Introduction

History, First Finance Principles, Probability Refresher

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

Financial Engineering – September, 2011

Outline

...

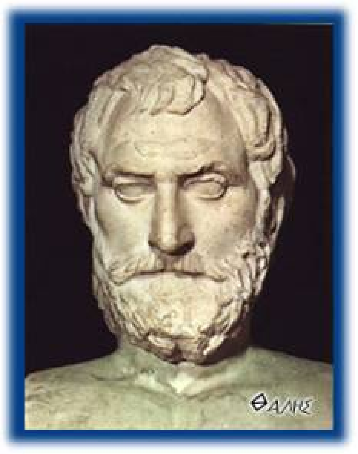

I'M EXHAUSTED. I THINK I'LL...
KEEP DOING THIS

Timeline

In Ancient Times

"Deducing from his knowledge of the stars that there would be a good crop of olives, while it was still winter and he had a little money to spare, used it to pay deposits on all the oil-presses in Miletus and Chios, thus securing their hire. This cost him only a small sum, as there were no other bidders. Then the time of the olive-harvest came and, as there was a sudden and simultaneous demand for oil-presses, he hired them out at any price he liked to ask." (Aristotle, The Politics, Book I,xi)

Thales of Miletus



Timeline

How did we get here?

- Risk and Chance in the Greek and Roman culture
- The Early Christianity: ”The good Christian should beware of mathematicians. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell.” (Augustinus, 354 - 430)

“Secularization” of Chance

1494 Luca Paccioli and his “brainteaser” (1494)

1550 Cardano: “liber de ludo alea”

1654 Chevalier de Méré and the letters of Pascal and Fermat.

1686 Newton’s Infinitesimal Calculus

1731 Daniel Bernoulli “Specimen Theoriae Novae de Mensura Sortis”.

Chevalier de Méré's Problem

Example

A 17th century gambler, the Chevalier de Méré, made it to history by turning to Blaise Pascal for an explanation of his unexpectd losses. Pascal combined his efforts with his friend Pierre de Fermat and the two of them laid out mathematical foundations for the theory of probability. Gamblers in the 1717 France were used to bet on the event of getting at least one 1 (ace) in four rolls of a dice. As a more trying variation, two die were rolled 24 times with a bet on having at least one double ace. According to the reasoning of Chevalier de Méré, two aces in two rolls are 1/6 as likely as 1 ace in one roll. (Which is correct.) To compensate, de Méré thought, the two die should be rolled 6 times. And to achieve the probability of 1 ace in four rolls, the number of the rolls should be increased four fold - to 24. Thus reasoned Chevalier de Méré who expected a couple of aces to turn up in 24 double rolls with the frequency of an ace in 4 single rolls. However, he lost consistently.

Explain ;)

Timeline

South Sea Company

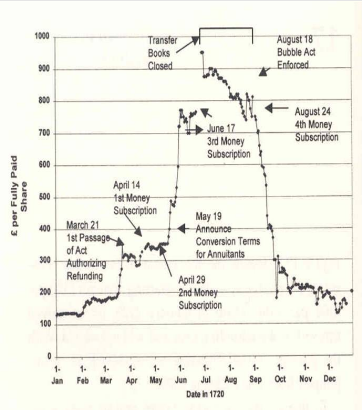


Figure 17.1
Daily South Sea Share Prices, 1720. Data courtesy of Larry Neal.

- The South Sea Company was a British joint stock company that traded in South America during the 18th century.
- Founded in 1711, the company was granted a monopoly to trade in Spain's South American colonies as part of a treaty during the War of Spanish Succession.
- In return, the company assumed the national debt England had incurred during the war.
- Speculation in the company's stock led to a great economic bubble known as the South Sea Bubble in 1720, which caused financial ruin for many.
- "I can calculate the movements of heavenly bodies, but not the madness of people." (Sir Isaac Newton, 1720)
- See <click>

Timeline

1827 Robert Brown and his view through the microscope.<click>

1900 Louis Bachelier and his ‘Théorie de la Speculation’.<click>

1905 Albert Einstein’s investigation on the theory of Brownian motions.

1923 Norbert Wiener developed a rigorous theory for Brownian motion.

1944 Itô’s stochastic version of Newton’s infinitesimal calculus.

“People in Finance make their living in part by Itô’s Lemma.” (Henry McKean)

Timeline

1950 The 1970 Nobel Laureate in Economics, Paul Samuelson, was responsible for setting the tone for subsequent generations of economists.

1952 Harry Markowitz was the first to propose a modern quantitative methodology for portfolio selection. This required knowledge of assets' volatilities and the correlation between assets.

1963 William Sharpe of Stanford, John Lintner of Harvard and Norwegian economist Jan Mossin independently developed a simple model for pricing risky assets, the CAPM.

1966 Eugene Fama concluded that stock prices were unpredictable and coined the phrase 'market efficiency'.

1968 Sobol, Faure, Hammersley, Haselgrove and Halton, etc. Many people were associated with the definition and development of quasi random number theory or low-discrepancy sequence theory.

1968 Ed Thorp's first claim to fame was that he figured out how to win at casino Blackjack (Beat the Dealer, the book that made Las Vegas change its rules.) His second claim to fame is that he invented and built, with Claude Shannon, the information theorist, the world's first wearable computer. His third claim to fame is that he used the 'correct' formulæ for pricing options.

Timeline

1973 Fischer Black, Myron Scholes and Robert Merton derived the Black–Scholes equation for options in the early seventies, publishing it in two separate papers in 1973 (Black and Scholes, 1973, and Merton, 1973). The date corresponded almost exactly with the trading of call options on the Chicago Board Options Exchange. Scholes and Merton won the Nobel Prize for Economics in 1997. Black had died in 1995.

1974 In 1974 Robert Merton introduced the idea of modelling the value of a company as a call option on its assets, with the company's debt being related to the strike price and the maturity of the debt being the option's expiration. Thus was born the structural approach to modelling risk of default, for if the option expired out of the money (i.e. assets had less value than the debt at maturity) then the firm would have to go bankrupt.

1977 Phelim Boyle related the pricing of options to the simulation of random asset paths. He showed how to find the fair value of an option by generating lots of possible future paths for an asset and then looking at the average that the option had paid off.

Timeline

1977 So far quantitative finance hadn't had much to say about pricing interest rate products. Some people were using equity option formula for pricing interest rate options, but a consistent framework for interest rates had not been developed. This was addressed by Vasicek.

1979 Boyle had shown how to price options via simulations, an intuitively reasonable idea, but it was John Cox, Stephen Ross and Mark Rubinstein, who gave option-pricing capability to the masses using binomial trees.

1979 Mike Harrison and David Kreps, in 1979, showed the relationship between option prices and advanced probability theory, originally in discrete time. Harrison and Stan Pliska in 1981 used the same ideas but in continuous time.

1986 The problem with the Vasicek model was that it was not able to get even bond prices right. Thomas Ho and Sang-Bin Lee found a way around this, introducing the idea of yield-curve fitting or calibration.

1992 Instead of modelling just a short rate and deducing the whole yield curve, David Heath, Robert Jarrow and Andrew Morton (HJM) modelled the random evolution of the whole yield curve.

1993 Steve Heston introduces the stochastic volatility model in continuous time.

1994 Dupire, Rubinstein, Derman and Kani, and others brought the Ho-Lee approach to the volatility surface by introducing "local volatility".

Timeline

1997 HJM still had two major drawbacks. It required the existence of a spot rate and it assumed a continuous distribution of forward rates. Alan Brace, Dariusz Gatarek and Marek Musiela (1997) introduced a model which only relied on a discrete set of rates – ones that actually are traded.

2000 The 1990s saw an explosion in the number of credit instruments available, and also in the growth of derivatives with multiple underlyings, e.g.,CDOs. A probabilistic pricing approach based on copulas was proposed by David Li (2000). The copula approach allows one to join together default models for individual companies in isolation to make a model for the probabilities of their joint default.

2002 Industry needs models that are both fast and match traded prices well. The interest-rate model of Pat Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward (2002), which has come to be called the SABR model, is a model for a forward rate and its volatility, both of which are stochastic.

2007 In early August 2007 several hedge funds using quant strategies experienced losses on such a scale as to bring the field of quant finance into disrepute. Trading of complex d'rives in obscene amounts using simplistic mathematical models almost brought the global financial market to its knees.

2011 Rating agencies, sovereign debt, interbank market, and and and!

Outline

Main Problems

Questions

1. Suppose we are given a financial derivative. What is the "fair" price for such a contract? [Pricing]
2. Suppose we are managing a large portfolio. How can we control the exposure to financial risks? [Hedging]
3. Suppose you have to achieve some financial targets with your portfolio. How can I minimize risk and costs of missing the target? [Risk Management]

Answers to Q1 – Easy, and wrong

A(1) Using arguments from operations research, a reasonable price for a contingent claim is obtained by computing the expected value of the discounted future cash-flows.

A(2) From the first economics courses, you know that the price is determined by demand and supply. Thus, the price for a contingent claim is where demand equals supply. As demand and supply are not easy to determine, we easily get lost in economic diffuse arguments and hardly can nail down a correct price for the contingent claim.

Main Ideas

So what?

Luckily, both of the above answers A(1) and A(2) are not correct - at least in a complete market setup.

We will analyze the above raised questions in detail. To this end, we will follow a handful of main ideas:

- A financial derivative is defined in terms of some underlying asset which already exists and is traded on the (liquid) market.
- The derivative cannot be priced arbitrarily. There exists an intimate relation between the price of the underlying and the derivative. Neglecting this relation gives rise to mispricing. This can hurt!
- The theory of derivatives is about pricing **relative** to the market price of the underlying. It is not concerned about pricing in absolute terms.

Continuous Interest Rate

Making life easy...

- Invest \$100 for a year at an annual rate of 10% (with one-year compounding). After one year, we get \$110.
- If we have a semi-annual rate of 10%, then we get $\$100 \times 1.05^2 = \$110.25$ after a year.
- If we have a 10% quarterly rate, then we get \$110.38 after one year.
- We can continue to increase the compounding frequency, say n, and calculate the resulting cash amount after one year **ad nauseam** ... until we reach the continuously compounded interest rate:

$$\$100 \times \lim_{n\to\infty}\left(1 + \frac{0.1}{n}\right)^n = \$100 \times e^{0.1} = \$110.52.$$

- We will make the convention that we write r (not R) for the continuously compounded interest rate, often referred to as "short rate".
- A practical question for you to think about: "What is the riskless interest rate"?

Stock Prices

- In these notes, we will - if not stated otherwise - think of the stock as an underlying.
- In addition to the above obvious option structures, we can find implicit options in many different situation.
- The stock is denoted by S. Then, S_t is the market price of the stock S at time t.
- For an option, the time of expiry is denoted by T. Hence, we call the difference $T - t$ the time-to-expiration or the time-to-maturity.
- The price of an equity is a stochastic variable, i.e., it is a random variable whose value changes over time.
- The evolution of the stock price is often assumed to be decomposable into a purely deterministic part and a random part.
  - The deterministic part evolves according the the stock's **drift term**, often labeled as μ. The instantaneous return μ is assumed to be a continuously compounded rate.
  - The random part is referred to as the **diffusion term**.
- Hence,

$$S_T = S_t e^{\text{“stock return”}} = S_t e^{\mu(T-t)+\text{“randomness”}}.$$

Stock Prices

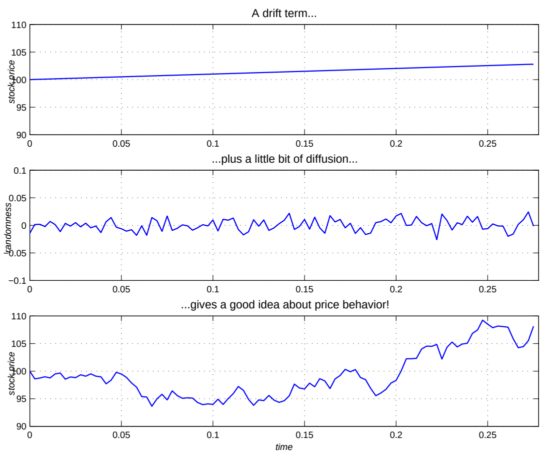


Figure: A probabilistic concept about stock price dynamics.

Dividends

- **Discrete dividends**, D. Market prices of equities drop by the present value of the dividend on the ex-dividend date.
- The second type is called **continuous dividends**. As in the case with interest rate, it is mathematically much more convenient to work with continuous variables.
- In an **infinitesimally** small interval of time, dt, the stock will lose dividend equal to an amount $\delta S_t dt$, where δ is the continuous dividend rate.
- Exponential decay in the growth of the stock. Therefore, if we assume that the stock price has no randomness at all, S evolves according to

$$dS_t = (\mu - \delta) S_t dt,$$

which has solution

$$S_T = S_t e^{(\mu - \delta)(T - t)}.$$

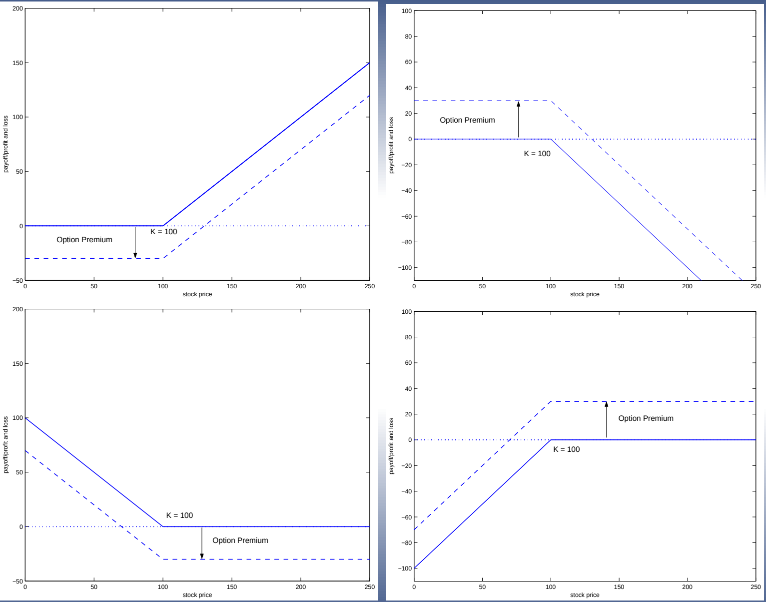
Word of Caution

In real life, dividends have a major impact on the pricing and are a source of instability for the implied volatility surface.

Call and Put Options

- Buying a (plain vanilla) call option at time t gives the right to buy the underlying (stock S, currency, interest rate contract, etc.)
  1. with current price S_t
  2. at the exercise price K
  3. at a future time T, $T \geq t$.
- By buying a put option at time t, the holder of the option has the right at maturity T to sell the stock at a specified price K.
- Whether an option will be exercised or not depends on whether it is profitable to exercise the option.
- An option is a "right", not an "obligation." (like forwards, futures, etc.). Having the character of a "right" makes things nonlinear!
- Options can be, among other things, European or American (and Asian, Russian, Parisian, etc.).

Payoff Diagrams



Some Common Terms

In the financial world, derivative theory is filled with jargon. Here are some loose terms we will often deal with:

- Premium
- Underlying
- Strike or exercise price
- Intrinsic value
- Time value
- In/Out/At-the-money
- Long/short position
- Volatility

Exotics and Structured Products

The most common contracts are still plain vanilla options, i.e., puts and calls. However, there are many other forms of options. They are just a variation of the exercise function and are often referred to as "exotics".

- Binaries and digital options
- Bermuda options
- Asian options
- Lookback options
- Trend derivatives
- Cuban options? (check out www.ssrn.com)
- et cetera

- Bull and bear spreads
- Straddles and strangles
- Butterflies and condors
- Risk reversal
- Calendar spreads

To get an overview of the myriads of different exotics and option strategies, you may want to visit the website of LIFFE.

Outline

"They have an expression on Earth that I believe applies to this situation - There ain't such thing as a free lunch."

Adam Warlock and Moondragon in Silver Surfer/Warlock: Resurrection 1 (1993), Marvel Comics

The Main Idea of Derivative Pricing

Question

How much would you pay me for the option below: 1, 2, 3, 4, 5, or 6?

Introduction

- Overview on value bounds of derivatives.
- Derived from no-arbitrage principles which are free of any distributional assumptions.
- Focus on American and European put and call options with the purpose of getting an intuition on how prices of derivatives are determined in financial markets.
- The all guiding principle is the **no arbitrage principle.**
- The principle of no arbitrage formalizes a very convincing economic argument: in a financial market it should not be possible to make a profit with zero net investment and without bearing any risk.

We will base our theory at least on the following set of assumptions:

- There are no transaction costs.
- Tax rate is zero.
- Lending and borrowing rates are the same and constant.
- Market participants are rational and share the same set of (full) information.

If not otherwise stated, the above assumptions apply on all the following propositions. In subsequent sections, this set of assumption will be expanded.

Call Options

Proposition

If the stock price is zero, then the price of an American call is also zero.

Proof.

Share prices reflect the present value of all future cash flows. □

Proposition

The lower bound of an American call option is given as $C_t \geq (S_t - K)^+$.

Proof.

Limited liabilities implies $C_t \geq 0$. If $S_t > K$ and $C_t < S_t - K$: Buy C_t and exercise the call. Cash flow is $S_t - K$ with a net profit of $S_t - K - C_t > 0$. Therefore, to avoid arbitrage $C_t \geq (S_t - K)$ must hold. □

Call Options

Proposition

An American call cannot be worth more than the underlying, i.e., $C_t \leq S_t$.

Proof.

1. Assume $C_t \geq S_t$. Take a short position in the American call option and go long in the stock. This portfolio gains a positive cash flow.
2. In case of exercising the call option, we are able to deliver the stock. □

Proposition

For a European call, c_t, we have $c_t \geq (S_t - e^{-r(T-t)}K)^+$.

Proof.

Consider the value of a long forward $f_t = S_t - e^{-r(T-t)}K$. Recall $c_t \geq 0$ and that the forward is an **obligation** whereas the option is a **right**. □

Call Options

Proposition

An American call option is worth at least as much as a European call option:

$$C_t \geq c_t.$$

Proof.

1. We could keep the American option until maturity.
2. However, the American option gives us the additional flexibility to exercise prior to the maturity date.
3. This flexibility needs to be reflected in a higher price for the American option. □

Call Options

Proposition

Consider a non-dividend paying stock and a positive interest rate. An American call option will never be exercised before maturity, i.e.,

$$C_t = c_t.$$

Proof.

1. We know $c_t \geq (S_t - Ke^{-r(T-t)})^+$ and $C_t \geq (S_t - K)^+$.
2. Since for $r > 0$ $(S_t - Ke^{-r(T-t)})^+ > (S_t - K)^+$ and $C_t \geq c_t$, we have $C_t > (S_t - K)^+$. Thus, one would rather sell the option than exercise it.
3. It is never optimal to exercise the American option at every instance t. $\square$

Put Options

Proposition

If the stock price is zero, then the price of an American put option corresponds to the exercise price, i.e., $S_t = 0 \quad \Rightarrow \quad P_t = K.$

Proof.

POC! □

Proposition

The lower bound of an American put option is zero or $K - S_t$*:*

$$P_t \geq (K - S_t)^+.$$

Proof.

Two steps:

- Let $K - S_t > P_t$. Then we buy the put option P_t and exercise it immediately. This strategy generates a profit of $K - S_t \leq P_t$.
- Because of limited liabilities the put option can never be negative. □

Put Options

Proposition

The upper bound of an American put option corresponds to the exercise price:

$$P_t \leq K.$$

Proof.

- Assume that $P_t > K$. Then, we sell P_t and invest the proceeds in the money market account $B_t = B_0 e^{rt}$ (say $r = 0$, then $B > K$).
- If exercised, the portfolio with the short position in P_t and the long position in the cash bond B can be written as $\Pi_t = B - (K - S_t)^+$.
- Then,

$$\Pi_t = \begin{cases} B & > 0, \quad \text{if} \quad S_t \geq K, \\ B - K + S_t & \geq 0, \quad \text{if} \quad S_t < K, \end{cases}$$

The last line follows from the fact that $B > K$. $\square$

Put Options

Proposition

Let S_t be a non-dividend paying stock. For a European put:

$$p_t \geq (Ke^{-r(T-t)} - S_t)^+.$$

Proof.

Limited liabilities assures that $p_t \geq 0$. The value of the short forward contract is $Ke^{-r(T-t)} - S_t$. The put option is at least worth the same, since it does not have to be exercised at maturity. □

Proposition

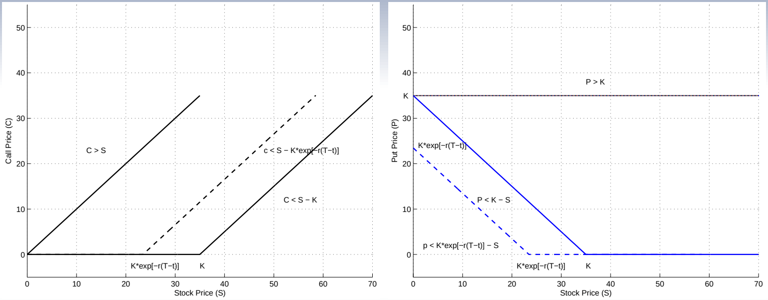
The price of an American put is at least the price of a European put, i.e.,

$$P_t \geq p_t.$$

Proof.

Homework. □

Call and Put Option Bounds



To see how it works...

Example

Assume $S_t = \$90$; $K = \$100$; $r = 3.25\%$; $T - t = 3$ months. Then, the lower bound is $Ke^{-r(T-t)} - S_t = \$9.20$. Set, e.g., $p_t = \$8.00$. This is not an arbitrage-free price for the European put option. Why?

1. Since we believe the option is underpriced, we buy the option and go long into the forward contract:

$$\begin{aligned} F(t, t+0.25) &= S_t e^{r \times (T-t)} \\ &= 90e^{0.0325 \times 0.25} \\ &= 90.73 \qquad (\neq K) \end{aligned}$$

2. The value of this contract is at the time t zero. We now take out a loan of

$$e^{-r(T-t)}(100 - 90.73) = 9.20.$$

The cash flow in time t is therefore:

Borrowing money	9.20
Long in Put Options	-8.00
Cash Flow in t	+1.20

3. And at maturity:

	$S_T \leq 100$	$S_T > 100$
Put Option	$100 - S_T$	0
Forward Contract	$S_T - 90.73$	$S_T - 90.73$
Redemption	-9.20	-9.20
Cash Flow in T	0	$S_T - 100 > 0$

We see from the payoff table above that there exists an arbitrage opportunity.

Outline

Put Call Parity

Proposition

The relationship between European options with the same underlying, the same strike price, and the same maturity is given as:

$$p_t = c_t + Ke^{-r(T-t)} - S_t$$

Proof

The proof goes as follows:

- Consider the two portfolios Π_1 and Π_2, where portfolio Π_1 contains a long position in a put option, and portfolio Π_2 consists of a long call, a long zero bond with face value K and a short position in the stock:

$$\begin{aligned} \Pi_1 &= p_t \\ \Pi_2 &= c_t + Ke^{-r(T-t)} - S_t \end{aligned}$$

Put Call Parity

Proof (cont'd)

- At time T, the values of the two portfolios are:

		$S_T \leq K$	$S_T > K$
$\Pi_1(T)$	$+$ Put	$K - S_T$	0
$\Pi_2(T)$	$+$ Call	0	$S_T - K$
	$+$ Bond	K	K
	- Stock	$-S_T$	$-S_T$
		$K - S_T$	0

- From $\Pi_1(T) = \Pi_2(T)$ it follows from the law of one price that $\Pi_1(t) = \Pi_2(t)$. $\square$

Note

The put-call-parity can also be rewritten as $c_t = S_t - Ke^{-r(T-t)} + p_t$:

- $S_t - Ke^{-r(T-t)}$ can be interpreted as a borrowed stock position and
- p_t can be understood as an insurance policy for bad states, (e.g. for $S_T < K$).

Put Call Parity

Proposition

For American Put and Call Options on S with exercise price K and the time-to-maturity $T-t$, the Put-Call Parity is:

$$S_t - K \leq C_t - P_t \leq S_t - Ke^{-r(T-t)}$$

Proof

We prove the proposition in two steps as follows:

- First, we look at the RHS of the inequality.
  - Consider the European Put-Call Parity

    $$c_t + Ke^{-r(T-t)} = S_t + p_t$$

    and recall $c_t = C_t$.
  - Then, $C_t + Ke^{-r(T-t)} = S_t + p_t$.
  - We know that $P_t \geq p_t$. Therefore,

$$\begin{aligned} C_t + Ke^{-r(T-t)} &\leq P_t + S_t \\ C_t - P_t &\leq S_t - Ke^{-r(T-t)} \end{aligned}$$

Put Call Parity

Proof (cont'd)

- In the second step, we need to show that the left part of the inequality

$$S_t - K \quad \leq \quad C_t - P_t$$

holds. This inequality has to be examined in two cases:

  - Put option will be exercised before maturity. Build the portfolio:

$$\Pi(t) \quad = \quad C_t - P_t - S_t + K \geq 0.$$

If the put option is exercised at time $t^* \in [t, T]$, then

$$\Pi(t^*) \quad = \quad C_{t*} - P_{t*} - S_{t*} + K \geq 0$$
$$\Pi(t^*) \quad = \quad C_{t*} - (K - S_{t*})^+ - S_{t*} + K \geq 0$$

A put option is only exercised before maturity if $K > S_{t*}$. This implicates that the LHS inequality must be true.

  - Put option will not be exercised before maturity. Then, same portfolio yields:

$\Pi(T)$	$S_T \leq K$	$S_T > K$
Long Call	0	$(S_T - K)$
Short Put	$-(K - S_T)$	0
Short Stock	$-S_T$	$-S_T$
Long K Cash	K	K
	0	0

$\square$

Cross Sectional Properties

Proposition (Monotonicity)

Consider a non-dividend payable stock and a positive interest rate. In the absence of arbitrage, then

$$
\begin{aligned}
C(S_t, K_1, t, T) &\geq C(S_t, K_2, t, T), \\
c(S_t, K_1, t, T) &\geq c(S_t, K_2, t, T), \qquad \forall K_2 \geq K_1, \\
P(S_t, K_1, t, T) &\leq P(S_t, K_2, t, T), \\
p(S_t, K_1, t, T) &\leq p(S_t, K_2, t, T), \qquad \forall K_2 \geq K_1, \\
P(S_t, K, t, T_2) &\geq P(S_t, K, t, T_1), \\
C(S_t, K, t, T_2) &\geq C(S_t, K, t, T_1), \\
c(S_t, K, t, T_2) &\geq c(S_t, K, t, T_1), \qquad \forall t \leq T_1 \leq T_2.
\end{aligned}
$$

Proof

The proof of the first two sets of inequalities follows directly from the monotonicity of the payoff profiles and the arbitrage-free pricing rule. The statement in the last set has to be proven separately for call and put options.

Cross Sectional Properties

Proof (cont'd)

- Call options (only check for European options): We prove by contradiction. Assume that the price of the call option is decreasing with decreasing time-to-maturity, i.e.,

$$c(S_t, K, t, T_2) \leq c(S_t, K, t, T_1) \qquad \forall t \leq T_1 \leq T_2.$$

Consider the portfolio, $\Pi_t = c_2 - c_1 \leq 0$. At time T_1 we have the following payoff table:

Π_t	Π_{T_1}	
	$S_{T_1} \leq K$	$S_{T_1} > K$
c_2	$c_2 \geq 0$	$c_2 \geq S_{T_1} - Ke^{-(T_2-T_1)r}$
$-c_1$	0	$-(S_{T_1} - K)$
	≥ 0	$\geq Ke^{-(T_2-T_1)r} > 0$

This opens up arbitrage possibilities.

Cross Sectional Properties

Proof (cont'd)

- American put: We know that it might be optimal to exercise an American put option early. So $P(S_t, K, t, T_2) \geq P(S_t, K, t, T_1)$ is obvious.
- European put: To see that monotonicity in time-to-maturity does not hold for the European put option is more subtle. Consider:

Π_t	Π_{T_1}	
	$S_{T_1} \leq K$	$S_{T_1} > K$
p_2	$p_2 \geq 0$	$p_2 \geq 0$
$-p_1$	$-(K - S_{T_1})$	0
	?	≥ 0

- We know that $P_t \geq p_t$. If t is a stopping time and the American put is exercised, $P_t \geq p_t$ is equivalent to $K - S_t \geq p_t$. Indeed, we observe such a situation when stock prices are low. As a consequence, we cannot uniquely determine the sign of the above portfolio in case of $S_{T_1} \leq K$.

Cross Sectional Properties

Proposition (Convexity and Homogeneity)

- *Convexity: The price of a call option (American and European) is convex wrt strike price. For* $K_1 < K_2 < K_3$

$$C(K_2) \leq \frac{K_3 - K_2}{K_3 - K_1} C(K_1) \quad + \quad \frac{K_2 - K_1}{K_3 - K_1} C(K_3)$$

$$K_2 - K_1 \quad \geq \quad C(K_1) - C(K_2)$$

$$(K_2 - K_1) e^{-r(T-t)} \quad \geq \quad c(K_1) - c(K_2)$$

- *Homogeneity: The price of a call or put option (American as well as European) has first order homogeneity wrt the stock and strike price:*

$$C(\alpha K, \alpha S) \quad = \quad \alpha C(K, S),$$

Hence

$$C(K, S) = \frac{\partial C(K, S)}{\partial S} S + \frac{\partial C(K, S)}{\partial K} K$$

Cross Sectional Properties

Proof - Convexity (Informal)

Let $K_1 < K_3, \alpha \in (0,1), \alpha \equiv \frac{K_3 - K_2}{K_3 - K_1}$. Then

$$K_2 = \frac{K_3 - K_2}{K_3 - K_1} K_1 + \frac{K_2 - K_1}{K_3 - K_1} K_3 = \alpha K_1 + (1-\alpha) K_3.$$

Now consider these option portfolios mentioned below

$$\begin{aligned} \Pi_1 &= C(\alpha K_1 + (1-\alpha)K_3) \\ \Pi_2 &= \alpha C(K_1) + (1-\alpha) C(K_3). \end{aligned}$$

Compare the payoff profiles of these two option portfolios in Figure (2). The illustration clearly shows that the payoff of the weighted option portfolio (Π_2) dominates the individual option (Π_1). Therefore, from the monotonicity, the absence of arbitrage dictates for every $\alpha \in (0,1)$:

$$C(\alpha K_1 + (1-\alpha)K_3) \quad \leq \quad \alpha C(K_1) + (1-\alpha) C(K_3).$$

Cross Sectional Properties

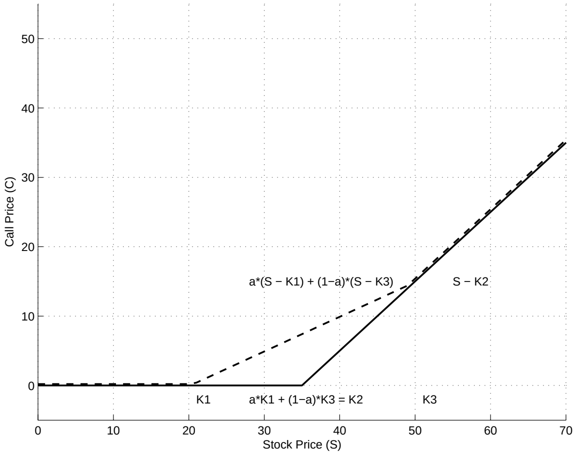


Figure: Payoff profiles of Call Combinations

Cross Sectional Properties

Proof (cont'd)

Now, consider the portfolio,

$$\Pi_b = C(K_1) - C(K_2).$$

In case of exercising at time $t^* \in (0, T]$ the payoff is

	$S_{t^*} \leq K_1$	$K_1 \leq S_{t^*} \leq K_2$	$S_{t^*} \geq K_2$
$C(K_1)$	0	$S_{t^*} - K_1$	$S_{t^*} - K_1$
$-C(K_2)$	0	0	$-(S_{t^*} - K_2)$
	0	$S_{t^*} - K_1 > 0$	$K_2 - K_1 > 0$

The payoffs in the three different cases are limited by $K_2 - K_1$. Therefore, the relationship $C(K_1) - C(K_2) \leq K_2 - K_1$ must hold for American options. For European options, we know that $c(K_1) \leq (S_t - K_1 e^{-r\tau})$ and $c(K_2) \leq (S_t - K_2 e^{-r\tau})$. Therefore, $c(K_1) - c(K_2) \leq K_2 e^{-r\tau} - K_1 e^{-r\tau}$. The same results hold, when the stock pays some dividends. □

Cross Sectional Properties

Proof - Homogeneity

Consider the portfolio

$$\Pi_c = C(\alpha K) - \alpha C(K)$$

In case of exercising at time $t^* \in (0, T]$ the payoff is

	$S_{t^*} \leq K$	$S_{t^*} > K$
$C(\alpha K)$	0	$\alpha S_{t^*} - \alpha K$
$-\alpha C(K)$	0	$\alpha(S_{t^*} - K)$
	0	0

The property of linear homogeneity implies a very important interpretation for the pricing function of options.

Recall that a function $f(x, y, z)$ is said to be homogeneous of degree k with respect to the variables x and y if the following condition holds

$$f(\alpha x, \alpha y, z) = \alpha^k f(x, y, z).$$

Cross Sectional Properties

Proof (con't)

Functions that are homogeneous of degree one are referred to as linearly homogeneous (used in economics for production functions with constant returns to scale.) The Swiss mathematician Leonhard Euler proved for linearly homogeneous functions the property

$$\frac{\partial f(x,y)}{\partial x}x + \frac{\partial f(x,y)}{\partial y}y = f(x,y), \tag{1}$$

called Euler's Rule. In the option pricing context:

$$\begin{aligned} C(S,K) &= S\frac{\partial C}{\partial S} + K\frac{\partial C}{\partial K} = C(S,K) \\ &= \phi S - \psi e^{-r(T-t)}K \end{aligned} \tag{2}$$

where equation (2) can be interpreted as a portfolio with a long position in stocks (ϕ) and a short position (ψ) in the zero bond with face value K. Obviously, if we knew the portfolio weights, then we get the price of a call option, which equals the value of a leveraged stock portfolio.

Outline

"At the first bend he crouched behind a boulder, pulled out his improbability automatic, took aim and actuated the possibiliballistic destabilizers."

The Third Sally or The Dragons of Probability
- Stanislav Lem

Overview

We review the following concepts:

- Probability space, σ-algebra, probability measure;
- Random variable and their distribution;
- Expectation, variance, covariance;
- Independence and conditional probability;
- Conditional expectation;
- Jensen's inequality;
- Filtration.

Probability

A systematic approach to ‘probability’

1. Probability as intuition (Koopman, "The axioms and algebra of intuitive probability", Annals of Mathematics 41, 269–292, 1940)
2. Probability as the ratio of possible events: We can calculate the a-priori probability of an event if we calculate the number of events relative to the number of all possible outcomes, e.g., if we through a dice.
3. Probability as the relative frequency of an event: We can measure probability by repeating an experiment n-times and count the number of events n_A. Then, asymptotically, we might define the probability for A using $\lim_{n\to\infty} = \frac{n_A}{n}$.
4. Probability as an axiomatic theory (Kolmogorov 1933).

Events and Probability

Definition

Let Ω be a non-empty set. A σ-algebra $\mathcal{F}$ is a family of subsets of Ω such that

- $\oslash \in \mathcal{F}$ and $\Omega \in \mathcal{F}$,
- If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$,
- If $A_1, A_2, \ldots \in \mathcal{F}$ where $A_1, A_2, \ldots$ is a countable sequence, then $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ also belong to $\mathcal{F}$.

Definition

Let $\mathcal{M}$ be a collection of subsets of Ω. The smallest σ-algebra containing $\mathcal{M}$ is denoted by

$$\sigma(\mathcal{M}) := \bigcap_{\mathcal{F} \supseteq \mathcal{M}} \mathcal{F} \quad | \mathcal{F} \quad \sigma - \text{algebra}.$$

Note that the intersection of σ-algebras is still a σ-algebra.

Events and Probability

Example

The Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ is the σ-algebra generated by the Euclidian topology of $\mathbb{R}^N$, i.e.,

$$\mathcal{B}(\mathbb{R}^N) = \sigma(\{A \mid A \text{ open set in } \mathbb{R}^N\}). \tag{3}$$

Note that

$$\mathcal{B}(\mathbb{R}^N) = \sigma(\mathcal{I}) = \sigma(\mathcal{J}),$$

where $\mathcal{I} = (\{]a, b[\mid a, b \in \mathbb{Q},\ a < b\})$ and $\mathcal{J} = (\{] - \infty, b[\mid b \in \mathbb{Q}\})$

Definition

An element of a σ-algebra $\mathcal{F}$ is called a measurable set.

Events and Probability

Definition

A measure $\mathbb{P}$ on $\mathcal{F}$ is a map $\mathbb{P} : \mathcal{F} \to \mathbb{R}^+$, such that:

- $\mathbb{P}(\emptyset) = 0$;
- $\mathbb{P}$ is countably additive that is, for every sequence of pairwise disjoint elements $(A_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$, we have $\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mathbb{P}(A_n)$.

If $\mathbb{P}(\Omega) < \infty$, we say that $\mathbb{P}$ is a finite measure. Further, if $\mathbb{P}(\Omega) = 1$, then we say that $\mathbb{P}$ is a probability measure.

Definition

- A measure space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ a σ-algebra on Ω and $\mathbb{P}$ a measure on $\mathcal{F}$.
- If $\mathbb{P}$ is a probability measure then $(\Omega, \mathcal{F}, \mathbb{P})$ is called probability space and the set Ω is called sample space.
- A function $\xi : \Omega \to \mathbb{R}^N$ is $\mathcal{F}$-measurable (or, simply, measurable) if $\xi^{-1}(H) \in \mathcal{F}$ for any $H \in \mathcal{B}$.
- If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then ξ is called a random variable.

Events and Probability

Remark

- We can think of every element ω of Ω as the result of an experiment or the state of a phenomenon, e.g., price of a stock.
- An element E of $\mathcal{F}$ is also called event and $\mathbb{P}(E)$ is called the probability of the event E.
- Example: If $\Omega = \mathbb{R}_{>0} :=]0, +\infty[$ is the sample space representing the possible prices of a risky asset, then $\mathbb{P}(]a, b[)$ represents the probability that the price is greater than a and smaller than b.
- We say that $E \in \mathcal{F}$ is a negligible event (certain) if $\mathbb{P}(E) = 0$ ($\mathbb{P}(E) = 1$). We denote by $\mathcal{N}_P$ the collection of the $\mathbb{P}$-negligible events.
- It is not restrictive to assume that $\mathbb{P}$ is complete, that is for any $A \subseteq E$ with $E \in \mathcal{N}_P$, we have $A \in \mathcal{N}_P$.

