0}^{p-1} \exp\left(2\mathrm{i}\pi h \left(\frac{k_l}{p^{l+1}} + \cdots + \frac{k_0}{p}\right)\right) \\ &= \sum_{l=0}^{r} \exp\left(2\mathrm{i}\pi h \left(\frac{a_r}{p^{r+1}} + \cdots + \frac{a_{l+1}}{p^{l+2}}\right)\right) \left(\sum_{k_l=0}^{a_l-1} \exp\left(2\mathrm{i}\pi h \frac{k_l}{p^{l+1}}\right)\right) \\ &\quad \times \prod_{j=0}^{l-1} \left(\sum_{k_j=0}^{p-1} \exp\left(2\mathrm{i}\pi h \frac{k_j}{p^{j+1}}\right)\right). \end{aligned}$$

If $l \geqslant v(h)+1$, then

$$\sum_{k_{v(h)}=0}^{p-1} \exp\left(2\mathrm{i}\pi h \frac{k_{v(h)}}{p^{v(h)+1}}\right) = 0,$$

since $v(h) \leqslant l-1$ and $p^{v(h)+1} \mid ph$.

If $l = v(h)$, then

$$\sum_{k_l=0}^{a_l-1} \exp\left(2\mathrm{i}\pi h \frac{k_l}{p^{l+1}}\right) = \frac{\exp(2\mathrm{i}\pi h a_l/p^{l+1})-1}{\exp(2\mathrm{i}\pi h/p^{l+1})-1} \quad \text{and}$$

$$\sum_{k_j=0}^{p-1} \exp\left(2\mathrm{i}\pi h \frac{k_j}{p^{j+1}}\right) = p, \quad 0 \leqslant j \leqslant l-1.$$

If $l \leqslant v(h)-1$, then

$$\sum_{k_l=0}^{a_l-1} \exp\left(2\mathrm{i}\pi h \frac{k_l}{p^{l+1}}\right) = a_l \quad \text{and} \quad \sum_{k_j=0}^{p-1} \exp\left(2\mathrm{i}\pi h \frac{k_j}{p^{j+1}}\right) = p, \ 0 \leqslant j \leqslant l-1.$$

Subsequently,

$$S_n(h) = \sum_{l=0}^{v(h)-1} a_l p^l \exp\left(2\mathrm{i}\pi h\left(\frac{a_r}{p^{r+1}} + \cdots + \frac{a_{l+1}}{p^{l+2}}\right)\right)$$
$$+ \frac{\exp(2\mathrm{i}\pi h a_{v(h)}/p^{v(h)+1})-1}{\exp(2\mathrm{i}\pi h/p^{v(h)+1})-1} p^{v(h)} \exp\left(2\mathrm{i}\pi h\left(\frac{a_r}{p^{r+1}} + \cdots + \frac{a_{v(h)+1}}{p^{v(h)+2}}\right)\right).$$

Now, if $l \leqslant v(h)-1$,

$$h\left(\frac{a_l}{p^{l+1}} + \cdots + \frac{a_0}{p}\right) \in \mathbb{N}.$$

Then, rewriting the above equation the proper way yields the announced result. $\square$

Corollary 1.2. (a) $n \leqslant p^{v(h)}-1 \Rightarrow S_n(h) = n\,\mathrm{e}^{2\mathrm{i}\pi h \xi_n}$;

(b) $n = a_r p^r$ *with* $r \geqslant v(h)+1 \Rightarrow S_n(h) = 0$;

(c) $\forall n \geqslant 1$, $\forall h \in \mathbb{Z}^*$,

$$|S_n(h)| \leqslant \min\left(\sum_{j=0}^{v(h)} a_j p^j,\ p^{v(h)+1} - \sum_{j=0}^{v(h)} a_j p^j\right) \leqslant \tfrac{1}{2} p^{v(h)+1};$$

(d) *If* $p=2$, *the first inequality in* (c) *holds as an equality.*

Proof. (a) and (b) are obvious when looking carefully at formula (1).

(c) Theorem 1.1 straightforwardly implies that $|S_n(h)| \leqslant \Sigma_{j=0}^{v(h)} a_j p^j$. If $n \geqslant p^{v(h)}$, then $r+1 \geqslant v(h)+1$, hence $\Sigma_{k=0}^{p^{r+1}-1} \mathrm{e}^{2\mathrm{i}\pi h \xi_k} = 0$ and

$$S_n(h) = -\sum_{k=n}^{p^{r+1}-1} \mathrm{e}^{2\mathrm{i}\pi h \xi_k} = -\mathrm{e}^{2\mathrm{i}\pi h \xi_n} - \sum_{k=0}^{p^{r+1}-2-n} \exp(2\mathrm{i}\pi h \xi_{p^{r+1}-k-1}). \tag{2}$$

Now, a simple computation shows that

$$\xi_{p^{r+1}-1-k} = 1 - \frac{1}{p^{r+1}} - \xi_k$$

whenever k lies in $\{0, \ldots, p^{r+1} - 1\}$. Then the substitution in (2) yields

$$S_n(h) = -\mathrm{e}^{2\mathrm{i}\pi h \xi_n} - \mathrm{e}^{-2\mathrm{i}\pi h / p^{r+1}} \overline{S_{p^{r+1}-1-n}(h)}.$$

Hence,

$$|S_n(h)| \leqslant 1 + |S_{p^{r+1}-1-n}(h)| \leqslant 1 + \sum_{j=0}^{v(h)} (p - 1 - a_j) p^j.$$

If $n \leqslant p^{v(h)} - 1$, $|S_n(h)| = n \leqslant p^{v(h)+1} - n$. The second inequality derives from $\min(a, b) \leqslant \frac{1}{2}(a + b)$.

