HOMEWORK 7

654 STOCHASTIC PROCESSES

Note for the current and future homework: due to a large class size and time constraints, not all exercises will be graded.

As you are the first class to go through these homework assignments, they may contain typos/ambiguities. Feel welcome to contact us if you see a typo or have doubt about other issues.

An exercise on independence. Let $(Z_t)_{t=1}^{\infty}$ be a sequence of iid random variables and let $f(\cdot,\cdot)$ be a random mapping representation for the matrix of transitions $P(\cdot,\cdot)$ on a finite set $\mathcal{X}$, so that for any $x, y \in \mathcal{X}$ and any $t \geq 1$,

$$P(x,y) = \mathbf{P}[f(x, Z_t) = y].$$

Let $0 = T_0 < T_1 < T_2 < T_3 < ...$ be integer valued random variables independent of $(Z_t)_{t=1}^{\infty}$.

For $u \in \mathcal{X}$, define $U_0 = u$ and

$$U_{k+1} = f(U_k, Z_{T_{k+1}}).$$

1. Show that $(U_k)_{k\geq 0}$ is a Markov Chain with transition P.

2. Let $m \geq k \geq 1$. Let $g : \mathcal{X}^{k+1} \to \{0,1\}$ be a nonrandom function. Show that the event $\{g(U_0, U_1, ..., U_k) = 1\}$ is independent of the event $\{T_1 = t_1, T_2 = t_2, ..., T_m = t_m\}$ for any nonrandom integers $t_1, t_2, ..., t_m$.

3. Let $\nu = \min\{k \geq 0 : U_k = 0\}$ be the first hitting time of 0 of the chain $(U_k)_{k=0}^{\infty}$. Prove that the event $\{\nu = q\}$ is independent from the random variables $T_1, ..., T_m$ for any $m \geq q \geq 1$.

4. Let $d \geq 2$ and assume that $\mathcal{X} = \{0, 1, 2, ..., n-1\}$. We define a Markov Chain (D_t) on $\mathcal{X}^d$ as follows. Let $(A_t, B_t, C_t)_{t=1}^{\infty}$ be iid random variables uniformly distributed on $\{-1, +1\} \times \{0, 1\} \times \{1, ..., d\}$. Let $x, y \in \mathcal{X}^d$ be two starting points with $x(1) < y(1)$. Set $X_0 = x$, $Y_0 = y$ and let

$$X_{t+1}(C_t) = B_t A_t + X_t(C_t),$$

$$Y_{t+1}(C_t) = \begin{cases} X_{t+1}(C_t) \text{ if } X_t(C_t) = Y_t(C_t) \text{ and} \\ Y_{t+1}(C_t) = (1 - B_t)A_t + Y_t(C_t) \text{ otherwise;} \end{cases}$$

and $X_{t+1}(c) = X_t(c), Y_{t+1}(c) = Y_t(c)$ for all coordinates $c \neq C_t$. That is, we pick a coordinate a random and then we only update this coordinate; if the two chains are equal on this coordinate then we update the two chains on this coordinate so that the equality is maintained; if the two chains are not equal we only update one of the chains at this coordinate, as in Section 5.3.3. Finally, define for every $c = 1, ..., d$, $D_t(c) = Y_t(c) - X_t(c)$.

a. What is the transition matrix of the Markov Chain $(D_t(1))_{t \geq 0}$? (You may answer with a graph where edges are labeled with transition probabilities).

b. Let $T_0 = 0$ and let $T_k = \min\{t > T_{k-1} : C_t = 1\}$ be the k-th time that coordinate 1 was picked in the previous question. What is the transition matrix of the Markov Chain $(D_{T_k}(1))_{k \geq 0}$?

c. Define f and $(Z_t)_{t\geq 0}$ such that $D_{T_k}(1)$ is equal to U_k defined above.

5. What is the distribution and expectation of $T_{k+1} - T_k$?

6. Use exercise 5.3 from the book to prove the relationship between $\mathbf{E}[\nu]$ and $\mathbf{E}[T_\nu]$.

Exercises from the book page 73.

- 5.1

- 5.2

- 5.3

- (Optional) 5.4

A Transformation of Markov Chain using Harmonic functions. *The techniques and results from Exercises 1.10, 1.12 and Proposition 2.1 may be useful.*

Let $\mathcal{X}$ be finite and consider a Markov Chain $(X_t)_{t=0}^{\infty}$ on $\mathcal{X}$ with transition matrix $P(\cdot,\cdot)$.

Let $A, B \subset \mathcal{X}$ be disjoint subsets of $\mathcal{X}$ such that $P(x,x) = 1$ for $x \in A \cup B$ and $P(x,x) < 1$ for $x \notin A \cup B$. The elements of $A \cup B$ are the only absorbing sites.

We assume that for any $x \in \mathcal{X} \setminus (A \cup B)$, there exists $y \in A$ such that y is accessible from x (i.e., there exists a path from x to y with positive probability with respect to $P(\cdot,\cdot)$).

1. Let $h : \mathcal{X} \to [0, +\infty)$ be a harmonic and positive on $\mathcal{X} \setminus (A \cup B)$. Define a matrix $\check{P}$ by
$$\check{P}(x,y) = P(x,y)h(y)/h(x)$$
for $x \in \mathcal{X} \setminus (A \cup B)$ and $y \in \mathcal{X}$. and $\check{P}(x,x) = 1$ for $x \in A \cup B$. Is $\check{P}$ a transition matrix?