Events and Probability

Example

With the definition of the probability measure, it is easy to show for $(\Omega, \mathcal{F}, \mathbb{P})$ and $E, F \in \mathcal{F}$ the following relations:

1. $\mathbb{P}(\emptyset) = 0$
2. $\mathbb{P}(E \cap F^c) = \mathbb{P}(E) - \mathbb{P}(E \cap F) \quad , \quad E, F \in \mathcal{F}$
3. $\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$
4. $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

and by induction

5. $\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mathbb{P}(E_i)$ if $E_i \cap E_j = \emptyset$, $\forall i \neq j$, and more generally, it holds that:
$$\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mathbb{P}(E_i).$$

Events and Probability

Example

Consider an experiment which consists of throwing a coin once. The event space is $\Omega = \{H, T\}$. The σ-algebra consists of the following sets $\{H\}, \{T\}, \Omega, \emptyset$. The probability is a set function and associates each event with a number between 0 and 1. If we have a fair coin, then

$$\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}, \quad \mathbb{P}(\Omega) = 1, \quad \mathbb{P}(\emptyset) = 0.$$

Example

Throw a dice once. Then, $\Omega = \{1, 2, 3, 4, 5, 6\}$. The σ-algebra has 2^6 sets such as, e.g., $\Omega, \emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{2, 4\}, \cdots$ The probability of these sets can be calculated using the probability axioms: Let $A = \{1\}$, $B = \{2, 5\}$. Then $\mathbb{P}(A) = \frac{1}{6}$ and $\mathbb{P}(B) = \mathbb{P}(\{2\}) + \mathbb{P}(\{5\}) = \frac{2}{6}$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + P(B) = \frac{1}{2}$, since $A \cap B = \emptyset$.

Random Variable

Definition

(Repetition) Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the $\mathcal{F}$-measurable function $\xi : \Omega \to \mathbb{R}$ is called a random variable.

Remark

For $B \in \mathcal{B}(\mathbb{R})$ the set $\{\omega : \xi(\omega) \in B\}$ must correspond to an event $E_B \in \mathcal{F}$, which requires that ξ can only be a random number if the inverse image $\xi^{-1}(B)$, $B \in \mathcal{B}(\mathbb{R})$, is again an event.

Definition

The σ-algebra $\sigma(\xi)$ generated by the random number $\xi : \Omega \to \mathbb{R}$ consists of all sets of the form $\{\xi \in B\}$, where $B \in \mathcal{B}(\mathbb{R})$.

Random Variable

Example

Randomly ask a person whether she/he has a brother or sister. Let $\xi \in \{0, 1\}$ such that $\xi = 0$ if the answer is no and $\xi = 1$ if it is yes. Hence, $\Omega = \{yes, no\}$, $\mathcal{F} = \{\emptyset, \Omega, \{yes\}, \{no\}\}$. Furthermore, $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$ and as an assumption we set $\mathbb{P}(\{yes\}) = 2/3$, $\mathbb{P}(\{no\}) = 1/3$. For ξ we have $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\xi \leq \infty) = \mathbb{P}(\Omega) = 1$, $\mathbb{P}(\xi = 0) = 1/3$, $\mathbb{P}(\xi = 1) = 2/3$. We have furthermore:

$$\begin{aligned}
\mathbb{P}(3 \leq \xi \leq 4) &= \mathbb{P}(\emptyset) = 0 \\
\mathbb{P}(0 \leq \xi < 1) &= \mathbb{P}(\{no\}) = 1/3 \\
\mathbb{P}(0 \leq \xi \leq 2) &= \mathbb{P}(\Omega) = 1 \\
\mathbb{P}(0 < \xi \leq 1) &= \mathbb{P}(\{yes\}) = 2/3.
\end{aligned}$$

ξ is a random variable.

Random Variable

Remark

The last example has shown that we can assign to each event a probability. However, there might be many events! So it is more convenient to construct a probability function, which entails all necessary information to determine the probability $\mathbb{P}(A)$ of some $A \in \mathcal{B}(\mathbb{R})$. This function is called the distribution function.

Definition

Every random variable $\xi : \Omega \to \mathbb{R}$ gives rise to a probability measure

$$\mathbb{P}_{\xi}(B) = \mathbb{P}\{\xi \in B\},$$

on $\mathbb{R}$ defined on the σ-algebra of Borel sets $B \in \mathcal{B}(\mathbb{R})$. We call $\mathbb{P}_{\xi}$ the distribution of ξ. The function $F_{\xi} : \mathbb{R} \to [0,1]$, defined by

$$F_{\xi}(x) = \mathbb{P}\{\xi \le x\},$$

is called the distribution function of ξ.

Random Variable

Remark

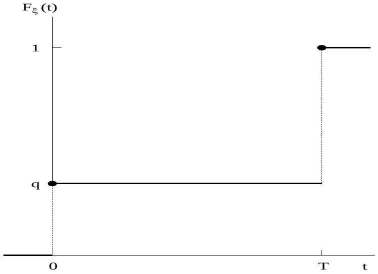
The distribution function has the properties

i) $F_\xi(\infty) = 1, \quad F_\xi(-\infty) = 0.$

ii) $x_1 \le x_2 \to F_\xi(x) \le F_\xi(x_2).$

iii) $F_\xi(x)$ right-continuous, i.e., for $\epsilon > 0$ we have $F_\xi(x) = \lim_{\epsilon \to 0} F_\xi(x+\epsilon)$

Example from credit risk: default risk assumed to be constant during $t \in [0, T]$. Default probability is q, $\Omega = \{D, ND\}$. ξ is time of default. We can plot $F_\xi(t)$ as follows:



Random Variable

Definition

If there is a Borel function $f_\xi : \mathbb{R} \to \mathbb{R}$ such that for every Borel set $B \subset \mathbb{R}$

$$\mathbb{P}\{\xi \in B\} = \int_B f_\xi(x)\, dx,$$

then ξ is said to be a random variable with absolutely continuous distribution and f_ξ is the density of ξ.

If there is a (finite or infinite) sequence of pairwise distinct real numbers $x_1, x_2, \ldots$ such that for any Borel set $B \subset \mathbb{R}$

$$\mathbb{P}\{\xi \in B\} = \sum_{x_i \in B} \mathbb{P}\{\xi = x_i\},$$

then ξ is said to have a discrete distribution with values $x_1, x_2, \ldots$ and mass $\mathbb{P}\{\xi = x_i\}$ at x_i (example: Poisson distribution)

Random Variable

Remark

- When ξ has a continuous distribution with density f_ξ, then
$$\frac{d}{dx}F_\xi(x) = f_\xi(x),$$
if f_ξ is continuous at x.
- If ξ has discrete distribution with values $x_1, x_2, \ldots,$ then F_ξ is constant on each interval $(s, t]$ not containing any of the x_i's and has jumps of size $\mathbb{P}\{\xi = x_i\}$ at x_i.

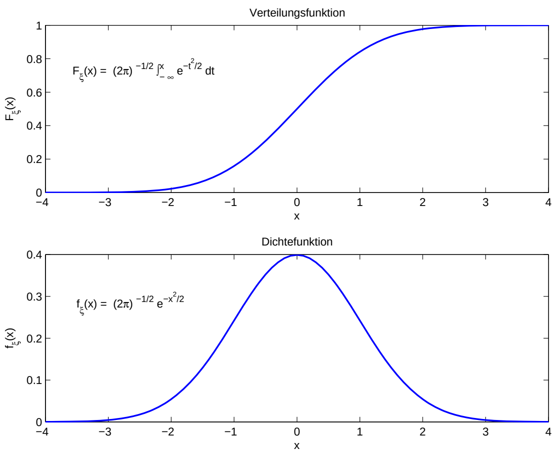
Example

The univariate standard Gaussian (normal) distribution is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

which we abbreviate $N(\mu, \sigma^2)$.

Random Variable



Random Variable

Definition

A random variable $\xi :\to \mathbb{R}$ is integrable, if $\int_\Omega |\xi| \, d\mathbb{P} < \infty$. Then

$$\mathbb{E}(\xi) = \int_\Omega \xi \, d\mathbb{P},$$

exists and is called the expectation of ξ.

Example

An indicator function $\mathbb{1}_A$ of a set A equals 1 on A and 0 on $\complement A$. For any event A $\mathbb{E}(\mathbb{1}_A) = \int_\Omega \mathbb{1}_A \, d\mathbb{P} = \mathbb{P}(A)$. We say that $\eta \; : \; \Omega \to \mathbb{R}$ is a step function if $\eta = \sum_{i=1}^n \eta_i \mathbb{1}_{A_i}$, with $\eta_1, \ldots, \eta_n$ real numbers $A_1, \ldots, A_n$ pairwise disjunct sets. Then

$$\mathbb{E}(\eta) = \sum_{i=1}^n \eta_i \int_\Omega \mathbb{1}_{A_i} \, d\mathbb{P} = \sum_{i=1}^n \eta_i \mathbb{P}(A_i).$$

Random Variable

Example

We can show that for every Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $h(\xi)$ is integrable, we have

$$\mathbb{E}(h(\xi)) = \int_{\mathbb{R}} h(x)\, d\mathbb{P}_{\xi}(x).$$

Remark

From the above example we learn:

- If ξ has continuous distribution:

$$\mathbb{E}(h(\xi)) = \int_{-\infty}^{\infty} h(x) f_{\xi}(x)\, dx.$$

- If ξ has discrete distribution with pairwise distinct values $x_1, x_2, \ldots,$

$$\mathbb{E}(h(\xi)) = \sum_i h(x_i)\mathbb{P}(\xi = x_i).$$

Random Variable

Example

1. Let the random variable ξ be normally distributed, $\xi \sim \mathcal{N}(\mu, \sigma^2)$. Calculate $\mathbb{E}(\xi)$ (Hint: $f_\xi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.
2. Calculate the $\mathbb{E}(\xi)$ under the assumption that ξ is Poisson-distributed with parameter a (Hint: $\mathbb{P}_\xi(k) = e^{-a}\frac{a^k}{k!}$).

Random Variable

Definition

A random number $\xi : \Omega \to \mathbb{R}$ is said to be square integrable if

$$\int_{\Omega} |\xi|^2 \; d\mathbb{P} < \infty.$$

Then, the function

$$\mathrm{Var}(\xi) = \int_{\Omega} (\xi - \mathbb{E}(\xi))^2 \; d\mathbb{P},$$

is called the variance of ξ.

The family of integrable and square-integrable random variables $\xi : \Omega \to \mathbb{R}$ are denoted by $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $L^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively.

Random Variable

Example

Show that a square-integrable random variable is also integrable by using the Schwarz inequality

$$(\mathbb{E}(\xi\eta))^2 \leq \mathbb{E}(\xi)^2\mathbb{E}(\eta)^2,$$

and an appropriately chosen η.

Example

Let ξ be normally distributed, $\xi \sim \mathcal{N}(0, \sigma^2)$. Calculate $\mathbb{E}(Y)$, $Y = \xi^2$.

Example

Show that if $\xi : \Omega \to \mathbb{R}$ is a non-negative square integrable random variable, then

$$\mathbb{E}(\xi^2) = 2\int_0^\infty t\mathbb{P}(\xi > t)dt.$$

(Hint: Express $\mathbb{E}(\xi^2)$ in terms of the distribution function $F_\xi(t)$ of ξ and then integrate by parts.

Conditional Expectation

Definition

Given a random variable ξ with satisfies $\mathbb{E}[|\xi|] < \infty$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some sub-sigma-algebra $\mathcal{G} \subset \mathcal{F}$, the *conditional expectation of X given $\mathcal{G}$* is defined as the almost-surely unique random variable $\mathbb{E}[X|\mathcal{G}]$ which satisfies the following two conditions:

- $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable,
- $\mathbb{E}[Y|Z] = \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]|Z]$ for every Z bounded and $\mathcal{G}$-measurable.

Properties

Conditional expectations have the following properties:

- $\mathbb{E}[aY + bZ|\mathcal{G}] = a\mathbb{E}[Y|\mathcal{G}] + b\mathbb{E}[Z|\mathcal{G}]$
- $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[Y]$
- $\mathbb{E}[YZ|\mathcal{G}] = Z\mathbb{E}[Y|\mathcal{G}]$, if Z is $\mathcal{G}$-measurable.
- $\mathbb{E}[Y|\mathcal{G}] = \mathbb{E}[Y]$, if Y is independent of $\mathcal{G}$.
- $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[Y|\mathcal{H}]$ if $\mathcal{H} \subset \mathcal{G}$.
- If $Y > 0$, then $\mathbb{E}[Y|\mathcal{G}] \geq 0$.

Filtration and Filtered Probability Space

Definition

A *stochastic process* is a function $\omega \to X(t,\omega)$ such that for any fixed t the mapping $\omega \to X(t,\omega)$ is a random variable on $(\omega, \mathcal{F}, \mathbb{P})$.

Definition

A *filtration* $(\mathcal{F}_t)$ is an increasing collection of sub-sigma algebras of $\mathcal{F}$, i.e. for $s \leq t$ we have $\mathcal{F}_s \subseteq \mathcal{F}_t$. In addition: $\mathcal{F}_\infty = \cup_{t \geq 0} \mathcal{F}_t$.

Definition

A *filtered probability space* $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$.

Definition

A process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}$ is said to be adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, if X_t is $\mathcal{F}_t$-measurable $\forall t \geq 0$.

Stochastic Calculus

Definition

Random Walk Let n be a positive integer. A binomial process can then be defined as:

1. $W_n(0) = 0$,
2. layer spacing $\frac{1}{n}$,
3. up and down movements process with $\frac{1}{\sqrt{n}}$,
4. probability measure is p $= \frac{1}{2}$.

In other words: If $\{X_1, X_2 ...\}$ is a sequence of independent binomial random variable taking values between $+1$ and -1 with equal probability then

$$W_n\left(\frac{i}{n}\right) = W_n\left(\frac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}}.$$

Stochastic Calculus

Properties of $W_n(1)$

1. W_1 can reach n+1 values,
2. The expected value of W_1 is zero,
3. The variance of $W_n(1)$ is one
   ($W_n(1)$ is a sum of n iid random variables with an expected value of zero and a variance of $\frac{1}{n}$).
4. Central-Limit-Theorem (CLT): For n $\rightarrow \infty$, the distribution function of $W_n(1)$ approximate to a normal distribution function. Or,

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right) \overset{CLT}{\Rightarrow} N(0,1)$$

   Therefore $W_n(1) \sim N(0,1)$.
5. The property of the distribution is independent of the history of the process (Markov-Property).

In fact, W_n converge to a Brownian Motion.

Stochastic Calculus

Definition

A *Brownian motion* $(W_t)_{t\geq 0}$ with respect to the probability measure $\mathbb{P}$ and to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a real-valued stochastic process such that W_t is $\mathcal{F}_t$-adapted at every time t and satisfies the following conditions:

- It starts at 0: $W_0 = 0$.
- The increments $W_t - W_s$ for all t and s such that $s \leq t$ are independent of $\mathcal{F}_s$ under $\mathbb{P}$ and follow a normal distribution: $W_t - W_s \sim \mathcal{N}(0, t-s)$.
- W has continuous trajectories , i.e. the function $t \to W_t(\omega)$ is continuous $\mathbb{P}$-almost surely for all $\omega \in \Omega$.

Stochastic Calculus

Properties of W_t

- W_t is continuous, but not differentiable (with probability 1),
- a Brownian Motion will reach each value somewhere along the way,
- a Brownian Motion is fractal,
- a Brownian Motion is also known as a Wiener Process,
- a Brownian Motion is one dimensional Gaussian Process.

Therefore a possible way to describe share price movements is,

$$S_t = \mu t + \sigma W_t$$

as Bachelier did. But, actually (Samuelson)

$$S_t = e^{(\mu t + \sigma W_t)},$$

is more appropriate.

Definition

A *martingale* with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$ is a real-valued stochastic process (M_t) such that for every time t,

- M_t is $\mathcal{F}_t$-adapted
- M is $\mathbb{P}$-integrable, i.e. $M_t \in \mathcal{L}^1(\mathbb{P})$ for all t
- For $s \leq t$:

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \mathbb{P} - \textit{almost surely}.$$

Intuition: A martingale is a fair game.

Exercise: Show that a Brownian motion is a martingale.

Stochastic Calculus

Refresher: Riemann Integral

- The main target in differential calculus is to find a rule or a procedure which leads to the gradient of a function in every point of the curve.
- The first idea to handle this problem is to express the marginal rate of the function with two points x and $x+h$ and let then h converge to zero.
- Formally:

$$m = \frac{f(x+h)-f(x)}{(x+h)-x}, \quad \lim_{h\to 0} = f'(x)$$

- Remarkably, for well-behaved functions this procedure is independent from which side we starting to calculate the limes.
- To calculate the integral of a function we can use the same rule:

$$\lim_{h\to 0} f(x+h)\left((x+h)-x\right) = \int_{-\infty}^{\infty} f(x)dx = F(x)$$

Also remarkable at this point is the fact

$$\lim_{h\to 0} f(x+h)\left((x+h)-x\right) = \lim_{h\to 0} f(x)\left((x+h)-x\right).$$

Stochastic Calculus

And in a stochastic environment?

Consider now the function

$$g(W_t) = aW_t$$

What is the integral of this function? Of course, we are tempted to use the described way above, and use therefore the Riemann-Stieltjes definition. This leads then to

$$\begin{aligned} \int_0^T aW_t dW_t &= a\left[\frac{1}{2}W_t^2\right]_0^T \\ &= \frac{1}{2}a\left(W_T^2 - W_0^2\right), \end{aligned}$$

which we will see later is wrong.

Stochastic Calculus

Definition

If f and all its derivatives exist in the region $[x, x+h]$, then

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \cdots + \frac{1}{n!}f^{(n)}(x)h^n + \cdots .$$

If f and all its derivatives up to order n exist in the region $[x, x+h]$, then it can be represented by Taylor series with Lagrange remainder

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(x)h^{n-1} + \frac{1}{n!}f^{(n)}(x^*)h^n,$$

where x^* is in $[x, x+h]$. For a function of two or more arguments the extension is obvious

$$\begin{aligned} F(x+h, y+k) &= F(x,y) + F_1(x,y)h + F_2(x,y)k \\ &+ \frac{1}{2}F_{11}(x,y)h^2 + \frac{1}{2}F_{22}(x,y)k^2 + F_{12}(x,y)hk \\ &+ \cdots + \frac{1}{n!}\left(\frac{\delta}{\delta x} + k\frac{\delta}{\delta y}\right)^n F(x,y) + \cdots . \end{aligned}$$

Stochastic Calculus

Remark

The Taylor Expansion is not only an approximation of a function. It is an exact calculus. But it can be used as a good method of approximation if we have some special constraints. Usually in deterministic calculus we are able to say that every component of the expansion that have a higher power then 2 can be ignored. That means

$$|x - x_0| > |x - x_0|^2 > |x - x_0|^3 > \dots$$

for a given x_0 in a near environment of x. Therefore in deterministic calculus is

$$f(x) \cong f(x_0 + \frac{\partial f}{\partial x_0}(x - x_0))n$$

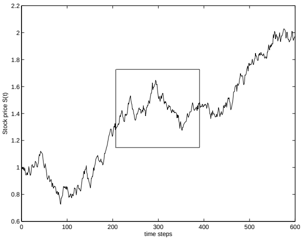
and if $(x - x_0) \to 0$ then

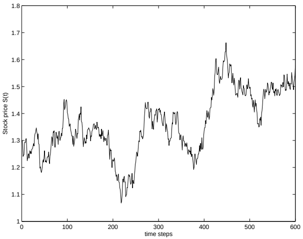
$$df(x) = \frac{\partial f}{\partial x_0} dx.$$

Stochastic Calculus

Main Objective

- Can we use the deterministic calculus concepts to solve an exercise in a stochastic environment? Answer: No, but why?





- Riemann-Stieltjes integral concept won't hold anymore. Hence, we need a new concept. Our target is to define an integral of

$$\int_{t_0}^{T} g(s)dW_s$$

for a function $g \in \mathcal{L}^2(t_0, T)$.

Stochastic Calculus

Let's start with a function g_N (simple process) which is constant on some interval parts $[t_i, t_{i+1})$. Then we define the **stochastic integral** as

$$\int_{t_0}^{T} g_N(s)dW(s) = \sum_{i=1}^{N-1} g_N(\tau_i)\left(W_{t_{i+1}} - W_{t_i}\right)$$

where $\tau_i \in [t_i, t_{i+1})$.

A general process g (not a simple process) has to comply with two other constraints:

1. the approximation of g with g_N has to be
$$\int_{t_0}^{T} \mathbb{E}\left((g_N(s) - g(s)^2\right) ds \to 0$$
2. For every n is $Z_N = \int_{t_0}^{T} g_N(s)dW_s$ a stochastic variable and it is $Z_N \to Z(\in \mathcal{L}^2)$ if $N \to \infty$.

Stochastic Calculus

Then we can define the stochastic integral as

$$\int_{t_0}^{T} g(s)dW_s = qm \lim_{N\to\infty} \int_{t_0}^{T} g_N(s)dW_s$$

where qm lim means the convergence in the quadratic mean.

A first step towards Itô Calculus

We can show that:

$$\int_{t_0}^{T} W_s dW_s = qm \lim_{N\to\infty} \sum_{i=0}^{N-1} W_{\tau_i}(W_{t_{i+1}} - W_{t_i})$$

$$= \frac{1}{2}(W_T^2 - W_{t_0}^2) + \left(\alpha - \frac{1}{2}\right)(T - t_0),$$

where $\tau_i = (1-\alpha)t_i + \alpha t_{i+1} \qquad i = 1, \ldots, N-1$ and $\alpha \in [0,1]$, that means the limes in the sense of the mean square depends on the choice of points where the limits start!

Stochastic Calculus

Proof

Let $T(N) = \{t_0 < t_1 < \ldots < t_N = T\}$ be a discrete time axle and $\delta_N = \max\{t_{i+1} - t_i\}$ the maximal interval size. Furthermore let $\{X_N\}_N$ defined as

$$
\begin{aligned}
X_N &:= \sum_{i=0}^{N-1} W_{\tau_i}(W_{t_{i+1}} - W_{t_i}) \\
&= \underbrace{\frac{1}{2}(W_{t_N}^2 - W_{t_0}^2)}_{1} - \underbrace{\frac{1}{2}\sum_{i=0}^{N-1}(W_{t_{i+1}} - W_{t_i})^2}_{2} \\
&+ \underbrace{\sum_{i=0}^{N-1}(W_{\tau_i} - W_{t_i})^2}_{3} + \underbrace{\sum_{i=0}^{N-1}(W_{t_{i+1}} - W_{\tau_i})(W_{\tau_i} - W_{t_i})}_{4}
\end{aligned}
$$

Now it is obvious that $\frac{1}{2}(W_{t_N}^2 - W_{t_0}^2)$ has to be a part of the limit value.

Stochastic Calculus

Proof (cont'd)

For $N \to \infty$ the single components converge to

1. $$\frac{1}{2}(W_{t_N}^2 - W_{t_0}^2) = \frac{1}{2}(W_T^2 - W_{t_0}^2),$$

2. $$\mathbb{E}\left[\sum_{i=0}^{N-1}(W_{t_{i+1}} - W_{t_i})^2\right] = \sum_{i=0}^{N-1} t_{i+1} - t_i = t_N - t_0 = T - t_0,$$

Stochastic Calculus

Proof (cont'd)

2.

$$
\begin{aligned}
var\left[\sum_{i=0}^{N-1}(W_{t_{i+1}} - W_{t_i})^2\right] &= \sum_{i=0}^{N-1} var\left[(W_{t_{i+1}} - W_{t_i})^2\right] \\
&= \sum_{i=1}^{N-1} 3(t_{i+1} - t_i)^2 - (t_{i+1} - t_i)^2 \\
&= \sum_{i=0}^{N-1} (t_{i+1} - t_i)^2 \\
&\leq 2\sum_{i=1}^{N-1}(t_{i+1} - t_i)\delta_N \\
&= 2(T - t_0)\delta_N \to^{N\to\infty} 0 \\
&\Rightarrow qm \lim_{N\to\infty} \sum_{i=0}^{N-1}(W_{t_{i+1}} - W_{t_i})^2 = T - t_0,
\end{aligned}
$$

Stochastic Calculus

Proof (cont'd)

3.

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{N-1}(W_{\tau_i} - W_{t_i})^2\right] &= \sum_{i=1}^{N-1}(\tau_i - t_i)^2 \\
var\left[\sum_{i=0}^{N-1}(W_{\tau_i} - W_{t_i})^2\right] &= \sum_{i=0}^{N-1}[(W_{\tau_i} - W_{t_i})] - (\tau_i - t_i)^2 \\
&= 2\sum_{i=0}^{N-1}(\tau_i - t_i)^2 \leq n(T - t_0)\delta_N \to 0 \\
&\Rightarrow qm \lim_{N\to\infty}\sum_{i=0}^{N-1}(W_{\tau_i} - W_{t_i})^2 = \sum_{i=0}^{N-1}(\tau_i - t_i),
\end{aligned}
$$

Stochastic Calculus

Proof (cont'd)

4.

$$\mathbb{E}\left[\sum_{i=0}^{N-1}(W_{t_{i+1}}-W_{\tau_i})(W_{\tau_i}-W_{t_i})\right]=0$$

$$var\left[\sum_{i=0}^{N-1}(W_{t_{i+1}}-W_{\tau_i})(W_{\tau_i}-W_{t_i})\right]$$

$$=\sum_{i=0}^{N-1}\mathbb{E}\left[(W_{t_{i+1}}-W_{\tau_i})^2(W_{\tau_i}-W_{t_i})^2\right]$$

$$=\sum_{i=1}^{N-1}(t_{i+1}-\tau_i)(\tau_i-t_i)\leq(T-t_0)\delta_N\to 0$$

$$\Rightarrow qm\lim_{N\to\infty}\sum_{i=1}^{N-1}(W_{t_{i+1}}-W_{\tau_i})(W_{\tau_i}-W_{t_i})=0.$$

Stochastic Calculus

Therefore we can write

$$qm \lim_{N\to\infty} \sum_{i=0}^{N-1} W_{\tau_i} \left(W_{t_{i+1}} - W_{t_i}\right)$$

$$= qm \lim_{N\to\infty} X_N = \frac{1}{2}\left(W_T^2 - W_{t_0}^2\right) - \frac{1}{2}(T - t_0) + \sum_{i=0}^{N-1}(\tau_i - t_i).$$

Since $\tau_i = \alpha t_{i+1} + (1-\alpha)t_i$, we can simplify to

$$qm \lim_{N\to\infty} X_N = \frac{1}{2}\left(W_T^2 - W_{t_{t_0}}^2\right) + \left(\alpha - \frac{1}{2}\right)(T - t_0).$$

Stochastic Calculus

Definition

For $a = 0$, e.g. $\tau_i = t_i$ and $i = 0, 1, \ldots, N-1$, which means that the inter points are as same as the left interval limit, is

$$\int_{t_0}^{T} W_u dW_u = \frac{1}{2}\left(W_T^2 - W_{t_0}^2\right) - \frac{1}{2}(T - t_0)$$

and is said to be the **Itô-Integral**. In a more special case $t_0 = 0$ and $W_0 = 0$ is

$$\int_{0}^{T} W_u dW_u = \frac{1}{2} W_T^2 - \frac{1}{2} T.$$

Stochastic Calculus

Direct Implications:

- $$\int_{t_0}^{T} (dW_u)^2 = \int_{t_0}^{T} dt \qquad or \qquad (dW_u)^2 = dt.$$

- From $\mathbb{E}\left[\sum_{i=0}^{N-1}(t_{i+1} - t_i)(W_{t_{i+1}} - W_{t_i})\right] = 0$ and

  $$var\left[\sum_{i=0}^{N-1}(t_{i+1} - t_i)(W_{t_{i+1}} - W_{t_i})\right] = \sum_{i=0}^{N-1}(t_{i+1} - t_i)^2 \le (T - t_0)\delta_N^2 \to 0$$

  follows that

  $$\int_{t_0}^{T} du dW_u = 0 \qquad or \qquad dt dW_u = 0.$$

- $$\int_{t_0}^{T} (du)^2 = \lim_{\delta N \to 0} \sum_{i=0}^{T} (t_{i+1} - t_i)^2 \le \lim_{\delta N \to 0} (T - t_0)\delta_N = 0$$

  or $(dt)^2 = 0$.

Stochastic Calculus

Definition

Let $f(t, x_1, \ldots, x_n) \in \mathcal{C}^{1,2}$ a function on $[t_0, T] \times \mathbb{R}^n$ and let $\{X_i(t)\}_t$ be a n-dimensional stochastic process, which is defined by stochastic differential equations as

$$dX_i(t) = a_i(t, X_i(t))dt + b_i(t, X(t_i))dW_t \qquad i = 1, \ldots, n.$$

Then, the process $f(t, X_1(t), \ldots, X_n(t))$ is also a solution of the stochastic integral equation and

$$df = f_t dt + \sum_{i=1}^{n} f_{x_i} dX_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{x_i x_j} dX_i dX_j,$$

where $dX_i dX_j = b_i b_j dt$, $\forall i, j \leq n$ or

x	dW	dt
dW	dt	0
dt	0	0

Stochastic Calculus

An explicit application...

- Given the geometric Brownian Motion process

$$dS = \mu S dt + \sigma S dW_t \tag{4}$$

we look for S_t.

- Rewrite and integrate:

$$\int_0^t \frac{dS}{S} = \mu t + \sigma W_t. \tag{5}$$

- In standard calculus we know the solution for

$$\int_0^t \frac{dS}{S} = lnS. \tag{6}$$

Hence, our best 'guess' for a function is $\ln(S)$.

- We therefore write $f(S) = ln(S)$ and use Itô's lemma to find

$$df = \frac{1}{S}(dS) + \frac{1}{2}\frac{1}{S^2}(dS)^2 \tag{7}$$

Stochastic Calculus

- Since $(dS)^2 = \sigma^2 S^2 dt$,

$$\int_0^t df = \int_0^t dln(S) = \int_0^t \frac{1}{S}(dS) - \int_0^t \frac{1}{2}\sigma^2 dt.$$

- Hence

$$\int_0^t dln(S) = \mu t + \sigma W_t - \frac{1}{2}\sigma^2 t$$

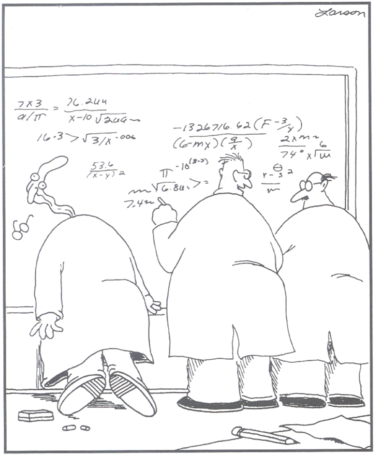
and therefore

$$ln(S) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + ln(S_0)$$

or,

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

Outlook



"Ha! Webster's blown his cerebral cortex."

Financial Engineering - Lecture 2

The Binomial Model

Markus Leippold University of Zurich
Chris Bardgett University of Zurich
Elise Gourier University of Zurich

29.09.2011

Q&A

- **How to get Matlab?**
  For students of the UZH, a student version can be downloaded from `http://www.id.uzh.ch/dl/sw/angebote.html`.
- **Registration**
  To get credits everyone needs to be registered (Kanzlei). For people with a special curriculum (PhD...), check with your program administration if the credits are accepted.
- **Slides, exercise sheets and solutions** are put online: `http://www.bf.uzh.ch/cms/studium/_248_1408.html`. If you have problems displaying this page, close your browser and re-open it.
- **Room** next week.

Outline

Most Fundamental concept

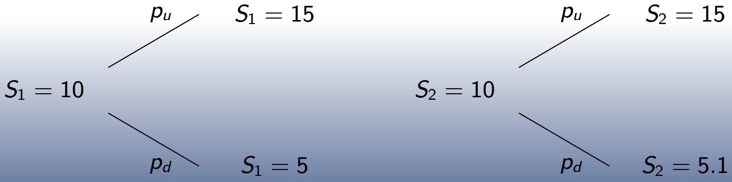
All pricing models are based on the **no arbitrage assumption**.

No arbitrage

No arbitrage means it is **not possible** to build a portfolio P, s.t.

- The value at time $t = 0$ is zero: $V_0(P) = 0$,
- The value at some time in the future $T > 0$ can be positive: $\mathbb{P}(V_T(P) > 0) > 0$ and cannot be negative $V_T(P) \geq 0$.

Example:



Usual assumptions

No market frictions

- Can buy/sell any fraction of shares,
- Can buy/sell unlimited amounts of shares (e.g. short sells),
- No bid/ask spread,
- No transaction costs.

We will always assume: **no arbitrage + no market frictions**.

Consequence: Law of one price (1)

Law of one price

If two portfolios P_1 and P_2 have the same value $V_T(P_1) = V_T(P_2)$ at some time in the future $T > 0$, then

$$\forall t \leq T, \quad V_t(P_1) = V_t(P_2).$$

Example: **Put-call parity (relative valuation put/call)**

- P_1: long K bonds and one call, short one put, $V_t(P1) = KB_t - P_t + C_t$,
- P_2: long one share, $V_t(P2) = S_t$.

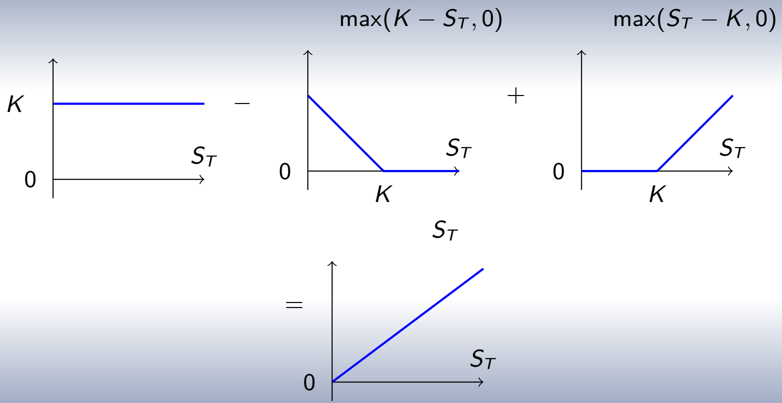
At maturity of options and bond

$$V_T(P1) = K - \max(K - S_T, 0) + \max(S_T - K, 0) = S_T = V_T(P2),$$

Therefore, P_1 and P_2 have same value at all times

$$KB_t - P_t + C_t = S_t.$$

Law of one price (2)



Static replication

Static replication

Find the value $V_t(P)$ of a portfolio P by building another portfolio at t (no rebalancing) $\tilde{P}$ such that $V_T(P) = V_T(\tilde{P})$ at some point in time in the future T. Then,

$$\forall t \leq T, \quad V_t(P) = V_t(\tilde{P}).$$

$\tilde{P}$ is called a (static) **replicating** portfolio.

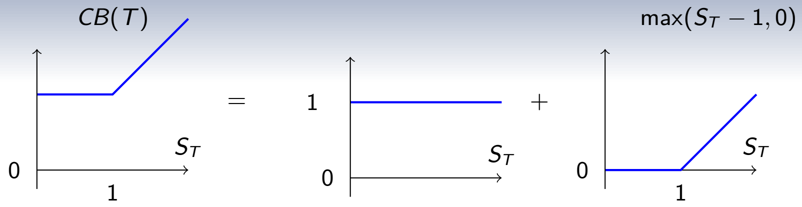
- **Model independent** (free of proba assumptions, still assume no frictions),
- **In general not possible to find** a static replicating portfolio.

Example: Convertible bond valuation $CB(t)$

- maturity $T = 2$ years, zero-coupon,
- conversion ratio $\kappa = 1$, European-style conversion,
- assume no default possible.

Static replication (2)

In 2 years, **choose bond or stock**: $CB(T) = \max(S_T, 1)$.



$$CB(t) = B(t) + C_1(t).$$

- B: zero coupon bond maturing in 2 years, notional $= 1$,
- C_1: European call option maturing in 2 years, strike $= 1$.

Dynamic replication

Dynamic replication

Find the value $V_t(P)$ of a portfolio P by building another portfolio $\tilde{P}$ at t and **readjusting it any time needed** such that $V_T(P) = V_T(\tilde{P})$ at some point in time in the future T. Then,

$$\forall t \leq T, \quad V_t(P) = V_t(\tilde{P}).$$

$\tilde{P}$ is called a (dynamic) **replicating** portfolio.

- **Model dependent**: probabilistic assumptions for the evolution of $\tilde{P}$, assume no frictions (more rebanlancing $\Rightarrow$ more frictions in reality),
- In **complete markets possible to find** a dynamic replicating portfolio,
- When markets are incomplete, no perfect dynamic replication and therefore no unique arbitrage-free price

Examples: Binomial model, Black-Scholes & extensions

Outline

The Binomial model: Motivation

Motivation:

No static replication of European options (and other products!)

$\Downarrow$

Need to introduce **model**

Why the binomial model?

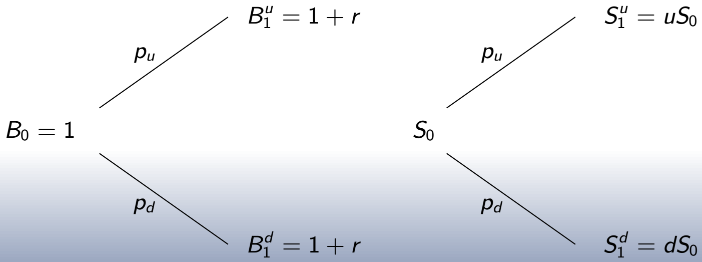
- **Discrete-time** model: easy to understand replication,
- **Converges** to the Black-Scholes continuous-time model,
- **Fast** pricing model for European/American options,
- Complete, etc.

One period: Introduction

Notation:

- S_t: stock price at time t,
- $t_0 = 0$ today, T: maturity, t: running time

Market: 1 stock, 1 bond, only **two possible states**.



Requirements: $S_0 > 0$, $r > -1$, $u > d > 0$.

One period: Definitions

Portfolio

The value $V_t(P)$ of a portfolio P composed of

- $x \in \mathbb{R}$ units of stock,
- $y \in \mathbb{R}$ units of bond,

is: $V_0 = xS_0 + yB_0 \quad , \quad V_1 = xS_1 + yB_1.$

Arbitrage portfolio

P is an *arbitrage portfolio* iff

- At time $t = 0$, $V_0 := 0$.
- At time $t = 1$, $V_1 \geq 0$, positive probability of making profits $\mathbb{P}(V_1 > 0) > 0$.

Interpretation: An **arbitrage** is a strategy where you can **make money out of nothing** $\Rightarrow$ **Build a model with No Arbitrage!**

One period: Conditions for NA

Are there **conditions** on r, u, d, p_u to enforce NA = No Arbitrage?