(d) If $a_{v(h)} = 0$, then

$$S_n(h) = \mathrm{e}^{2\mathrm{i}\pi h \xi_n} \sum_{j=0}^{v(h)-1} a_j 2^j.$$

If $a_{v(h)} = 1$, then $n \geqslant p^{v(h)}$ and

$$S_n(h) = -\mathrm{e}^{2\mathrm{i}\pi h \xi_n} - \exp\left\{2\mathrm{i}\pi h\left(1/p^{r+1} + \xi_{p^{r+1}-1-n}\right)\right\} \sum_{j=0}^{v(h)-1} (1 - a_j) 2^j,$$

since $1 - a_{v(h)} = 0$. Hence,

$$S_n(h) = -\mathrm{e}^{2\mathrm{i}\pi h \xi_n}\left(2^{v(h)+1} - 2^{v(h)} - \sum_{j=0}^{v(h)-1} a_j 2^j\right). \qquad \square$$

2. Applications to numerical integration

There are essentially two applications of the computation of exponential sums for a uniformly distributed sequence. One is the estimation of its diaphony, the other is related to the numerical integration of smooth periodic functions. The first application looks rather natural considering (one of) the definition(s) of the diaphony (due to Zinterhof [21]) of n points $x_0, \ldots, x_{n-1}$ lying in [0, 1]:

$$F_n = \frac{1}{n}\left(2 \sum_{h=1}^{+\infty} \frac{1}{h^2} \left| \sum_{k=0}^{n-1} \mathrm{e}^{2\mathrm{i}\pi h x_k} \right|^2\right)^{1/2}. \tag{3}$$

For instance, this expansion of the diaphony had already been used (see [1]) in order to give an optimal asymptotic rate of convergence for the diaphony of the 2-adic Van der Corput sequence,

$$\limsup_n \frac{nF_n}{\sqrt{\ln(n)}} = \frac{\pi}{3\sqrt{\ln 2}},$$

based on exponential sums, which may be found in [3,20] by combinatorial arguments. However, notice that if the rate of convergence $\sqrt{\ln n}/n$ is optimal among one-dimensional sequences, the constant of this very sequence is not (see [3], for instance).

The second application had already been explored too, but only for the $\{n\alpha\}$-sequences. Xiao [20] was the first to introduce this idea for the "arithmetic" sequences like the Van der Corput ones.

The forthcoming subsection will point out that the diaphony is actually not only one more characterization of uniformly distributed sequences (see [21]) but is also directly connected to numerical integration (see also [22]).

2.1. Diaphony and numerical integration: a new formula

Let $x_0,\dots,x_{n-1}$ be n points in [0, 1] and $H^1 := \{f\in L^2([0,1]) \mid f'\in L^2([0,1])\} = \{f\in L^2([0,1]) \mid \sum_{h\in\mathbb{Z}^*} h^2 |c(h)|^2 < +\infty\}$.

Proposition 2.1. *For every $f\in H^1$, set $S_n^f := (f(x_0)+\cdots+f(x_{n-1}))/n$. Then*

$$\left|S_n^f + (f(1)-f(0))\left(\frac{1}{2} - \frac{x_0+\cdots+x_{n-1}}{n}\right) - \int_0^1 f(u)\,\mathrm{d}u\right| \leqslant \frac{F_n}{2\pi}\sigma(f'), \tag{4}$$

with

$$\sigma(f') := \sqrt{\int_0^1 f'^2(u)\,\mathrm{d}u - \left(\int_0^1 f'(u)\,\mathrm{d}u\right)^2}.$$

Furthermore, (4) *stands as an equality iff*

$$f(y) = \alpha + \beta y + \gamma \int_0^y \left(\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{[0,u]}(x_k)\right)\mathrm{d}u.$$

Proof. Let

$$\mu_n(\mathrm{d}y) = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k} - \mathrm{d}y$$

be the signed measure with respect to the $x_0,\dots,x_{n-1}$. Integrating by parts, we get

$$\begin{aligned}\int_{[0,1]} f(y)\mu_n(\mathrm{d}y) &= \mu_n([0,1])f(1) - \int_{[0,1]}\mu_n([0,y])\,\mathrm{d}f(y)\\ &= -\int_{[0,1]}\mu_n([0,y])f'(y)\,\mathrm{d}y.\end{aligned} \tag{5}$$

Applying (5) with $f(y) - y(f(1)-f(0)) - \int_0^1 f(u)\,\mathrm{d}u$ yields

$$\begin{aligned}&\left|S_n^f + (f(1)-f(0))\left(\frac{1}{2} - \frac{x_0+\cdots+x_{n-1}}{n}\right) - \int_0^1 f(u)\,\mathrm{d}u\right|\\ &\quad= \left|\int_0^1 \mu_n([0,y])(f'(y) - (f(1)-f(0)))\,\mathrm{d}y\right|.\end{aligned}$$

Now, using the probabilistic covariance inequality (and its equality case ...), we get

$$\left|\int_0^1 \mu_n([0, y])(f'(y) - (f(1) - f(0)))\,\mathrm{d}y\right|$$
$$= \left|\int_0^1 \mu_n([0, y]) f'(y)\,\mathrm{d}y - \int_0^1 \mu_n([0, y])\,\mathrm{d}y \int_0^1 f'(y)\,\mathrm{d}y\right|$$
$$\leqslant \left(\int \mu_n([0, y])^2\,\mathrm{d}y - \left(\int \mu_n([0, y])\,\mathrm{d}y\right)^2\right)^{1/2} \sigma(f').$$

The Parseval identity applied to $\mu_n([0, \cdot\,])$ and another integration by parts yield

$$\int_0^1 \mu_n([0, y])^2\,\mathrm{d}y = \left(\int_0^1 \mu_n([0, y])\,\mathrm{d}y\right)^2 + 2\sum_{h \geqslant 1} \left|\int_0^1 \mu_n([0, y])\,\mathrm{e}^{2\mathrm{i}\pi h y}\,\mathrm{d}y\right|^2$$
$$= \left(\int_0^1 \mu_n([0, y])\,\mathrm{d}y\right)^2 + 2\sum_{h \geqslant 1} \left|-\frac{1}{2\mathrm{i}\pi h n}\sum_{k=0}^{n-1} \mathrm{e}^{2\mathrm{i}\pi h x_k}\right|^2.$$

We finally obtain the so-called Koksma formula that completes the proof:

$$\left(\frac{F_n}{2\pi}\right)^2 = \int_0^1 \mu_n([0, y])^2\,\mathrm{d}y - \left(\int_0^1 \mu_n([0, y])\,\mathrm{d}y\right)^2. \qquad \square$$

Remark 2.2. Formula (4) shows that it is possible to numerically compute an integral with a *nonsymmetric* infinite uniformly distributed sequence with an $\mathrm{O}(\sqrt{\ln n}\,/n)$-error. Furthermore, if f is periodic, that is if $f(1) = f(0)$, the additional term

$$(f(1) - f(0))\left(\frac{1}{2} - \frac{x_0 + \cdots + x_{n-1}}{n}\right)$$

vanishes and (4) becomes:

$$\left|\frac{f(x_0) + \cdots + f(x_{n-1})}{n} - \int_0^1 f(u)\,\mathrm{d}u\right| \leqslant \frac{F_n}{2\pi}\,\|f'\|_2,$$

which is more accurate than the L^2-Koksma–Hlawka inequality.

Remark 2.3. If we set $x_k = k/n$, $0 \leqslant k \leqslant n-1$, equation (4) turns into

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) + \frac{f(1) - f(0)}{2n} - \int_0^1 f(u)\,\mathrm{d}u\right|$$
$$\leqslant \frac{1}{\sqrt{12}\,n}\sqrt{\int_0^1 f'^2(u)\,\mathrm{d}u - \left(\int_0^1 f'(u)\,\mathrm{d}u\right)^2},$$

which is the error bound in the trapezoïds integration method.