2. In the rest of the problem, we set $h(x) = \mathbf{P}_x(\tau_A < \tau_B)$ where $\tau_A = \min\{t \geq 0 : X_t \in A\}$ and $\tau_B = \min\{t \geq 0 : X_t \in B\}$. $h(x)$ is the probability, starting from x, to hit A before hitting B.

   a. Prove that h is positive on $\mathcal{X} \setminus (A \cup B)$.
   b. Prove that h is harmonic on $\mathcal{X} \setminus (A \cup B)$.

3. For h as in the previous question and $x \notin A \cup B$, prove that

$$\check{P}(x, y) = \mathbf{P}_x[X_1 = y | \tau_A < \tau_B].$$

4. Describe in words the probability $\check{P}(x, y)$ obtained in the previous question.

5. Explain why $h(x) = \mathbf{P}_x(\tau_A < \tau_B)$ satisfies

   a. $h(x) = 1$ for $x \in A$ and $h(z) = 0$ for $z \in B$;
   b. h is harmonic at x for every $x \notin A \cup B$;

   Prove that $h(\cdot)$ is the unique solution of the linear system of equations given by a. and b. Equations in a. above are sometimes referred to as the boundary conditions.

6. If $\mathcal{X} = \{0, 1, ..., n\}$ and $P(\cdot, \cdot)$ is the Gambler's ruin transition matrix (see Section 2.1), $A = \{n\}$ and $B = \{0\}$, what is the linear system of equations satisfied by $h(\cdot)$? Recall the values of $h(\cdot)$ in this simple case.

In more complicated cases, we can ask a computer to solve the linear system to obtain the values of h on $\mathcal{X}$, compute $\check{P}$, and finally, simulate a Markov Chain with transition matrix $\check{P}$ until it is absorbed by an element of A. For instance, in https://bellecp.github.io/teaching/2019-Spring-Stochastic-Processes/conditional-walk.gif, the transition matrix $P(\cdot, \cdot)$ is a simple random walk on the 2d lattice (with diagonal moves allowed), the set B is the union of the square boundary and the obstacle in the middle. The set A is the bottom left corner. The chain is started from the top right corner. The code to generate this Markov Chain is available at https://gist.github.com/bellecp/58b93c7ff16ee84d9f7c79ed06837ad6#file-6-check-p-transform-2d-lattice-ipynb.

i. Download the notebook, inspect the code, and by adding Markown cells around the code cells, explain what each portion of the code is doing. (For instance, explain where each matrices $P, \check{P}$ are constructed, where the sets A and B are defined, where the linear system is constructed and solved, where the starting point is defined, etc. You may break the python cells into smaller cells to better explain what the code is doing.)

ii. *A random walk conditioned to hit the top of the square.* Here, B is the union of the left, right and bottom boundary of the square, while A is the top boundary of the square. (No more obstacle in the middle of the square). Compute $P, \check{P}$ and sample 5 independent copies of a random walk started from $(n/2, 1)$ with transition matrix $\check{P}$. Draw the 5 walks on 5 different pictures.

iii. *Teleportation sites.*

Consider the above picture. Here, A contains only one point which is the center of the square, in gray. The other gray points are in the set B; this includes the square boundaries, the diagonal band and the circle. The dark point at the bottom-left is the starting point. Outside of $A \cup B$, the matrix $P(\cdot, \cdot)$ is a simple random walk as in the previous question with a slight modification at the blue and orange points of the top right corner: we set $P(blue, green) = 1$ and $P(orange, red) = 1$ where *red* is the colored point within the circle and *green* is the colored point at the bottom. Compute $P, \check{P}$ and sample 5 independent copies of a random walk started from the dark point (bottom left) with transition matrix $\check{P}$. Draw the 5 walks on the same picture, each with its own color. Do not try to reconstruct exactly the positions of the gray and colored points, but simply respect the general layout. *Hint: Your code should produce trajectories similar to the picture*
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iv. *Design your own maze.* Choose a starting point, an exit point (the set A), the obstacles (the set B) and design your own maze. For instance, here is one: https://bellecp.github.io/teaching/2019-Spring-Stochastic-Processes/conditional-maze.gif.

v. (Optional) *Markov Chains on the 3d lattice.* Let $n = 8$. Let $\mathcal{X}$ be a 3d lattice of size $n \times n \times n$ ("the cube"), i.e, $\mathcal{X} = \{(i, j, k) \in \mathbb{Z}^3 : 0 \le \min(i, j, k) \le \max(i, j, k) < n\}$. Let P be a simple random walk on the lattice where the 6 possible moves are up/down/east/west/north/south. Let A be the top face of the cube, while B is the union of the remaining 5 faces. Let $(n/2, n/2, 1)$ be the starting point (at the almost bottom of the cube). Compute $P, \check{P}$ and sample 5 independent copies of a Markov Chain with respect to $\check{P}$, started at the starting point; draw these 5 random walks.

vi. (Optional) Same as the previous question; but try to increase n until your laptop cannot execute the code and produce the pictures in less than 4 minutes. To improve the performance of your code, you may use `numba.jit` to precompile parts of your code and make it faster (specifically, the `for` loops in python are very slow!). See some previous homework solutions for an example of use of `numba.jit`.