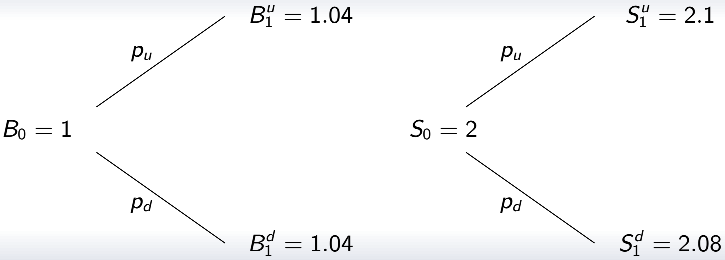
Example:

$$
\begin{tikzcd}[column sep=large]
 & B_1^u = 1.04 \\
B_0 = 1 \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\
 & B_1^d = 1.04
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large]
 & S_1^u = 2.1 \\
S_0 = 2 \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\
 & S_1^d = 2.08
\end{tikzcd}
$$

One period: Conditions for NA

Are there **conditions** on r, u, d, p_u to enforce NA $=$ No Arbitrage?

Example:



Arbitrage: Buy one stock, sell 2 bonds! Make 2 cents :-) Reason for arbitrage: **one asset outperforms the other in all states**.

One period: Conditions for NA

NA condition in the one-period binomial model

$$\text{No Arbitrage} \Leftrightarrow 0 < d < 1 + r < u$$

Proof:

($\Rightarrow$) Contraposition (i.e. not having "$0 < d < 1 + r < u$" $\Rightarrow$ Arbitrage). Since the model itself requires $0 < d < u$, we study 2 cases:

- $0 < 1 + r \leq d < u$: The bond always performs worse (or equal) than the stock $\Rightarrow$ buy the stock and short the bond.
- $0 < d < u \leq 1 + r$: The bond always performs better (or equal) than the stock $\Rightarrow$ buy the bond and short the stock.

One period: Conditions for NA

$(\Leftarrow)$ Assume $0 < d < 1 + r < u$.

Consider a portfolio P, for which $V_0(P) = 0$ and $V_t(P) = xS_t + yB_t$. We want to show that it is **not possible** to have **simultaneously** $\{V_1 \geq 0\}$ and $\{\mathbb{P}(V_1 > 0) > 0\}$.

First $V_0 = xS_0 + y = 0 \Rightarrow y = -xS_0 \Rightarrow V_1 = xS_1 + (-xS_0)B_1$.

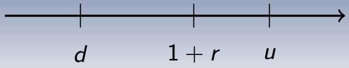
$$\textit{up state:} \quad V_1^u = xuS_0 - x(1+r)S_0 = xS_0(u - (1+r)),$$
$$\textit{down state:} \quad V_1^d = xdS_0 - x(1+r)S_0 = xS_0(d - (1+r)),$$

If $x = 0$, then $V_t \equiv 0$,

If $x \neq 0$, $0 < d < 1 + r < u$ guaranties that V_1^u **and** V_1^d **are of opposite sign** (you must loose money in some states) $\Rightarrow$ NA.

One period: Martingale measure

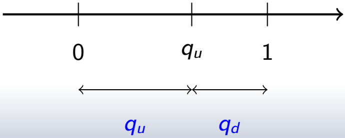
$NA \Rightarrow 1 + r \in (d, u)$:



We can write $1 + r$ as a **convex combination** of d and u

$$1 + r = q_d d + q_u u$$

where $q_u, q_d \in [0, 1]$ and $q_u + q_d = 1$.



where

$$q_u = \frac{(1 + r) - d}{u - d} \quad ; \quad q_d = \frac{u - (1 + r)}{u - d}.$$

One period: Martingale measure

$$q_u, q_d \in [0,1] \text{ and } q_u + q_d = 1$$
$$\Downarrow$$
Can interpret $\{q_d, q_u\}$ as a **new probability measure** $\mathbb{Q}$!

The measure $\mathbb{Q}$

- **does not depend on actual probabilities**, has nothing to do with the real world,
- **only depends** on the distance between $d, (1+r), u$!

Note that

$$\mathbb{E}_{\mathbb{Q}}[\frac{1}{1+r}S_1] = \frac{1}{1+r}S_0(\underbrace{q_d d + q_u u}_{=1+r}) = S_0.$$

In words: The discounted expectation under the risk-neutral measure of the stock price tomorrow is equal to the stock price today. Not true under $\mathbb{P}$!

The artificial measure $\mathbb{Q}$ is called **risk-neutral measure**.

One period: Martingale measure

Martingale in the one-period tree

Process $(X)_{t\geq 0}$ is a martingale under a probability measure $\mathbb{Q}$ iff

$$\mathbb{E}_{\mathbb{Q}}[X_1|X_0] = X_0.$$

Interpretation: If X is a martingale, the estimated value for the future value X_1 at time $t = 0$ is its current value X_0.

Example: Take $X_t := \frac{S_t}{(1+r)^t}$ and the risk neutral measure $\mathbb{Q}$. We know that $\mathbb{E}_{\mathbb{Q}}[\frac{1}{1+r}S_1] = S_0$.

$\Rightarrow$ **Discounted stock price is a martingale under risk-neutral measure**.

Martingale measure in the one-period model

The probability measure $\mathbb{Q}$ is a martingale measure iff

$$\mathbb{E}_{\mathbb{Q}}[\frac{1}{1+r}S_1|S_0] = S_0.$$

One period: FTAP

1st Fundamental theorem of asset pricing

No Arbitrage

$\Updownarrow$

There exists a martingale measure $\mathbb{Q}$ equivalent to the historical measure $\mathbb{P}$.

Remark 1: $\mathbb{P}$ and $\mathbb{Q}$ are equivalent iff for all event ω, $\mathbb{P}(\omega) = 0 \Leftrightarrow \mathbb{Q}(\omega) = 0$. For the one-period model, the only null-set is $\emptyset$.

Remark 2: This theorem is of **major importance**. It holds for all models used in practice.

Remark 3: Martingale measures are extremely useful.

- Their existence **guarantees NA**,
- They will give us the **no-arbitrage price(s) of any asset**!

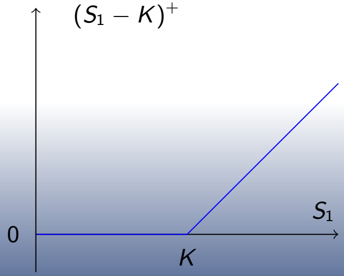
One period: Pricing European options

Aim: Pricing options!

$\longrightarrow$ **Dynamic replication through an example**.

Consider a European call option on S, $S_0 = 100$.

- Maturity: $T = 1$,
- Strike $K = 101$.



One period: Pricing European options

Given $r = 4\%$, $u = 1.1$, $d = 0.92$,

$$B_0 = 1 \quad \begin{cases} B_1^u = 1.04 \\ B_1^d = 1.04 \end{cases} \qquad\qquad S_0 = 100 \quad \begin{cases} S_1^u = 110 \\ S_1^d = 92 \end{cases}$$

Find C_0

$$C_0 = ? \quad \begin{cases} C_1^u = 9 \\ C_1^d = 0 \end{cases}$$

One period: Pricing European options

Idea: **Replicate the option using the underlying assets** at disposal.

Consider a portfolio P composed of Δ stocks and α bonds. $V_t(P)$ is characterized by

- $V_0 = \alpha + 100\Delta$,
- $V_1^u = 1.04\alpha + 110\Delta$,
- $V_1^d = 1.04\alpha + 92\Delta$.

P **replicates the option iff**

$$V_1^u = 1.04\alpha + 110\Delta = 9 = H_1^u$$
$$V_1^d = 1.04\alpha + 92\Delta = 0 = H_1^d$$

2 linearly independent equations with 2 unknowns $\Rightarrow$ **one unique solution**: $\alpha = -\frac{46}{1.04} \approx -44.23$ and $\Delta = \frac{1}{2}$.

$\Rightarrow P = \{$-44.23 bonds, 0.5 stock $\}$ replicates the option C.

Law of one price $\Rightarrow C_0 = V_0(P) = \alpha \times 1 + \Delta \times S_0 \approx 5.77$.

One period: Pricing European options

Remarks and questions:

- We have **not** used probabilities p_u, p_d,
- When a trader sells the option C, he can **hedge** his position by borrowing 44.23CHF and buy 0.5 share (worth 50CHF $= 44.23 + C_0$). Hedging eliminates all his risk (assuming the binomial model is correct),
- The **key argument** is "2 linearly **independent** equations with 2 unknowns",
- Can we price / hedge other financial instruments than European options?

One period: Contingent claims

European style contingent claim

A **European style contingent claim** is a financial product which can only be exercised at maturity $t = 1$.

Example: European put, European calls, Forwards. Not American options.

Attainable contingent claim

A European style contingent claim with payoff $H_1(S_1)$ is said to be **attainable**/reachable iff it is possible to construct a **replicating** / **hedging** portfolio with value V_t composed of the riskless asset and the stock such that $H_1 = V_1$.

Example: The previous European call option struck at $K = 101$. Hedging portfolio $P = \{$-44.23 bonds, 0.5 stock $\}$.

One period: Complete market

Complete market

A **market** is said to be **complete** when all European style contingent claims are attainable.

Completeness of the one-period binomial model

$$\text{NA} \Rightarrow \text{the market is complete.}$$

Interpretation: You can **price/hedge all European contingent claims** in the binomial model.

One period: Complete market

Proof: Assume NA. This implies $0 < d < 1 + r < u$.

Consider an arbitrary contingent claim H with payoff $H(S_1)$ at time $t = 1$.

Can we find a portfolio P composed of x units of bonds B and y shares S such that

$$V_1^u = H_1^u$$
$$V_1^d = H_1^d.$$

We have to solve the system

$$V_1^u = x(1+r) + yuS_0 = H_1^u,$$
$$V_1^d = x(1+r) + ydS_0 = H_1^d.$$

The condition $d < 1 + r < u$ guaranties that there **always exists a unique solution**

$$x = \frac{1}{1+r}\frac{uH_1^d - dH_1^u}{u-d}, \qquad y = \frac{1}{S_0}\frac{H_1^u - H_1^d}{u-d}.$$

One period: Complete market

Remarks:

- It is possible to hedge any contingent claim in this model. The **reasons are** that there are at least **two assets for two possible future states** and **the two assets are linearly independent**.
- Completeness is an attractive feature because we get a hedging portfolio for each contingent claim, BUT it also means that **contingent claims are superfluous** since we can trade in the stock/bond and replicate it.
- **In practice the market is not complete**, i.e. it is not possible to hedge any contingent claim only trading a risk-free bond and the stock. The model is overly simplistic.

One period: Martingale pricing

Martingale pricing

In the one-period binomial model, no arbitrage implies that the price H_0 of any contingent claim H_1 is given by

$$H_0 = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[H_1]$$

where Q is the martingale measure given by $q_u = \frac{(1+r)-d}{u-d}$, $q_d = \frac{u-(1+r)}{u-d}$.

Proof: The replicating portfolio for H was earlier given by

$$x = \frac{1}{1+r}\frac{uH_1^d - dH_1^u}{u-d}, \qquad y = \frac{1}{S_0}\frac{H_1^u - H_1^d}{u-d}.$$

which yields

$$H_0 = x + yS_0 = \frac{1}{1+r}\left(q_u H_1^u + q_d H_1^d\right).$$

One period: Martingale pricing

Note: KEY THEOREM.

Interpretation: Under a martingale measure, the price of **any asset** is given by its discounted payoff.

Remarks:

- This theorem **does not hold for the historical** (real world) **measure** $\mathbb{P}$, only for martingale measures,
- Even though the proof relies on the result from dynamic replication, **this theorem allows to price a contingent claim without mentioning replication**.

One period: Completeness

2nd Fundamental theorem of asset pricing

In the absence of arbitrage, the binomial model is complete iff there exists a **unique** equivalent martingale measure $\mathbb{Q}$.

Useful: This allows to **link replication** (completeness) **to the martingale measure**. It is often easier to work with martingale measures than with portfolios.

Multi-period: Motivation

Improvement: From $t_0 = 0$ to maturity T there are

- **more than one move of the underlying** stock S
- **more than 2 states in the economy**

$\Rightarrow$ **Increase the number of periods** in the binomial model.

Multi-period: Non-recombining tree

S_0

p_{u_1}: $S_1^{u_1} = u_1 S_0$

p_{u_2}: $S_2^{u_1 u_2} = u_2 S_1^{u_1} = u_2 u_1 S_0$

p_{d_2}: $S_2^{u_1 d_2} = d_2 S_1^{u_1} = d_2 u_1 S_0$

p_{d_1}: $S_1^{d_1} = d_1 S_0$

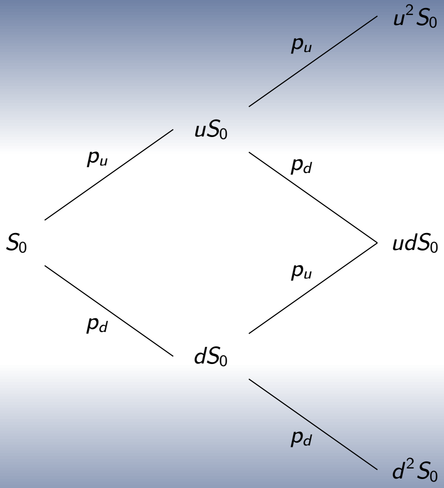
p_{u_3}: $S_2^{d_1 u_3} = u_3 S_1^{d_1} = u_3 d_1 S_0$

p_{d_3}: $S_2^{d_1 d_3} = d_3 S_1^{d_1} = d_3 d_1 S_0$

2-period: $\sum_1^2 2^i$ probabilities, $\sum_1^2 2^i$ moves $\Rightarrow$ 12 parameters.

N-period: $\sum_1^N 2^i$ probabilities, $\sum_1^N 2^i$ moves $\Rightarrow$ $2^{N+2} - 4$ parameters.

Multi-period: Recombining tree



N-period: 2 probabilities, 2 moves $\Rightarrow$ 4 parameters :-)

Multi-period: Description

- Period of interest $[t_0, T] = [0, T]$,
- Divide $[0, T]$ into N equal-length periods $[t_k, t_{k+1}]$, with $t_k = kT/N$ (for $0 \leq k \leq N$),
- 2 assets (risk-free bond, stock) with prices B_k and S_k at t_k.

It is a **collection of N one-period binomial models**.

Bond: The annual interest on the bond is r. So the one period interest rate is $\tilde{r} = rT/N$.

$$B_{k+1} = (1 + \tilde{r})B_k.$$

When $N \to \infty$, we have $B_N = \left(1 + r/\frac{N}{T}\right)^N \sim_{N\to\infty} e^{rT}$ (continuous compounding).

Multi-period: Description

Stock:

- At time t_0, $S = S_0$. $S_t = S_0$ for $t \in [t_0, t_1)$,
- At time $t = t_1$, the stock S jumps from S_0 to S_1. $S_t = S_1$ for $t \in [t_1, t_2)$, etc.

Stock distributional properties:

The **probabilistic evolution of the stock does not depend on its past**, only on its present $\mathbb{P}(S_{k+1}|S_0, ..., S_k) = \mathbb{P}(S_{k+1}|S_k)$.

- At time t_1, S_1 can take $1+1$ values: uS_0, dS_0, $\mathbb{P}(S_1^u) = p_u$, $\mathbb{P}\left(S_1^d\right) = p_d$
- At time t_2, S_2 can take $2+1$ values: $u^2 d^0 S_0, u^1 d^1 S_0, u^0 d^2 S_0$, $\mathbb{P}\left(S_2 = u^2 S_0\right) = p_u^2$, $\mathbb{P}(S_2 = udS_0) = 2p_u p_d$, $\mathbb{P}\left(S_2 = d^2 S_0\right) = p_d^2$
- At time t_N, S_N can take $N+1$ values: $u^i d^{N-i} S_0, \forall 0 \leq i \leq N$, $\mathbb{P}(S_T = S_0 u^i d^{N-i}) = \binom{N}{i} p_u^i p_d^{N-i}$.

Multi-period: Events & Filtration

Example: Two-period model. Define elementary events: $\omega_1 := \{uu\}$, $\omega_2 := \{ud\}$, $\omega_3 := \{du\}$, $\omega_4 := \{dd\}$.

Notice: $\omega_2 \neq \omega_3$ even though $S_1(\omega_2) = S_1(\omega_3)$.

Events

- $\Omega := \{\emptyset, \omega_1, \omega_2, \omega_3, \omega_4\}$,
- σ-algebra (set of all possible events) $\mathcal{F} := 2^\Omega$ (unions and complementary of Ω elements)

Filtration (**evolution of available information**): $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$

- $\mathcal{F}_0 = \{\Omega, \emptyset\}$ (at time $t = 0$, we cannot yet distinguish between any of the events),
- $\mathcal{F}_1 := \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ (at $t = 1$, we can only distinguish between events $\{S_1 = S_1^u\} = \{\omega_1, \omega_2\}$ and $\{S_1 = S_1^d\} = \{\omega_3, \omega_4\}$ but cannot yet distinguish between ω_1 and ω_2 for instance)
- $\mathcal{F}_2 = \mathcal{F}$.

Multi-period: Trading strategy

Trading strategy

A *trading strategy*/portfolio strategy $\{\mathcal{P}\}_t$ is a discrete time stochastic process that is composed of $x_k \in \mathbb{R}$ units of stock and $y_k \in \mathbb{R}$ units of bond at time k. We are only allowed to change our portfolio just after the stock price has moved, i.e. at times k^+. Making the analogy with a continuous time process, we have in more detail

$$\{\mathcal{P}\}_t = (x_t, y_t) = (x_k, y_k) \text{ for } t \in (k-1, k].$$

x_k and y_k are only allowed to depend on $S_0, ..., S_{k-1}$. The (continuous time) *value* $V_t(\mathcal{P})$ of this portfolio is therefore

$$V_0 := x_0 S_0 + y_0 B_0$$

$$V_{0^+} := x_1 S_0 + y_1 B_0 \quad , \quad V_t = x_1 S_0 + y_1 B_0 \ , \forall t \in (0,1) \quad , \quad V_1 := x_1 S_1 + y_1 B_1,$$

(the trading strategy remains unchanged on $(0, 1]$)

$$V_k := x_k S_k + y_k B_k, \ \ V_{k^+} := x_{k+1} S_k + y_{k+1} B_k, \ \ V_t = x_{k+1} S_k + y_{k+1} B_k, \ \ \forall t \in (k, k+1).$$

Multi-period: Trading strategy

In words:

- To keep things simple, we see that $V_k := x_k S_k + y_k B_k$, $\forall k = 0, \dots N$,
- V_k is the value of the portfolio with weights (x_k, y_k) with which **we arrive** at time k,
- At time k^+, we adjust our portfolio to (x_{k+1}, y_{k+1}) and cannot change it until time $k+1$.

Multi-period: Self-financing strategies

We are interested in strategies where we **do not need to inject/remove capital**.

Self-financing strategy

A trading strategy is **self-financing** if we have

$$x_k S_k + y_k B_k = V_k = V_{k+} = x_{k+1} S_k + y_{k+1} B_k \quad \forall k = 0, ..., N-1.$$

In words:

- When rebalancing our portfolio $(x_k, y_k) \to (x_{k+1}, y_{k+1})$, **the value** of our portfolio does not change,
- The value of the portfolio can only change due to movements in the assets B and S.

Multi-period: No Arbitrage

Arbitrage portfolio

A (self-financing) trading strategy (x, y) is **an arbitrage** iff

$$V_0 = 0,$$
$$P(V_N \geq 0) = 1,$$
$$P(V_N > 0) > 0.$$

NA condition in the multi-period binomial model

No arbitrage in the N-period binomial model

$\Updownarrow$

No arbitrage in each one-period sub models

$\Updownarrow$

$$0 < d < 1 + \tilde{r} < u.$$

Consequence: N-period model has **same properties** as 1-period model.

Multi-period: Martingale measure

Martingale measure in multi-period

A *martingale measure* (or risk-neutral measure) is a probability measure $\mathbb{Q}$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{1+\tilde{r}}S_{k+1}|S_k\right] = S_k \quad \forall k.$$

So, a measure $\mathbb{Q}$ is a martingale measure in the N-period model iff it is a martingale measure for each sub-period model.

Consequence: NA $\Leftrightarrow$ existence of an equivalent martingale measure $\mathbb{Q}$. From the one period model, we know that

$$q_u = \frac{(1+r)-d}{u-d} \quad ; \quad q_d = \frac{u-(1+r)}{u-d}.$$

All theorems from the one-period also hold: **completeness, unique martingale measure**.

Multi-period: Binomial algorithm

Binomial algorithm

Under no-arbitrage, the price H_k^i at time $t_k = kT/N$ of any contingent claim with payoff $H_N = H(S_N)$ at time T is given by the algorithm:

$$H_N^i = H(S_0 u^i d^{N-i}) \ , \ \forall i = 0, ..., N,$$

$$H_k^i = \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{(1+\tilde{r})} H_{k+1} \middle| S_k = S_0 u^i d^{k-i} \right] = \frac{1}{1+\tilde{r}} \left(q_u H_{k+1}^{i+1} + q_d H_{k+1}^i \right) \ ,$$

$$\forall k = N-1, ..., 0 \ , \ \forall i = 0, ..., k$$

Multi-period: Binomial algorithm

Consequence: $H_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{(1+\tilde{r})^N} H(S_T)\right]$.

At time t_N, there are $N+1$ nodes defined by the number of times i the stock goes up $\{S_N = u^i d^{N-i} S_0\}$. There are $\binom{N}{i}$ paths leading to this node. All paths are independent one another and has risk-neutral probability $q_u^i q_d^{N-i}$.

So $\mathbb{Q}(S_N = u^i d^{N-i} S_0) = \binom{N}{i} q_u^i q_d^{N-i}$

Binomial option pricing formula

The price at time $t = 0$ of a European contingent claim H with payoff $H(S_T)$ at time T is given in the multi-period binomial model by the **binomial option pricing formula**

$$H_0 = \frac{1}{(1+\tilde{r})^N} \sum_{i=0}^{N} \binom{N}{i} q_u^i q_d^{N-i} H(S_0 u^i d^{N-i}).$$

Multi-period: conversion $N \to \infty$

Focus on a **call option** C_t: maturity T, strike K

$$C_0 = \frac{1}{(1+\tilde{r})^N} \sum_{i=0}^{N} \binom{N}{i} q_u^i q_d^{N-i} \max(u^i d^{N-i} S_0 - K, 0).$$

It makes sense to **choose** $u^N S_0 > K$. There exists $p \in \{0, ..., N\}$ such that for all $i \geq p$, $u^i d^{N-i} S_0 - K > 0$.

Interpretation of p: S has to go up **at least** p times so that the option expires with a strictly positive payoff.

We can write

$$C_0 = \frac{1}{(1+\tilde{r})^N} \sum_{i=p}^{N} \binom{N}{i} q_u^i q_d^{N-i} (u^i d^{N-i} S_0 - K, 0)$$

$$= S_0 \underbrace{\sum_{i=p}^{N} \binom{N}{i} \frac{1}{(1+\tilde{r})^N} q_u^i q_d^{N-i} u^i d^{N-i}}_{:=I_1} - K \frac{1}{(1+\tilde{r})^N} \underbrace{\sum_{k=p}^{N} \binom{N}{i} q_u^i q_d^{N-i}}_{:=I_2)}.$$

Multi-period: conversion $N \to \infty$

$I_2 = \sum_{i=p}^{N} \binom{N}{i} q_u^i q_d^{N-i} = \mathbb{Q}\left(S_N \geq S_p\right) = \mathbb{Q}\left(S_N > K\right)$,

$I_1 =? \to$ Use **change of measure** technique. Define new probabilities of going up and down:

$$\tilde{q}_u := \frac{uq_u}{1+\tilde{r}} \quad , \quad \tilde{q}_d := \frac{dq_d}{1+\tilde{r}} \quad , \quad \tilde{q}_u, \tilde{q}_d > 0 \quad , \quad \tilde{q}_u + \tilde{q}_d = 1.$$

So,

$$\begin{aligned} I_1 &= \sum_{i=p}^{N} \binom{N}{i} \frac{1}{(1+\tilde{r})^N} q_u^i q_d^{N-i} u^i d^{N-i} \\ &= \sum_{i=p}^{N} \binom{N}{i} \tilde{q}_u^i \tilde{q}_d^{N-i} = \tilde{\mathbb{Q}}\left(S_N \geq S_p\right) = \tilde{\mathbb{Q}}\left(S_N > K\right). \end{aligned}$$

Finally,

$$C_0 = S_0 \tilde{\mathbb{Q}}\left(S_N > K\right) - K \frac{1}{(1+\tilde{r})^N} \mathbb{Q}\left(S_N > K\right),$$

$N \to \infty$: S_N becomes **log-normally distributed** under $\mathbb{Q}$ (provided $u, d \to 1$).

Multi-period: Calibration

Question: How do we **estimate** u, d, q_u, q_d?

Calibration **relies on convergence** of log returns $\log(S_T/S_0)$ to a $\mathcal{N}(\mu T, \sigma^2 T)$.

- Estimate σ empirically assuming $N \to \mathit{infty}$ (as if we were in the Black-Scholes model),
- In the binomial model, the variance of $\log(S_T/S_0)$ is $Nq_u(1-q_u)\left(\log\left(\frac{u}{d}\right)\right)^2$,
- We want $u, d \to 1$ for the continuous approximation to make sense. Usual choice: $u = 1/d$.

This leads to

$$u = e^{\sigma\sqrt{T/N}}$$

$$d = e^{-\sigma\sqrt{T/N}}$$

$$p_u = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{N}}.$$

Financial Engineering - Lecture 3

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

06.10.2011

University of Zurich

Q&A

- **Room**: Same room next time.
- **Mailing list**
  $\Rightarrow$ To receive emails with the room number and updates regarding exercises / slides.

History

- Mathematical model to price European options.
- Introduced by **Fischer Black** and **Myron Scholes**, and **Robert Merton** in a different paper in 1973.
- Has been extensively used in practice.
- Merton and Scholes received the **Nobel Prize** in Economics in 1997.
- **Beware of the assumptions!**

Link with the Binomial Model

Distribution of the log-returns in the Binomial Model:

Discretization $t_k = \frac{kT}{N}$. Returns can be expressed as:

$$\frac{S_T}{S_0} = \frac{S_{t_N}}{S_{t_0}} = \frac{S_{t_N}}{S_{t_{N-1}}} \frac{S_{t_{N-1}}}{S_{t_{N-2}}} \dots \frac{S_{t_2}}{S_{t_1}} \frac{S_{t_1}}{S_{t_0}}.$$

$$\log\left(\frac{S_T}{S_0}\right) = \sum_{k=1}^{N} \log\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) =: \sum_{k=1}^{N} X_k.$$

Link with the Binomial Model

Central Limit Theorem

Let $X_1, ..., X_N$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let us denote their sum by V_N:

$$V_N = \sum_{k=1}^{N} X_k.$$

Then:

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{V_N - \mu N}{\sigma\sqrt{N}} \leq z\right) = \Phi(z)$$

where Φ denotes the cumulative distribution function of the standard normal distribution, i.e. $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$.

Link to the Binomial Model

Log-returns under $\mathbb{P}$ satisfy:

$$\mathbb{E}_{\mathbb{P}}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \hat{\mu}N$$

$$Var_{\mathbb{P}}\left(\log\left(\frac{S_T}{S_0}\right)\right) = \hat{\sigma}^2 N.$$

Using the Central Limit Theorem:

$$\mathbb{P}\left(\frac{\log(S_T/S_0) - n\hat{\mu}}{\hat{\sigma}\sqrt{N}} \leq x\right) \to \Phi(x)$$

when $N \to \infty$.

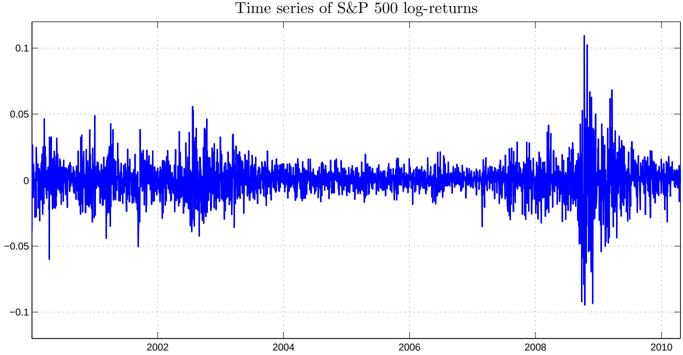
Assumptions specific to this model

These assumptions add to the usual assumptions (**no arbitrage** + **no market frictions**)!

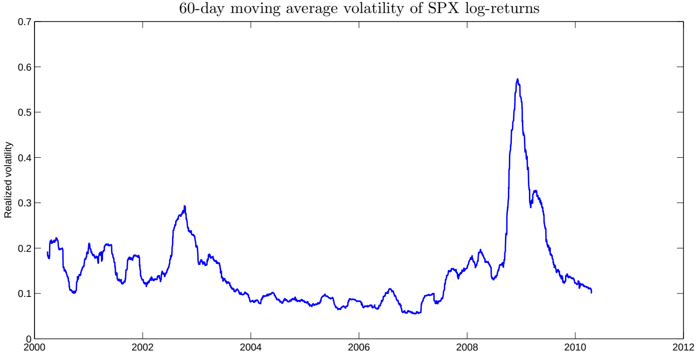
Assumptions of Black Scholes model

1. Constant and known **interest-rates**,
2. Lognormal stock prices $\Leftrightarrow$ Normal instantaneous returns,
3. Constant **volatility**,
4. One random factor, the stock price which evolves continuously.

What do stock returns look like in reality?

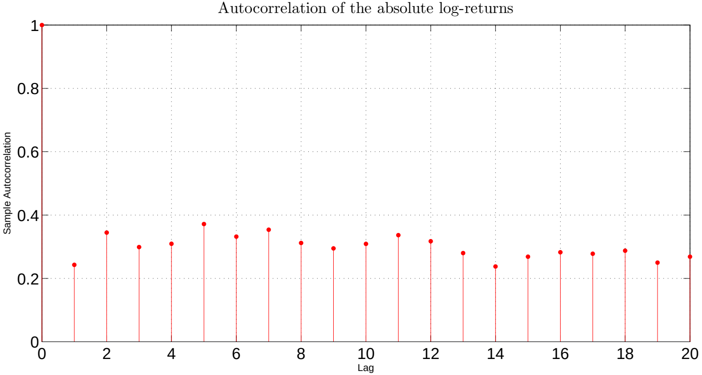


Assumption of constant volatility

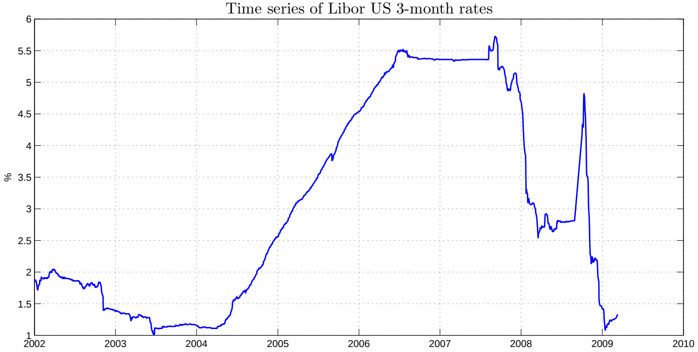


Assumption of constant volatility

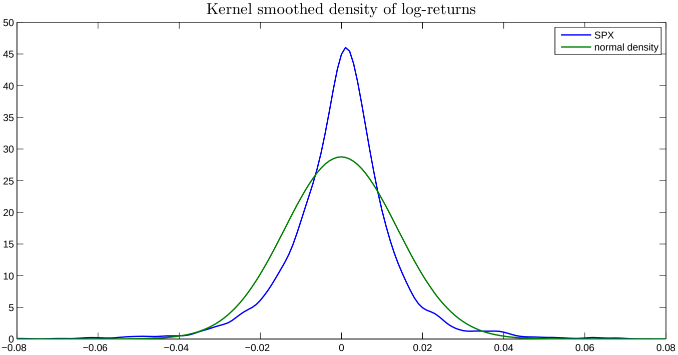
"*Large changes tend to be followed by large changes, and small changes tend to be followed by small changes*" (Mandelbrot).



Assumption of constant interest-rates



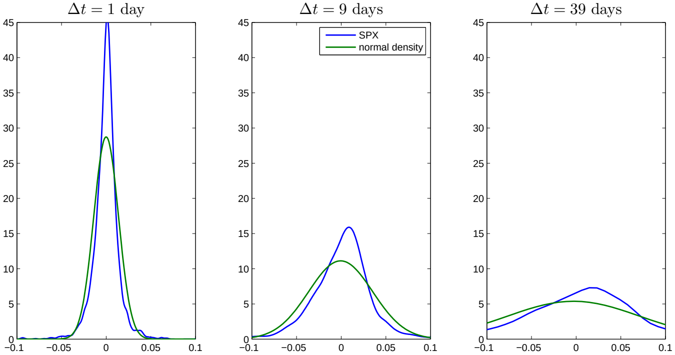
Normality of log-returns



$\Rightarrow$ The density of the log-returns of the S&P 500 is **much heavier-tailed than the normal density**.

Brief discussion of the assumptions

When the sampling frequency **decreases**, the distribution of returns looks more and more like a normal distribution.



PDE approach

Let us consider at time t a stock S_t and a riskless bond B_t which earns a constant interest r.

The stock and bond prices evolve as follows:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1}$$

$$\frac{dB_t}{B_t} = rdt \tag{2}$$

$\Rightarrow$ What do the terms represent ?

PDE approach

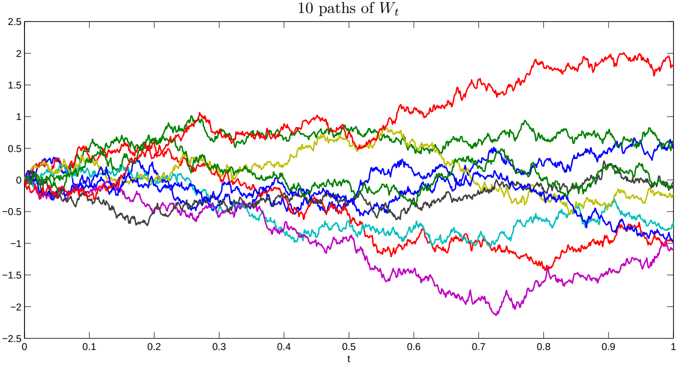
Brownian Motion

A *Brownian motion* $(W_t)_{t\geq 0}$ with respect to the probability measure $\mathbb{P}$ and to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a real-valued stochastic process such that W_t is $\mathcal{F}_t$-adapted at every time t and satisfies the following conditions:

- It starts at 0: $W_0 = 0$,
- The increments $W_t - W_s$ for all t and s such that $s \leq t$ are independent of $\mathcal{F}_s$ under $\mathbb{P}$ and follow a normal distribution: $W_t - W_s \sim \mathcal{N}(0, t-s)$.
- W has continuous trajectories , i.e. the function $t \to W_t(\omega)$ is continuous $\mathbb{P}$-almost surely for all $\omega \in \Omega$.

Remark: W_t is nowhere differentiable.

PDE approach



PDE approach

S_t as defined in 1 is called **Geometric Brownian Motion**.

Exercise: Show that $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$.

Hint: Itô's lemma...

Itô's lemma

When to use it? Every time you want to derive the dynamics of a function of a stochastic process $f(X_t, t)$!

Itô's formula for Brownian motions

Let $W = (W_t)_{t \geq 0}$ be a Brownian motion and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be twice differentiable in w and once differentiable in t. With probability 1, we have for all $t \geq 0$:

$$\begin{aligned} f(t, W_t) =& f(0, W_0) + \int_0^t \frac{\partial f}{\partial w}(s, W_s) dW_s + \\ & \int_0^t \frac{\partial f}{\partial t}(s, W_s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial w^2}(s, W_s) ds. \end{aligned}$$

or equivalently:

$$df(t, W_t) = \frac{\partial f}{\partial w}(t, W_t) dW_t + \frac{\partial f}{\partial t}(t, W_t) dt + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(t, W_t) dt.$$

Expression of S_t

With $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} = f(t, W_t)$ we have:

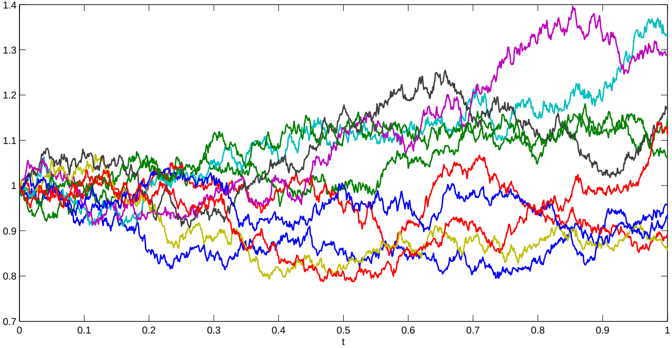
- $\dfrac{\partial f}{\partial t}(t, W_t) = (\mu - \dfrac{\sigma^2}{2}) S_t$,
- $\dfrac{\partial f}{\partial w}(t, W_t) = \sigma S_t$ and
- $\dfrac{\partial^2 f}{\partial w^2}(t, W_t) = \sigma^2 S_t$.

Hence:

$$
\begin{aligned}
dS_t &= (\mu - \frac{\sigma^2}{2}) S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 S_t dt \\
&= S_t(\mu dt + \sigma dW_t).
\end{aligned}
$$

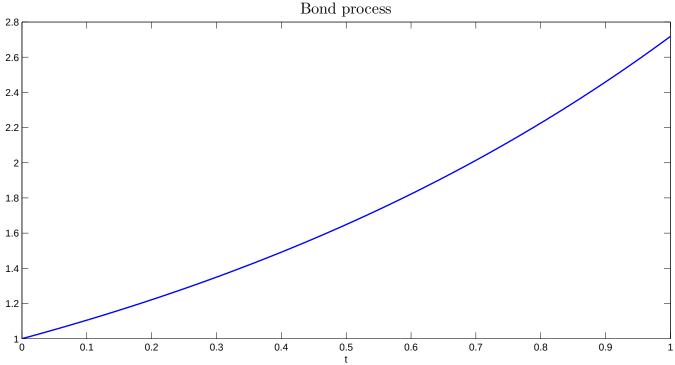
Geometric Brownian Motion

10 paths of S_t with $\mu = 5\%$ and $\sigma = 20\%$



Bond process

No random part: $B_t = e^{rt}$.



Back to PDE approach

Let us consider a call option with price C_t.

Idea: Using static replication, invest in the underlying to hedge the risk.

$\Rightarrow$ Dynamics of the option price $C_t = C(t, S_t)$?