Remark 2.4. Inequality (4) can be extended to higher dimensions. A general proof will be given in a forthcoming paper.

2.2. Application to $O((\ln(n))^{1/2}/n)$-numerical integration: diaphony of the Van der Corput sequences

Inequality (4) of Proposition 2.1 may be read as a stopping rule provided that one has a numerical explicit upper bound of the diaphony F_n. When taking the Van der Corput sequence into account, such an upper bound can be computed from the results of Section 1. As an example the computation of an explicit upper bound of the diaphony is given below. Chaix and Faure obtained, by a combinatorial method, an explicit and optimal upper bound for a much wider class of sequences (see [3] for a first announcement).

Proposition 2.5. *Let F_n be the diaphony of the first n terms of the p-adic Van der Corput sequence.*
(a) *Let $r = [\ln n/\ln p]$. Then, $\forall n \geqslant 1$,*

$$F_n \leqslant \frac{\pi\sqrt{p^2-1}}{\sqrt{3}\, n}\left(\sum_{l=0}^{r}\left(\min\left(\left\{\frac{n}{p^{l+1}}\right\}, 1-\left\{\frac{n}{p^{l+1}}\right\}\right)\right)^2 + \frac{1}{p^2-1}\left(\frac{n}{p^{r+1}}\right)^2\right)^{1/2}$$

and

$$\phi(n) = \max_{1\leqslant k\leqslant n} kF_k \leqslant \frac{\pi\sqrt{p^2-1}}{2\sqrt{3}}\left(r+\frac{p^2+3}{p^2-1}\right)^{1/2} = O\left(\frac{\sqrt{\ln n}}{n}\right).$$

(b) $$\limsup_n \frac{nF_n}{\sqrt{\ln n}} \leqslant \frac{\pi\sqrt{p^2-1}}{\sqrt{12\ln p}}.$$

(c) *When $p = 2$, the first inequality stands as an equality.*

Proof. (a), (b)

$$(nF_n)^2 \leqslant 2\sum_{h=1}^{+\infty}\frac{1}{h^2}\left|\min\left(\sum_{j=0}^{v(h)} a_j p^j,\ p^{v(h)+1} - \sum_{j=0}^{v(h)} a_j p^j\right)\right|^2.$$

Since $h = p^{v(h)}k$, $k \in \mathbb{N} - p\mathbb{N}$, and

$$\sum_{k\in\mathbb{N}-p\mathbb{N}}\frac{1}{k^2} = \frac{p^2-1}{p^2}\sum_{k\in\mathbb{N}^*}\frac{1}{k^2} = \frac{p^2-1}{p^2}\frac{\pi^2}{6},$$

one gets

$$\begin{aligned}(nF_n)^2 &\leqslant \tfrac{1}{3}\pi^2(p^2-1)\sum_{l=0}^{+\infty}\left|\min\left(\left\{\frac{n}{p^{l+1}}\right\}, 1-\left\{\frac{n}{p^{l+1}}\right\}\right)\right|^2\\ &\leqslant \tfrac{1}{3}\pi^2(p^2-1)\left(\sum_{l=0}^{r}\min\left(\left\{\frac{n}{p^{l+1}}\right\}, 1-\left\{\frac{n}{p^{l+1}}\right\}\right)^2 + \frac{1}{p^2-1}\left(\frac{n}{p^{r+1}}\right)^2\right)\\ &\leqslant \tfrac{1}{3}\pi^2(p^2-1)\left(\frac{r+1}{4}+\frac{1}{p^2-1}\right) \quad \left(\text{with } r = \left[\frac{\ln n}{\ln p}\right]\right).\end{aligned}$$

(c) is an obvious consequence of (a) and Corollary 1.2(d). □

When $p=2$, some additional computations would lead to Proïnov and Atanassov's result (see [1] or [20]): $\pi/(3\sqrt{\ln p}) \approx 1.257\ldots$.

2.3. Application to O$(1/n)$*-numerical integration*

Let us first introduce some notation. Every Borel function f in $L^2([0, 1], \mathrm{d}x)$ can be expanded in Fourier series by the formula

$$f \overset{L^2}{=} \sum_{h \in \mathbb{Z}} c(h)\, \mathrm{e}^{2\mathrm{i}\pi h}.$$

Let us define the set $\mathscr{C}_v = \{f \in L^2([0, 1], \mathbb{R}) \mid \Sigma_{h \geqslant 1} |c(h)|\, p^{v(h)} < +\infty\} \subset \mathscr{C}([0, 1], \mathbb{R})$. All the functions in $\mathscr{C}_v$ can be numerically integrated with a p-adic Van der Corput sequence at an O$(1/n)$-rate. More precisely, we have the next result.

Proposition 2.6. $\forall f \in \mathscr{C}_v$, $\forall n \geqslant 1$,

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(\xi_k) - \int_0^1 f(u)\, \mathrm{d}u \right| \leqslant \frac{p}{n} \sum_{h \geqslant 1} |c(h)|\, p^{v(h)}.$$

Proof. This result is a straightforward consequence of Corollary 1.2(c):

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(\xi_k) - \int_0^1 f(u)\, \mathrm{d}u \right| &\leqslant \frac{1}{n} \sum_{h \in \mathbb{Z}^*} |c(h)|\, |S_n(h)| \\ &\leqslant \frac{1}{n} \sum_{h \in \mathbb{Z}^*} |c(h)|\, \tfrac{1}{2} p^{v(h)+1} \\ &\leqslant \frac{p}{n} \sum_{h \geqslant 1} |c(h)|\, p^{v(h)}. \qquad \square \end{aligned}$$

Several criteria derive from this proposition to decide whether or not a function belongs to $\mathscr{C}_v$.

Criterion 2.7. $\Sigma_{h \geqslant 1} h |c(h)| < +\infty \Rightarrow f \in \mathscr{C}_v$.

Thus, a function f belongs to $\mathscr{C}_v$ as soon as the Fourier expansion of its derivative is absolutely convergent. The proof is obvious. The next criteria may look a bit more unexpected. They improve Xiao's first results in [20].

Criterion 2.8. *$c(h) = \mathrm{O}(1/|h|^{\alpha})$, $\alpha > 1 \Rightarrow f \in \mathscr{C}_v$ and*

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(\xi_k) - \int_0^1 f(u)\, \mathrm{d}u \right| \leqslant K \frac{p(p^{\alpha}-1)}{n(p^{\alpha}-p)} \left(\sum_{k \geqslant 1} \frac{1}{k^{\alpha}} \right), \quad \textit{if } |c(h)| \leqslant \frac{K}{|h|^{\alpha}}.$$

Proof.

$$\sum_{h\geqslant 1}\frac{p^{v(h)}}{h^{\alpha}}=\sum_{l\geqslant 0}\sum_{k\in\mathbb{N}-p\mathbb{N}}\frac{1}{k^{\alpha}p^{l(\alpha-1)}}=\frac{p^{\alpha}-1}{p^{\alpha}-p}\sum_{k\geqslant 1}\frac{1}{k^{\alpha}}. \qquad \square$$

Criterion 2.9. $c(h)=\mathrm{O}(1/(|h|\ln^{\beta}|h|))$, $\beta>2\Rightarrow f\in\mathscr{C}_v$.

Proof.

$$\sum_{h\geqslant 2}\frac{p^{v(h)}}{h\,\ln^{\beta}(h)}=\sum_{k\in\mathbb{N}-p\mathbb{N},k\neq 2}\frac{1}{k\,\ln^{\beta}(k)}+\sum_{l\geqslant 1}\sum_{k\in\mathbb{N}-p\mathbb{N}}\frac{1}{k(l\,\ln\,p+\ln\,k)^{\beta}}.$$

If $\beta\leqslant 1$, the single sum on the right-hand side is divergent. If $\beta>1$, the single sum is convergent and the double sum behaves like $\iint \mathrm{d}x\,\mathrm{d}y/x(y\ln p+\ln x)^{\beta}$ in the neighbourhood of $(+\infty,+\infty)$, which, in turn, behaves like $\iint \mathrm{d}x\,\mathrm{d}y/(y\ln p+x)^{\beta}$. The latter is convergent iff $\beta>2$. $\square$

Carrying on the process of Criterion 2.9 yields the following criterion.

Criterion 2.9′. *Let us denote*

$$\ln_n(x)=\underbrace{\ln(\ln(\ldots\ln(x)))}_{n}.$$

Then,

$$c(h)=\mathrm{O}\left(\frac{1}{|h|\,\ln^2|h|\,\ln_2^{\beta_1}|h|}\right),\quad \beta_1>1\ \Rightarrow\ f\in\mathscr{C}_v,$$

$$c(h)=\mathrm{O}\left(\frac{1}{|h|\,\ln^2|h|\,\ln_2|h|\ldots\ln_{n-1}|h|\,\ln_n^{\beta_n}|h|}\right),\quad \beta_n>1\ \Rightarrow\ f\in\mathscr{C}_v.$$

The last criterion is rather different from the others: it shows that if f is symmetric with respect to $\frac{1}{2}$, its smoothness may be slightly weakened.

Criterion 2.10. *Let $f(x)=g(x\wedge(1-x))$ with g a real-valued function having a derivative (in the distribution sense) with finite variation over* $[0,\frac{1}{2}]$. *Then f is in $\mathscr{C}_v$.*

Proof. For every h in $\mathbb{Z}^*$, the symmetry implies

$$c(h)=2\mathscr{R}\left(\int_0^{1/2}g(x)\,\mathrm{e}^{2\mathrm{i}\pi hx}\,\mathrm{d}x\right).$$

Successive integrations by parts yield

$$c(h)=2\mathscr{R}\left(\frac{1}{2\mathrm{i}\pi h}\int_0^{1/2}g(x)\,\mathrm{e}^{2\mathrm{i}\pi hx}\,\mathrm{d}x\right)$$

$$=2\mathscr{R}\left(\frac{1}{2\mathrm{i}\pi h}\left(g(\tfrac{1}{2})(-1)^h-g(0)-\frac{1}{2\mathrm{i}\pi h}\int_0^{1/2}g'(x)\frac{\mathrm{d}\,\mathrm{e}^{2\mathrm{i}\pi hx}}{2\mathrm{i}\pi h}\right)\right)$$

$$=\frac{2}{4\pi^2h^2}\mathscr{R}\left(g'(\tfrac{1}{2})(-1)^h-g'(0)-\int_0^{1/2}\mathrm{e}^{2\mathrm{i}\pi hx}\,\mathrm{d}g'\right).$$

Hence,

$$|c(h)| \leqslant \frac{1}{2\pi^2 h^2}\left(|g'(\tfrac{1}{2})| + |g'(0)| + \int_{]0,1/2]} |\mathrm{d}g'| \right) = \mathrm{O}\left(\frac{1}{h^2}\right). \qquad \square$$

Remark 2.11. There is no inclusion between H^1 and $\mathscr{C}_v$: $f = \sum_{h \in \mathbb{Z}^*} \mathrm{e}^{2\mathrm{i}\pi h \cdot}/|h|^{\alpha} \in \mathscr{C}_v$ as soon as $\alpha > 1$ (see Criterion 2.7) but $f \in H^1$ iff $\alpha > \frac{3}{2}$.

On the opposite,

$$f = \sum_{h \in \mathbb{Z}^*} \frac{p^{v(h)}}{h^2(1+v(h))} \mathrm{e}^{2\mathrm{i}\pi h \cdot} \in H^1 \quad \text{and} \quad f \notin \mathscr{C}_v$$

since

$$\sum_{h \in \mathbb{N}^*} h^2 \left(\frac{p^{v(h)}}{(1+v(h))h^2} \right) = \left(\sum_{l \in \mathbb{N}} \frac{1}{(1+l)^2} \right) \sum_{k \in \mathbb{N} - p\mathbb{N}} \frac{1}{k^2} < +\infty$$

and

$$\sum_{h \in \mathbb{N}^*} \frac{p^{v(h)}}{(1+v(h))h^2} p^{v(h)} = \left(\sum_{l \in \mathbb{N}} \frac{1}{1+l} \right) \sum_{k \in \mathbb{N} - p\mathbb{N}} \frac{1}{k^2} = +\infty.$$

3. The generalized Kakutani transform and its applications to sequences with low discrepancy

3.1. Introduction

The idea of introducing a (uniquely) ergodic transform to study the VdC sequence is originally due, independently, to Hellekalek [7] and Lambert [10]. In both cases it appears as an illustration of the theorem about the rotations of a compact abelian group (see [13,17]): a rotation $x \to x + g$ defined on a compact abelian group G is uniquely ergodic with the Haar probability as unique invariant probability as soon as $\mathbb{Z}g$ is dense in G.

Lapeyre and Pagès [11] introduced some sequences obtained from a wider class of transforms, called *generalized Kakutani transforms*. We will briefly recall their definition below and study their properties, essentially from a numerical point of view. To carry out this program, we need first to define, for any fixed integer $p \geqslant 2$, the p-adic pseudo-addition $\oplus_p$.