Itô's lemma

Using Itô's lemma again :-)

Itô's formula in the Black-Scholes framework

Let $S = (S_t)_{t \geq 0}$ be a Geometric Brownian motion with drift μ and diffusion σ and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be twice differentiable in S and once differentiable in t. With probability 1, we have for all $t \geq 0$:

$$\begin{aligned} f(t, S_t) =& f(0, S_0) + \int_0^t \frac{\partial f}{\partial S}(s, S_s) dS_s + \\ & \int_0^t \frac{\partial f}{\partial t}(s, S_s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial S^2}(s, S_s) \sigma^2 S_s^2 ds. \end{aligned}$$

or equivalently:

$$df(t, S_t) = \frac{\partial f}{\partial S}(t, S_t) dS_t + \frac{\partial f}{\partial t}(t, S_t) dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(t, S_t) \sigma^2 S_t^2 dt.$$

Derivation of the option price dynamics

$$
\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}\sigma^2 S_t^2 dt \\
&= \left[\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}\sigma^2 S_t^2\right] dt + \frac{\partial C_t}{\partial S}\sigma S_t dW_t
\end{aligned}
$$

Idea: Construct a risk-free portfolio π with one option and α units of stock.

$$
\begin{aligned}
d\pi_t &= \alpha_t dS_t + dC_t \\
&= \left(\alpha_t \mu S_t + \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}\sigma^2 S_t^2\right) dt + \left(\alpha_t \sigma S_t + \frac{\partial C_t}{\partial S}\sigma S_t\right) dW_t
\end{aligned}
\tag{3}
$$

Construction of a replicating portfolio

Risk-free $\Rightarrow$ Should have the same dynamics as the bond:

$$d\pi_t = \pi_t r dt. \tag{4}$$

- Term in $W_t = 0$:

$$\alpha_t = -\frac{\partial C_t}{\partial S} =: -\Delta_t \tag{5}$$

$\Rightarrow$ Like in the Binomial Model, to replicate the payoff / hedge the risk of the option one needs to sell Δ units of the underlying.

$\Rightarrow$ Delta-hedging ... we will see this later!

Construction of a replicating portfolio 2

- Term in dt:

$$\frac{\partial C_t}{\partial t} + rS_t \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S^2} = rC_t. \tag{6}$$

- Black-Scholes PDE satisfied by all derivatives on the underlying S (under the Black-Scholes assumptions).
- Terminal condition for a call option:

$$C(S_T, T) = \max(0, S_T - K)$$

Role of risk preferences

Important remark: The Black-Scholes PDE does not involve the drift term μ.

"*The warrant value did not depend on the stock's expected return, or on any other asset's expected return. That fascinated me.*" (Black)

Equity risk premium

The term $\frac{\mu - r}{\sigma}$ is called the equity risk premium.

$\Rightarrow$ Reflects the expectations of investors, who want a higher return when they take a higher risk

BUT...

Complete market

In the Black-Scholes model the market is **complete**, i.e. one can replicate any instrument using the underlying and the bond.

$\Rightarrow$ Possibility to hedge the risk of options.

$\Rightarrow$ Options are therefore priced using only the risk-free rate!

How to solve the Black-Scholes PDE

- Changes of variables: $T' = \frac{1}{2}\sigma^2(T-t)\ ;\ x = \log S$
- Change of objective function:

$$H(x,t) = e^{\alpha x + \beta T'} C(x, T') \tag{7}$$

with

$$\alpha = \frac{m-1}{2} \quad ; \quad \beta = \frac{(m+1)^2}{4} \qquad m = \frac{r}{\frac{1}{2}\sigma^2}$$

H satisfies:

$$\frac{\partial H}{\partial T'}(x, T') - \frac{\partial^2 H}{\partial x^2}(x, T') = 0 \tag{8}$$

which is the **heat equation**.

How to solve the Black-Scholes PDE

Boundary condition:

$$H(x,0) = e^{\frac{1}{2}(m-1)x}(e^x - K)^+ \tag{9}$$

But in order to get a solution we first first to solve the equation with a simpler boundary condition:

Fundamental solution

The *Fundamental Solution F* is the solution of the problem:

$$\frac{\partial H^0}{\partial T'}(x,T') - \frac{\partial^2 H^0}{\partial x^2}(x,T') = 0$$
$$H^0(x,0) = \delta(x)$$

$\delta(x)$ is the Dirac function ($\delta(0) = \infty$ and $\delta(x) = 0\ \forall x \neq 0$).

How to solve the Black-Scholes PDE

Exercise: Show that $H_0 = \frac{1}{\sqrt{4\pi T'}} e^{-\frac{x'}{4T'}}$ is the fundamental solution.

Next step: How to derive the solution of the heat equation with our boundary condition from the fundamental solution

Solution to the heat equation

The solution of the general heat equation is given by the convolution of the solution of the Fundamental Solution and the boundary condition of the problem to solve.

$$H(x, T') = \int_{\mathbb{R}} H^0(x-y, T')H(y,0)dy = H^0(x, T') * H(x,0) \qquad (10)$$

How to solve the Black-Scholes PDE

Proof: Let us show that
$H(x, T') = \int_{\mathbb{R}} H^0(x-y, T')H(y,0)dy = H^0(x, T') * H(x,0)$ satisfies the heat equation:

$$\frac{\partial}{\partial T'} H(x, T') = \int_{\mathbb{R}} \frac{\partial}{\partial T'} H^0(x-y, T')H(y,0)dy$$
$$\frac{\partial^2}{\partial x^2} H(x, T') = \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} H^0(x-y, T')H(y,0)dy$$

Hence,

$$\begin{aligned} H'_T(x, T') - H_{xx}(x, T') &= \int_{\mathbb{R}} \left(\frac{\partial}{\partial T'} H^0(x-y, T') - \frac{\partial^2}{\partial x^2} H^0(x-y, T') \right) H(y,0)dy \\ &= 0 \end{aligned}$$

since H^0 is a fundamental solution ($\frac{\partial}{\partial T'} H^0(x-y, T') = \frac{\partial^2}{\partial x^2} H^0(x-y, T')$).

Let us check that the boundary condition is satisfied. We use the fact that $\lim_{T' \to 0} H^0(x, T') = \delta(x)$.

How to solve the Black-Scholes PDE

Last step: Simplify the expression of $H(x, T')$. New changes of variables give:

$$C_t = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$

with

$$d_\pm = \frac{\ln \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}.$$

$\Rightarrow$ Black-Scholes formula for call options.

PDE approach: Summary of steps

Derive the BS call price with the PDE approach

1. Write the dynamics of the call option
2. Build a portfolio π composed of α units of underlying and one option.
3. Compute α_t such that the portfolio is risk-free (term in front of the Brownian motion $= 0$)
4. Infer the Black-Scholes PDE from the return of the risk-free portfolio (same as the one of the bond)
5. Transform the PDE into the Heat Equation and write the boundary condition
6. Find a Fundamental Solution of the Heat Equation
7. The solution of the Heat Equation is the convolution of the Fundamental Solution and the boundary condition
8. After some changes of variables, get the Black-Scholes call price.

Replication of a call option

Recall: $\Delta_t = \frac{\partial C_t}{\partial S}$.

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S^2} \sigma^2 S_t^2 dt \\ &= \Delta_t dS_t + \frac{1}{r} \frac{1}{B_t} \left(\frac{\partial C_t}{\partial t} + \frac{1}{2} \frac{\partial^2 C_t}{\partial S^2} \sigma^2 S_t^2 \right) dB_t \\ &= \Delta_t dS_t + \frac{1}{r} \frac{1}{B_t} \left(rC_t - rS_t \frac{\partial C_t}{\partial S} \right) dB_t \\ &= \Delta_t dS_t + \frac{C_t - S_t \Delta_t}{B_t} dB_t \end{aligned} \tag{11}$$

$\Rightarrow$ Replicating porfolio $\mathcal{P} = \Delta_t$ units of stock and $\frac{C_t - S_t \Delta_t}{B_t}$ bonds!

Self-financing property

Self-financing portfolio

Let us consider a portfolio composed of α_t units of the underlying and β_t bonds, where $\{\alpha_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ are adapted processes. This portfolio is said to be *self-financing* if, with probability 1, for every $t \in [0, T]$,

$$\alpha_t S_t + \beta_t B_t = \alpha_0 S_0 + \beta_0 B_0 + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s. \tag{12}$$

Furthermore, the portfolio $\{(\beta_t, \alpha_t)\}_{0\le t\le T}$ *replicates* a European-style derivative security with value V_T at time T if with probability 1,

$$V_T = \alpha_T S_T + \beta_T B_T.$$

Self-financing property

Question: Is $\mathcal{P}$ self-financing? Value at time t:

$$V_t(\mathcal{P}) = \Delta_t S_t + \beta_t B_t = C_t \tag{13}$$

Denote $\beta_t = \frac{C_t - S_t \Delta_t}{B_t}$ and integrate equation (11):

$$dC_t = \Delta_t dS_t + \beta_t dB_t$$

$$\Updownarrow$$

$$C_t = C_0 + \int_0^t \Delta_s dS_s + \int_0^t \beta_s dB_s \tag{14}$$

Use (13) for $t = T$ and $t = 0$ and insert in (14):

$$\Delta_t S_t + \beta_t B_t = \Delta_0 S_0 + \beta_0 B_0 + \int_0^t \Delta_s dS_s + \int_0^t \beta_s dB_s$$

$\Rightarrow$ Self-financing portfolio!

Martingale approach

Martingale

A *martingale* with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$ is a real-valued stochastic process (M_t) such that for every time t,

- M_t is $\mathcal{F}_t$-adapted
- M is $\mathbb{P}$-integrable, i.e. $M_t \in \mathcal{L}^1(\mathbb{P})$ for all t
- For $s \leq t$:

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \;\; \mathbb{P} - \textit{almost surely}.$$

Intuition: Fair game.

Martingale approach

First Fundamental Theorem of Asset Pricing (FTAP)

No arbitrage $\Leftrightarrow$ There exists a probability measure $\mathbb{Q}$ with $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$ such that the discounted price process of every tradable asset is a $\mathbb{Q}$-martingale.

Martingale measure

Such a measure $\mathbb{Q}$ is called *Equivalent Martingale Measure* or *risk-neutral measure*.

Example of martingales: discounted stock price, discounted option price...

Call option valuation

Consequence of the FTAP

The value of any derivative can be calculated by discounting its final payoff under the risk-neutral measure.

- No need to know the investors' expectations / risk preferences (cf Black-Scholes PDE)
- Possibility to calculate the fair price of an asset by discounting under the historical measure, using a risk-adjusted interest-rate (real options).

Call option valuation

Hence the call option price can be written as:

$$
\begin{aligned}
C_t &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}C_T|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T>K|\mathcal{F}_t)
\end{aligned}
$$

$\Rightarrow$ Need to calculate $\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t]$ and $\mathbb{Q}(S_T>K|\mathcal{F}_t)$.

Expression of S_t under $\mathbb{Q}$

Under $\mathbb{P}$:

$$\frac{dS_t}{S_t} = \mu dt + \sigma d\tilde{W}_t$$

Question: Dynamics of S_t under $\mathbb{Q}$?

Change of measure from $\mathbb{P}$ to $\mathbb{Q}$

If W_t is a Brownian motion under $\mathbb{P}$, then $\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$ is a Brownian motion under $\mathbb{Q}$.

Expression of S_t under $\mathbb{Q}$

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t \\ &= rdt + \sigma\left(\frac{\mu - r}{\sigma}dt + dW_t\right) \\ &= rdt + \sigma d\tilde{W}_t.\end{aligned}$$

$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$ Brownian motion under $\mathbb{Q}$.

- Change of probability measure: changes the probability of paths so that $\mathbb{E}^Q\left[\frac{dS_t}{S_t}\right] = rdt \Rightarrow$ Risk-neutral investor.
- Itô's formula gives the expression of S_t under $\mathbb{Q}$:

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t} \quad \forall t \in [0, T]$$

Back to the call option price

$$
\begin{aligned}
\mathbb{Q}(S_T > K|\mathcal{F}_t) &= \mathbb{Q}(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)} > K|\mathcal{F}_t) \\
&= \mathbb{Q}\left(\tilde{W}_T - \tilde{W}_t > \frac{\ln\left(\frac{K}{S_t}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma}|\mathcal{F}_t\right) \\
&= 1 - \mathbb{Q}\left(\tilde{W}_T - \tilde{W}_t \leq \frac{\ln\left(\frac{K}{S_t}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}|\mathcal{F}_t\right) \\
&= 1 - \Phi\left(\frac{\ln\left(\frac{K}{S_t}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi\left(-\frac{\ln\left(\frac{K}{S_t}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi\left(\frac{\ln\left(\frac{S_t}{K}\right) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi(d_2)
\end{aligned}
$$

Back to the call option price

Recall:

$$C_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K|\mathcal{F}_t)$$

Remaining term to calculate:

$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\mathbb{E}^{\mathbb{Q}}[e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)}\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t]$$

Problem: $e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)}$ and $\mathbf{1}_{\{S_T>K\}}$ are not independent...

Change of measure and Girsanov theorem

Define:

$$Z_t = e^{-\frac{\sigma^2}{2}t+\sigma\tilde{W}_t} = \exp\left(\sigma\tilde{W}_t - \frac{1}{2}[\sigma\tilde{W}_t]_t\right) = \mathcal{E}(\sigma\tilde{W}_t).$$

- $\mathcal{E}(\sigma\tilde{W}_t)$ Doléans exponential (or stochastic exponential) of $\sigma\tilde{W}_t$ or stochastic exponential
- $[\sigma\tilde{W}_t]_t$ quadratic variation of $\sigma\tilde{W}_t$:

$$[\sigma\tilde{W}]_t = \sigma^2[\tilde{W}]_t = \sigma^2 t.$$

Change of measure and Girsanov theorem

Girsanov theorem for Brownian motions

Let us consider $Z_t = \mathcal{E}(X)_t$ where X_t is $\sigma \tilde{W}_t$. Z_t is a martingale under $\mathbb{Q}$ and defines a change of measure from $\mathbb{Q}$ to $\mathbb{Q}^*$, with:

$$\left.\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right|_T = Z_T$$

Z_t is called Radon-Nykodym derivative of the change of measure from $\mathbb{Q}$ to $\mathbb{Q}^*$. Furthermore, as $\tilde{W}_t$ is a standard Brownian motion under $\mathbb{Q}$, under $\mathbb{Q}^*$, $W_t^* = \tilde{W}_t - \sigma t$ is a standard Brownian motion.

Finally, for every $\mathcal{F}_t$ measurable and integrable random variable X we have:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{Z_T}{Z_t} X | \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^*}[X|\mathcal{F}_t].$$

Back to the calculation of the call option price

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}[S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\tilde{W}_t-\tilde{W}_t)} \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t e^{r(T-t)} \mathbb{E}^{\mathbb{Q}}[\frac{Z_T}{Z_t} \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t e^{r(T-t)} \mathbb{E}^{\mathbb{Q}^*}[\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t e^{r(T-t)} \mathbb{Q}^*(S_T > K|\mathcal{F}_t)
\end{aligned}
$$

$\Rightarrow$ Need to calculate $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$.

Back to the calculation of the call option price

Same procedure as before: to calculate $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$:

- Write down the dynamics of S_t under $\mathbb{Q}^*$
- Use Itô's formula to find the expression of S_T as a function of S_t
- Replace in $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$ and use the normality of the Brownian motion

Back to the calculation of the call option price

Dynamics of S_t under $\mathbb{Q}^*$ given that $W_t^* = \tilde{W}_t - \sigma t$ is a $\mathbb{Q}^*$-Brownian motion:

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \sigma d\tilde{W}_t \\ &= rdt + \sigma dW_t^* + \sigma^2 dt \\ &= (r + \sigma^2)dt + \sigma dW_t^*\end{aligned}$$

Itô's lemma gives:

$$S_T = S_t e^{(r+\frac{\sigma^2}{2})(T-t)+\sigma(W_T^* - W_t^*)}$$

Back to the calculation of the call option price

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] &= S_t e^{r(T-t)}\mathbb{Q}^*(S_t e^{(r+\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)} > K|\mathcal{F}_t) \\
&= S_t e^{r(T-t)}\mathbb{Q}^*\left(\frac{(W_T^*-W_t^*)}{\sqrt{T-t}} > \frac{\ln\left(\frac{K}{S_t}\right)-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}|\mathcal{F}_t\right) \\
&= S_t e^{r(T-t)}\left(1-\mathbb{Q}^*\left(\ldots \le \frac{\ln\left(\frac{K}{S_t}\right)-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}|\mathcal{F}_t\right)\right) \\
&= S_t e^{r(T-t)}\left(1-\Phi\left(\frac{\ln\left(\frac{K}{S_t}\right)-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= S_t e^{r(T-t)}\Phi\left(-\frac{\ln\left(\frac{K}{S_t}\right)-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= S_t e^{r(T-t)}\Phi\left(\frac{\ln\left(\frac{S_t}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= S_t e^{r(T-t)}\Phi(d_1)
\end{aligned}
$$

Back to the calculation of the call option price

Putting everything together we get the Black-Scholes formula for call options:

$$C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{15}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{16}$$

For put options:

$$P_t = K e^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1).$$

Martingale approach: Summary of steps

Derive the Black-Scholes price of an option with the martingale approach

1. Apply the FTAP to write the call option as the discounted payoff under $\mathbb{Q}$
2. Decompose the price into
$$C_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K|\mathcal{F}_t)$$
3. Solve for $\mathbb{Q}(S_T > K|\mathcal{F}_t)$ by writing the dynamics of S_t under $\mathbb{Q}$ and using the normality of the Brownian motion.
4. Using a change of measure from $\mathbb{Q}$ to $\mathbb{Q}^*$ write $\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t]$ as:
$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$$
5. Solve for $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$ (same procedure as step 3).

From the PDE approach to the martingale approach

Feyman-Kac formula

Let S_t be the price of a stock, which satisfies the following SDE under the probability measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = rdt + \sigma d\tilde{W}_t$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. Then the Black-Scholes PDE (6) replacing C_t by the more general derivative value V_t with payoff $h(S_T)$) has the following solution:

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}h(S_T)|\mathcal{F}_t].$$

Financial Engineering - Lecture 4

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

06.10.2011

University of Zurich

Black-Scholes model with dividends

Recall from last week: two approaches to value options in the Black-Scholes framework:

1. **The PDE approach**
   - Construct a portfolio such that the risk of the option is hedged
   - The expected return of this portfolio is the risk-free rate $\rightarrow$ Black-Scholes PDE
   - Transform into the Heat Equation and solve.
2. **The martingale approach**
   - Use the FTAP to write the option price as discounted expected payoff under $\mathbb{Q}$
   - Use Itô's formula and Girsanov theorem to solve.

Black-Scholes model with dividends

What changes here? Almost nothing :-)

Under $\mathbb{P}$, the stock evolves as follows:

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma dW_t$$

q continuous dividend yield.

Remark: In practice: cash dividends but...

- Unknown amount, based on the success of the company
- Uncertainty about the date.

Black-Scholes model with dividends

When did Microsoft pay out dividends?

Figure: Stock price of Microsoft

Black-Scholes model with dividends

Itô's formula:

$$dC_t = \left[\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}(\mu - q)S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}\sigma^2 S_t^2\right] dt + \frac{\partial C_t}{\partial S}\sigma S_t dW_t.$$

Like last week, we build a portfolio π with α units of stock and 1 option:

$$\begin{aligned} d\pi_t = \quad & \alpha_t dS_t + dC_t + \alpha_t q S_t dt \\ = \quad & \left(\alpha_t(\mu - q)S_t + \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}(\mu - q)S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}\sigma^2 S_t^2 + \alpha_t q S_t\right) dt + \\ & \left(\alpha_t \sigma S_t + \frac{\partial C_t}{\partial S}\sigma S_t\right) dW_t. \end{aligned}$$

Black-Scholes model with dividends

- Risk-free portfolio $\rightarrow$ no random term:

$$\alpha_t = -\frac{\partial C_t}{\partial S}$$

- Expected return $= r \Rightarrow$ Black-Scholes PDE with dividends:

$$\frac{\partial C_t}{\partial t} + (r - q)\frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S^2} = rC_t$$

Black-Scholes model with dividends

To solve the equation: similar to last week with $r - q$ instead of r. Final result:

$$C_t = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{1}$$

$$P_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1) \tag{2}$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{4}$$

Black-Scholes model with dividends

Martingale approach: Under $\mathbb{Q}$ we have

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma d\tilde{W}_t$$

Itô's formula gives the solution to the SDE:

$$S_T = S_t e^{(r-q-\frac{\sigma^2}{2})(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)}. \tag{5}$$

The FTAP gives:

$$\begin{aligned}
C_t &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T - K)\mathbf{1}_{S_T>K}|\mathcal{F}_t] \\
&= e^{-r(T-t)}\left(\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{S_T>K}|\mathcal{F}_t] - K\mathbb{Q}(S_T > K|\mathcal{F}_t)\right)
\end{aligned}$$

Replace S_T following 5 and solve like last week with $r - q$ instead of r.

Delta of an option

Delta

The *delta* of an option is the sensitivity of the option price V (call or put) with respect to changes in the underlying price:

$$\Delta_t = \frac{\partial V_t}{\partial S_t}$$

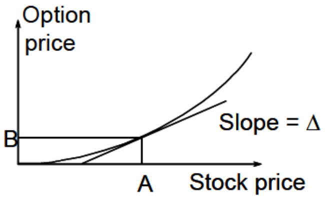


Figure: Delta of an option

Delta of an option

In Black-Scholes model:

$$\Delta_t^{BS}(Call) = e^{-q(T-t)}\Phi(d_1).$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

- $0 < \Delta_t^{BS}(Call) < 1$
- When S_t increases, $\Delta_t^{BS}(Call)$ increases (by monotonicity of the cdf)
- When $T - t \to 0$, $d_1 \to -\infty$ if $S_t < K$ and ∞ if $S_t > K$: the delta as a function of the stock price gets closer to a step function.
- Some traders use the delta as probability to end in-the-money but this is WRONG (do the math)

Delta of an option

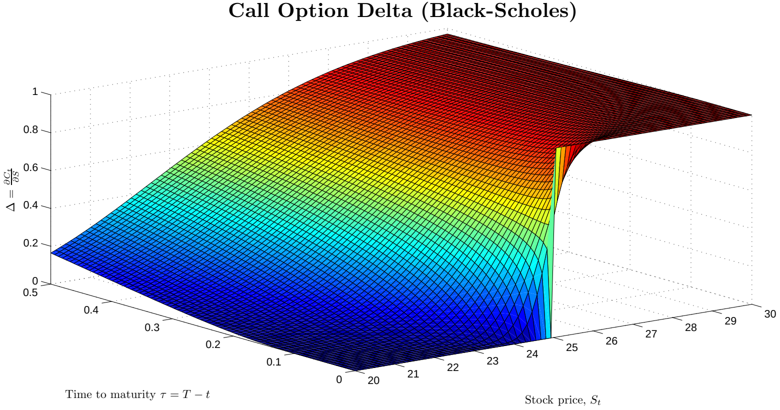


Figure: Delta of a call option as a function of the stock price and time-to-maturity

Delta-hedging

Idea: We want the total delta of a portfolio with 1 option and N shares of stock to be 0.

$$V(\mathcal{P}) = C - NS$$

$$\frac{\partial V(\mathcal{P})}{\partial S} = \frac{\partial C}{\partial S} - N = 0$$

or equivalently $N = \frac{\partial C}{\partial S} = \Delta$.

Practical issues:

- Rebalancing is costly
- Discrete trading

Gamma of an option

Gamma

The *gamma* of an option with price V is the sensitivity of the option delta with respect to the underlying price:

$$\Gamma_t = \frac{\partial^2 V_t}{\partial S_t^2}$$

- Measures the curvature / convexity of the option price.
- Put-Call parity :

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$
$$\Rightarrow \frac{\partial^2 C_t}{\partial S_t^2} - \frac{\partial^2 P_t}{\partial P_t^2} = 0$$
$$\Rightarrow \Gamma_t(Put) = \Gamma_t(Call)$$

.

Gamma of an option

In Black-Scholes framework for a call option:

$$\Gamma_t^{BS}(Call) = \frac{e^{-q(T-t)}\phi(d_1)}{S_t\sigma\sqrt{T-t}} = \frac{e^{-q(T-t)-\frac{1}{2}d_1^2}}{S_t\sigma\sqrt{2\pi(T-t)}}$$

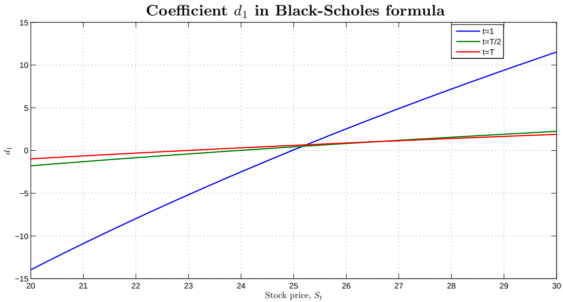


Figure: d_1 in Black-Scholes formula

Gamma of an option

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Properties of gamma:

- Non-negative (by convexity of call prices)
- When $S_t \to \infty$, $d_1 \to \infty$ hence the numerator of $\Gamma_t^{BS}(Call) \to 0$, its denominator $\to \infty$ which gives $\Gamma_t^{BS}(Call) \to 0$.
- When $S_t \to 0$, $d_1 \to -\infty$ hence the numerator of $\Gamma_t^{BS}(Call) \to 0$, the denominator also but because of the square in the numerator it dominates the denominator. Hence $S_t \to 0$.
- When $T - t \to 0$ and $S_t = K$, $d_1 \to 0$, thus the numerator of $\Gamma_t^{BS}(Call) \to 1$ and the denominator $\to 0$, thus $\Gamma_t^{BS}(Call) \to \infty$. Consequence: impossible to delta-hedge.

Gamma of an option

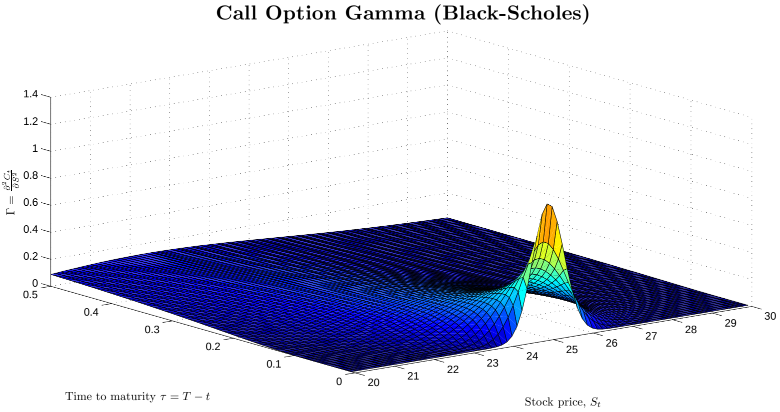


Figure: Gamma of a call option as a function of the stock price and time-to-maturity

Gamma of an option

How to relate an option price, its delta and gamma

$\rightarrow$ Taylor development.

$$
\begin{aligned}
C(S+\Delta S, t+\Delta t) = C(S,t) + & \left.\frac{\partial C}{\partial t}\right|_{S,t} \Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,t} \Delta S \\
& + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,t} \frac{(\Delta S)^2}{2} + O((\Delta S)^3) \\
= C(S,t) + & \left.\frac{\partial C}{\partial t}\right|_{S,t} \Delta t + \Delta_t \Delta S + \Gamma_t \frac{(\Delta S)^2}{2} + O((\Delta S)^3)
\end{aligned}
$$

Delta-hedging $\Leftrightarrow$ Ignoring the convexity term!

Gamma of an option

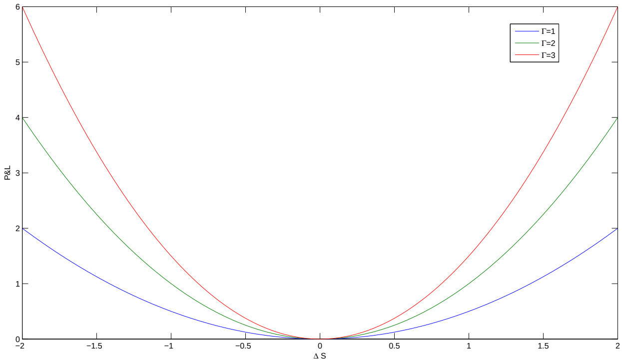


Figure: Profit and loss of a portfolio with a call long and delta units of the underlying short

Robustness of delta-hedging

Assume we sell a call option with price C_t, and delta-hedge by holding Δ_t units of S. **What happens if the Black-Scholes model is not well-specified?**

True model:

$$\frac{dS_t}{S_t} = \alpha(t,\omega)dt + \beta(t,\omega)dW_t \tag{6}$$

where α_t, β_t are $\mathcal{F}_t$- adapted bounded processes.

Adapted process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space. Then the stochastic process $(X_t)_{0\leq t\leq T}$ is said to be $\mathcal{F}_t$-adapted if for every t, X_t is $\mathcal{F}_t$-measurable.

Intuition: Cannot see into the future. Also called *non-anticipating process*.

Robustness of delta-hedging

Using equation 6, the option price satisfies:

$$dC_t = \Delta_t dS_t + \left(\frac{\partial C}{\partial t} + \frac{1}{2} \beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

If we think that the Black-Scholes model is **the right model**, to hedge ourselves:

$\Rightarrow$ Replicating portfolio $\mathcal{P}$ for an option $= \Delta_t$ units of S and $\frac{C_t - S_t \Delta_t}{B_t}$ bonds.

$$dV_t(\mathcal{P}) = \Delta_t dS_t + (V_t(\mathcal{P}) - \Delta_t S_t) r dt.$$

How wrong is our hedge?

Hedging error

Hedging error $=$ price of the replicating portfolio - option price

$$V_t(\mathcal{P}) - C_t =: Z_t.$$

Robustness of delta-hedging

Implied volatility

The *implied volatility* of an option is the value of the volatility parameter in the Black Scholes equation that makes the value of the option $C_t^{BS}(\sigma_{imp})$ in that model match the market price C_t^{obs} of the option:

$$C_t^{BS}(\sigma_{imp}) = C_t^{obs}$$

- Value of the volatility to use to "make the Black-Scholes formula" right (more details in the next lecture)
- Bijection with the stock price $\Rightarrow$ Traders quote one or the other.

Robustness of delta-hedging

The hedging error satisfies:

$$
\begin{aligned}
dZ_t &= dV_t(\mathcal{P}) - dC_t \qquad (7)\\
&= \left(rV_t(\mathcal{P}) - rS_t\Delta_t - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt\\
&= \left(rV_t(\mathcal{P}) - rS_t\Delta_t - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \Gamma_t \right) dt\\
&= \left(rZ_t + rC_t - rS_t\Delta_t - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \Gamma_t \right) dt \qquad (8)
\end{aligned}
$$

$\Rightarrow$ We found the dynamics of the hedging error.

Robustness of delta-hedging

How to relate the hedging error to the error on the volatility?

Using Black-Scholes PDE (always true when using the implied vol), we get:

$$rC_t - rS_t\Delta_t - \frac{\partial C}{\partial t} = \frac{1}{2}\sigma_{imp}^2 S_t^2 \Gamma_t^2.$$

Hence

$$dZ_t = rZ_t + \frac{1}{2}S_t^2\Gamma^2(\sigma_{imp}^2 - \beta_t^2).$$

Since $Z_0 = 0$, the final hedging error is obtained by solving the above ODE:

$$Z_T = \int_0^T e^{rT}\frac{1}{2}S_t^2\Gamma_t^2(\sigma_{imp}^2 - \beta_t^2)dt.$$

Main factors impacting the hedging error:

- the Gamma of the option
- the position of the Black-Scholes implied volatility of the option with respect to the "true" volatility function of the asset

Vega of an option

Vega

The *vega* of an option with price V is the sensitivity of the option price with respect to changes in the volatility:

$$vega_t = \frac{\partial V_t}{\partial \sigma}.$$

In Black-Scholes model:

$$vega_t^{BS} = S_t e^{-q(T-t)} \phi(d_1) \sqrt{T-t}$$

Put-Call parity $\Rightarrow$ $vega_t(Put) = vega_t(Call)$.

Vega of an option

Properties of vega

- Sometimes written ν
- Strictly positive
- When $S_t \to 0$, $d_1 \to -\infty$, $\phi(d_1) \to 0$ and hence $vega_t^{BS} \to 0$.
- When $S_t \to \infty$, $d_1 \to \infty$, hence $\phi(d_1) \to 0$ exponentially. $S_t \to \infty$ only linearly and is dominated by $\phi(d_1)$. Hence $vega_t^{BS} \to 0$.
- Maximum for at-the-money options.

Vega of an option

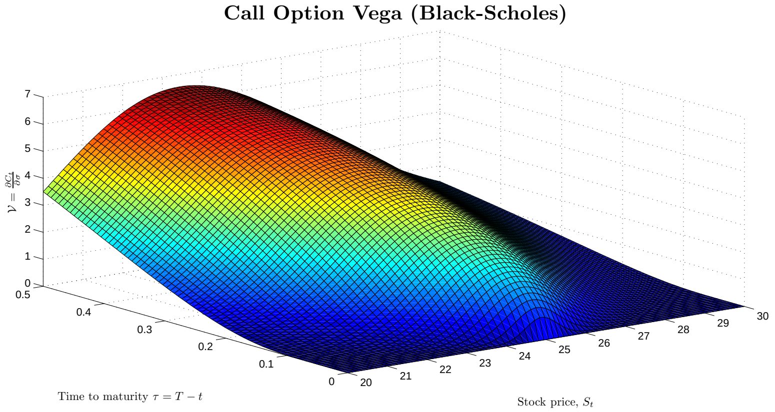


Figure: Vega of a call option as a function of the stock price and time-to-maturity

Vega-hedging

Idea: How to make the option price insensitive to movements in the volatility

- Technique similar to delta-hedging: aims at being covered against market movements
- Impossible to vega-hedge an option by investing in the underlying
- Volatility not tradable (at least for now)
- Invest in other options or derivatives.

Theta of an option

Theta

The *theta* of an option with value V is the sensitivity of the option price with respect to time:

$$\theta_t = \frac{\partial V}{\partial t}$$

- Almost always negative (positive for ITM put options without dividends or ITM call options with a very high dividend rate)
- Large and negative for at-the-money options
- Large magnitude when $T - t \to 0$

Theta of an option

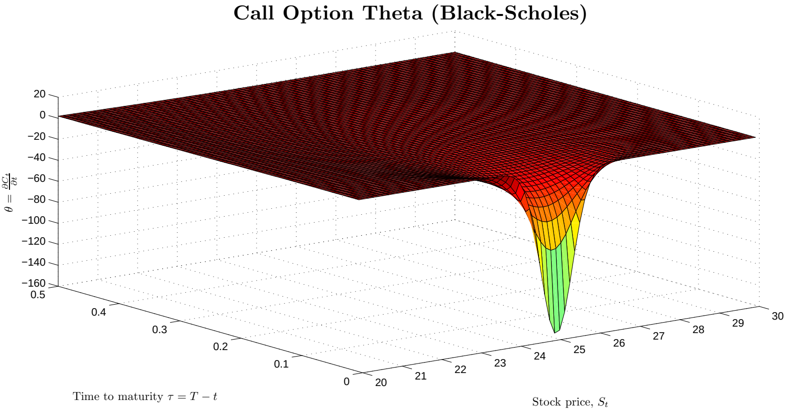


Figure: Theta of a call option as a function of the stock price and time-to-maturity

Rho of an option

Rho

The *rho* of an option is the sensitivity of the option price with respect to the interest-rate:

$$rho_t = \frac{\partial V}{\partial r}$$

In Black-Scholes setting:

$$rho_t^{BS}(Call) = K(T-t)e^{-r(T-t)}\Phi(d_2) \geq 0$$

$$rho_t^{BS}(Put) = -K(T-t)e^{-r(T-t)}\Phi(-d_2) \leq 0.$$

Intuition:

- For call options, the replicating strategy involves **selling bonds**: if r is larger the bond price decreases.
- For put options, it involves **buying bonds**: if r is larger the long position in bonds decreases in value.

Rho of an option

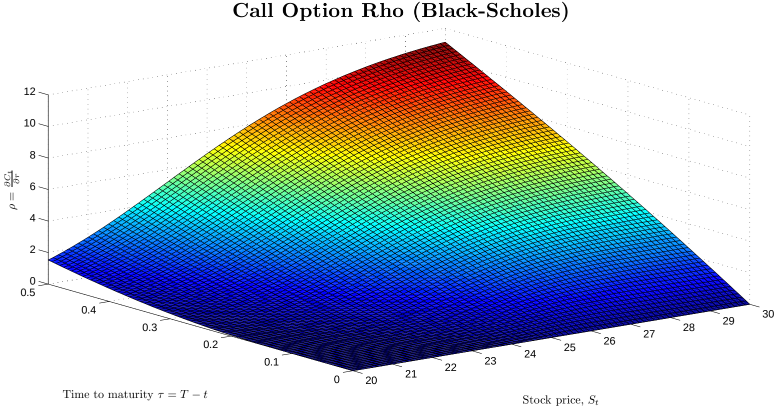


Figure: Rho of a call option as a function of the stock price and time-to-maturity

Breeden-Litzenberger formula

Why is it important? Provides a link between the risk-neutral probability distribution of the stock price at time T and current call prices.

Breeden-Litzenberger formula

The density function of the stock price at time T, denoted by $f_T^S(s)$ is obtained from call prices using:

$$f_T^S(K) = e^{r(T-t)} \frac{\partial^2 C_t}{\partial K^2}$$

In practice: Need a continuum of strikes to infer the second derivative $\frac{\partial^2 C_t}{\partial K^2}$.

Breeden-Litzenberger formula

Proof:

1. Write the call price as an integral using the definition of the expectation

$$\begin{aligned} C_t &= \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+] \\ &= e^{-r(T-t)} \int_{\mathbb{R}} (s-K)\mathbf{1}_{s>K} f_T^S(s)ds \\ &= e^{-r(T-t)} \int_K^{\infty} (s-K) f_T^S(s)ds \end{aligned} \tag{9}$$

Breeden-Litzenberger formula

2. Differentiate once with respect to K

Leibniz rule

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \frac{\partial b(\alpha)}{\partial\alpha} f(b(\alpha),\alpha) - \frac{\partial a(\alpha)}{\partial\alpha} f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

Breeden-Litzenberger formula

We apply the Leibniz rule to calculate $\frac{\partial}{\partial K}\int_K^\infty (s-K)f_T^S(s)ds$:

$$\frac{\partial}{\partial K}\int_K^\infty (s-K)f_T^S(s)ds = 0 - 1(K-K)f_T^S(K) + \int_K^\infty \frac{\partial}{\partial K}(s-K)f_T^S(s)ds$$
$$= -\int_K^\infty f_T^S(s)ds$$

Hence, denoting by F_T^S the c.d.f. of S_T:

$$e^{r(T-t)}\frac{\partial C_t}{\partial K} = -\int_K^\infty f_T^S(S)dS = -(1-F_T^S(K))$$
$$= F_T^S(K) - 1 = \int_0^K f_T^S(S)dS - 1.$$

Breeden-Litzenberger formula

3. Differentiate a second time with respect to K

$$e^{r(T-t)}\frac{\partial^2 C_t}{\partial K^2} = f_T^S(K)$$

4. Same result with put option prices:

$$e^{r(T-t)}\frac{\partial^2 P_t}{\partial K^2} = f_T^S(K)$$

Remark: Watch out the notations...