Definition 3.1. For any x, $y \in [0, 1]$ the *pseudo-sum* $x \oplus_p y$ is defined as the addition from the left to the right with carrying over of the regular p-adic expansions of x and y (if $x = 1$, set $x = \overline{0, (p-1)\ldots(p-1)\ldots}^{p}$).

Thus, if $p = 10$, then $0{,}123\,33\ldots \oplus_{10} 0{,}142\,777\ldots = 0{,}265\,0111\ldots$. Unfortunately, the result usually does not naturally display as a regular expansion: $0{,}123\,444\ldots \oplus_{10} 0{,}051\,555\ldots = 0{,}174\,999\ldots = 0{,}175$. That especially means that $x \oplus_p y$ has no global associative property.

However, if y is a $[0, 1[$-valued p-adic rational, $x \oplus_p y$ appears as a regular expansion (see [11] for more details).

Definition 3.2. Let $\mathbb{Q}_p$ be the set of p-adic rationals and

$$y = \frac{y_0}{p} + \cdots + \frac{y_s}{p^{s+1}} \in \mathbb{Q}_p \cap [0, 1[.$$

Then the $T_{p,y}$ transform displays as

$$\forall x \in [0, 1] \quad T_{p,y}(x) = x \oplus_p y.$$

Theorem 3.3. *Let* $y \in \mathbb{Q}_p \cap [0, 1[$,

$$y = \frac{y_0}{p} + \cdots + \frac{y_s}{p^{s+1}}.$$

Then, $T_{p,y}$ *is uniquely ergodic iff* $\gcd(y_0, p) = 1$.

The proof of this result — not reproduced here (see [15]) — derives from the fact that the $T_{p,y}$ transforms may be seen as representations on [0, 1] of the rotations on the compact abelian group $\mathbb{Z}(p) := \{0, \ldots, p-1\}^{\mathbb{N}}$ endowed with the product topology and the addition component by component with carrying over (denoted $\oplus_p$ too). $\mathbb{Z}(p)$ is known as the group of p-adic integers; its Haar probability displays as

$$\lambda = \left(\frac{1}{p} \sum_{k=0}^{p-1} \delta_k\right)^{\oplus \mathbb{N}}.$$

Basically, the measure-preserving representing morphism is given by

$$\Phi : (\mathbb{Z}(p), \lambda) \to ([0, 1], \mathrm{d}x)$$
$$\omega \to \sum_{k \geqslant 0} \frac{\omega_k}{p^{k+1}}.$$

As usual, some technicalities related to the regularity of the p-adic expansions in [0, 1] make the proof less straightforward than expected. However, the crucial fact is that, for any $y \in \mathbb{Q}_p \cap [0, 1[$, $\Phi^{-1}(y)\mathbb{Z}(p)$ is dense in $\mathbb{Z}(p)$ iff $\gcd(y_0, p) = 1$. A detailed proof is provided in [15].

Corollary 3.4. $\forall y \in \mathbb{Q}_p \cap [0, 1[$,

(a) $\forall f \in \mathscr{C}([0, 1], \mathbb{R})$,

$$\sup_{x \in [0,1]} \left| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T_{p,y}^k(x) - \int_0^1 f(u)\, \mathrm{d}u \right| \to 0;$$

(b) $\forall x \in [0, 1]$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_{p,y}^k}(x) \Rightarrow \mathrm{d}x,$$

i.e., $(T_{p,y}^k(x))_{k \geqslant 0}$ *is u.d.*

Remark 3.5. If we set $y = 1/p$, then $T_{p,1/p}$ (denoted T_p from now on) is the p-adic Von Neumann–Kakutani transform (also called additioning machine when $p = 2$). It is usually defined by its algorithmic definition: $\forall x \in [0, 1]$,

$$T_p(x) := \frac{1 + x_{u(x)}}{p^{u(x)+1}} + \sum_{k \geqslant u(x)+1} \frac{x_k}{p^{k+1}},$$

with $u(x) := \min\{k \mid x_k \neq p-1\}$. Such a formula, summed up in computer programming terms, displays as

"While $x_k = p - 1$ do $x_k \leftarrow 0$ else $x_k \leftarrow x_k + 1$ Stop".

Eventually, notice that $T_p^k(0) = 1/p \oplus_p \cdots \oplus_p 1/p = \xi_k$ is the kth term of the p-adic VdC sequence which subsequently belongs to the above family of u.d. sequences.

Remark 3.6. If

$$y = \frac{y_0}{p} + \cdots + \frac{y_s}{p^{s+1}},$$

then $T_{p,y} = T_p^m$ with $m = y_0 + \cdots + y_s p^s$ (see [15] for a proof). So a sequence $(T_{p,y}^k(x))_{k \geqslant 0}$ is actually "m-arithmetically sampled" from the sequence $(T_p^k(x))_{k \geqslant 0}$.

3.2. Discrepancy and diaphony of the sequences $(T_{p,y}^k(x))_{k \geqslant 0}$

Since the sequences $(T_{p,y}^k(x))_{k \geqslant 0}$ are uniformly distributed on [0, 1] and the well-known VdC sequence is one of them, it is most natural to study their discrepancy and diaphony. Actually, we will show that these moduli behave uniformly with respect to the starting point x.

3.2.1. Discrepancy of the sequences $(T_{p,y}^k(x))_{k \geqslant 0}$

Theorem 3.7. *For every $n \geqslant 1$, if $r := [\ln n / \ln p]$,*

$$\sup_{\substack{y \in [0,1] \cap \mathbb{Q}_p \\ \gcd(y_0, p) = 1}} D_n^*\left(T_{p,y}^k(x)\right) \leqslant \frac{1}{n}\left(1 + (p-1)\max\left(\sum_{l=0}^{r}\left(1 - \left\{\frac{n}{p^{l+1}}\right\}\right), \sum_{l=0}^{r}\left\{\frac{n}{p^{l+1}}\right\}\right)\right)$$

$$\leqslant \frac{1}{n}\left(1 + (p-1)\left[\frac{\ln(pn)}{\ln p}\right]\right).$$

This result shows that these well-known upper bounds of the discrepancy of the VdC (see [8,9]) are relevant for any sequence of the family. This result had already been announced in [11]. Its proof consists in a string of three lemmas.

First let us define the "upside-down" (or radical inverse) function by $\forall n \geqslant 1$, $\forall x \in [0, 1]$,

$$R_n(x) := \sum_{k=0}^{n-1} x_k p^k, \quad \text{where } \sum_{k \geqslant 0} \frac{x_k}{p^{k+1}} \text{ is the regular } p\text{-adic expansion of } x.$$

Lemma 3.8. *Let $x, y \in [0, 1]$, $x = \sum_k x_k / p^{k+1}$, $y = \sum_k y_k / p^{k+1}$ their regular p-adic expansion. Then*

$$x < y \iff \exists k \geqslant 1\ \exists \beta \in \{0, \ldots, y_{k-1} - 1\}\ R_k(x) = R_{k-1}(y) + \beta p^{k-1}.$$

The proof of this lemma is obvious.

Lemma 3.9. *If* $x \in [0, 1]$, $y \in [0, 1[\cap \mathbb{Q}_p$, $R_n(x \oplus_p y) \equiv R_n(x) + R_n(y) \bmod p^n$ *for every* $n \geqslant 1$.

Proof. The very definition of $\oplus_p$ yields that $x \oplus_p y := \sum_{k \geqslant 0} z_k/p^{k+1}$ with the z_k's satisfying $\forall n \geqslant 1$,

$$\sum_{k=0}^{n-1} z_k p^k \equiv \sum_{k=0}^{n-1} x_k p^k + \sum_{k=0}^{n-1} y_k p^k \bmod p^n.$$

As soon $\sum_{k \geqslant 0} z_k/p^{k+1}$ is the regular expansion of $x \oplus_p y$, $R_n(x \oplus_p y) = \sum_{0 \leqslant k \leqslant n-1} z_k p^k$. Now, since the expansion of x is regular and $y \in \mathbb{Q}_p$ (the y_k's are equal to 0 for k large enough), it is obvious that the z_k's cannot be indefinitely equal to $p-1$. $\square$

It follows from Lemma 3.9 that $\forall x \in [0, 1]$, $\forall y \in [0, 1[\cap \mathbb{Q}_p$, $\forall n \geqslant 1$, $\forall k \geqslant 0$,

$$R_n\left(T^k_{p,y}(x)\right) \equiv R_n(x) + kR_n(y) \bmod p^n. \tag{6}$$

Eventually, let us recall an obvious result about congruence equations.

Lemma 3.10. $x \equiv a \bmod p$ *has* $[n/p] + \theta$, $\theta \in \{0, 1\}$, *solutions in* $\{0, \ldots, n-1\}$.

Proof of Theorem 3.7. Lemma 3.13 and (6) yield that, for every $x \in [0, 1]$, $y \in [0, 1] \cap \mathbb{Q}_p$, $a \in [0, 1]$,

$$\begin{aligned} &T^k_{p,y}(x) < a \iff \\ &\quad \exists l \geqslant 1\ \exists \beta \in \{0, \ldots, a_{l-1} - 1\}\ R_l(x) + kR_l(y) \equiv R_{l-1}(a) + \beta p^{l-1} \bmod p^l. \end{aligned}$$

Subsequently,

$$\begin{aligned} &\operatorname{card}\left\{k \in \{0, \ldots, n-1\} \mid T^k_{p,y}(x) < a\right\} \\ &\quad = \operatorname{card}\{k \in \{0, \ldots, n-1\} \mid \exists l \geqslant 1, \exists \beta \in \{0, \ldots, y_{k-1} - 1\} \text{ with} \\ &\qquad\qquad kR_l(y) \equiv R_{l-1}(a) + \beta p^{l-1} - R_l(x) \bmod p^l\}. \end{aligned}$$

$R_l(y)$ is invertible in $\mathbb{Z}/p^l\mathbb{Z}$ since $\gcd(y_0, p^l) = 1$; hence the same k cannot satisfy two different congruences and

$$\begin{aligned} &\operatorname{card}\{k \in \{0, \ldots, n-1\} \mid kR_l(y) \equiv R_{l-1}(a) + \beta p^{l-1} - R_l(x) \bmod p^l\} \\ &\quad = \operatorname{card}\{k \in \{0, \ldots, n-1\} \mid k \equiv R_{l-1}(a) + \beta p^{l-1} - R_l(x) \bmod p^l\}. \end{aligned}$$

Then,

$$\begin{aligned} &\operatorname{card}\left\{k \in \{0, \ldots, n-1\} \mid T^k_{p,y}(x) < a\right\} \\ &\quad = \sum_{l=1}^{r+1} \sum_{\beta=0}^{a_{l-1}-1} \operatorname{card}\{k \in \{0, \ldots, n-1\} \mid k \equiv R_{l-1}(a) + \beta p^{l-1} - R_l(x) \bmod p^l\} \\ &\quad\quad + \epsilon(r+1, a, x), \end{aligned}$$

with $\epsilon(r+1, a, x) \in \{0, 1\}$. Hence

$$\begin{aligned} &\operatorname{card}\left\{k \in \{0, \ldots, n-1\} \mid T^k_{p,y}(x) < a\right\} \\ &\quad = \sum_{l=1}^{r+1} \left(a_{l-1}\left[\frac{n}{p^l}\right] + \sum_{\beta=0}^{a_{l-1}-1} \theta(l, a, x, \beta) \right) + \epsilon(r+1, a, x), \end{aligned}$$

with all the $\theta(l, a, x, \beta)$ in $\{0, 1\}$. Subsequently,

$$\begin{aligned}
&\operatorname{card}\left\{k \in \{0,\ldots,n-1\} \mid T_{p,y}^k(x) < a\right\} - na \\
&\quad\leqslant \sum_{l=1}^{r+1} a_{l-1}\left(1 + \left[\frac{n}{p^l}\right] - \frac{n}{p^l}\right) + 1 \\
&\quad\geqslant \sum_{l=1}^{r+1} a_{l-1}\left(\left[\frac{n}{p^l}\right] - \frac{n}{p^l}\right) - n \sum_{l \geqslant r+2} \frac{a_{l-1}}{p^l}.
\end{aligned}$$

The combination of these two inequalities eventually leads to

$$\begin{aligned}
&\sup_{a \in [0,1]} \left|\operatorname{card}\left\{k \in \{0,\ldots,n-1\} \mid T_{p,y}^k(x) < a\right\} - na\right| \\
&\quad\leqslant 1 + (p-1) \max\left(\sum_{l=0}^{r}\left(1 - \left\{\frac{n}{p^{l+1}}\right\}\right), \sum_{l=0}^{r}\left\{\frac{n}{p^{l+1}}\right\}\right). \qquad \square
\end{aligned}$$

Remark 3.11. The exact value of the discrepancy $D_n^*(T_{p,y}^k(x))$ is of course a function of the initial value x. Some additional — but heuristic — computations were achieved for the p-adic Kakutani transform (that is T_p). They emphasized that whenever $p = 2$, 3 or $p \geqslant 10$, the lowest values were obtained with the starting value $x = \frac{1}{5}$. Furthermore, exact computations with a sorting algorithm on a computer, up to $n = 5 \cdot 10^4$ with many initial values, empirically strengthened our opinion.