Breeden-Litzenberger formula

How can we use the Breeden-Litzenberger formula for static hedging?

Let us consider an instrument with a European payoff $H(S_T)$ which is twice differentiable. The price of this instrument at time t is denoted by V_t.

$$\begin{aligned} V_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H(S_T)] \\ &= e^{-r(T-t)} \int_0^{\infty} H(s) f_T^S(s) ds \end{aligned}$$

Separate the integral at the strike level $K = \bar{K}$:

$$V_t = e^{-r(T-t)} \left[\int_0^{\bar{K}} H(s) f_T^S(s) ds + \int_{\bar{K}}^{\infty} H(s) f_T^S(s) ds \right]$$

Breeden-Litzenberger formula

Apply Breeden-Litzenberger formulas with $K = s$:

$$f_T^S(s) = e^{r(T-t)} \frac{\partial^2 C_t}{\partial K^2}(K = s) \text{ with } s \geq \bar{K}$$

$$f_T^S(s) = e^{r(T-t)} \frac{\partial^2 P_t}{\partial K^2}(K = s) \text{ with } s \leq \bar{K}$$

$$V_t = \int_0^{\bar{K}} \frac{\partial^2 P_t}{\partial K^2}(K = s)H(s)ds + \int_{\bar{K}}^{\infty} \frac{\partial^2 C_t}{\partial K^2}(K = s)H(s)ds$$

Integration by parts

Let f and g be differentiable on $[a, b]$. Then:

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx.$$

Breeden-Litzenberger formula

Integrate by parts twice $\int_a^b \frac{\partial^2 V_t}{\partial K^2}(K=s)H(s)ds$ for $V = C$ or P.

$$\int_a^b \frac{\partial^2 V_t}{\partial K^2}(K=s)H(s)ds = \left[\frac{\partial V_t}{\partial K}(K=s)H(s)\right]_a^b - \int_a^b \frac{\partial V_t}{\partial K}(K=s)\frac{\partial H}{\partial s}ds$$
$$= \left[\frac{\partial V_t}{\partial K}(K=s)H(s)\right]_a^b - \left[V_t\frac{\partial^2 H}{\partial s^2}\right]_a^b + \int_a^b V_t\frac{\partial H}{\partial s}ds$$

We infer V_t:

$$V_t = \left[\frac{\partial P_t}{\partial K}(K=s)H(s)\right]_0^{\bar{K}} - \left[P_t(K=s)\frac{\partial H}{\partial s}\right]_0^{\bar{K}} + \int_0^{\bar{K}} P(K=s)\frac{\partial^2 H}{\partial s^2}ds$$
$$+ \left[\frac{\partial C_t}{\partial K}(K=s)H(s)\right]_{\bar{K}}^{\infty} - \left[C_t(K=s)\frac{\partial H}{\partial s}\right]_{\bar{K}}^{\infty} + \int_{\bar{K}}^{\infty} C_t(K=s)\frac{\partial^2 H}{\partial s^2}ds$$

Breeden-Litzenberger formula

Boundary conditions:

- A put option with strike 0 has no value:

$$\frac{\partial P_t}{\partial K}(K = 0) = 0$$
$$P_t(K = 0) = 0$$

- A call option with infinite strike has no value either:

$$\frac{\partial C_t}{\partial K}(K = \infty) = 0$$
$$C_t(K = \infty) = 0$$

Breeden-Litzenberger formula

We obtain:

$$H_t = H(S_T = \bar{K}) \left(\frac{\partial P_t}{\partial K}(K = \bar{K}) - \frac{\partial C_t}{\partial K}(K = \bar{K}) \right) + \int_0^{\bar{K}} P_t(K = s) \frac{\partial^2 H}{\partial s^2} ds$$
$$+ \frac{\partial H}{\partial s}(S_T = \bar{K}) \left(C_t(K = \bar{K}) - P_t(K = \bar{K}) \right) + \int_{\bar{K}}^{\infty} C_t(K = s) \frac{\partial^2 H}{\partial s^2} ds$$

Put-call parity:

$$P_T(K = \bar{K}) - C_t(K = \bar{K}) = \bar{K} e^{-r(T-t)} - S_t$$

We differentiate it:

$$\frac{\partial P_t}{\partial K}(K = \bar{K}) - \frac{\partial C_t}{\partial K}(K = \bar{K}) = e^{-r(T-t)}$$

Breeden-Litzenberger formula

Decomposition of European payoff

$$H_t = H(S_T = \bar{K})e^{-r(T-t)} + \frac{\partial H}{\partial s}(S_T = K)\left(S_t - \bar{K}e^{-r(T-t)}\right)$$
$$+ \int_0^{\bar{K}} P_t(K = s)\frac{\partial^2 H}{\partial s^2}ds + \int_{\bar{K}}^{\infty} C_t \frac{\partial^2 H}{\partial s^2}ds$$

- Constant riskfree payoff $H(\bar{K})$ discounted at the riskfree rate
- Forward contract with delivery price $\bar{K}$
- Combination of puts with strikes below $\bar{K}$ with densities given by $\frac{\partial^2 H}{\partial s^2}$.
- Combination of calls with strikes above $\bar{K}$

Practical issue: Needs an infinity of options with a continuum of strikes.

Static Hedging with the Breeden-Litzenberger formula

Summary of steps

1. Take an arbitrary derivative with European payoff $H(S_T)$ and write its value V_t in integral form
2. Decompose the integral into 2 integrals: from 0 to $\bar{K}$ and from $\bar{K}$ to ∞.
3. Apply Breeden-Litzenberger formula for puts in the first integral and for calls in the second integral to transform f_T^S into derivatives of put/call prices.
4. Integrate by parts so that the second derivatives "disappear"
5. Simplify using boundary conditions
6. Simplify again using the put-call parity

What we obtain: **decomposition of the derivative** price into a **risk-free part**, a **forward contract**, **put** and **call options**.

Extension of B-S to multi-assets

Exchange option

An *exchange option* is an option to exchange one asset for another at maturity T. The payoff is:

$$H_T = \max(S^1_T - S^2_T, 0).$$

What is the price of such a derivative in B-S setting?

Extension of B-S to multi-assets

Under $\mathbb{Q}$:

$$\frac{dS_t^1}{S_t^1} = rdt + \sigma_1 dW_t^1 \tag{10}$$

$$\frac{dS_t^2}{S_t^2} = rdt + \sigma_2 dW_t^2. \tag{11}$$

$$\mathbb{E}^{\mathbb{Q}}[dW^1 dW^2] = \rho dt.$$

The solutions of these SDEs are given by:

$$S_t^1 = S_0^1 e^{\left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_t^1} \quad ; \quad S_t^2 = S_0^2 e^{\left(r - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_t^2}.$$

Extension of B-S to multi-assets

Under $\mathbb{Q}$:

$$\frac{dS_t^1}{S_t^1} = rdt + \sigma_1 dW_t^1 \tag{12}$$

$$\frac{dS_t^2}{S_t^2} = rdt + \sigma_2 dW_t^2. \tag{13}$$

$$\mathbb{E}^{\mathbb{Q}}[dW^1 dW^2] = \rho dt.$$

The solutions of these SDEs are given by:

$$S_t^1 = S_0^1 e^{\left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_t^1},$$

$$S_t^2 = S_0^2 e^{\left(r - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_t^2}$$

Extension of B-S to multi-assets

The FTAP gives us the price of the exchange option:

$$
\begin{aligned}
C_t^{ex} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}H_T|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S_T^2 \max\left(\frac{S_T^1}{S_T^2}-1,0\right)|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T^2 \max\left(\frac{S_T^1}{S_T^2}-1,0\right)|\mathcal{F}_t\right] \\
&= S_t^2 e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[e^{(r-\frac{\sigma_2^2}{2})(T-t)+\sigma_2(W_T^2-W_t^2)} \max\left(\frac{S_T^1}{S_T^2}-1,0\right)|\mathcal{F}_t] \\
&= S_t^2\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{\sigma_2^2}{2}(T-t)+\sigma_2(W_T^2-W_t^2)} \max\left(\frac{S_T^1}{S_T^2}-1,0\right)|\mathcal{F}_t\right]
\end{aligned}
$$

Change of measure

$$Z_t = e^{-\frac{\sigma_2^2}{2}t + \sigma_2 W_t^2}.$$

- Z_t is a martingale and defines a change of measure from $\mathbb{Q}$ to $\tilde{\mathbb{Q}}$
- Under $\tilde{\mathbb{Q}}$, $W_t^2 - \sigma_2^2 t$ is a standard Brownian motion
- If X is $\mathcal{F}_t$-adapted and integrable:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{Z_T}{Z_t}X|\mathcal{F}_t\right] = \mathbb{E}^{\tilde{\mathbb{Q}}}[X|\mathcal{F}_t].$$

We get:

$$C_t^{ex} = S_t^2 \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\max\left(\frac{S_T^1}{S_T^2} - 1, 0\right)|\mathcal{F}_t\right].$$

Dynamics of S_t^1/S_t^2

We define:

$$Y_t = \frac{S_t^1}{S_t^2}$$

Under $\mathbb{Q}$:

$$\begin{aligned} dY_t &= \frac{1}{S_t^2} dS_t^1 - \frac{S_t^1}{(S_t^2)^2} dS_t^2 + \frac{S_t^1}{(S_t^2)^3} (\sigma_2)^2 (S_t^2)^2 dt - \frac{1}{(S_t^2)^2} \rho\sigma_1\sigma_2 dt \\ &= \frac{1}{S_t^2} S_t^1 (rdt + \sigma_1 dW_t^1) - \frac{S_t^1}{(S_t^2)^2} S_t^2 (rdt + \sigma_2 dW_t^2) + \frac{S_t^1}{(S_t^2)^3} \sigma_2^2 (S_t^2)^2 dt - \frac{1}{(S_t^2)^2} \rho\sigma_1\sigma_2 \end{aligned}$$

Hence:

$$\frac{dY_t}{Y_t} = \sigma_1 dW_t^1 - \sigma_2 dW_t^2 + (\sigma_2^2 - \rho\sigma_1\sigma_2) dt.$$

Decomposition of correlated Brownian Motions

W^1 and W^2 are correlated with correlation coefficient ρ. We write:

$$W_t^1 = \rho W_t^2 + \sqrt{1-\rho^2}\, W_t^{2\perp}$$

where $W_t^{2\perp}$ is a $\mathbb{Q}$-Brownian motion independent of W_t^2.

Under $\tilde{\mathbb{Q}}$, we have:

$$d\tilde{W}_t^2 = dW_t^2 - \sigma_2 dt \;\Rightarrow dW_t^2 = d\tilde{W}_t^2 + \sigma_2 dt$$
$$dW_t^1 = \rho dW_t^2 + \sqrt{1-\rho^2}\, dW_t^{2\perp} = \rho d\tilde{W}_t^2 + \rho\sigma_2 dt + \sqrt{1-\rho^2}\, dW_t^{2\perp}$$

Decomposition of correlated Brownian Motions

Change of measure for independent Brownian motions

Suppose that B_t and $B_t^{\perp}$ are independent Brownian motions on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{Q})$ and that ψ is an adapted process such that $\int_0^T \psi_s^2 ds < \infty$. Define:

$$\Lambda_t = \exp\left(\int_0^t \psi_s dB_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right).$$

Define a measure $\bar{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_T)$ by taking

$$\left.\frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}}\right|_{\mathcal{F}_T} = \Lambda_T$$

and a process $\bar{B}$ by $d\bar{B}_t = dB_t - \psi_t dt$. Then $\bar{B}_t$ and $B_t^{\perp}$ are independent Brownian motions under measure $\bar{\mathbb{Q}}$.

Decomposition of correlated Brownian Motions

According to this lemma, $\tilde{W}_t^2$ and $W_t^{2\perp}$ are independent $\tilde{\mathbb{Q}}$- Brownian motions. Define

$$d\tilde{W}_t^1 = \rho d\tilde{W}_t^2 + \sqrt{1-\rho^2}dW_t^{2\perp}$$

$$dW_t^1 = d\tilde{W}_t^1 + \rho\sigma_2 dt$$

and thus:

$$\begin{aligned}\frac{dY_t}{Y_t} &= \sigma_1 d\tilde{W}_t^1 + \sigma_1\rho\sigma_2 dt - \sigma_2 d\tilde{W}_t^2 - \sigma_2^2 dt + \sigma_2^2 dt - \rho\sigma_1\sigma_2 dt \\ &= \sigma_1 d\tilde{W}_t^1 - \sigma_2 d\tilde{W}_t^2\end{aligned}$$

Decomposition of correlated Brownian Motions

We can rewrite:

$$\frac{dY_t}{Y_t} = \sigma dW_t.$$

with

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Using Black-Scholes formula for $K = 1$ and $r = 0$ we get:

$$C_t^{ex} = S_t^2 \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\max\left(Y_T - 1, 0\right) | \mathcal{F}_t\right] = S_t^2 (Y_t \Phi(\tilde{d}_1) - \Phi(\tilde{d}_2)) = S_t^1 \Phi(\tilde{d}_1) - S_t^2 \Phi(\tilde{d}_2)$$

with

$$\tilde{d}_1 = \frac{\ln\left(\frac{Y_t}{1}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{S_t^1}{S_t^2}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$\tilde{d}_2 = d_1 - \sigma\sqrt{T-t}.$$

Financial Engineering - Lecture 5

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

20.10.2011

University of Zurich

Aim of lecture

BS distributional assumptions: The stock S_T is **log-normally distributed** under $\mathbb{P}$ and $\mathbb{Q}$.

Aim of lecture:

- To which extent is this assumption wrong?
- What is the implied volatility smile? How to process options' data?
- What distributions does the smile infer?

Estimation of BS model under $\mathbb{P}$

Under $\mathbb{P}$:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

$$\Downarrow$$

$$d \ln S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\Downarrow$$

$$\ln \frac{S_{t+\Delta t}}{S_t} \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)\Delta t, \sigma^2 \Delta t)$$

When $\Delta t = 1$ year, σ **is the annual std. dev. of the log-returns**.

Not enough annual data!

Estimation of BS model under $\mathbb{P}$

Realized daily volatility

The **realized daily volatility** RV of an index / stock S over a period of N days is the standard deviation of the daily log returns r_i:

$$RV_1 = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \overline{r})^2}$$

with

$$r_i = \log\left(\frac{S_{i+1}}{S_i}\right) \approx \frac{\Delta S_i}{S_i}.$$

$\hat{\sigma}$ = Annual volatility = $\sqrt{365}$ × Realized daily volatility.

Change to the martingale measure

Under $\mathbb{Q}$ the volatility σ **does not change**:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

$\Rightarrow$ We would like to use $\hat{\sigma}$ for pricing:

Consider a call option traded on the market (K, T), we use BS formula $C_{BS}(S_t, t, r, K, T, \hat{\sigma})$. Unfortunately we find

$$C_{BS}(S_t, t, r, K, T, \hat{\sigma}) \neq C_{Mkt}(t, S_t, K, T)$$

$\Rightarrow$ Something wrong with BS dynamics... What can we learn?

Start from output: What is the value of σ that matches market prices?

Implied volatility (IV): Definition

Implied volatility (IV)

The **implied volatility** of an option is the value of the volatility parameter in the Black Scholes equation that makes the value of the option $C_{BS}(S, K, t, T, r, \sigma_{imp})$ in the model match the market price C_{Mkt} of the option:

$$C_{BS}(S, t, T, r, \sigma_{imp}) = C_{Mkt}$$

Questions:

- Does σ_{imp} always exist?
- Does σ_{imp} depend on the option chosen? $\rightarrow \sigma_{imp}(K, T)$
- Is σ_{imp} the same for a put (K, T) and a call (K, T)?

Implied volatility (cont.)

NA $\Rightarrow$ Implied volatility exists

No arbitrage bounds on **data**:

$$(S_t - Ke^{-r(T-t)})^+ \leq C_{Mkt}(t, S_t, T, K) \leq S_t.$$

Attainable prices in BS:

$$C_{BS}(S_t, t, T, r, \sigma = 0) = (S_t - Ke^{-r(T-t)})^+ \quad \text{lower bound}$$
$$C_{BS}(S_t, t, T, r, \sigma = \infty) = S_t \quad \text{upper bound.}$$

$\sigma \mapsto C_{BS}(S_t, t, T, r, \sigma)$ is continuous, strictly increasing ($\frac{\partial C_{BS}}{\partial \sigma} > 0$).
Intermediate value theorem $\Rightarrow$ There **exists** a **unique** σ_{imp} for each option.

Implied volatility (cont.)

NA $\Rightarrow$ Implied volatility is the same of call and puts

Market call and put satisfy Put call parity

$$P_{Mkt}(t, T, K) + S_t = C_{Mkt}(t, T, K) + KB(t, T)$$

&

BS call and put satisfy Put call parity **only if same σ used**

$$P_{BS}(t, T, K, \sigma_{imp}^{Put}) + S_t = C_{BS}(t, T, K, \sigma_{imp}^{Put}) + KB(t, T)$$

$$\Downarrow$$

$$\sigma_{imp}^{Call}(K, T) = \sigma_{imp}^{Put}(K, T)$$

Implied volatility surface IVS

For different options (K_1, T_1) and (K_2, T_2), it turns out

$$\sigma_{imp}(K_1, T_1) \neq \sigma_{imp}(K_2, T_2).$$

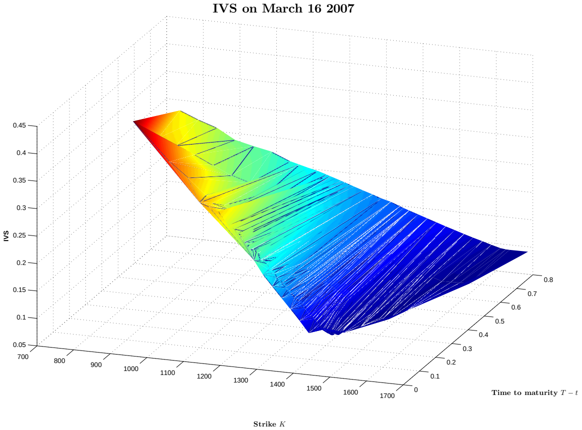
This was not always the case (before 1987)!

Financial terms:

- The **implied volatility surface**: $(K, T) \mapsto \sigma_{imp}(K, T)$,
- **IV slice**: evolution w.r.t. strike $K \mapsto \sigma_{imp}(K, T)$,
- **Term structure of IVs**: evolution w.r.t. maturity $T \mapsto \sigma_{imp}(K, T)$

Implied volatility surface

Example: March 16, 2007, S&P500 index, $S_0 = 1386.95$.



Smile, skew.

S&P500 index options

S&P500 index: 500 biggest capitalizations in the US.

- **Huge market** for options (many strikes / maturities),
- Approximation of **continuous dividend yield**,
- Individual stocks $\rightarrow$ American options, cash dividends.

S&P500 index options

Example: March 16, 2007. Option Metrics.

t_0	S_0	T	Last traded date	K	Bid	Offer	Volume	Open interest	P/C flag
⋮				⋮					⋮
20070316	1386.95	20070421	20070316	950	0.05	0.15	28	5181	1
20070316	1386.95	20070421	20070315	950	439.3	441.3	0	10	0
⋮				⋮					⋮
20070316	1386.95	20070421	20070316	1400	27.5	28	4581	161704	1
20070316	1386.95	20070421	20070316	1400	19.4	20	4374	160735	0
⋮				⋮					⋮
20070316	1386.95	20070421	NaN	1575	180.4	182.4	0	0	1
20070316	1386.95	20070421	20070316	1575	0.05	0.1	351	902	0
⋮				⋮					⋮

On that date: 13 maturities (<3Y), most T<1Y, less strikes traded as $T \to 3Y$, 170 options with T=April.

S&P500 index options

Indicators of liquidity / trading activity: bid-ask spread, volume, open interest.

- **In the money** (ITM) option: Exercising now gives a profit.
  Example: Call options with $K < S_0$, put options with $K > S_0$.
- **At the money** (ATM) option: Exercising now gives ≈ 0.
  Example: Call and put options with $K \approx S_0$.
- **Out the money** (OTM) option
  Example: Call options with $K > S_0$, put options with $K < S_0$.

Liquid options (most traded): ATM and OTM options.

ITM option prices provided **misleading information**

$\Rightarrow$ delete ITM options from sample.

Data issues

Issues:

- ITM options are rarely traded,
- Open-interest or volume $= 0$, $T = 1$ day, $T = 2Y$,
- Minimum tick: 0.05, rounding errors,

Solution: Delete :-)

Data issues

We want to use $C_{BS} = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = C_{Mkt}$,

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Issues:

- S&P500 index (SPX) price is closing price, but option price not necessarily from closing $\Rightarrow$ **option prices are not recorded with correct value of underlying**
- How do we **estimate** the SPX index non-constant **dividends**?

Data issues

Solution: Use forwards to hide dividends. For each t, T, we have

$$F_t(T) = S_t e^{(r-q)(T-t)}.$$

$F_t(T)$: value you are willing to pay at t to have S_T at time T.
As $t \to T$, $F_t(T) \to S_T \Rightarrow$ **option on S_T is an option on $F_t(T)$**.

$$C_t = e^{-r_{t,T}(T-t)}\left(F_t(T)\Phi(d_1) - K\Phi(d_2)\right)$$

$$P_t = e^{-r_{t,T}(T-t)}\left(K\Phi(-d_2) - F_t(T)\Phi(-d_1)\right)$$

$$d_1 = \frac{\ln\left(F_t(T)/K\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

dividends have disappeared :-)

Remark: Estimate $r \longrightarrow r_{t,T}$ from Libor rates.

Data issues: Non-synchronized quotes

Idea: **Imply the underlying value** that best fits option values using put-call parity.

Consider **ATM forward** put-call parity:

$$
\begin{aligned}
C_t^{Mkt}(F_t(T), K \approx F_t(T), T, r_{t,T}) &+ Ke^{-r_{t,T}(T-t)} \\
&= P_t^{Mkt}(F_t(T), K \approx F_t(T), T, r_{t,T}) + F_t(T)e^{-r_{t,T}(T-t)},
\end{aligned}
$$

- Why ATM? $\rightarrow$ **both** put and call options are liquid and contain information. Otherwise, one option is OTM and one ITM,
- Why put-call parity with underlying the forward? $\rightarrow$ to **avoid dividend estimation**.

Finally, we have all parameters!

Get the IVs

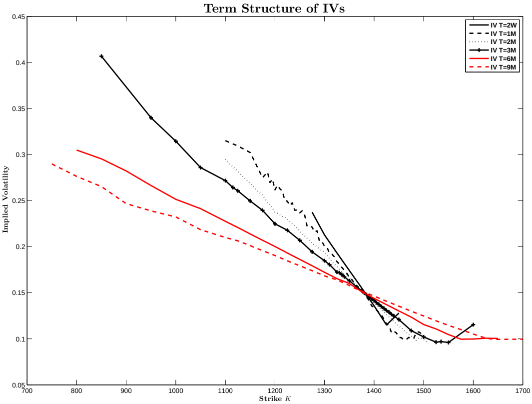
Standard practice $(C_{bid}^{Mkt}, C_{ask}^{Mkt}) \to C_{mid}^{Mkt}$ and back out $\sigma_{imp}(K, T)$

$$C_{BS}(t, F_t(T), T, K, r_{t,T}, \sigma_{imp}) = C_{Mkt-mid}(t, K, T).$$

Usual algorithm: Newton-Raphson.

IV slices as a function of strike K

Term structure of IVs. $S_0 = 1386.95$



IV slices as a function of strike K

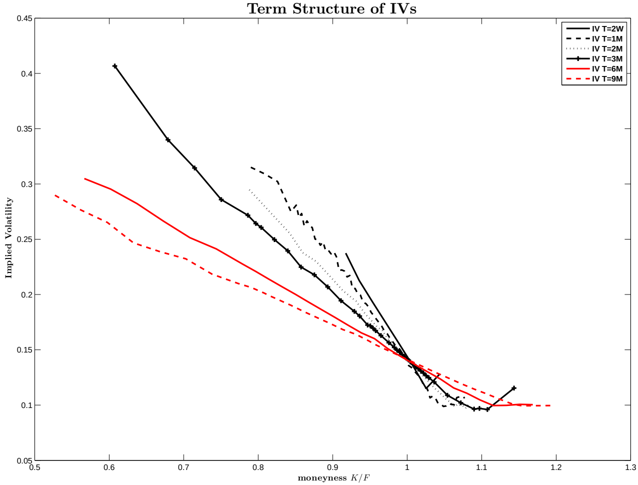
Standard practice: plot σ_{imp} as a function of $K/F_t(T)$.

Moneyness: $m = K/F_t(T)$.

$\Rightarrow$ Allows to **plot σ_{imp} independently of the value of the index**.

- Can compare the IVS on different dates,
- Generally, IVS changes less over time when plotted w.r.t. (log-) moneyness.

IV slices as a function of moneyness



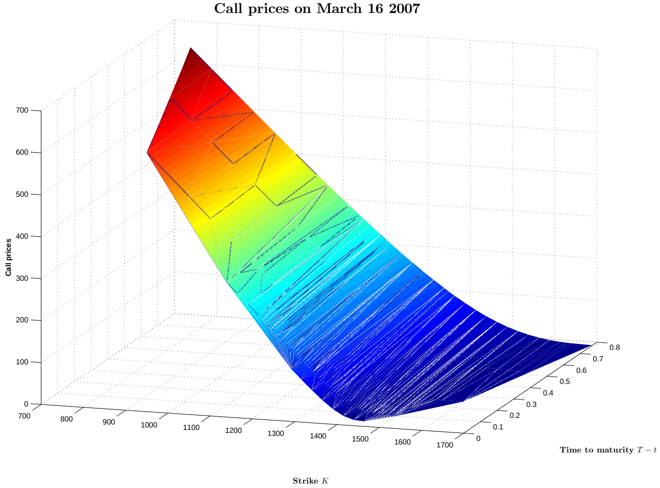
IV slices as a function of moneyness

Some properties: (not always valid)

- For T fixed, the IV is generally a **decreasing function of moneyness** except possibly for high moneyness. There is a high level of dependence between IVs with different strikes,
- The **slope gets steeper as** $T \searrow$. As **maturity increases, the implied volatilities decrease when** $m < 1$. There is a high level of dependence for different maturities,
- The minimum volatility as a function of strike usually occurs ATM or for m slightly larger than 1,
- The IV slices (seen as strings in $\mathbb{R}^3$) seem to intersect at around a moneyness of $m \approx 1$ and revolve around this intersection point.

Why work with IVs

Call price surface:



Why work with IVs

Why work with IVs:

- $C_{Mkt}(K,T) \in [0\$, 600\$]$ depending on index level but $\sigma_{imp}(K,T) \in [5\%, 50\%]$ (most of the time). **Easier to compare IVs than prices** for traders,
- Compare option prices on different dates.
- Compare option prices in **different countries** (even if different currencies), across **different asset classes**,
- **Shape of IVS** directly linked to **risk-neutral distribution**.

Given that we work with mid-prices, non-synchronized prices, **is our IVS arbitrage free?**

Model independent conditions for NA

To avoid static arbitrage:

$$-e^{-r_{t,T}(T-t)} \leq \frac{\partial C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K} \leq 0, \tag{1}$$

$$\frac{\partial^2 C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2} \geq 0, \tag{2}$$

$$\max\left(e^{-q_{t,T}(T-t)}S_t - e^{-r_{t,T}(T-t)}K\right) \leq C(S_t, t, K, T, r_{t,T}, q_{t,T}) \leq e^{-q_{t,T}(T-t)}S_t, \tag{3}$$

$$\frac{\partial C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial T} \geq 0. \tag{4}$$

Find $q_{t,T}$ piecewise constant dividend yield using $F_t(T) = S_t e^{(r_{t,T} - q_{t,T})(T-t)}$.

Intuition for NA constraint (1)

Constraint: $-e^{-r_{t,T}(T-t)} \leq \dfrac{\partial C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K} \leq 0.$

Intuition: Consider $K_1 < K_2$, $C(t, T, K_1)$, $C(t, T, K_2)$.

- Call (K_1, T) allows you to buy S_T for K_1,
- Call (K_2, T) allows you to buy S_T for $K_2 > K_1$.

Call (K_1, T) is a better deal (whatever S_T is),
NA $\Rightarrow C(t, T, K_1) > C(t, T, K_2)$.

$$\frac{\partial C(t, K, T)}{\partial K} \leq 0.$$

Intuition for NA constraint (1)

Constraint: $-e^{-r_{t,T}(T-t)} \leq \dfrac{\partial C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K} \leq 0.$

Intuition: Consider $K_1 < K_2$, $P(t, T, K_1)$, $P(t, T, K_2)$.

- Put (K_1, T) allows you to sell S_T for K_1,
- Put (K_2, T) allows you to sell S_T for $K_2 > K_1$.

Put (K_2, T) is a better deal (whatever S_T is),
NA $\Rightarrow P(t, T, K_2) > P(t, T, K_1)$.

$$\frac{\partial P(t, K, T)}{\partial K} \geq 0.$$

Differentiate put-call parity:

$$\frac{\partial P(t, K, T)}{\partial K} = \frac{\partial C(t, K, T)}{\partial K} + e^{-r_{t,T}(T-t)} \geq 0.$$

Intuition for NA constraint (2)

Constraint: $\dfrac{\partial^2 C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2} \geq 0.$

Intuition 1: Breeden-Litzenberger formula.

$$e^{r_{t,T}(T-t)} \frac{\partial^2 C(t, K, T, S_t)}{\partial K^2} = f^{\mathbb{Q}}_{S_T|S_t}(S_T = K)$$

$$f_{S_T|S_t}(S_T = K) \geq 0 \quad \Leftrightarrow \quad \frac{\partial^2 C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2} \geq 0.$$

Intuition for NA constraint (2)

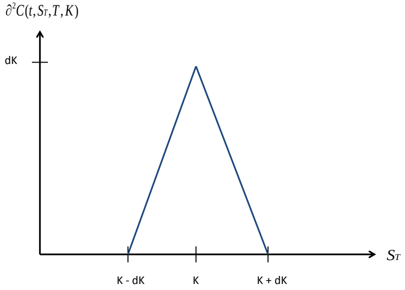
Constraint: $\dfrac{\partial^2 C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2} \geq 0.$

Intuition 2: Butterfly spreads.

$$\frac{\partial^2 C(t, S_t, T, K)}{\partial^2 K} \approx_{dK \to 0} \frac{C(K + dK) - 2C(K) + C(K - dK)}{dK^2}.$$

Intuition for NA constraint (2)

Payoff $\partial^2 C(t, S_T, T, K) \approx C(K + dK) - 2C(K) + C(K - dK)$



Butterfly spread: Buy $C(K + dK)$ and $C(K - dK)$, sell 2 $C(K)$.

Intuition for NA constraint (2)

Payoff $\dfrac{\partial^2 C(S_T, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2}$ has triangular shape.

- Base $= 2dK$,
- Height $= 1/dK$.

Payoff area $= 1 > 0 \Rightarrow$ price positive.

When $dK \to 0$, payoff of area is $1_{S_T = K}$. **Behaves like a state contingent security**.

Payoff $\dfrac{\partial^2 C(S_T, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2}$ is **non-negative**

$\Rightarrow$ **price** today $\dfrac{\partial^2 C(S_t, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2}$ is **non-negative**.

Intuition for NA constraint (2)

Remark: **Static hedging** using butterfly spreads.

Payoff of butterfly spread $\dfrac{\partial^2 C(S_T, t, K, T, r_{t,T}, q_{t,T})}{\partial K^2}$ is a hat function.

$\Rightarrow$ Linear combinations of butterfly spread can **generate any** piecewise linear payoff H.

$\Rightarrow$ **Can price/hedge any European style contingent claims with** (infinitesimal) **butterfly spreads**: Static Hedging using the Breeden-Litzenberger formula.

Continuous-time version:

$$V_t(H) = \int_0^\infty \frac{\partial^2 C_t}{\partial K^2}(K = s)H(s)ds$$

NA constraints (3)-(4)

Constraint:
$$\max\left(e^{-q_{t,T}(T-t)}S_t - e^{-r_{t,T}(T-t)}K\right) \leq C(S_t, t, K, T, r_{t,T}, q_{t,T}) \leq e^{-q_{t,T}(T-t)}S_t.$$

$\longrightarrow$ No Arbitrage conditions on European options.

Constraint: $\frac{\partial C(S_t,t,K,T,r_{t,T},q_{t,T})}{\partial T} \geq 0.$

Technical.

If necessary: **Model independent** **asymptotic constraints:** For $K \to 0, \infty$, $T \to 0$.

Build a complete IVS

The real **difficulty** is not to force the data to follow NA conditions, it is to build a **whole surface** that satisfies NA!

Example: Fengler (2005). See script for more ref.

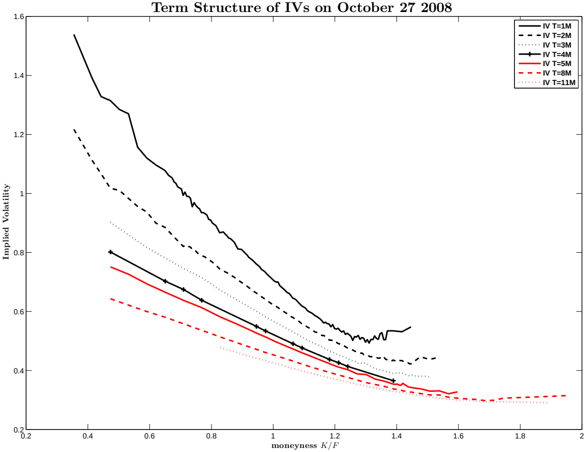
Need for a complete surface:

- To **quote OTC vanilla** (European) options,
- **Structured products** are often decomposed into elementary securities such as vanilla options with non-quoted strikes $\Rightarrow$ accurate pricing/hedging of structured products depend on good construction of IV surface.

Dynamics of the smile

The smile is **dynamic** $\rightarrow \sigma_{imp}(t, T, K)$.

Different market scenario:



Dynamics of the smile

Remarks:

- IVs are decreasing with maturity T,
- No intersection between IV slices,
- Level of IVs!
- Liquid moneyness traded $\in [0.4, 1.6]$!

High price of puts (m = 0.4, T = 1M) $\Rightarrow$ "$\mathbb{Q}(S_T < 0.4S_t|S_t) >> 0$"!

$\Rightarrow$ **Moneyness traded, level/shape of IV** gives us information about $\mathbb{Q}(S_T < mS_t|S_t)$.

More about IV dynamics

- **IVs rise very fast** and **decrease slowly**,
- **IVs are mean reverting**: random oscillations around a reference value.

More about IV dynamics

Drivers of the smile:

- **Fear** leads to high demand of OTM put options,
- **Leverage effect**: volatility is negatively correlated to the stock price, i.e. when $S_t \searrow \sigma \nearrow$.
  Explanation: If $S_t \searrow$, then the debt-to-equity ratio $\nearrow$, increasing the risk (σ) of the firm.
- etc.

What can we infer from the smile?

Black-Scholes framework: returns are $\mathbb{Q}$ **normally distributed** $\Rightarrow$ **IVS** is **flat**.

Existence of non-flat IVS $\Rightarrow$ **returns are non-normal under** $\mathbb{Q}$.

Question: Can we use option prices / the IV skew to infer $\hat{f}^{\mathbb{Q}}_{S_T|S_t}(S_T)$?

Yes, to some extent:

$$e^{r_{t,T}(T-t)} \left. \frac{\partial^2 C^{Mkt}(t, K, T, S_t, r_{t,T}, q_{t,T})}{\partial K^2} \right|_{K=S_T} = \hat{f}^{\mathbb{Q}}_{S_T|S_t}(S_T = K)$$

Method: Assume $\hat{f}^{\mathbb{Q}}_{S_T|S_t}$ unknown and use market data to compute left hand side term.

Implied risk-neutral distribution

Starting point: $r_{t,T}, q_{t,T}$. Call options $C^{Mkt}(K_i, T_j)$, $j \in \{1, ...M\}$, $i \in \{1, ..., N_j\}$.

Compute empirical estimator of

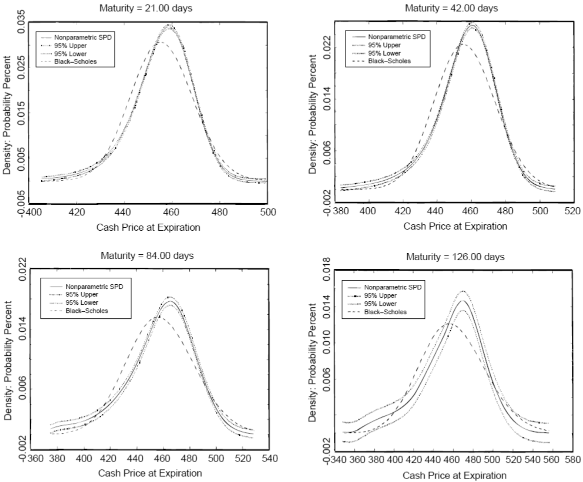
$$\left.\frac{\partial^2 C^{Mkt}(t, K, T, S_t, r_{t,T}, q_{t,T})}{\partial K^2}\right|_{K=S_T} = \hat{f}^{\mathbb{Q}}_{S_T|S_t}(S_T = K)$$

Difficulty:

- Enforce NA,
- Daily data is hardly enough to build one slice, (whole surface!),
- Try to a model independent estimation of C^{Mkt} $\longrightarrow$ non-parametric approaches.

Implied risk-neutral distribution (RND)

Some results: **Implied risk-neutral density** of the SPX, forward looking 1,2,4,8 months. $S_t = 455$.



Implied risk-neutral distribution

- RND is **heavy tailed**, kurtotic
  $\Rightarrow$ option prices indicate that **extreme movements of index possible** much more than the log-normal distribution predicts,
- RND is **skewed**: A lot of $\mathbb{Q}$ mass for small S
  $\Rightarrow$ option prices indicate that extreme drops are more likely than extreme increases,
- Sometimes bump in left tail
  Some interpret as **jump** possibility.

$\Rightarrow$ **smile & RNDs give us**

- The market sentiment,
- Ideas **how to improve Black-Scholes**.

Unfortunately **does not give us information on the path of the underlying**.