3.2.2. Diaphony of the sequences $(T_p^k(x))_{k \geqslant 0}$

Unfortunately we could not reach any (uniform) bound for the diaphony of the sequences $(T_{p,y}^k(x))_{k \geqslant 0}$ as we failed in computing the corresponding exponential sums. However, for the sole Kakutani transform T_p we could manage to work out another way round. Indeed we succeeded in proving that the diaphony F_n^x of the sequence $T_p^k(x)$, $k \geqslant 0$, was vanishing to 0 uniformly with respect to x with an $O(\sqrt{\ln n} /n)$ rate.

Let $F_n := F_n^0$ denote the diaphony of the p-adic VdC sequence. Recall (Proposition 2.5(a)) that

$$\phi(n) = \max_{1 \leqslant k \leqslant n} kF_k \leqslant \frac{\pi\sqrt{p^2-1}}{2\sqrt{3}}\left(\left[\frac{\ln(pn)}{\ln p}\right] + \frac{p^2+3}{p^2-1}\right)^{1/2}.$$

Theorem 3.12. $\forall n \geqslant 1$,

$$\sup_{x \in [0,1]} F_n^x \leqslant \frac{\sqrt{3}}{n}\phi(p^{r+1}) \leqslant \frac{\pi\sqrt{p^2-1}}{2n}\left(\left[\frac{\ln(pn)}{\ln p}\right] + \frac{2(p^2+1)}{p^2-1}\right)^{1/2}.$$

Proof. If $x = 1$, $\forall k \geqslant 1$, $T_p^k(x) = \xi_{k-1}$ since $T_p(1) = 0$. Hence $F_n^1 = F_{n-1}$ as soon as $n \geqslant 2$ and $F_1^1 = \sqrt{\frac{2}{6}\pi^2} = \pi/\sqrt{3}$. If $x \in [0, 1[$ and $n \in \{0,\ldots,p^{r+1}-1\}$, then $x \in [\xi_l, \xi_l + 1/p^{r+1}[$, i.e., $x = \xi_l + y$ with $0 \leqslant y < 1/p^{r+1}$. Now $T_p^k(x) = (\xi_l + y) \oplus_p \xi_k$ and some easy computations yield

$$T_p^k(x) = \xi_{k+l} + y, \quad \text{if } k + l \leqslant p^{r+1},$$

and

$$T_p^k(x) = \xi_{k+l-p^{r+1}} + \left(y \oplus_p \frac{1}{p^{r+2}}\right), \quad \text{if } k + l \leqslant p^{r+1}.$$

Hence,

$$\sum_{k=0}^{n-1} e^{2i\pi h T_p^k(x)} = \sum_{k=l}^{(n+l)\wedge(p^{r+1}-1)} e^{2i\pi h y}\, e^{2i\pi h \xi_k}$$
$$+ \sum_{k=0}^{(n+l-p^{r+1})_+-1} \exp\left(2i\pi h\left(y \oplus_p \frac{1}{p^{r+2}}\right)\right) e^{2i\pi h \xi_k},$$

with $m_+ := \max(m, 0)$.

$$\left|\sum_{k=0}^{n-1} e^{2i\pi h T_p^k(x)}\right|^2 \leqslant 3\left(\left|S_{(n+l)\wedge p^{r+1}}(h)\right|^2 + |S_l(h)|^2 + |S_{(n+l-p^{r+1})_+}(h)|^2\right),$$

subsequently,

$$(nF_n^x)^2 \leqslant 3\left(((n+l)\wedge p^{r+1})F^2_{(n+l)\wedge p^{r+1}} + (lF_l)^2 + ((n+l-p^{r+1})_+)F_{(n+l-p^{r+1})_+}\right)$$
$$\leqslant 3\phi^2(p^{r+1}),$$
$$F_n^x \leqslant \frac{\pi\sqrt{p^2-1}}{2n}\left(r + \frac{2(p^2+1)}{p^2-1}\right)^{1/2}. \qquad \square$$

3.3. $O(1/n)$-integration with the $(T_p^k(x))_{k\geqslant 0}$ sequences

Let us recall (Remark 3.6) that if

$$y = \frac{y_0}{p} + \cdots + \frac{y_s}{p^{s+1}} \quad \text{and} \quad g := y_0 + \cdots + y_s p^s,$$

then $T_{p,y} = T_p^g$. The aim of this section is to point out that all the orbits $(T_p^k(x))_{k\geqslant 1}$ of the p-adic Kakutani transform T_p integrate with an $O(1/n)$ rate the same smooth periodic functions as the p-adic VdC sequence. We will also show that any function with finite variation integrated at an $O(1/n)$ rate by a given orbit of a $T_{p,y}$ is actually integrated at the same rate by all the orbits of this transform (provided that $\gcd(y_0, p) = 1$).

Lemma 3.13. *Let $r \in \mathbb{N}$ and*

$$I_k := [T_p^k(0), T_p^k(0) + \frac{1}{p^{r+1}}[, \quad 0 \leqslant k \leqslant p^{r+1}-2,$$
$$I_{p^{r+1}-1} := \left[T_p^{p^{r+1}-1}(0), T_p^{p^{r+1}-1}(0) + \frac{1}{p^{r+1}}\right] = \left[1 - \frac{1}{p^{r+1}}, 1\right].$$

The I_k make up a partition of the unit interval satisfying:

$$\forall k \in \{0, \ldots, p^{r+1}-1\} \; T_p(I_k) \subset I_{\overline{k+1}^{p^{r+1}}}$$

(equality holds iff $k \neq p^{r+1} - 2$).

Proof. If $k \in \{0, \ldots, p^{r+1}-2\}$, T_p is monotonous with slope 1 on I_k. Hence

$$T_p(I_k) = [T_p^{k+1}(0),\, T_p^{k+1}(0) + \frac{1}{p^{r+1}}[\subset I_{k+1} \quad (= \text{iff } k \neq p^{r+1}-2).$$

$$T\left(I_{p^{r+1}-1}\right) = \left[0,\, \frac{1}{p^{r+1}}\right] = I_0 = I_{\overline{p^{r+1}-1+1}^{p^{r+1}}}$$

because of carrying over in $\oplus_p$. Eventually, the I_k make up a partition of [0, 1] since

$$\left\{T_p^k(0),\, 0 \leqslant k \leqslant p^{r+1}-1\right\} = \left\{\xi_k,\, 0 \leqslant k \leqslant p^{r+1}-1\right\} = \left\{\frac{k}{p^{r+1}},\, 0 \leqslant k \leqslant p^{r+1}-1\right\}. \quad \square$$

Lemma 3.14. *Let $f:[0, 1] \to \mathbb{R}$ be a function with finite variation and y a p-adic rational with* $\gcd(y_0, p) = 1$. *Then, if $g = y_0 + \cdots + y_s p^s$,*

$$\forall x, u \in [0, 1]\ \forall n \in \mathbb{N}^* \quad \left|\sum_{k=0}^{n-1} f\left(T_{p,y}^k u\right)\right| \leqslant \int_{]0,1]} |\mathrm{d}f| + 2 \sup_{l \geqslant 1}\left|\sum_{k=0}^{l-1} f \circ T_{p,y}^k(x)\right|.$$

Proof. According to Lemma 3.13 there exist unique integers $\lambda(x)$ and $\lambda(y)$ in $\{0, \ldots, p^{r+1}-1\}$ such that $x \in I_{\lambda(x)}$ and $y \in I_{\lambda(y)}$. The equation $gk \equiv \lambda(x) - \lambda(y) \bmod p^{r+1}$ has a unique solution $\tilde{k}$ in $\{0, \ldots, p^{r+1}-1\}$, hence Lemma 3.13 yields $T^{g\tilde{k}}(x) \in I_{\overline{g\tilde{k}+\lambda(x)}^{p^{r+1}}} = I_{\lambda(y)}$, subsequently:

$$\begin{aligned}
\left|\sum_{k=0}^{n-1} f\left(T_p^{kg} y\right)\right| &\leqslant \left|\sum_{k=0}^{n-1} f\left(T_p^{kg} y\right) - f\left(T_p^{kg}\left(T_p^{\tilde{k}g} y\right)\right)\right| + \left|\sum_{k=\tilde{k}}^{\tilde{k}+n-1} f\left(T_p^{kg} x\right)\right| \\
&\leqslant \int_{]0,1]} |\mathrm{d}f| + \left|\sum_{k=0}^{\tilde{k}+n-1} f\left(T_p^{kg} x\right)\right| + \left|\sum_{k=0}^{\tilde{k}-1} f\left(T_p^{kg} x\right)\right| \\
&\leqslant \int_{]0,1]} |\mathrm{d}f| + 2 \sup_{l \geqslant 1}\left|\sum_{k=0}^{l-1} f \circ T_p^{kg}(x)\right|. \quad \square
\end{aligned}$$

Theorem 3.15. *Let $f:[0, 1] \to \mathbb{R}$ be a function with finite variation.*

(a) *The three following propositions are equivalent:*

(i) $\exists x \in [0, 1]$ *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(T_{p,y}^k x\right) - \int_0^1 f(u)\,\mathrm{d}u = \mathrm{O}\left(\frac{1}{n}\right);$$

(ii) $\forall x \in [0, 1]$,

$$\sup_{x \in [0,1]}\left|\frac{1}{n}\sum_{k=0}^{n-1} f\left(T_{p,y}^k x\right) - \int_0^1 f(u)\,\mathrm{d}u\right| = \mathrm{O}\left(\frac{1}{n}\right);$$

(iii) $$\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_{kg}) - \int_0^1 f(u)\,\mathrm{d}u = \mathrm{O}\left(\frac{1}{n}\right), \quad \xi_k = T_p^k(0).$$

(b) $$\sum_{h \geqslant 1} |c(h)|\, p^{v(h)} < +\infty \ \Rightarrow\ \sup_{x \in [0,1]}\left|\frac{1}{n}\sum_{k=0}^{n-1} f\left(T_p^k x\right) - \int_0^1 f(u)\,\mathrm{d}u\right| = \mathrm{O}\left(\frac{1}{n}\right).$$

Proof. (a) The equivalence straightforwardly derives from Lemma 3.14 applied to $f - \int_0^1 f(u)\,\mathrm{d}u$.

(b) T_p corresponds to the case $g = 1$. Then, the result is an instant corollary of (a) and Proposition 2.6. □

This theorem emphasizes that the initial "seed" in the numerical integration of functions in $\mathscr{C}_v$ has no influence on the asymptotic rate of convergence.

Let us notice however that some numerical simulations with periodic functions displayed, as for the discrepancy, that $x = \frac{1}{5}$ remains the most performing seed whenever $p = 2$, 3 or $p \geqslant 10$.

Theorem 3.15 does not yield any practical result for the iterated transform $T_{p,y}$. This lack ensues from our inability to compute the corresponding exponential sums. We are conjecturing that these sums are also bounded except, maybe, for some h "p-adically" linked with $g = y_0 + \cdots + y_s p^s$. Numerical simulations are as positive as possible on that matter.

As a conclusion for this section let us mention that the hypotheses in Theorem 3.15(a) are also equivalent to the existence of a Riemann integrable function ϕ such that $f = \phi - \phi \circ T_{p,y}$. Such functions are usually called coboundaries (see [12,15] for a complete proof).

4. Conclusion

The purpose of this paper was to show on a concrete example how introducing uniquely ergodic transforms on the unit interval [0, 1] easily yields a great number of u.d. sequences with similar properties as the "seminal" ones. Analogous results had been already established with the $\{n\alpha\}$ sequences (see [14], for example) a long time ago.

However, when they are actually implemented over a "human" range of values (that is no more than two or three million), some orbits turn out to be much more efficient than the original one. So is the case, for example, of the sequence $(T_2^k(\frac{1}{5}))_{k \geqslant 0}$: if

$$\delta_n(\xi) = \frac{1}{n} \sum_{1 \leqslant k \leqslant n} \frac{k}{\ln(k+1)} D_k^*(\xi)$$

denotes the "average constant" of ξ over $1, \ldots, n$, one verifies that $\delta_{50\,000}(\mathrm{VdC}_2) \approx 0.416$ and $\delta_{50\,000}(T_2^k(\frac{1}{5})) \approx 0.291$. Furthermore, the discrepancy of $(T_2^k(\frac{1}{5}))_{k \geqslant 0}$ is lower than the VdC_2 one for 99.84% of the values of $n \leqslant 50\,000$. Similar results were verified on all the smooth periodic functions tested (sin, cos or $x \wedge (1 - x)$). Actually this sequence successfully competes either with the $\{n\sqrt{2}\}$ sequence — the best $\{n\alpha\}$ one — or with the best one-dimensional scrambled VdC sequences (see [3,4]) on current numerical tests (see [20]) over this range of values.

In higher dimensions similar behaviours are empirically verified when some multi-dimensional extensions of the VdC sequences — the Halton sequences — are implemented to compute integrals of smooth periodic functions (see [2,16]). These sequences are made up with p_i-adic VdC sequences, p_i ith prime number.

So far, no theoretical results about the diaphony or the class of $\mathrm{O}(1/n)$-integrated functions were established for the Halton sequences and their related uniquely ergodic transform.

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