Financial Engineering - Lecture 6

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

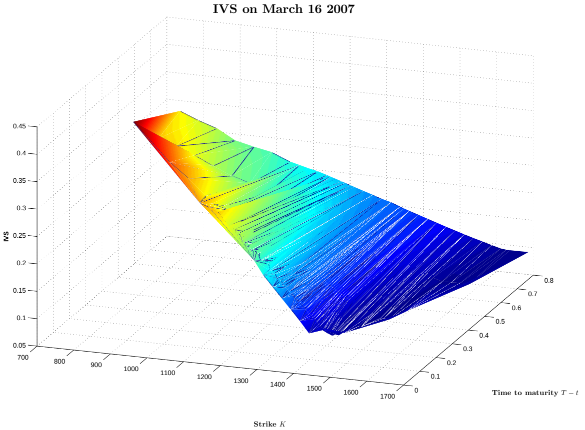
Elise Gourier
University of Zurich

27.10.2011

University of Zurich

Recall: Implied volatility surface

S&P500 index, $S_0 = 1386.95$.



Implied volatility: $C_{BS}(S_t, t, r, K, T, \sigma_{imp}(K, T)) = C_{Mkt}(t, S_t, K, T)$.

Recall: Implied volatility surface

The existence of a non-flat implied volatility surface (IVS) **violates Black-Scholes assumptions**.

Implementing the Breeden-Litzenberger formula gives us an idea of how to improve the **log-normal** dynamics:

- Add **heavy tails** $\rightarrow$ and make **extreme movements of index possible**,
- Add **skewness**: under $\mathbb{Q}$, extreme drops in the stock are more likely than extreme increases,
- Add **jumps**.

Stochastic volatility

Stochastic volatility: Adding **fat tails** and **skewness**.

BS $\mathbb{Q}$ dynamics:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t.$$

Idea: $\sigma \rightarrow \sigma_t$.
Randomizing σ gives uncertainty in the future distribution of the stock price.

Example:

$$\frac{dS_t}{S_t} = rdt + \sigma(S_t, t)dW_t,$$
$$d\sigma(S_t, t) = \mu_\sigma(S_t, t, \sigma)dt + v(S_t, t, \sigma)dZ_t,$$

where

- σ **can be specified as a function of** t**,** S_t **and other stochastic processes**,
- Z is a Brownian motion possibly correlated to W following $d[W, Z]_t = \rho dt$.

Today's lecture: Local volatility

Local volatility (LV): under $\mathbb{P}$

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dW_t,$$

where $(S, t) \mapsto \sigma(S, t)$ is a **function** or the stock price and time.

Beware: Do not confuse $\sigma_{imp}(K, T)$ and the LV function $\sigma(S, t)$!
The LV function $\sigma(S, t)$ cannot depend on (K, T).

Aim of lecture: Understand the **properties of a LV model**.

Example of local volatility (LV)

Example of parametric LV: CEV model

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \alpha S_t^{\beta-1} dW_t, \qquad \alpha, \beta \geq 0,$$

$\Rightarrow \sigma(S,t) = \sigma(S) := \alpha S^{\beta-1}$.

Remarks:

- σ is a **function**: $\sigma(x) = \alpha x^{\beta-1}$ defined for a variable $x > 0$.
  When we plug in a stochastic variable $S_t > 0$, then $\sigma(S_t)$ **becomes stochastic**.

  $\Rightarrow$ The randomness in $\sigma(S_t)$ is **entirely determined by** S_t.
- $\beta = 1$: Black-Scholes, $\beta = 0$: Bachelier,
  $0 < \beta < 1$ introduces **leverage**, i.e. $S \searrow \Leftrightarrow$ volatility $\nearrow$.
- Thanks to the leverage effect **this model generates an implied volatility skew**. Fit to market is **not perfect** (parametric fit).

General local volatility model

Local volatility (LV): under $\mathbb{P}$,

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)d\tilde{W}_t,$$

where σ can be **any function** (continuous, positive, Lipschitz & linear growth conditions).

No arbitrage: NA $\Leftrightarrow$ there exists a martingale measure $\mathbb{Q}$, where

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma(S_t, t)dW_t,$$

where r_t (resp. q_t) is the instantaneous interest rate (resp. dividend yield), W a $\mathbb{Q}$ Brownian motion.

Under $\mathbb{Q}$, the discounted stock price is a martingale.

Completeness and pricing

Remark: In a local volatility setting, the martingale measure is **unique**. The **market** is therefore **complete**. **All European style contingent claims can be hedged using the stock and a risk-free bond**.

Intuition for completeness: The **only randomness** in the LV model is due to the **stock price** $\Rightarrow$ Trading the stock can therefore remove all risks (randomness).

Fair value V_t at time t of a payoff $h(S_T)$ at T is

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)ds} h(S_T)|\mathcal{F}_t].$$

Kolmogorov backward equation

Price $V(t, S)$ at time t with stock price S satisfies the **Black-Scholes PDE** where σ constant is replaced by $\sigma(S, t)$:

$$\frac{\partial V}{\partial t}(S,t) + (r-q)\frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) = rV(S,t),$$

with terminal condition $V(T, S) = h(S)$.

Backward equation: value is propagated backwards in time from T (where we know $V(T, S) = h(S)$ for all $S \in \mathbb{R}_+$) to its value $V(0, S)$ today $t = 0$ that we want to find out.

For arbitrary given $\sigma(S, t) \rightarrow$ **cannot solve PDE in closed form** $\Rightarrow$ no adapted BS formula. **Once you know** $\sigma(S, t) \rightarrow$ lots of **numerical techniques**

to approximate the price $V(0, S)$.

Questions on the LV model

Questions:

- **How do we specify** the LV function $\sigma(S, t)$?
- Can the LV model **fit the IV surface**? Is there a **link between the LV function** $\sigma(S, t)$ to the **IV surface** $\sigma_{imp}(K, T)$?
- What are the **properties** of a LV model? Are the **dynamics of the LV model realistic**?

Choice of LV function

Parametric: **Choose a function form** for the LV function, **e.g.** CEV model, $\sigma(S) := \alpha S^{\beta-1}$.

- Few (2) unknown parameters,
- **Calibration** (estimation) is **easy**,
- **Approximate fit**, i.e. model prices are "close" to market option prices.

Non-parametric: Do not specify a function form for the LV function. **Let the option prices** ($\Leftrightarrow$ implied volatilities) **imply** all the values $\sigma(S_i, t_j)$ for a chosen grid of points.

- Many (depending on precision you want to achieve) unknown parameters,
- **Calibration** can be (very) **difficult**,
- **Fit is exact**, i.e. model prices $=$ market prices for all strikes/maturities.

We discuss non-parametric LV functions.

Input and tools

Starting point: We have a **smooth** and arbitrage-free **call price surface** (equivalently smooth IVs) as a function of K and T, i.e. at least $\mathcal{C}^2$ w.r.t. K and $\mathcal{C}^1$ w.r.t. T.

There are two fundamental equations that have at our disposal: Backward Kolmogorov equation and Fokker-Planck (FP) equation.

We will use the FP equation to imply the LV function from option prices.

Kolmogorov backward equation

Backward equation: Over time, call prices $C(S,t,K,T)$ satisfy

$$\frac{\partial C}{\partial t}(S,t) + (r-q)\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) = rC(S,t),$$

with terminal condition $C(S,t=T,K,T) = (S-K)^+$.

Intuition: Fix an option (K,T). The backward equation tells us how the price of $C(S,t,K,T)$ evolves with time from maturity $t=T$ to today t_0.

Since the dynamics of the options ($\frac{\partial C}{\partial t}$, etc.) are unknown, **we can not find** $\sigma(S,t)$ **from this equation**.

Fokker-Planck equation

Notation: $f(0, S_0, T, S)$ is the **model** risk-neutral density of the stock price going from $(0, S_0)$ to (T, S).
The stock price value at time $t = 0$ is fixed to S_0 and observable $\Rightarrow$ can write $f(S_0, 0, S, T) = f(S, T)$, $f(0, S_0, T, S)$ **must depend on the LV function**.

Fokker-Planck equation: Risk-neutral densities $f(S, T)$ satisfy

$$\frac{\partial f}{\partial T}(S, T) = -\frac{\partial}{\partial S}\left((r_T - q_T)Sf(S, T)\right) + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma(S, T)^2 S^2 f(S, T)\right),$$

with initial condition $f(S, 0) = f(S_0, 0, S, 0) = \delta_{S_0}(S)$: Dirac mass centered in S_0.

Also called **Forward** Kolmogorov equation: **At time** $t = 0$ the **density of the stock price is known** $f(S, 0) = \delta_{S_0}(S)$. This equation describes the evolution of the probability density up to time T: $f(S, T)$.

Infer the LV function from market prices

Recall **Breeden-Litzenberger's formula:**

$$f(K,T) = f(0,S_0,S=K,T) = e^{r_{0,T}T}\frac{\partial^2 C}{\partial K^2}(0,S_0,K,T).$$

We can plug the derivative of the call price $C(0,S_0,K,T) = C(K,T)$ into the Fokker-Planck equation and integrate twice:

$$\frac{\partial C}{\partial T}(K,T) = \frac{1}{2}\sigma(K,T)^2K^2\frac{\partial^2 C}{\partial K^2}(K,T) - q_T C(K,T) - K(r_T - q_T)\frac{\partial C}{\partial K}(K,T).$$

Remark: This equation relates the price of calls today for different strikes K and different maturities T to the LV function. (compare to the Backward equation!)

Infering the LV function: Dupire's formula

Dupire's formula

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T}(K,T) + q_T C(K,T) + (r_T - q_T) K \frac{\partial C}{\partial K}(K,T)}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}(K,T)}$$

Interpretation: $\sigma(K,T)$: **local volatility that will prevail at time** $t = T$ **if** $S_T = K$.

Steps:

1. Treat options' data,
2. Compute a smooth estimator of call prices,
3. Use Dupire's formula to infer the LV function $\sigma(K,T)$

Proof: See both proofs in script.

Remarks on Dupire's formula

Remarks:

- Distinguish the backward and forward equation.
  - Backward equation gives information about option prices **dynamics for a fixed option** (K, T),
  - Forward equation gives information about **relation between all option prices at one point in time** (t_0, S_0). No information about dynamics.

- Option prices today entirely determine the LV function, i.e. **static information of the smile determines the dynamics of the stock price process, i.e. determine the dynamics of the smile**. This is a crucial property of the LV model.

Remarks on Dupire's formula

More remarks:

- The option prices today determine entirely the risk-neutral distributions $f(S_0, t_0, S_T, T)$ $(\forall T > 0)$ and vice-versa (Breeden-Litzenberger), i.e. the risk-neutral distributions entirely determine the LV function and vice-versa.
  Given a set of *marginal* **distributions** $f(S_0, t_0, S_T, T)$, $\forall T > 0$, there **exists a unique LV function** such that $f(S_0, t_0, S_T, T)$ is the risk-neutral distribution of the stock at time T.

- Assume $r = q = 0$, $\sigma^2(K, T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$. We can approximate

$$\frac{\partial C}{\partial T} = \frac{C(S_0, 0, K, T + dT) - C(S_0, t_0, K, T)}{dT}, \text{ calendar spread}$$
$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S_0, 0, K + dK, T) - 2C(S_0, 0, K, T) + C(S_0, 0, K - dK, T)}{dK^2},$$

butterfly spread

Remarks on Dupire's formula

Technical remarks:

- For the Dupire formula to make sense the **right hand side must be positive**.
  - $\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}(K,T)$ proportional to risk-neutral density,
  - No arbitrage conditions show that numerator must be positive,
- Difficult to implement. Denominator $\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}(K,T) \to 0$ when $K \to 0$ and $K \to \infty$.
  - Use implied volatilities instead of option prices,
  - Use PDE numerical methods to solve the Fokker-Planck equation.

Local volatilities as a function of IVs

Dupire's formula

$$\sigma(K,T)^2 = \frac{\sigma_{imp}^2 + 2\sigma_{imp} T\left(\frac{\partial \sigma_{imp}}{\partial T} + (r_T - q_T)K\frac{\partial \sigma_{imp}}{\partial K}\right)}{\left(1 - \frac{Ky}{\sigma_{imp}}\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + K\sigma_{imp} T\left(\frac{\partial \sigma_{imp}}{\partial K} - \frac{1}{4}K\sigma_{imp} T\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + K\frac{\partial^2 \sigma_{imp}}{\partial K^2}\right)}$$

where y is the log- forward moneyness: $y = \ln(K/F_T)$.

Questions answered so far

Some answers:

- **How do we specify** the LV function $\sigma(S, t)$?
  $\longrightarrow$ Dupire formula!
- Can the LV model **fit the IV surface**?
  $\longrightarrow$ Yes, The LV function is constructed to match all vanilla prices. The IV surface is reproduced.

How does the IV surface and LV function compare?

Comparison

Start with IV surface and compute the LV function using Dupire's formula

Example 1: Flat IV surface.

If $\sigma_{imp}(K,T) = \sigma_{imp}$, then $\sigma(S,t) = \sigma_{imp}$ and the dynamics is

$$\frac{dS_t}{S_t} = rdt + \sigma_{imp} dW_t.$$

Comparison

Example 2: Maturity dependent IV surface.

If $\sigma_{imp}(K,T) = \sigma_{imp}(T)$, then the LV only depends on time

$$\begin{aligned}\sigma^2(T) &= \sigma^2_{imp}(T) + 2T\sigma_{imp}(T)\sigma'_{imp}(T)\\ &= \sigma^2_{imp}(T) + T\left(\sigma^2_{imp}(T)\right)'\\ &= \left(T\sigma^2_{imp}\right)'(T)\end{aligned}$$

which means that local variances are the derivative of the total BS implied variance.

$$\sigma^2_{imp}(T) = \frac{1}{T}\int_0^T \sigma^2(t)dt.$$

Interpretation: The implied variance for maturity T is the **average of the local variance from today** $t = 0$ **to this maturity** T.

To price options with maturity T, **plug** $\sigma^2_{imp}(T) = \frac{1}{T}\int_0^T \sigma^2(t)dt$ **into the BS formula**.

Comparison

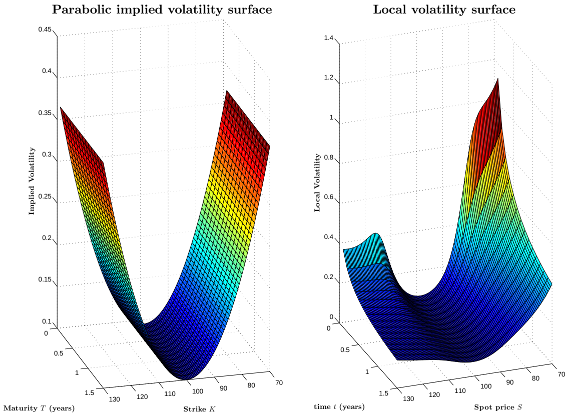
Example 3: Strike dependent IV surface.

When $\sigma_{imp}(K,T) = \sigma_{imp}(K)$, then $\sigma(S,t)$ **depends on stock price and time**.

We consider characteristic shapes of the IV surface (parabola, hyperplane) and look at corresponding LV function.

Parabolic implied volatility surface

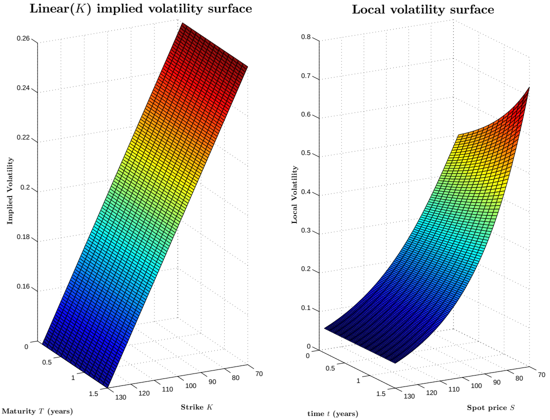
Parabolic implied volatility surface as a function of strike. $S_0 = 100$, $r = q = 0$



Comments: level, asymmetry, flattening of LV function.

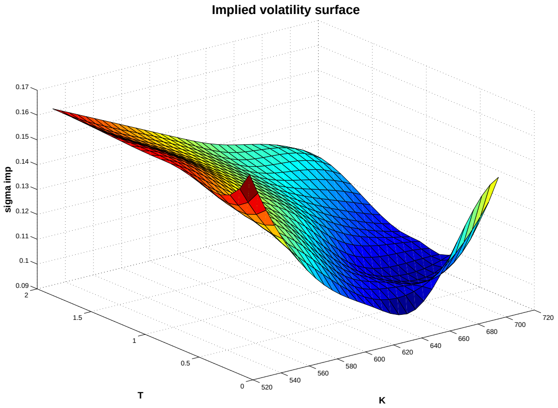
Affine implied volatility surface

Affine implied volatility surface as a function of strike. $S_0 = 100$, $r = q = 0$



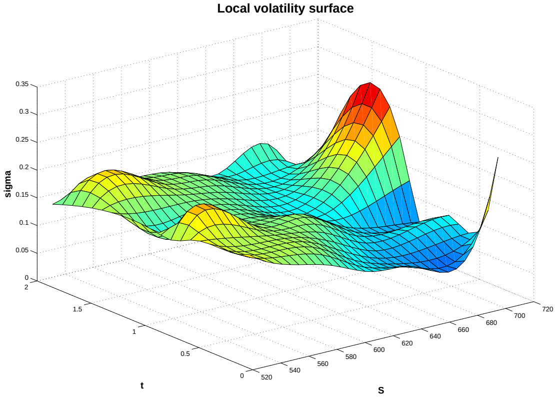
Comments: slight non-linearity, asymmetry.

Using some real data



Comments: smoothed with cubic splines.

Using some real data



Comments: **difficult to interpret!** Due to interpolation?

What we like

- Local volatility process is merely a function of the stock price process (and time) $\Rightarrow$ many useful properties of the Black-Scholes model remain valid.
  In particular **all European style contingent claims are hedgeable**, the martingale measure is unique (complete market).
- For a given smooth implied volatility surface, there exists a **unique local volatility function that will allow to replicate all option prices**.
- Only one stochastic factor (S), **pricing derivatives is fast**. PDE methods solving Backward equation or trees.

What we dislike

- **LV function very sensitive to interpolation method** (for σ_{imp}),
- **Dynamics** of LV model.

The implied dynamics

Issue:

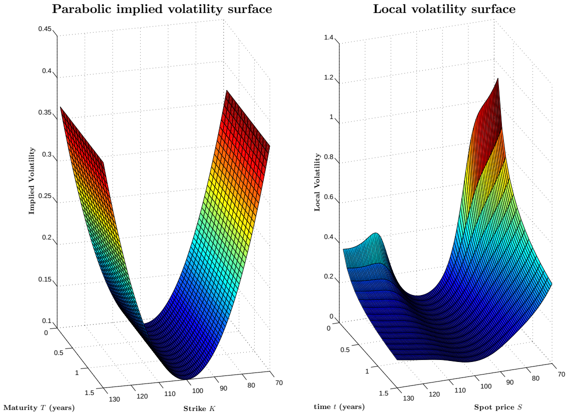
The shape of the smile today gives the LV function (Dupire formula),

LV function gives dynamics of prices (Kolmogorov backward equation),

$\Rightarrow$ The LV model claims: **The shape of the smile today determines the shape of the smile at any point in the future**!

The implied dynamics

Example:



Often LV flattens $\Rightarrow$ IV flattens over time.

What we dislike: the implied dynamics

Hagan et al. have shown:

Assume that the LV function $\sigma(S)$. Then

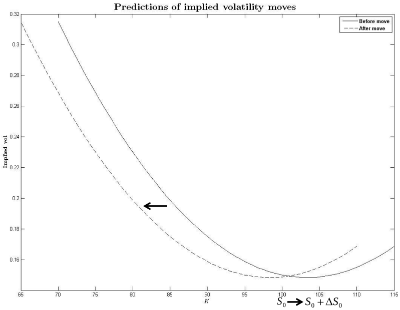
$$\sigma_{imp}(K, S_0) \approx \sigma(\frac{1}{2}[S_0 + K]).$$

If the stock prices moves from S_0 to $S_0 + \Delta S_0$, then

$$\sigma_{imp}(K, S_0 + \Delta S_0) = \sigma(\frac{1}{2}[S_0 + \Delta S_0 + K]) = \sigma_{imp}(K + \Delta S_0, S_0).$$

In a LV model: the stock price and implied volatility move in the **different directions**.

What we dislike: the implied dynamics



In reality: smile moves in the **same direction** as stock price.

Conclusion

Many empirical tests show that LV models **hedge worse** than Black-Scholes (because of LV dynamics).

LV model:

- Can replicate very accurately vanilla option prices,
- Dynamics are severely flawed: appears to hedge worse than Black-Scholes.

Financial Engineering - Lecture 7

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

03.11.2011

University of Zurich

Recall: Local Volatility Models

Goal: Represent the implied volatility surface

Advantages of local vol. models:

- By construction, reproduce well the volatility surface at some time t
- One-factor models $\Rightarrow$ Complete market
- Easy to understand.

BUT: They do not predict the right dynamics for the implied volatility surface

Recall: Implied volatility surface

Why should we care?

- When pricing products which are sensitive to future volatility levels (forward starting options)
- When pricing products on volatility
- For hedging purposes.

$\rightarrow$ **Stochastic volatility models !**

Stochastic volatility

General form of stochastic volatility models

$$dS_T = \mu_S(S_T, V_t, t)dt + \sqrt{V_t}dZ_t$$
$$(1)$$
$$dV_t = \mu_V(S_t, V_t, t)dt + \sigma_V(S_t, V_t, t)W_t$$
$$(2)$$
$$\mathbb{E}[dW_t dZ_t] = \rho dt \qquad (3)$$

Two-factor model $\Rightarrow$ The market is **not complete** anymore.

Market incompleteness

Recall: Completeness means that **perfect replication** of the option's payoff using the bond and the underlying is **possible**.

With stochastic volatility models: volatility is **not tradable** $\Rightarrow$ Cannot be hedged with bonds and stocks.

Implications:

- More than one risk-neutral measure
- $\Rightarrow$ More than one arbitrage-free price!
- Attempts to "complete" the market with options (not very efficient), and portfolios of variance / volatility-based instruments.

Relationship between implied and instantaneous (local / stochastic) volalitility

Assume $r = 0$ and a general continuous dynamics for S:

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

Goal: Find a relationship between $\sigma(t, S_t)$ and $\sigma_{imp}(K, T)$.

Proposition (Keller & Teichmann, 2009)

There exists a unique positive deterministic function $\bar{\sigma}_{K,T}(t)$, such that:

$$\mathbb{E}^{\mathbb{Q}}[C_{BS}(t, S_t, K, T, \bar{\sigma}_{K,T}(t))] = C_0(K, T) \quad \forall t \in [0, T). \tag{4}$$

Proof: See lecture notes.

Relationship between implied and instantaneous (local / stochastic) volalitility

At $t = 0$:

$$C_{BS}(0, S_0, K, T, \bar{\sigma}_{K,T}(0)) = C_0(K, T)$$

$\Rightarrow \bar{\sigma}_{K,T}(0)$ is the Black-Scholes implied volatility of the call C.

Black-Scholes forward implied variance (Gatheral)

$$v_{K,T}(t) = -\frac{\partial}{\partial t}(\bar{\sigma}^2_{K,T}(t).(T-t)).$$

Representation of the implied variance:

$$\sigma^2_{\text{imp}}(K, T) = \bar{\sigma}^2_{K,T}(0) = \frac{1}{T}\int_0^T v_{K,T}(t)dt.$$

This gives us $\sigma^2_{\text{imp}}(K, T)$ as a function of $v_{K,T} \Rightarrow$ we need $v_{K,T}$ as a function of $\sigma(t, S_t)$.

Relationship between implied and instantaneous (local / stochastic) volalitility

Denote

$$f(t, S_t) = C_{BS}(t, S_t, K, T, \bar{\sigma}_{K,T}(t))$$

Differentiation with respect to t gives:

$$\begin{aligned}\frac{\partial f}{\partial t}(t, S_t) = \quad & \frac{\partial C_{BS}}{\partial t}(t, S_t, K, T, \bar{\sigma}_{K,T}(t)) \\ & + \frac{\partial C_{BS}}{\partial \sigma}(t, S_t, K, T, \bar{\sigma}_{K,T}(t)) \frac{\partial \bar{\sigma}_{K,T}(t)}{\partial \bar{\sigma}^2_{K,T}(t)} \frac{\partial \bar{\sigma}^2_{K,T}(t)}{\partial t} \qquad (5)\end{aligned}$$

Relationship between implied and instantaneous (local / stochastic) volalitility

Black-Scholes PDE with $r = 0 \Leftrightarrow$ Relationship between Theta and Gamma

$$\frac{\partial C_{BS}}{\partial t} = -\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_{BS}}{\partial S^2}.$$

Relationship between Gamma and Vega:

$$\frac{\partial C_{BS}}{\partial \sigma} = \sigma S_t^2 (T - t) \frac{\partial^2 C_{BS}}{\partial S^2}.$$

Proof: Exercise

Relationship between implied and instantaneous (local / stochastic) volalitility

$$\frac{\partial \bar{\sigma}_{K,T}(t)}{\partial \bar{\sigma}^2_{K,T}(t)} = \frac{1}{2\bar{\sigma}_{K,T}(t)}$$

and

$$\begin{aligned} v_{K,T}(t) &= -\frac{\partial}{\partial t}(\bar{\sigma}^2_{K,T}(t).(T-t)) \\ &= -\frac{\partial}{\partial t}\bar{\sigma}^2_{K,T}(t)(T-t) + \frac{\partial}{\partial t}(\bar{\sigma}^2_{K,T}(t) \end{aligned}$$

$$\Rightarrow \frac{\partial \bar{\sigma}^2_{K,T}(t)}{\partial t} = \frac{1}{T-t}(\bar{\sigma}^2_{K,T}(t) - v_{K,T}(t)).$$

Relationship between implied and instantaneous (local / stochastic) volalitility

We plug these expressions in (5):

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t, S_t) = & \quad \frac{\partial C_{BS}}{\partial t}(t, ..., \bar{\sigma}_{K,T}(t)) + \frac{\partial C_{BS}}{\partial \sigma}(t, ..., \bar{\sigma}_{K,T}(t)) \frac{\partial \bar{\sigma}_{K,T}(t)}{\partial \bar{\sigma}^2_{K,T}(t)} \frac{\partial \bar{\sigma}^2_{K,T}(t)}{\partial t} \\
= & \quad -\frac{1}{2}\bar{\sigma}^2_{K,T}(t) S_t^2 \frac{\partial^2 C_{BS}}{\partial S^2}(t, ..., \bar{\sigma}_{K,T}(t)) + \\
& \quad \bar{\sigma}_{K,T}(t) S_t^2 (T-t) \frac{\partial^2 C_{BS}}{\partial S^2}(t, ..., \bar{\sigma}_{K,T}(t)) \frac{1}{2\bar{\sigma}_{K,T}(t)} \\
& \quad \frac{1}{T-t}(\bar{\sigma}^2_{K,T}(t) - v_{K,T}(t)) \\
= & \quad -\frac{1}{2} v_{K,T}(t) S_t^2 \frac{\partial^2 C_{BS}}{\partial S^2}(t, S_t, K, T, \bar{\sigma}_{K,T}(t)) \\
= & \quad -\frac{1}{2} v_{K,T}(t) S_t^2 \frac{\partial^2 f}{\partial S^2}(t, S_t)
\end{aligned}
$$

Relationship between implied and instantaneous (local / stochastic) volalitility

Apply Itô's lemma to f:

$$f(T,S_T)-f(\tau,S_\tau)=\int_\tau^T\frac{\partial f}{\partial S}dS_t+\int_\tau^T\frac{\partial f}{\partial t}dt+\int_\tau^T\frac{1}{2}\sigma^2(t,S_t)S_t^2\frac{\partial^2 f}{\partial S^2}dt$$

Take expectations:

$$\mathbb{E}^{\mathbb{Q}}[f(T,S_T)-f(\tau,S_\tau)]=C_0(K,T)-\mathbb{E}^{\mathbb{Q}}[f(\tau,S_\tau)]=0.$$

Relationship between implied and instantaneous (local / stochastic) volalitility

$$
\begin{aligned}
0 &= \mathbb{E}^{\mathbb{Q}}\left[\int_\tau^T \frac{\partial f}{\partial S}dS_t + \int_\tau^T \frac{\partial f}{\partial t}dt + \int_\tau^T \frac{1}{2}\sigma^2(t,S_t)S_t^2\frac{\partial^2 f}{\partial S^2}dt\right]\\
&= \int_\tau^T \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial f}{\partial S}dS_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\sigma^2(t,S_t)S_t^2\frac{\partial^2 f}{\partial S^2}dt\right]
\end{aligned}
$$

Let us solve separately:

$$
\int_\tau^T \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial f}{\partial S}dS_t\right] = \int_\tau^T \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial f}{\partial S}\sigma(t,S_t)d\tilde{W}_t\right] = \int_\tau^T \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial f}{\partial S}\sigma(t,S_t)d\tilde{W}_t|\mathcal{F}_t\right]\right]
$$

But $\frac{\partial f}{\partial S}\sigma(t,S_t)$ is $\mathcal{F}_t$-adapted, hence:

$$
\int_\tau^T \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial f}{\partial S}dS_t\right] = \int_\tau^T \frac{\partial f}{\partial S}\sigma(t,S_t)\mathbb{E}^{\mathbb{Q}}\left[d\tilde{W}_t|\mathcal{F}_t\right] = 0.
$$

Relationship between implied and instantaneous (local / stochastic) volalitility

$$
\begin{aligned}
0 &= \int_{\tau}^{T} \mathbb{E}^{\mathbb{Q}} \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 f}{\partial S^2} \right] dt \\
&= \int_{\tau}^{T} \mathbb{E}^{\mathbb{Q}} \left[-\frac{1}{2} v_{K,T}(t) S_t^2 \frac{\partial^2 f}{\partial S^2}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 f}{\partial S^2} \right] dt \\
&= \frac{1}{2} \int_{\tau}^{T} \mathbb{E}^{\mathbb{Q}} \left[\left(\sigma^2(t, S_t) - v_{K,T}(t)\right) S_t^2 \frac{\partial^2 f}{\partial S^2}(t, S_t) \right] dt
\end{aligned}
$$

Relationship between implied and instantaneous (local / stochastic) volalitility

τ is taken arbitrarily in $[0, \tau) \Rightarrow$ the integrand must be 0 for all $t \in [0, T)$.

$$
\begin{aligned}
v_{K,T}(t) &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\sigma^2(t, S_t) S_t^2 \frac{\partial^2 f}{\partial S^2}(t, S_t)\right]}{\mathbb{E}^{\mathbb{Q}}\left[S_t^2 \frac{\partial^2 f}{\partial S^2}(t, S_t)\right]} \\
&= \frac{\mathbb{E}^{\mathbb{Q}}\left[\sigma^2(t, S_t) S_t^2 \Gamma_{BS}(S_t, \bar{\sigma}_{K,T}(t))\right]}{\mathbb{E}^{\mathbb{Q}}\left[S_t^2 \Gamma_{BS}(S_t, \bar{\sigma}_{K,T}(t))\right]}
\end{aligned}
$$

Relationship between implied and instantaneous (local / stochastic) volalitility

$$\frac{d\mathbb{G}_t}{d\mathbb{Q}} = \frac{S_t^2 \Gamma_{BS}(S_t, \bar{\sigma}_{K,T}(t))}{\mathbb{E}^{\mathbb{Q}}\left[S_t^2 \Gamma_{BS}(S_t, \bar{\sigma}_{K,T}(t))\right]} \tag{6}$$

$$v_{K,T}(t) = \mathbb{E}^{\mathbb{G}_t}[\sigma^2(t, S_t)]$$

Relationship between implied and instantaneous (local / stochastic) volalitility

$$\sigma^2_{\text{imp}}(K, T) = \frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{G}_t}[\sigma^2(t, S_t)]dt.$$

$\Rightarrow$ **Implied vol = time-average of weighted expectations of the instantaneous volatility function** $\sigma^2(t, S_t)$.

Heston model

Dynamics under $\mathbb{P}$:

$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t} dZ_t$$
$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dW_t$$
$$d[Z, W]_t = \rho dt$$

- Stochastic variance process
- Cox-Ingerson-Ross process (see next slides)

Variance process

Properties of Cox-Ingerson-Ross process: $dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t$

- Mean-reverting with **level of mean reversion** θ and **speed of mean reversion** κ
- Volatility parameter *volvol* η
- Positive process (square root)
- Popular in interest-rate theory to model r_t

Variance process

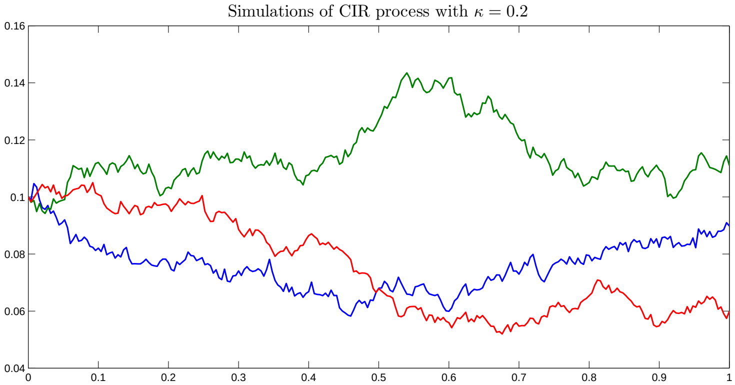


Figure: Paths of a CIR process with $\kappa = 0.2$, $\theta = 0.1$ and $\eta = 0.1$

Variance process

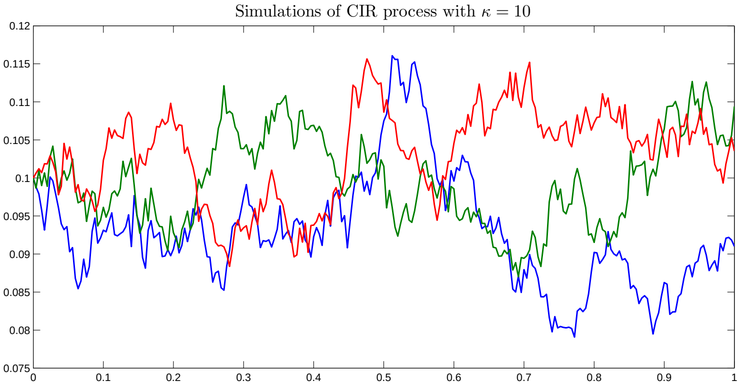


Figure: Paths of a CIR process with $\kappa = 10$, $\theta = 0.1$ and $\eta = 0.1$

Variance process

Feller condition

If $\eta^2 < 2\kappa\theta$ then $\mathbb{P}(\exists t \ s.t. \ V_t = 0) = 0$, i.e. the volatility will not reach 0 $\mathbb{P}$-almost surely.

- Intuition: η cannot be too large, θ and κ cannot be too small.
- In theory: Desired property
- In practice: is usually not satisfied by the calibrated parameters.

Leverage effect

The correlation coefficient is usually found to be **strongly negative**.

Leverage effect

Negative shocks in the stock price are followed by positive shocks in the volatility.

Idea:

Bad news $\Rightarrow$ The stock price decreases $\Rightarrow$ The ratio Debt/Equity increases $\Rightarrow$ The firm is riskier $\Rightarrow$ The volatility increases (and vice-versa).

Influence of ρ on the implied volatility surface

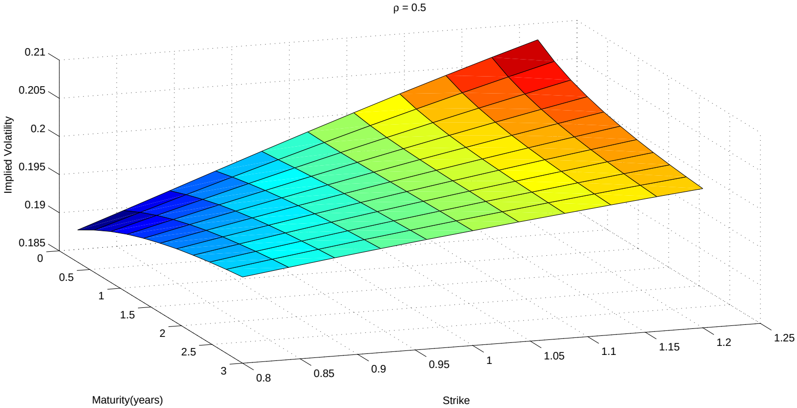


Figure: Implied volatility in the Heston model with $\rho = 0.5$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$

Influence of ρ on the implied volatility surface

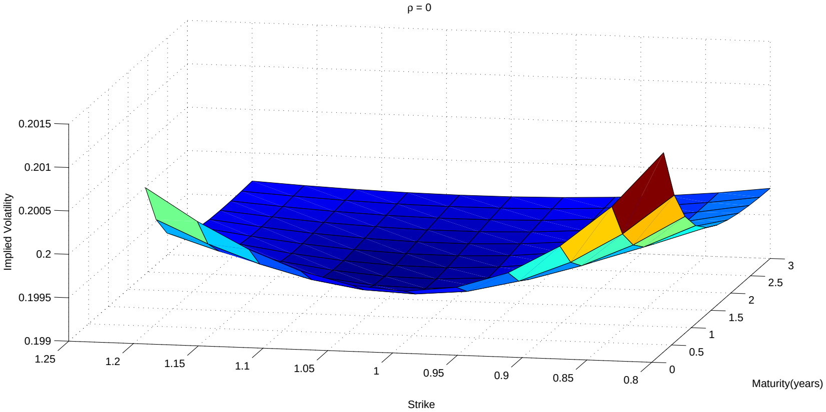


Figure: Implied volatility in the Heston model with $\rho = 0$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$

Influence of ρ on the implied volatility surface

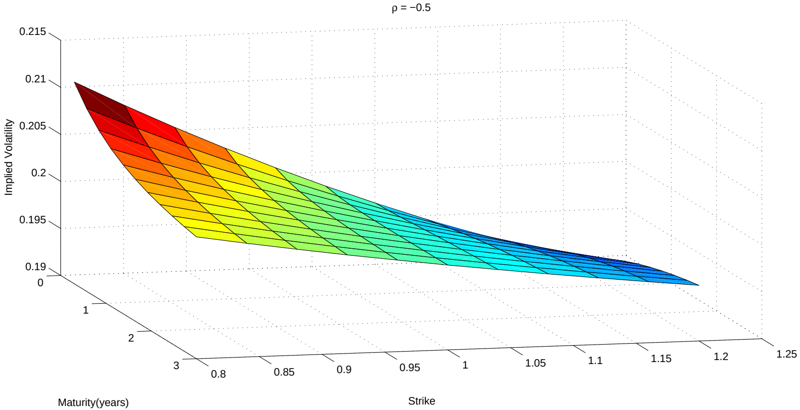


Figure: Implied volatility in the Heston model with $\rho = -0.5$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$

Influence of σ on the implied volatility surface

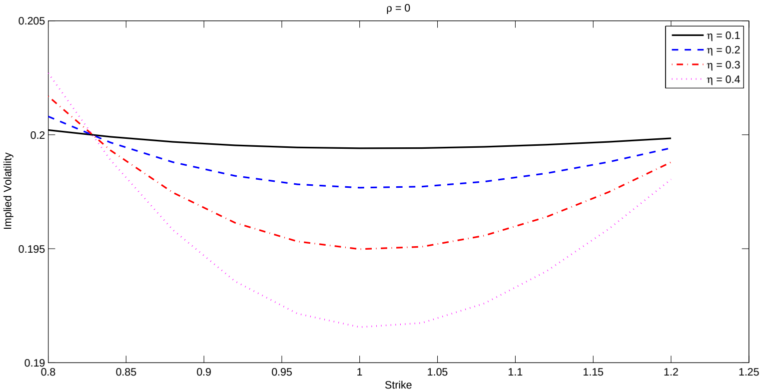


Figure: Implied volatility in the Heston model with $\rho = 0$, $\kappa = 2$, $\theta = 0.04$

Influence of σ on the implied volatility surface

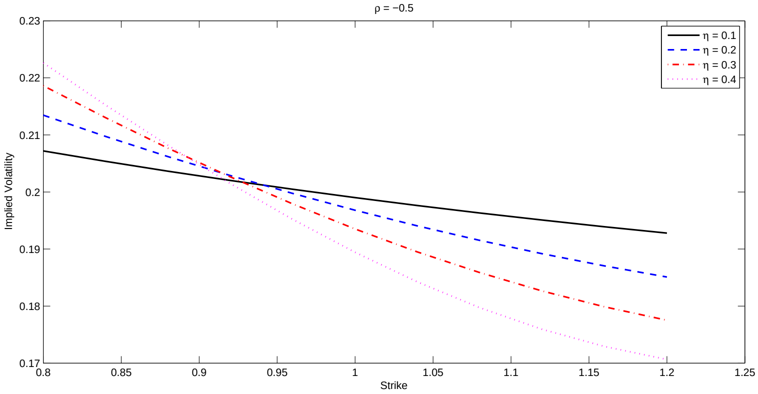


Figure: Implied volatility in the Heston model with $\rho = -0.5$, $\kappa = 2$, $\theta = 0.04$

Influence of σ on the implied volatility surface

To remember:

- ρ controls the **asymmetry** of the implied volatility smile
- η controls the kurtosis of the stock price distribution and impacts the curvature and asymmetry.

Stock price dynamics under $\mathbb{Q}$

Change of measure: Similar to Black-Scholes. Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T = \mathcal{E}\left(\int_0^T \frac{r-\mu_s}{\sigma dW_s}\right).$$

Under $\mathbb{Q}$, $\tilde{Z}_t = Z_t + \frac{\mu_t - r}{\sqrt{V_t}}$ is a Brownian motion.

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t}d\tilde{Z}_t.$$

Stock price dynamics under $\mathbb{Q}$

Goal: Derive the pricing PDE.

Difference compared to Black-Scholes: The risk of an option cannot be hedged using the underlying and the bond $\Rightarrow$ Need for another option.

1. Write the dynamics of the call option.

$$
\begin{aligned}
dC &= C_t dt + C_S dS_t + \frac{1}{2} C_{SS} d\langle S\rangle_t + C_V dV_t + \frac{1}{2} C_{VV} d\langle S\rangle_t + C_{SV} d\langle S, V\rangle_t \\
&= C_t dt + C_S dS_t + \frac{1}{2} C_{SS} V_t S_t^2 dt + C_V dV_t + \frac{1}{2} C_{VV} \eta^2 V_t dt + C_{SV} \rho\eta V_t S_t dt
\end{aligned}
$$

Derivation of the pricing PDE

1. Build a risk-free portfolio:

$$\pi_t = C_t^1 - a_t C_t^2 - b_t S_t$$

Apply Itô's formula to derive the dynamics of the portolio:

$$\begin{aligned}
d\pi_t = & C_t^1 dt + C_S^1 dS_t + \frac{1}{2} C_{SS}^1 V_t S_t^2 dt + C_V^1 dV_t + \frac{1}{2} C_{VV}^1 \eta^2 V_t dt + C_{SV}^1 \rho\eta V_t S_t dt \\
& - a_t \Big(C_t^2 dt + C_S^2 dS_t + \frac{1}{2} C_{SS}^2 V_t S_t^2 dt + C_V^2 dV_t + \\
& \frac{1}{2} C_{VV}^2 \eta^2 V_t dt + C_{SV}^2 \rho\eta V_t S_t dt \Big) \\
& - b_t dS_t
\end{aligned}$$

Derivation of the pricing PDE

2. Risk-free portfolio: return $= r$
$\Rightarrow d\pi_t = r\pi_t dt = r(C^1(t,S_t) - a_t C^2(t,S_t) - b_t S_t)dt.$

$$
\begin{aligned}
r(C^1 - a_t C^2 - b_t S_t)dt =& C^1_t dt + C^1_S r S_t dt + \frac{1}{2} C^1_{SS} V_t S_t^2 dt - C^1_V \kappa(V_t - \theta)dt \\
&+ \frac{1}{2} C^1_{VV} \eta^2 V_t dt + C^1_{SV} \rho\eta V_t S_t dt \\
&- a_t C^2_t dt - a_t C^2_S r S_t dt - \frac{1}{2} a_t C^2_{SS} V_t S_t^2 dt \\
&+ a_t C^2_V \kappa(V_t - \theta)dt - \frac{1}{2} a_t C^2_{VV} \eta^2 V_t dt \\
&- a_t C^2_{SV} \rho\eta V_t S_t dt - b_t r S_t dt \\
&+ (C^1_S - a_t C^2_S - b_t)\sqrt{V_t} S_t d\tilde{Z}_t \\
&+ (C^1_V - a_t C^2_V)\eta\sqrt{V_t} d\tilde{W}_t \qquad (7)
\end{aligned}
$$

Derivation of the pricing PDE

3. Compute the weights of the portfolio such that it is risk-free: no random terms.

$$a_t = \frac{C_V^1}{C_V^2}$$

$$b_t = C_S^1 - \frac{C_V^1}{C_V^2} C_S^2.$$

Derivation of the pricing PDE

4. Infer the PDE from the return of the portfolio = risk-free return. After dividing (7) by C_V^1:

$$
\begin{aligned}
0 = & \frac{1}{C_V^1}\left(-rC^1 + C_t^1 + C_S^1 rS_t + \frac{1}{2}C_{SS}^1 V_t S_t^2 - C_V^1\kappa(V_t-\theta)\right. \\
& \left. + \frac{1}{2}C_{VV}^1\eta^2 V_t + C_{SV}^1\rho\eta V_t S_t\right) \\
& + \frac{1}{C_V^2}\left(rC^2 - C_t^2 - C_S^2 rS_t - \frac{1}{2}C_{SS}^2 V_t S_t^2 + C_V^2\kappa(V_t-\theta)\right. \\
& \left. - \frac{1}{2}C_{VV}^2\eta^2 V_t - C_{SV}^2\rho\eta V_t S_t\right)
\end{aligned} \tag{8}
$$

Derivation of the pricing PDE

Infinitesimal generator

Let $X : [0, \infty) \times \Omega \to \mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ satisfying the following SDE:

$$dX_t = b(X_t)dt + \Sigma(X_t)dB_t$$

For any f which is twice differentiable with continuous second derivatives, the infinitesimal generator of X is the operator $\mathcal{A}$ given by:

$$\mathcal{A}f(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n}(\Sigma(x)\Sigma(x)^T)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

In Heston model:

$$\mathcal{A}_{Heston} = rS_t\frac{\partial}{\partial S} + \frac{1}{2}V_tS_t^2\frac{\partial^2}{\partial S^2} - \kappa(V_t - \theta)\frac{\partial}{\partial V} + \frac{1}{2}\eta^2 V_t\frac{\partial^2}{\partial V^2} + \rho\eta V_tS_t\frac{\partial^2}{\partial S\partial V}$$

Derivation of the pricing PDE

Equation (8) becomes:

$$\frac{1}{C_V^1}\left(-rC^1 + C_t^1 + \mathcal{A}_{Heston}C^1\right) = \frac{1}{C_V^2}\left(-rC^2 + C_t^2 + \mathcal{A}_{Heston}C^2\right) = \Phi(S, V, t).$$

Any option price satisfies:

$$C_V\Phi(S_t, V_t, t) = -rC + C_t + \mathcal{A}_{Heston}C \tag{9}$$

Market price of volatility risk

What is $\Phi(S_t, V_t, t)$? Apply Itô under $\mathbb{P}$:

$$\frac{dC_t}{C_t} = \mu_C dt + \sigma_{CS} dZ_t + \sigma_{CV} dW_t$$

Use the definition of the infinitesimal generator, the Heston pricing PDE and derive:

$$\mu_C - r = \frac{\sigma_{CS}}{\sqrt{V_t}}(\mu_t - r) + \frac{\sigma_{CV}}{\sqrt{V_t}}\Phi(S_t, V_t, t).$$

Risk premium of the option C = weighted average of the **market price of equity risk** $\mu_t - r$ and the **market price of volatility risk** $\Phi(S_t, V_t, t)$.

Market price of volatility risk

How to choose $\Phi(S_t, V_t, t)$?

- In theory: any choice is arbitrage-free: due to **incompleteness**.
- In practice: convenient choice to solve the PDE:

$$\Phi(S_t, V_t, t) = \Phi(V_t) = \lambda V_t.$$

Heston pricing PDE:

$$-rC + C_t + rSC_S + \frac{1}{2}VS^2 C_{SS} + [\kappa(\theta - V_t) - \lambda V_t]C_V + \frac{1}{2}\eta^2 VC_{VV} + \rho\eta VSC_{SV} = 0 \tag{10}$$

Transformation of Heston PDE

How to solve the PDE? Remember Black-Scholes $\Rightarrow$ transformed into the Heat Equation.

5. We make some changes of variables (don't learn them by heart!)
   - $\tau = T - t$,
   - $x = \ln\left(S_t e^{r(T-t)}\right) = \ln\left(S_t e^{r\tau}\right)$,
   - $C = e^{-r(T-t)} F$.

   We obtain:

$$-F_\tau + \frac{1}{2} V(F_{xx} - F_x) - (\kappa(V - \theta) + \theta V) F_V + \frac{1}{2}\eta^2 V F_{VV} + \rho\eta V F_{Vx} = 0$$

   Boundary condition:

$$F(x, V, 0) = \max(e^x - K, 0).$$

Transformation of Heston PDE

Issue: We obtained the Heston pricing PDE. But this time we could not transform it into the Heat Equation...

Solution: **Fourier Transform**

Fourier Transform

The *Fourier transform* $\hat{f}(k)$ of a function $f(x)$ is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

if the integral is well-defined. The *inverse Fourier transform* is:

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(k) dk$$

Fourier Transform

Why does it help to use Fourier transforms? Impact on derivatives of functions under certain conditions:

$$\hat{f}_x(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \frac{\partial f(x)}{\partial x} dx$$
$$= \left[\frac{1}{\sqrt{2\pi}} e^{ikx} f(x) \right]_{-\infty}^{\infty} - \frac{ik}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(x) dx$$
$$= -ik\hat{f}(k)$$

More generally:

$$\widehat{\frac{\partial^p f}{\partial x^p}}(k) = (-ik)^p \hat{f}(x) \tag{11}$$

$\Rightarrow$ **Fourier transforms make derivatives disappear!**

Fourier Transformed PDE

Apply Fourier transform to the PDE to get:

$$-\hat{F}_\tau + \frac{1}{2} V i k \hat{F} - \frac{1}{2} k^2 V \hat{F} + (\kappa(\theta - V) - \lambda V)\hat{F}_V + \frac{1}{2}\eta^2 V \hat{F}_{VV} - i k \rho \eta V \hat{F}_V = 0 \quad (12)$$

What happens to the boundary condition?

Boundary Condition

$$
\begin{aligned}
\hat{F}(k,V,0) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}F(x,V,0)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}\max(e^{x}-K,0)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}(e^{x}-K,0)\mathbf{1}_{e^{x}\geq 1}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\ln K}^{\infty} e^{ikx}(e^{x}-K)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(ik+1)x}}{ik+1}-K\frac{e^{ikx}}{ik}\right]_{\ln K}^{\infty} && (13) \\
&= \frac{1}{\sqrt{2\pi}}\frac{K^{ik+1}}{ik-k^{2}}, \quad \mathcal{I}m(k)>1 && (14)
\end{aligned}
$$

Convergence of the complex exponential

e^{ikx}, with k and x being real numbers, defines the unit circle in the complex plane.

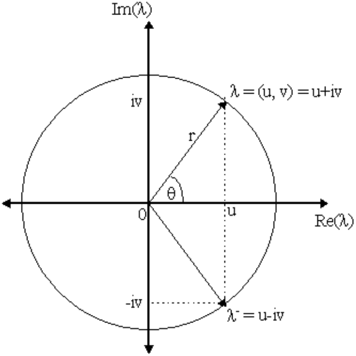


Figure: Unit circle with $r = 1$

$\Rightarrow$ Not converging when x tends to infinity.

Convergence of the complex exponential

Take k complex and write $k = \mathcal{R}e(k) + i\mathcal{I}m(k)$. We have:

$$ikx = -\mathcal{I}m(k)x + i\mathcal{R}e(k)x$$

$$e^{ikx} = e^{-\mathcal{I}m(k)x} e^{i\mathcal{R}e(k)x}.$$

For e^{ikx} to converge when $x \to \infty$ we need $e^{-\mathcal{I}m(k)x}$ to go to 0, or equivalently $\mathcal{I}m(k)x > 0$.

Solve the Heston PDE

6. Find a Fundamental Solution $\hat{F}^0(k, V, \tau)$

Fundamental Solution

The solution $\hat{F}^0(k, V, \tau)$ of the PDE (12) with the boundary condition

$$\hat{F}(k, V, 0) = 1$$

is called Fundamental Solution.

7. Option price recovered by:

$$F(x, V, \tau) = \int_{ia-\infty}^{ia+\infty} e^{-ikx} \hat{F}(k, V, 0) \hat{F}^0(k, V, \tau) dk \quad , \quad \mathcal{I}k > 1, a_- < a < a_+ \tag{15}$$

To find the fundamental solution we make the Ansatz:

$$\hat{F}^0(k, V, \tau) = e^{C(k,\tau)+D(k,\tau)V} \tag{16}$$

$$C(k, 0) = D(k, 0) = 0$$

Solve the Heston PDE

We obtain:

$$D(k,\tau) = r_1 \frac{1-e^{d\tau}}{1-ge^{d\tau}}. \tag{17}$$

$$C(k,\tau) = \kappa\theta \left(r_1\tau - \frac{2}{\eta^2} \log\left(\frac{1-ge^{d\tau}}{1-g} \right) \right) \tag{18}$$

Singularities in Heston model

Complex logarithm

The solution of the equation $e^w = z$, with $w, z = re^{i\phi} = r(\cos\phi + i\sin\phi) \in \mathbb{C}$ is called the complex logarithm. It is defined as:

$$\log z = \log|z| + i(\arg(z) + 2\pi n), \quad n \in \mathbb{Z}. \tag{19}$$

where $\arg(z) = \phi$ is the argument of the complex number and $|z| = r$ is its radius.

Problem: By continuity we want $\log e^{i\phi} = i\phi$ for all ϕ. In particular, $\log e^{2\pi i}$ should be equal to $2\pi i$. However, because $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$, we have $\log e^{2\pi i} = \log 1 = 0 \neq 2\pi i$.

$\Rightarrow$ Branches.

Singularities in Heston model

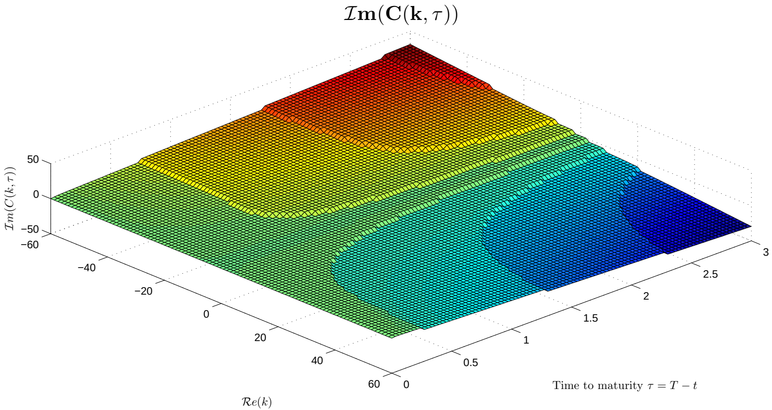


Figure: Discontinuities in the Fundamental Solution of Heston model. The parameters used are: $\kappa = 6.21$, $\theta = 0.019$, $\eta = 0.61$, $\rho = -0.7$, $T = 3$.

Singularities in Heston model

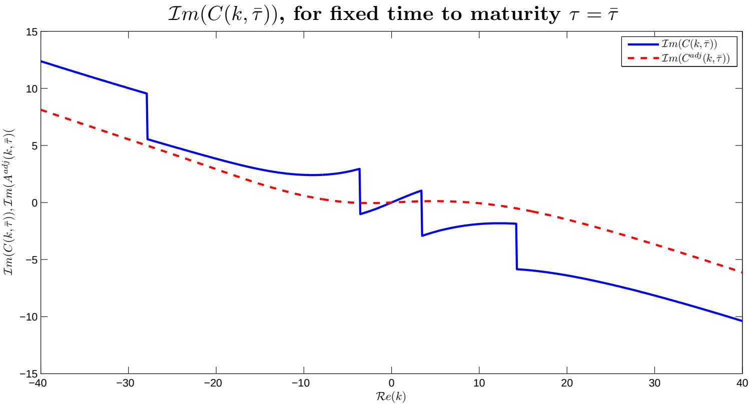


Figure: Comparison of the original formula for $C(k,\tau)$ with the adjusted formula $C^{adj}(k,\tau)$.

Simulation of Heston model

- **Euler discretization**

$$V_{i+1} = V_i - \kappa(V_i - \theta)\Delta t + \eta\sqrt{V_i}\sqrt{\Delta t}Z$$

$Z \sim \mathcal{N}(0,1)$.

BUT: V may become negative $\Rightarrow$ Set to 0 or to -V when it happens.

- **Milstein discretization**

$$V_{i+1} = V_i - \kappa(V_i - \theta)\Delta t + \eta\sqrt{V_i}\sqrt{\Delta t}Z + \frac{\eta^2}{4}\Delta t(Z^2 - 1)$$

Less negative values & Better convergence.

Heston model in practice

Pros:

- Exhibits **mean-reversion** of volatility
- Easy to implement (**semi closed-form** expression for prices)
- Represents well the **volatility surface**
- Captures its **dynamics** better than local vol. models.

Cons:

- **Initial value** of volatility not observable
- Smile too **flat** for short maturities
- **Feller condition** often not satisfied in practice.

Calibration of the Heston model

"Market implied" approach for model calibration: Find parameters such that the model prices are as close as possible to the market prices.

$\Rightarrow$ **Inverse problem formulation.**

Example: Least-Squares approach:

$$\min_{\Theta} LS(\Theta) = \sum_{i=1}^{N} \left(C_i^{obs} - C_i(K_i, T_i, \Theta) \right)^2$$

Problem: Favors in-the-money options.

Calibration of the Heston model

Two alternatives:

- Consider relative pricing errors:

$$\min_{\Theta} LS(\Theta) = \sum_{i=1}^{N} \left(\frac{C_i^{obs} - C_i(K_i, T_i, \Theta)}{C_i^{obs}} \right)^2$$

- Consider implied volatilities

$$\min_{\Theta} LS(\Theta) = \sum_{i=1}^{N} (\sigma_{imp,i}^{obs} - \sigma_{imp,i}(K_i, T_i, \Theta))^2$$

Even better: add **weights** (liquidity)

$$\min_{\Theta} LS(\Theta) = \sum_{i=1}^{N} \omega_i (\sigma_{imp,i}^{obs} - \sigma_{imp,i}(K_i, T_i, \Theta))^2$$

Calibration of the Heston model

Non-convex optimization: How to handle the issue of **local minima**?

- Use **non gradient-based algorithms**: Differential Evolution Algorithm
- **Regularize** the objective function to make it convex:

$$LS^{reg}(\Theta) = \sum_{i=1^N} \omega_i(\sigma^{obs}_{imp,i} - \sigma_{imp,i}(K_i, T_i, \Theta))^2 + \alpha p(\Theta).$$

Financial Engineering - Lecture 8

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

10.11.2011

University of Zurich

To start

- Preparation of the exam
- Projects: please send us the table of contents of your report and status update (5 lines max) before Monday 14 November, midnight.
- Questions
- Data

Recall: Models seen so far

Black-Scholes model:

- Complete market: every claim can be replicated using the underlying and the bond
- Closed-form expression for the option price, derived using for example the PDE or the martingale approach
- No representation of the implied volatility surface!

Recall: Models seen so far

Black-Scholes model:

- Complete market: every claim can be replicated using the underlying and the bond
- Closed-form expression for the option price, derived using for example the PDE or the martingale approach
- No representation of the implied volatility surface!

Recall: Models seen so far

Black-Scholes model:

- Complete market: every claim can be replicated using the underlying and the bond
- Closed-form expression for the option price, derived using for example the PDE or the martingale approach
- No representation of the implied volatility surface!

Recall: Models seen so far

Local volatility models:

- The volatility is a deterministic function of time and the underlying
- No new source of randomness, market still complete
- Accurate representation of the implied volatility surface at time 0
- No closed-form expression for the option price anymore $\rightarrow$ Need for numerical methods
- Wrong prediction of the dynamics of the implied vol surface!

Recall: Models seen so far

Local volatility models:

- The volatility is a deterministic function of time and the underlying
- No new source of randomness, market still complete
- Accurate representation of the implied volatility surface at time 0
- No closed-form expression for the option price anymore $\rightarrow$ Need for numerical methods
- Wrong prediction of the dynamics of the implied vol surface!

Recall: Models seen so far

Local volatility models:

- The volatility is a deterministic function of time and the underlying
- No new source of randomness, market still complete
- Accurate representation of the implied volatility surface at time 0
- No closed-form expression for the option price anymore $\rightarrow$ Need for numerical methods
- Wrong prediction of the dynamics of the implied vol surface!

Recall: Models seen so far

Stochastic vol models:

- Volatility modelled as a new stochastic process correlated with the underlying process
- Better prediction of the dynamics of the implied vol
- Semi-closed form expression for the call option price
- New source of randomness: the market is not complete anymore!
- Smile of volatility too flat for short maturities

Goal: Find a model which allows for steeper smiles at short maturities.

Recall: Models seen so far

Stochastic vol models:

- Volatility modelled as a new stochastic process correlated with the underlying process
- Better prediction of the dynamics of the implied vol
- Semi-closed form expression for the call option price
- New source of randomness: the market is not complete anymore!
- Smile of volatility too flat for short maturities

Goal: Find a model which allows for steeper smiles at short maturities.

Recall: Models seen so far

Stochastic vol models:

- Volatility modelled as a new stochastic process correlated with the underlying process
- Better prediction of the dynamics of the implied vol
- Semi-closed form expression for the call option price
- New source of randomness: the market is not complete anymore!
- Smile of volatility too flat for short maturities

Goal: Find a model which allows for steeper smiles at short maturities.

Recall: Models seen so far

Everything should be made as simple as possible, but not simpler.

Albert Einstein

Counting processes

Counting process

Let T_t be an increasing sequence of random times. The process N_t defined by

$$N_t = \begin{cases} n & \text{if } t \in [T_n, T_{n+1}] \\ +\infty & \text{otherwise} \end{cases}$$

is called **counting process**. Equivalently, we have:

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\}} = \sum_{n \geq 1} n \mathbf{1}_{\{T_n \leq t < T_{n+1}\}}.$$

Stochastic integral with respect to a counting process:

$$\int_0^t C_s dN_s = \sum_{n=1}^{\infty} C_{T_n} \mathbf{1}_{\{T_n \leq t\}}.$$

Poisson processes

Poisson process

A **Poisson process** $(N_t)_{t\geq 0}$ is a continuous-time counting process such that:

- $N_0 = 0$
- For every s, t, $N_{t+s} - N_t$ is independent of $\mathcal{F}_t$ where $\mathcal{F}_t$ is the filtration generated by N, i.e. increments are independent
- For every s, t, the random variable $N_{t+s} - N_t$ has the same law as N_s, i.e. increments are stationary.

- Special time of counting process
- Similarities with Brownian motion

Poisson processes

Some properties of Poisson processes:

- The distribution of N_t is a Poisson distribution with intensity λt:

$$\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

- Two first moments:

$$\mathbb{E}[N_t] = \lambda t \quad ; \quad Var(N_t) = \lambda t.$$

- The $\tau_i = T_{i+1} - T_i$ are exponentially distributed with parameter λ:

$$\mathbb{P}(\tau_i \leq y) = 1 - e^{-\lambda y},$$

Example: Total number of trams up to time t.

Poisson processes

One more important property:

The trajectories of a Poisson process are piecewise constant with jump size almost surely equal to 1:

$$\lim_{\Delta t \to 0} \mathbb{P}(N_{t+\Delta t} - N_t = 1 | N_{t+\Delta t} \neq N_t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_{t+\Delta t} - N_t = 1 \ \& \ N_{t+\Delta t} \neq N_t)}{\mathbb{P}(N_{t+\Delta t} \neq N_t)}$$

By definition:

$$\mathbb{P}(N_{t+\Delta t} - N_t = k) = \mathbb{P}(N_{\Delta t} = k) = \frac{e^{-\lambda \Delta t} (\lambda \Delta t)^k}{k!}.$$

Poisson processes

Hence:

$$\begin{aligned}
\lim_{\Delta t \to 0} \mathbb{P}(N_{t+\Delta t} - N_t = 1 | N_{t+\Delta t} \neq N_t) &= \lim_{\Delta t \to 0} \frac{e^{-\lambda \Delta t} \lambda \Delta t}{\sum_{k=1}^{\infty} \frac{e^{-\lambda \Delta t}(\lambda \Delta t)^k}{k!}} \\
&= \lim_{\Delta t \to 0} \frac{\lambda \Delta t}{\sum_{k=0}^{\infty} \frac{(\lambda \Delta t)^k}{k!} - 1} \\
&= \lim_{\Delta t \to 0} \frac{\lambda \Delta t}{e^{\lambda \Delta t} - 1} \\
&= \lim_{\Delta t \to 0} \frac{\lambda \Delta t}{\lambda \Delta t + o((\Delta t)^2)} = 1
\end{aligned}$$

Poisson processes

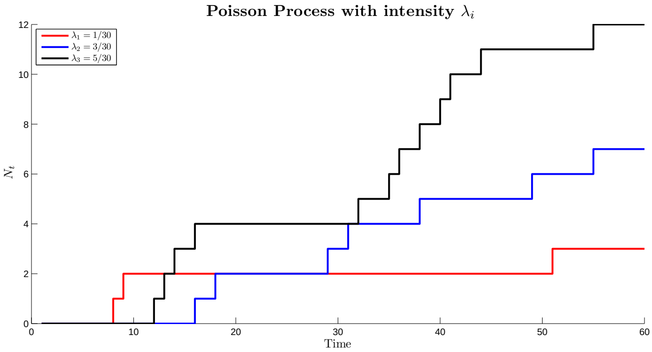


Figure: Paths of three Poisson processes with intensities $\lambda_1 = 1/30$, $\lambda_2 = 3/30$ and $\lambda_3 = 5/30$.

Characteristic function

Characteristic function

The characteristic function of a random variable X is the function $\varphi : \mathbb{R} \to \mathbb{C}$ such that:

$$\varphi(u) = \mathbb{E}[e^{iuX}].$$

Why is it useful?

> *The shortest path between two truths in the real domain passes through the complex domain.*
>
> *J. Hadamard*

... and also, later on, for calibration!

Characteristic function

For a Poisson process:

$$\begin{aligned}\varphi^{Poi}(u) = \mathbb{E}[e^{iuN_t}] &= \sum_{n=0}^{\infty} e^{iun}\mathbb{P}(N_t = n)\\ &= \sum_{n=0}^{\infty} e^{iun} e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\exp(\lambda t e^{iu})\\ &= \exp(\lambda t(e^{iu} - 1)).\end{aligned}$$

Martingale property

The process $M_t = N_t - \lambda t$ is a martingale:

$$\begin{aligned}\mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}[N_t|\mathcal{F}_s] - \lambda t = \mathbb{E}[N_t - N_s|\mathcal{F}_s] + \mathbb{E}[N_s|\mathcal{F}_s] - \lambda t \\ &= \mathbb{E}[N_t - N_s] + N_s - \lambda t = \mathbb{E}[N_{t-s}] + N_s - \lambda t \\ &= \lambda(t-s) + N_s - \lambda t = N_s - \lambda s = M_s.\end{aligned}$$

λt is said to *compensate* N_t.

Application to finance

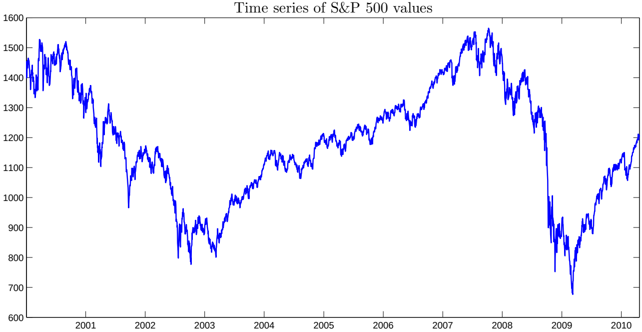


Figure: Time series of S&P 500.

Compound Poisson Process

Major restriction of Poisson process: Jump size $= 1$ almost surely

$\Rightarrow$ Compound Poisson process

Compound Poisson process

Let N be a Poisson process with parameter λ and Y_i a sequence of independent random variables with density function f. The process

$$X_t = \sum_{i=1}^{N_t} Y_i$$

is called **compound Poisson process**.

Compound Poisson Process

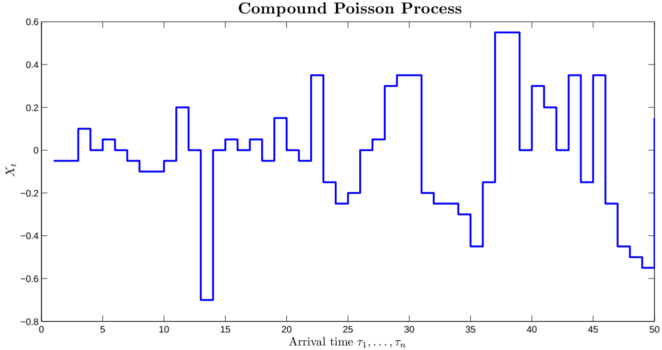


Figure: Path of a compound Poisson process

Compound Poisson Process

Characteristic function of the compound Poisson process:

$$\varphi^{CPoi}(u) = \mathbb{E}[e^{iuX}] = \mathbb{E}\left[e^{iu\sum_{k=1}^{N_t} Y_k}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{k=1}^{N_t} Y_k}|N_t = n\right]\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[\prod_{k=1}^{N_t} e^{iuY_k}|N_t = n\right]\right].$$

Because the Y_k are independent and identically distributed this is equal to:

$$\mathbb{E}\left[\mathbb{E}\left[\prod_{k=1}^{N_t} e^{iuY_k}|N_t = n\right]\right] = \mathbb{E}\left[\prod_{i=k}^{n} \mathbb{E}\left[e^{iuY_k}\right]\right]$$

Compound Poisson Process

Because the Y_i are independent and identically distributed this is equal to:

$$\begin{aligned}
\mathbb{E}[e^{iuX}] &= \mathbb{E}\left[\prod_{i=1}^{n}\mathbb{E}\left[e^{iuY_i}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuY_i}\right]^n\right] \\
&= \sum_{n=0}^{\infty}\mathbb{E}\left[e^{iuY_i}\right]^n\mathbb{P}(N_t = n) = e^{-\lambda t}\sum_{n=0}^{\infty}\mathbb{E}\left[e^{iuY_i}\right]^n\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\exp\left(\mathbb{E}[e^{iuY_i}]\lambda t\right) = \exp\left(\lambda t\left(\mathbb{E}\left[e^{iuY_i}\right] - 1\right)\right) \\
&= \exp\left(\lambda t\left(\int_{\mathbb{R}} e^{iuy}f(y)dy - 1\right)\right) = \exp\left(\lambda t\left(\int_{\mathbb{R}} e^{iuy}f(y)dy - \int_{\mathbb{R}} f(y)dy\right)\right) \\
&= \exp\left(\lambda t\int_{\mathbb{R}}(e^{iuy} - 1)f(y)dy\right).
\end{aligned}$$

Merton model

Dynamics of the stock price under $\mathbb{P}$:

$$S_t = S_0 \exp(L_t)$$

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k.$$

- N_t Poisson process
- Y_k follows a normal distribution: $Y_k \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.

What are the dynamics of S_t?

Itô's formula for jump processes

Itô-Doeblin formula for jump processes with a finite number of jumps

Let $(X_t)_{\{t\geq 0\}}$ be a jump process with a finite number of jumps and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. We denote by X^c the continuous part of X. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s^c + \frac{1}{2}\int_0^t f''(X_{s-})d[X^c]_s + \sum_{0<s\leq t}[f(X_s) - f(X_{s-})] \tag{1}$$

Itô's formula applied in the Merton model

We apply Itô-Doeblin formula to $\exp(L_t)$:

$$\begin{aligned}
\exp(L_t) = & \quad \exp(0) + \int_0^t \exp(L_{s-})(\mu ds + \sigma dW_s) \\
& + \frac{1}{2}\int_0^t \exp(L_{s-})\sigma^2 ds + \sum_{0\le s\le t}[\exp(L_s) - \exp(L_{s-})] \\
= & \quad 1 + \mu\int_0^t S_{s-}\, ds + \sigma\int_0^t S_{s-}\, dW_s \\
& + \frac{\sigma^2}{2}\int_0^t S_{s-}\, ds + \sum_{0\le s\le t} S_0(\exp(L_s) - \exp(L_{s-})).
\end{aligned}$$

$$\begin{aligned}
\exp(L_s) - \exp(L_{s-}) &= (\exp(L_{s-} + J_s) - \exp(L_{s-}))\mathbf{1}_{N_s \neq N_{s-}} \\
&= \exp(L_{s-})(\exp(J_s) - 1)\mathbf{1}_{N_s \neq N_{s-}}.
\end{aligned}$$

Itô's formula applied in the Merton model

$$
\begin{aligned}
\frac{S_t}{S_0} = \quad & 1 + \mu \int_0^t S_{s-}\, ds + \sigma \int_0^t S_{s-}\, dW_s \\
& + \frac{\sigma^2}{2} \int_0^t S_{s-}\, ds + \sum_{0 \le s \le t} S_0 \exp(L_{s-})(\exp(J_s) - 1)\mathbf{1}_{N_s \neq N_{s-}} \\
= \quad & 1 + (\mu + \frac{\sigma^2}{2}) \int_0^t S_{s-}\, ds + \sigma \int_0^t S_{s-}\, dW_s + \int_0^t S_{s-}(\exp(J_s) - 1) dN_s
\end{aligned} \tag{2}
$$

$$
dS_t = (\mu + \frac{\sigma^2}{2}) S_{t-}\, dt + \sigma S_{t-}\, dW_t + S_{t-}(\exp(J_t) - 1) dN_t.
$$

Market completeness

Is the market complete? NO!

$\Rightarrow$ Need for additional assumptions to get the dynamics of S under the risk-neutral measure.

Assumptions of Merton(1976):

- The volatility and jump statistics remain unchanged, i.e. only the drift of the original process changes.
- Under the risk-neutral measure, the process $e^{-rt}S_t$ is a martingale.

Market incompleteness

Under $\mathbb{Q}$:

$$\frac{dS_t}{S_{t-}} = \alpha dt + \sigma d\tilde{W}_t + (e^{J_t} - 1)d\tilde{N}_t$$

We need the martingale condition to derive α.

Dynamics of the asset price under a risk-neutral measure

$$
\begin{aligned}
e^{-rt}S_t = {} & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs} dS_s^c + \sum_{0\le s\le t} [e^{-rs}S_s - e^{-rs}S_{s-}] \\
= {} & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs} dS_s^c + \sum_{0\le s\le t} e^{-rs}(S_s - S_{s-}) \\
= {} & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs}(\alpha S_{s-} ds + \sigma S_{s-} d\tilde{W}_s) \\
& + \sum_{0\le s\le t} e^{-rs} S_0 (e^{L_s} - e^{L_{s-}}) \\
= {} & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs}(\alpha S_{s-} ds + \sigma S_{s-} d\tilde{W}_s) \\
& + \sum_{0\le s\le t} e^{-rs} S_0 e^{L_{s-}} (e^{J_s} - 1)
\end{aligned}
$$

Dynamics of the asset price under a risk-neutral measure

$$
\begin{aligned}
e^{-rt}S_t = \quad & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs}(\alpha S_{s-}ds + \sigma S_{s-}d\tilde{W}_s) \\
& + \sum_{0\leq s\leq t} e^{-rs}S_{s-}(e^{J_s}-1) \\
= \quad & S_0 - r\int_0^t e^{-rs}S_s ds + \int_0^t e^{-rs}(\alpha S_{s-}ds + \sigma S_{s-}d\tilde{W}_s) \\
& + \int_0^t e^{-rs}S_{s-}(e^{J_s}-1)d\tilde{N}_s
\end{aligned}
$$

We differentiate:

$$
d(e^{-rt}S_t) = (-re^{-rt}S_t + e^{-rt}\alpha S_{t-})dt + e^{-rt}\sigma S_{t-}d\tilde{W}_t + e^{-rt}S_{t-}(e^{J_t}-1)d\tilde{N}_t
$$

Dynamics of the asset price under a risk-neutral measure

Equivalently:

$$
\begin{aligned}
\frac{d(e^{-rt}S_t)}{e^{-rt}S_{t-}} &= (-r+\alpha)dt + \sigma d\tilde{W}_t + (e^{J_t}-1)d\tilde{N}_t \\
&= (-r+\alpha)dt + \sigma d\tilde{W}_t + (e^{J_t}-1)(d\tilde{N}_t - \tilde{\lambda}dt) + (e^{J_t}-1)\tilde{\lambda}dt
\end{aligned}
$$

Martingale condition:

$$
\begin{gathered}
\mathbb{E}^{\mathbb{Q}}[d(e^{-rt}S_t)] = 0. \\
-r + \alpha + (e^{J_t}-1)\tilde{\lambda} = 0 \\
\alpha = r - \tilde{\lambda}(e^{J_t}-1).
\end{gathered}
$$

Dynamics of S_t under $\mathbb{Q}$:

$$
\frac{dS_t}{S_{t-}} = (r - \tilde{\lambda}(e^{J_t}-1))dt + \sigma d\tilde{W}_t + (e^{J_t}-1)d\tilde{N}_t
$$

Hedging in the Merton model

Portfolio like in Black-Scholes model:

$$\pi_t = C - \alpha S_t$$

Infinitesimal change in the portfolio value: we apply Itô's formula:

$$\begin{aligned} d\pi_t = \quad & dC_t - \alpha_t dS_t \\ = \quad & \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t^c + \frac{1}{2}\frac{\partial C^2}{\partial S^2} d[S^c]_t + [C_t - C_{t-}]dN_t - \alpha_t dS_t \\ = \quad & \left(\frac{\partial C}{\partial t} + (\frac{\partial C}{\partial S} - \alpha_t)(\mu + \frac{\sigma^2}{2})S_{t-} + \frac{1}{2}\frac{\partial C^2}{\partial S^2}\sigma^2 S_{t-}^2 \right) dt \\ & + (\frac{\partial C}{\partial S} - \alpha_t)\sigma S_{t-} dW_t + [C_t - C_{t-} - \alpha_t(e^{J_t} - 1)S_t]dN_t \end{aligned}$$

It is impossible to hedge the risk coming from the Brownian motion and the one coming from the jumps!

Merton jump-diffusion PDE

Additional assumption:

- Jump term $\tilde{N}_t$ is uncorrelated with the market
- Diversifiable $\Rightarrow$ No risk premium

$$\mathbb{E}^{\mathbb{P}}[d\pi_t] = r\pi_t dt.$$

Merton jump-diffusion PDE

Expectation of the portfolio gain:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[d\pi_t] &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_{t-}^2\right) dt + \mathbb{E}^{\mathbb{P}}\left[(C_t - C_{t-} - \frac{\partial C}{\partial S}(e^{J_t} - 1)S_t)dN_t\right] \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_{t-}^2\right) dt + \mathbb{E}^{\mathbb{P}}\left[C_t - C_{t-} - \frac{\partial C}{\partial S}(e^{J_t} - 1)S_t\right] \lambda dt \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial C^2}{\partial S^2}\sigma^2 S_{t-}^2 + \mathbb{E}^{\mathbb{P}}\left[C_t - C_{t-}\right]\lambda - \frac{\partial C}{\partial S}\mathbb{E}^{\mathbb{P}}[e^{J_t} - 1]S_t\lambda\right) dt
\end{aligned}
$$

Merton jump-diffusion PDE

Resulting PDE:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\frac{\partial C^2}{\partial S^2}\sigma^2 S_{t-}^2+\mathbb{E}^{\mathbb{P}}\left[C_t-C_{t-}\right]\lambda-\frac{\partial C}{\partial S}\mathbb{E}^{\mathbb{P}}[e^{J_t}-1]S_t\lambda=r(C_t-\frac{\partial C}{\partial S}S_t)$$

Equivalently:

$$\frac{\partial C}{\partial t}+r\frac{\partial C}{\partial S}S_t+\frac{1}{2}\frac{\partial C^2}{\partial S^2}\sigma^2 S_{t-}^2+\mathbb{E}^{\mathbb{P}}\left[C_t-C_{t-}\right]\lambda-\frac{\partial C}{\partial S}\mathbb{E}^{\mathbb{P}}[e^{J_t}-1]S_t\lambda=0$$

with $\mathbb{E}^{\mathbb{P}}[e^{J_t}-1]=e^{\mu_J+\frac{1}{2}\sigma_J^2}-1$.

$\Rightarrow$ Price of a European option can be derived as $\sum_{k=1}^{\infty}\omega_k C^{BS}(\sigma_k)$.

Lévy processes

Goal: Larger richness of jump features

Lévy process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space which satisfies the usual conditions. Let $T \in [0, \infty]$. A càdlàg, adapted, real valued stochastic process $L = (L_t)_{0 \le t \le T}$ is called a **Lévy process** if it satisfies the usual conditions:

- $L_0 = 0$
- L has **independent increments**, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \le s < t \le T$
- L has **stationary increments**, i.e. for any $0 \le s, t \le T$ the distribution of $L_{t+s} - L_t$ does not depend on t
- L is **stochastically continuous**, i.e. for every $0 \le t \le T$ and $\epsilon > 0$, $\lim_{s \to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$.

Example of Lévy processes?

Lévy processes

Examples of Lévy processes:

- Linear drift bt,
- Brownian motion,
- Poisson process,
- Compound Poisson process.

Infinite divisibility

The distribution of a random variable X is **infinitely divisible** if for all $n \in \mathbb{N}$ there exists i.i.d. random variables $X_1^{(1/n)}, ..., X_n^{(1/n)}$ such that:

$$X \stackrel{d}{=} X_1^{(1/n)} + ... + X_n^{(1/n)}.$$

Lévy processes

Characteristic triplet

The distribution of a random variable X is infinitely divisible if and only if there exists a triplet (b, c, ν) with $b \in \mathbb{R}$, $c \in [0, \infty)$ and ν a measure satisfying

$$\nu(\{0\}) = 0$$

and

$$\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

such that

$$\mathbb{E}[e^{iuX}] = \exp\left(ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx) \right).$$

Lévy processes

$(L_t)_{0\le t\le T}$ Lévy process :

$$L_t = L_{\frac{t}{n}} + \left(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\right) + ... + \left(L_t - L_{\frac{(n-1)t}{n}}\right).$$

By definition, increments are independent and stationary:

$\Rightarrow$ L_t **infinitely divisible**

$\Rightarrow$ L_t **entirely determined by the distribution of** L_1

Lévy-Khintchine formula

Lévy-Khintchine formula

Let L_t be a Lévy process. Then its characteristic function satisfies:

$$\mathbb{E}[e^{iuL_t}] = e^{t\kappa(i)}$$

where κ is called the *characteristic exponent* and can be written as:

$$\kappa(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx). \tag{3}$$

- (b, c, ν) Lévy triplet
- Crucial for calibration!

Lévy measure

Lévy measure

The **Lévy measure** is a measure on $\mathbb{R}$ which satisfies:

$$\nu(\{0\}) = 0$$

and

$$\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

Intuitively, for a closed set $A \in \mathbb{R}$ with $0 \notin A$, $\nu(A) =$ **average number of jumps** of L in the time interval $[0, 1]$ whose sizes fall in A.

Random measure of jumps

Random measure of jumps

The **random measure of jumps** of the process μ^L is defined for a set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ as

$$\mu^L(\omega; t, A) = \#\{0 \leq s \leq t; \Delta L_s(\omega) \in A\}.$$

Intuitively: counts the jumps of the process L which have size in A up to time t.

Link between the Lévy measure and the random measure of jumps

$$\nu(A) = \mathbb{E}[\mu^L(1, A)].$$

Examples

Example 1: Brownian Motion

$$\varphi^{BM}(u) = \mathbb{E}[e^{iuW_t}] = e^{\frac{1}{2}Var(iuW_t)} = e^{-\frac{u^2}{2}t}$$

$\Rightarrow$ Lévy triplet: $(0, 1, 0)$.

Example 2: Poisson process (intensity λ)

$$\varphi^{Poi}(u) = \mathbb{E}[e^{iuN_t}] = \exp(\lambda t(e^{iu} - 1))$$

$\Rightarrow$ Lévy triplet: $(0, 0, \nu)$ with $\nu(\{1\}) = \lambda$.

Examples

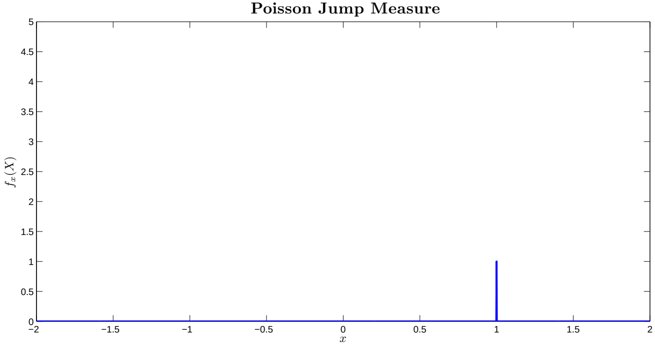


Figure: Density function of the Lévy measure of the standard Poisson process

Examples

Example 3: Compound Poisson process (jump sizes with density f and intensity λ)

$$\varphi^{CPoi}(u) = \mathbb{E}[e^{iuX_t}] = \exp(\lambda t \int_{\mathbb{R}} (e^{iuy} - 1) f(y) dy).$$

$\Rightarrow$ Lévy triplet: $(0, 0, \nu)$ with $\nu(dx) = \lambda f(dx)$.

Examples

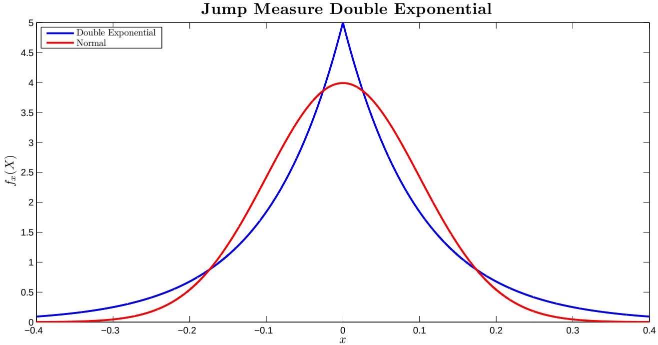


Figure: Density function of the Lévy measure of a compound Poisson process with double-exponentially and normally distributed jumps

Examples

Example 4: Jump diffusion model

$$L_t = bt + \sigma W_t + \left(\sum_{i=1}^{N_t} Y_i - t\lambda\kappa \right)$$

$$\begin{aligned}
\varphi^{JD}(u) &= \mathbb{E}\left(e^{iuL_t}\right) = \mathbb{E}\left(\exp\left[iu(\gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i - t\lambda\kappa) \right] \right) \\
&= e^{iu\gamma t}\mathbb{E}\left(\exp\left[\sigma W_t + \sum_{i=1}^{N_t} Y_i - t\lambda\kappa \right] \right) \\
&= e^{ut(i\gamma - \frac{1}{2}u\sigma^2)}\mathbb{E}\left(e^{\sum_{i=1}^{N_t} Y_i} \right) e^{-iut\lambda\kappa} \qquad (4)
\end{aligned}$$

Examples

$$
\begin{aligned}
\mathbb{E}\left(e^{iu\sum_{i=1}^{N_t} Y_i}\right) &= \mathbb{E}\left[\mathbb{E}\left(e^{iu\sum_{i=1}^{N_t} Y_i} | N_t = n\right)\right] \\
&= \sum_{n\geq 0} \mathbb{E}\left(e^{iu\sum_{i=1}^{N_t} Y_i} \underbrace{\mathbb{P}(N_t = n)}_{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}\right) \\
&= e^{-\lambda t} \exp\left(\sum_{n\geq 0} \left(\mathbb{E}[e^{iuY}]\lambda t\right)^n / n!\right) \\
&= e^{\lambda t(\mathbb{E}[e^{iuY}]-1)} \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\varphi^{JD}(u) &= \exp\left(iu\gamma t - 1/2\sigma^2 u^2 t + \lambda t(\mathbb{E}[e^{iuY}] - 1 - iu\mathbb{E}[Y])\right) \\
&= \exp\left(t(iu\gamma - 1/2\sigma^2 u^2 + \lambda \int_{\mathbb{R}} (e^{iux} - 1 - iux)dF_X)\right) \qquad (6)
\end{aligned}
$$

Lévy-Itô decomposition

Lévy-Itô decomposition

Let us consider a Lévy triplet (b, σ, ν) with $b \in \mathbb{R}$, $\sigma \in [0, \infty)$, and ν is a measure such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with four independent Lévy processes

- $L^{(1)}$ is a constant drift,
- $L^{(2)}$ is a Brownian motion,
- $L^{(3)}$ is a compound Poisson process and
- $L^{(4)}$ is a square integrable pure jump martingale with an almost surely countable number of jumps of magnitude less than 1 on each finite time interval,

such that the Lévy process L with characteristic triplet (b, σ, ν) can be written as

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}.$$

Lévy-Itô decomposition

$L^{(3)}$ can be written as:

$$L_t^{(3)} = \sum_{0 \leq s \leq t} \Delta L_s^{(3)} \mathbf{1}_{\{|\Delta L_s| \geq 1\}} = \int_0^t \int_{|x| \geq 1} x \mu^L(ds, dx)$$

What we need to remember from Itô-Lévy decomposition:

$$L_t = bt + \sqrt{c} W_t + \int_0^t \int_{|x| \geq 1} x \mu^L(ds, dx) + \int_0^t \int_{|x|<1} x(\mu^L - \nu^L)(ds, dx).$$

Canonical representation:

$$L_t = bt + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(ds, dx).$$

L_t martingale $\Leftrightarrow b = 0$.

Properties of Lévy processes

How to choose between Lévy processes?

Activity

Let L be a Lévy process with triplet (b, c, ν).

- If $\nu(\mathbb{R}) < \infty$, then almost all paths of L have a finite number of jumps on every compact interval, i.e. on every union of closed intervals of $\mathbb{R}$. In this case, L is said to have **finite activity**.
- Conversely, if $\nu(\mathbb{R}) = \infty$, then almost all paths of L have an infinite number of jumps on every compact interval. The process is said to have **infinite activity**.

Properties of Lévy processes

Which Lévy processes have finite activity?

- Continuous Lévy processes
- Poisson and compound Poisson processes

For processes with finite activity:

- The **Lévy-Itô decomposition** is as follows:

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x\mu^L(ds, dx).$$

- Characteristic exponent:

$$\kappa(u) = ibu - \frac{u^2\sigma}{2} + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx). \tag{7}$$

Example of a process with infinite activity

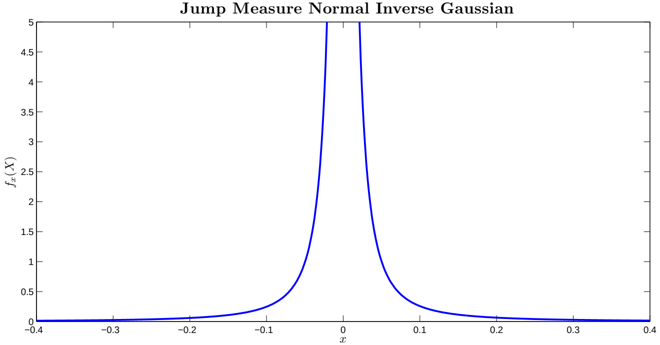


Figure: Density function of the Lévy measure of a NIG (Normal Inverse Gaussian) process

Properties of Lévy processes

Variation

Let L be a Lévy process with triplet (b, c, ν).

- If $c = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$, then almost all paths of L have **finite variation**.
- If $c \neq 0$ or $\int_{|x| \leq 1} |x| \nu(dx) = \infty$, then almost all paths of L have **infinite variation**.

Properties of Lévy processes

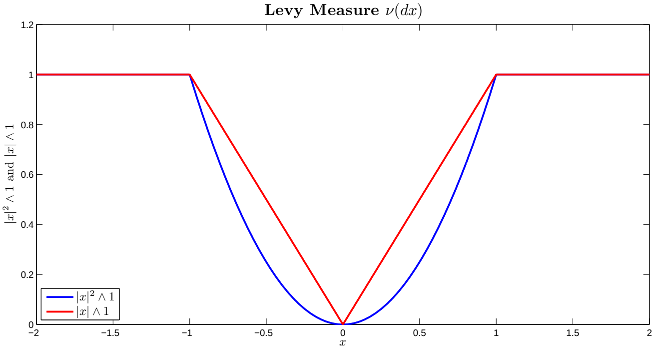


Figure: By definition the Lévy measure has to integrate $\min(x^2, 1)$ (blue line). If it integrates $\min(x, 1)$ (red line) then it has finite variation.

Properties of Lévy processes

Existence of moments

Let L be a Lévy process with triplet (b, c, ν). Then L_t has finite p^{th} moment $(\mathbb{E}[|L_t|^p] < \infty)$ if and only if $\int_{|x|\geq 1} |x|^p \nu(dx) < \infty$.

For option pricing: **We need the existence of the first moment!**

Financial Engineering - Lecture 9

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

17.11.2011

University of Zurich

Outline

1. **Lévy Synopsis**
2. Tractable Lévy Models
3. Changing Measure
4. Lévy Evidence
5. Time Change
   - Some General Thoughts
   - Changing Time
   - Option Pricing

Lévy's Main Idea

- The original construction of Lévy processes was rather heuristic.
- Lévy 's main idea: combine Brownian motion and a compound Poisson process.
- From this starting point, Lévy generalized the resulting characteristic function, essentially discussing the existence of an integral.
- Let

$$\begin{aligned} Z_t &= X_t + Y_t \\ X_t &= \mu t + \sigma W_t \\ Y_t &= \sum_{k=0}^{N_t} Y_{\tau_k} - \lambda t \mathbb{E}[Y_{\tau_k}], \end{aligned}$$

with W_t a Brownian motion and N_t a pure jump process and independent jump sizes Y_{τ_k}.

Combining Jumps and Brownian Motion

- We know

$$\begin{aligned}\varphi_X(u) &= \exp\left[t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right)\right]\\ \varphi_Y(u) &= \exp\left[\lambda t\int_{z=-\infty}^{\infty}\left(e^{iuz}-1-iuz\right)dF(z)\right].\end{aligned}$$

- Combining both characteristic functions yields the de Finetti expression:

$$\begin{aligned}\varphi_Z(u) &= \exp\left[t\left(i\mu u - \frac{1}{2}\sigma^2 u^2 + \lambda\int_{z=-\infty}^{\infty}\left(e^{iuz}-1-iuz\right)dF(z)\right)\right]\\ \varphi_Z(u) &= \exp(t\kappa(u)),\end{aligned}$$

where $\kappa(u)$ is called the stochastic exponent.

Adding More Components...

- So far, we have only combined a Brownian motion with a compound jump process.
- What happens if we superpose additional components?
- In the case of Brownian motions the answer is easy, since the sum of Brownian motions will result in another Brownian motion.
- Superposing jumps is a little bit more difficult.
- Consider mixing two jumps like:

$$
\begin{aligned}
I &= \lambda_1 t \int_{z=-\infty}^{\infty} \left(e^{iuz} - 1 - iuz\right) dF_1(z) \\
&\quad + \lambda_2 t \int_{z=-\infty}^{\infty} \left(e^{iuz} - 1 - iuz\right) dF_2(z) \\
&= \int_{z=-\infty}^{\infty} \left(e^{iuz} - 1 - iuz\right) \nu(dz),
\end{aligned}
$$

with $\nu(dz) = \lambda_1 dF_1(z_1) + \lambda_2 dF_2(z_2)$.

...Some Problems May Arise:

- Note:
  1. There is no reason that for given λ_1 and λ_2, the probability mass integrates up to 1.
  2. λ_1, λ_2, dF_1, dF_2 are all positive, thus the measure $\nu(dz)$ will be positive.
  3. Due to possible discontinuities at $z = 0$, the integrals should be considered as being $\int_{-\infty}^{0} + \int_{0}^{\infty}$.
- We can interpret $\nu(z)$ as the measure measuring the average number of jumps of size z in a unit of time.
- Given its importance $\nu(z)$ is called the Lévy measure, which opens up the question of existence of the integral.

Lévy's contribution

- Lévy's contribution is to spell out the conditions under which the integral I is well defined.
- Two first reasonable assumptions are
  - i) ν is positive
  - ii) $\int_z \nu(dz) < \infty$, i.e., the mass associated with each z is finite.
- Obviously, we can allow z to be of the form $z \in \mathbb{R}$. However, two problems arise:
  1. Jumps become infinitely active.
  2. Jumps are too large.
- We already solved the problem of infinite activity and too large jumps. We briefly revisit.

Infinite Activity

- When activity can become infinite, we need to interpret the integral with caution, i.e.,

$$\begin{aligned}&\int_{z=-\infty}^{\infty}\left(e^{iuz}-1-iuz\right)\nu(dz)\\&=\lim_{a\to 0,a>0}\int_{z=-\infty}^{-a}\left(e^{iuz}-1-iuz\right)\nu(dz)\\&+\lim_{a\to 0,a>0}\int_{z=a}^{\infty}\left(e^{iuz}-1-iuz\right)\nu(dz).\end{aligned}$$

- When activity of jumps increases, jump sizes decrease. Hence, we can tackle integrability by looking at neighborhood of 0, for convenience say in the neighborhood $[-1,1]$.

Infinite Activity (Con'd)

- First, look at the Taylor series of the integrand around zero:

$$\begin{aligned} e^{iuz} - 1 - iuz &= 1 - \frac{z^2}{2} + i\left[uz - \frac{(uz)^3}{3!}\right] - 1 - iuz + o(z^3) \\ &= -\frac{z^2}{2} + o(z^2). \end{aligned}$$

- Hence, as long as $\int z^2 \nu(dz)$ exists, I is integrable.
- When $\int z^2 \nu(dz) < \infty$, $\nu(z) \to \infty$, and $z \to 0$, the process is said to be of infinite activity.
- We can regroup the conditions for the existence of I as

$$\text{iii)} \quad : \quad \int_z \min(1, z^2)\nu(dz) < \infty.$$

To compensate or not to compensate?

- So far, we used a compensated Poisson process. Assume for now the measure $\nu(z) = z^{-\alpha}$ for small α. Then, $d\nu/dz = -\alpha z^{-\alpha-1}$ and

$$\int_{-1}^{1} z^2 \nu(dz) = \lim_{a\to 0, a>0} \int_{z=-1}^{-a} -\alpha z^{1-\alpha} dz + \lim_{a\to 0, a>0} \int_{z=a}^{1} -\alpha z^{1-\alpha} dz.$$

- The right-most integral equals

$$\lim_{a\to 0, a>0} \int_{z=a}^{1} -\alpha z^{1-\alpha} dz = \frac{-\alpha}{2-\alpha}(1 - a^{2-\alpha})$$

- This integral is well defined as $a^{2-\alpha} \to 0$ if $\alpha < 2$.
- Without compensating, the Taylor expansion of the integrand is

$$e^{iuz} - 1 = iuz + o(z).$$

- Doing the same calculations as above, we see that the integrability condition is $\alpha < 1$. Hence, the compensated Poisson process allows for larger jump intensity than the uncompensated Poisson process!

Dealing with Large Jumps

- Next question: We have now to examine the case when jumps are less frequent, but large.
- We will focus on the upper tail. Lower tail considerations follow same arguments.
- The integral, by definition, is finite if

$$\lim_{b\to\infty}\int_a^b \left(e^{iuz}-1-iuz\right)\nu(dz)<\infty, \quad \text{for some } a>0.$$

- To investigate the finiteness of the integral, we have to consider each component.
  - Note first that as long as $\int_a^b \nu(dz)$ exists, which holds if *iii)* is true, $\nu(z)$ will converge to zero for large z. Hence, since $e^{iuz}-1$ remains bounded, the first part of the integral exists.
  - However, if $\nu(dz)$ has no finite moments, (e.g. Cauchy or Pareto density), then the integral $\int_a^b z\nu(dz)$ ceases to exist for large b. What to do?

Dampening Large Jumps

- Lévy modified the definition of the exponent to guarantee that the integral will also remain finite for large jumps by introducing

$$c(z) = \frac{1}{1+z^2}$$

to dampen out the tails.

- By setting $\int_a^\infty zc(z)\nu(dz) = \int_a^\infty \frac{z}{1+z^2}\nu(dz) < \infty$, the integral is now well defined as long as also *iii)* holds.
- Other choices of the dampening function are possible. For instance:

$$c(z) = 1_{\{|z|<1\}}(z)$$

- Note however, with the introduction or change of the dampening function, we need to modify the mean to get a similar Lévy process.

Putting things together...

Theorem (Lévy-Khintchine Formula)

The law of a random variable X is infinitely divisible iff there exists a triplet (μ, σ, ν), with $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$ and a measure satisfying $\nu(\{0\}) = 0$ and $\int (1 \wedge |x|^2)\nu(dx) < \infty$, such that

$$\varphi_X(u) = \exp\left[\left(iu\mu(c) - \frac{1}{2}u^2\sigma^2 + \int \left(e^{iux} - 1 - iuc(x)\right)\nu(dx)\right) t\right],$$

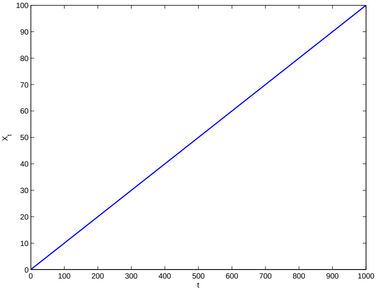
where $c(x)$ is an appropriate dampening function. The triplet $(\mu(c), \sigma, \nu)$ is called the **Lévy triplet** *and the exponent*

$$\kappa(u) = iu\mu(c) - \frac{1}{2}u^2\sigma^2 + \int \left(e^{iux} - 1 - iuc(x)\right)\nu(dx)$$

is called the **Lévy exponent**. *Moreover, $\mu(c)$ is called the drift term, σ the Gaussian coefficient, and ν the* **Lévy measure**.

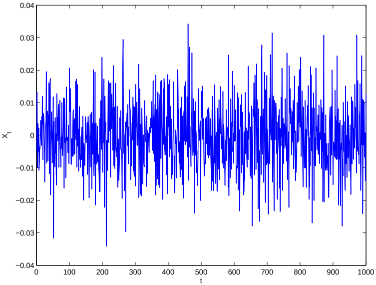
Example

- Linear Drift



$$
\begin{aligned}
X_t &= bt, \\
\varphi_X(u) &= \exp\left(iubt\right)
\end{aligned}
$$

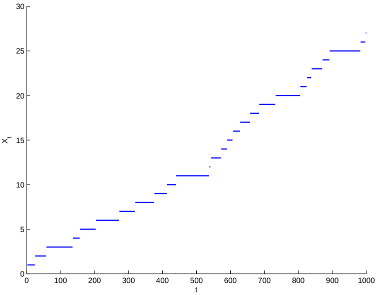
- Brownian Motion



$$
\begin{aligned}
X_t &= \sigma W_t, \\
\varphi_X(u) &= \exp\left(-\frac{1}{2}u^2\sigma^2 t\right)
\end{aligned}
$$

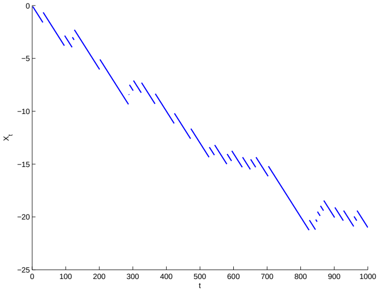
Example

- Poisson Process



$$X_t = \sum_{k=1}^{N_t} J_k,$$
$$\varphi_X(u) = \exp\left((e^{iu} - 1)\lambda t\right)$$

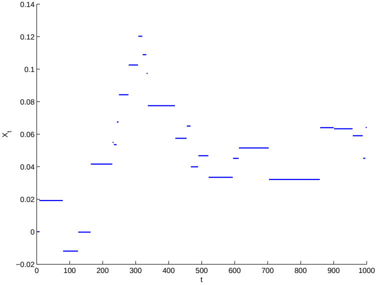
- Compensated Poisson Process



$$X_t = \sum_{k=1}^{N_t} J_k - t\lambda x,$$
$$\varphi_X(u) = \exp\left((e^{iu} - 1 - iu)\lambda t\right)$$

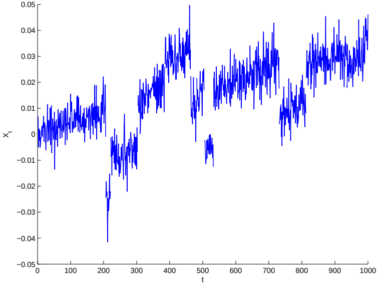
Example

- Compound Poisson Process



$$X_t = \sum_{k=1}^{N_t} J_k,$$
$$\varphi_X(u) = \exp\left(\mathbb{E}\left[e^{iuJ} - 1\right]\lambda t\right)$$

- Lévy Jump Diffusion



$$X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k$$
$$-\lambda t \mathbb{E}[J]; \quad \varphi_X(u) = \dots$$

Outline

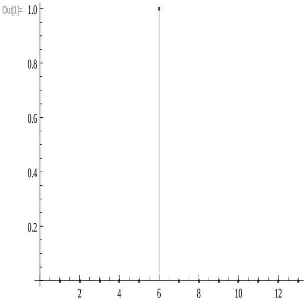
Activity and Variation

Proposition

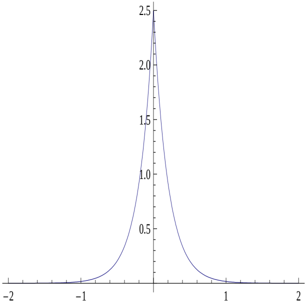
Let X be a Lévy process with triplet (μ, σ, ν).

1. *If $\nu(\mathbb{R}) < \infty$, then almost all paths of X have a finite number of jumps on every compact interval. In that case, the Lévy process has* **finite activity**.
2. *If $\nu(\mathbb{R}) = \infty$, then almost all paths of X have an infinite number of jumps on every compact interval. In that case, the Lévy process has* **infinite activity**.
3. *If $\sigma = 0$ and $\int_{|x|\leq 1} |x| \, \nu(dx) < \infty$, then almost all paths of X have* **finite variation**.
4. *If $\sigma \neq 0$ or $\int_{|x|\leq 1} |x| \, \nu(dx) = \infty$, then almost all paths of X have* **infinite variation**.

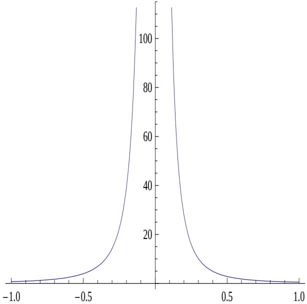
Examples of Lévy Measures



Standard Poisson process.



Compound Poisson process with double exponential distributed jumps.



α-Stable process.

Tractable Examples

- Compound Poisson jumps (Merton '76): Large but rate events.

$$\nu(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left(-\frac{(x-\mu_J)^2}{2v_J}\right).$$

- Dampened power law (DPL)

$$\nu(x) = \begin{cases} \lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0 \quad \lambda, \beta_\pm > 0, \\ \lambda \exp(-\beta_- \left|x\right|) \left|x\right|^{-\alpha-1}, & x > 0 \quad \alpha \in [-1, 2], \end{cases}$$

  - **Finite activity** when $\alpha < 0 : \int_{\mathbb{R}^0} \nu(x) dx < \infty$. Compound Poisson, i.e., large and rare events.
  - **Infinite activity** when $\alpha \geq 0$: Both small and large jumps. Jump frequency increases with declining jump size, and approaches infinity as $x \to 0$.
  - **Infinite variation** when $\alpha \geq 1$: many small jumps.

$\Rightarrow$ DPL flexible enough to account for market movements of all magnitudes, from small movements to market crashes.

Special Cases of DPL

- **α-stable law**: No exponential dampening, $\beta_{\pm} = 0$. (Carr & Wu, Finite Moment Log Stable Process and Option Pricing, Journal of Finance, 2003, 58(2), 753-777.)
  - Without exponential dampening, return moments greater than α are no longer well defined.
  - Characteristic function takes different form to account for singularity.
- **Variance gamma (VG) model**: $\alpha = 0$, (Madan, Carr, Chang, 1998, The Variance Gamma Process and Option Pricing, European Finance Review, 2(1), 79-105.)
- **Double exponential model**: $\alpha = -1$, (Kou 2002, A Jump-Diffusion Model for Option Pricing, Management Science, 48(8), 1086-1101.)

Other Lévy Examples

Other examples:

- The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
- The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
- The Meixner process (Schoutens (2003))
- ...

Bottom line:

- All tractable in terms of analytical characteristic exponents $\kappa(u)$.
- We can use FFT to generate the density function of the innovation (for model estimation).
- We can also use FFT to compute option values.

Outline

Lévy Martingales

Proposition

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy exponent κ and assume that $\mathbb{E}(e^{uX_t}) < \infty$, $u \in \mathbb{R}$. The process $M = (M_t)_{t\geq 0}$ defined as

$$M_t = \frac{e^{uX_t}}{\mathbb{E}\left(e^{uX_t}\right)} = \frac{e^{uX_t}}{e^{t\kappa(u)}}$$

is a **martingale**.

Proof

Given that $\kappa(u)$ is finite by definition, $\mathbb{E}(|M_t|)$ is finite. For $0 \leq s < t$, we have

$$\mathbb{E}\left[e^{u(X_t - X_s) - \kappa(u)(t-s)} | \mathcal{F}_s\right] = e^{\kappa(u)(t-s) - \kappa(u)(t-s)} = 1.$$

Girsanov for Lévy Processes**

Theorem

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with triplet (μ, σ, ν) under $\mathbb{P}$ that satisfies $(\mathcal{M})$. It has the canonical decomposition

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{\mathbb{R}} x \left(\mu^X - \nu^X\right)(ds, dx),$$

(A1) *Assume $\mathbb{P} \sim \mathbb{Q}$ and that $(\mathcal{M})$ holds under both. Then, there exists a deterministic process β and a measurable non-negative deterministic process Y, satisfying*

$$\int_0^t \int_{\mathbb{R}} |x\,(Y(s,x) - 1)|\, \nu(dx) ds < \infty, \quad \int_0^t (\sigma^2 \beta_s^2) ds < \infty,$$

such that the density process $\eta = (\eta_t)_{t\geq 0}$ has the form

Girsanov for Lévy Processes**

Theorem (cont.)

$$
\begin{aligned}
\eta_t &= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|\mathcal{F}_t\right] \\
&= \exp\left[\int_0^t \beta_s \sigma \, dW_s - \frac{1}{2}\int_0^t \beta_s^2 \sigma^2 ds\right] \\
&\quad + \int_0^t \int_{\mathbb{R}} (Y(s,x) - 1)\left(\mu^X - \nu^X\right)(ds, dx) \\
&\quad - \int_0^t \int_{\mathbb{R}} (Y(s,x) - 1 - \log Y(s,x))\, \mu^X (ds, dx)
\end{aligned}
$$

(A2) *Conversely, if η is a positive martingale of the above form, then it defines a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, such that $\mathbb{P} \sim \mathbb{Q}$.*

Girsanov for Lévy Processes**

Theorem (cont.)

(A3) *In both cases, we have that* $\widetilde{W}_t = W_t - \int_0^t \beta_s ds$ *is a* $\mathbb{Q}$*-BM,* $\widetilde{\nu}^X(ds, dx) = Y(s,x)\nu^X(ds, dx)$ *is the* $\mathbb{Q}$*-compensator of* μ^X *and* X *has the canonical decomposition under* $\mathbb{Q}$*:*

$$X_t = \widetilde{\mu} t + \sigma \widetilde{W}_t + \int_0^t \int_{\mathbb{R}} x \left(\mu^X - \widetilde{\nu}^X\right)(ds, dx),$$

where

$$\widetilde{\mu} t = \mu t + \int_0^t \sigma^2 \beta_s ds + \int_0^t \int_{\mathbb{R}} x \left(Y(s,x) - 1\right) \nu^X(ds, dx).$$

Girsanov's Theorem: Caveats**

The process X is not necessarily a Lévy process under the new measure $\mathbb{Q}$; it depends on the tuple (β, Y). We have the following cases:

(G1): if (β, Y) are deterministic and independent of time, then X remains a Lévy process under $\mathbb{Q}$; its triplet is $(\widetilde{\mu}, \sigma, Y \cdot \nu)$.

(G2): if (β, Y) are deterministic but depend on time, then L becomes a process with independent (but not stationary) increments under $\mathbb{Q}$, often called an additive process.

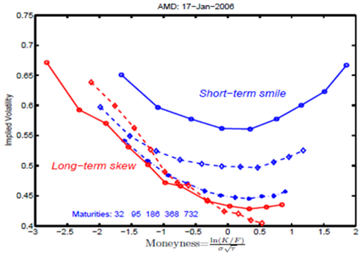
(G3): if (β, Y) are stochastic processes, i.e., neither deterministic nor independent of time, then X is a semimartingale under $\mathbb{Q}$.

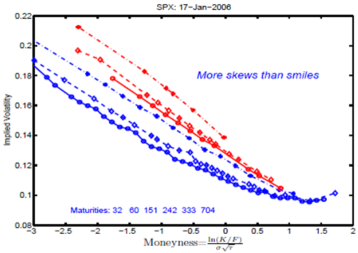
Outline

General Evidence on Lévy Return Innovations

- Credit risk: (compound) Poisson process
  The whole intensity-based credit modeling literature...
- Market risk: Infinite-activity jumps
  - Evidence from stock returns (CGMY (2002)): The α estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better.
  - Li, Wells, & Yu (2006): Infinite-activity jumps cannot be approximated by finite-activity jumps.
- The role of diffusion (in the presence of infinite-variation jumps)
  - Not big, difficult to identify (CGMY (2002), Carr & Wu (2003)).
  - Generate correlations with diffusive activity rates (Huang & Wu (2004)).
  - The diffusion (σ^2) is identifiable in theory even in presence of infinite-variation jumps (Ait-Sahalia (2004), Ait-Sahalia & Jacod 2005).

From Smiles and Skews

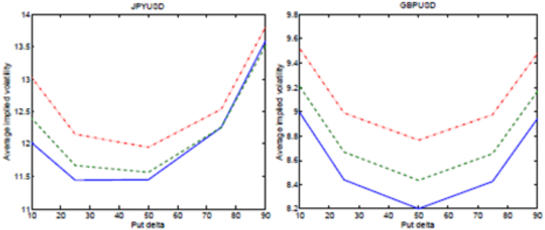




Implied Volatility Skews and Smiles of Equity.

Implied Volatility Skews on Index (SPX).

Implied Volatility Smiles on Currencies



Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)

But beware: Also institutional restrictions may cause stylized facts which are not directly related to the underlyings' dynamics: Leippold & Su, 2011, Collateral Smile, (click here) for the paper.

The role of jumps at very short maturities

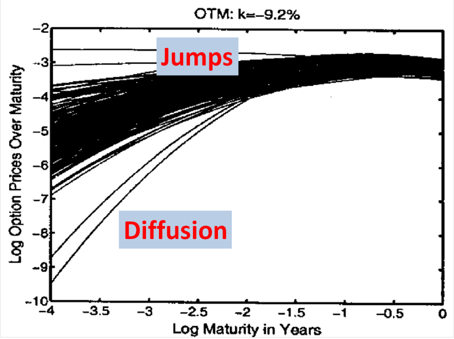
- Implied volatility smiles (skews) $\Leftrightarrow$ non-normality (asymmetry) for the risk-neutral return distribution.

$$IV(d) \approx ATMV\left(1 + \frac{Skew}{6}d + \frac{Kurt}{24}d^2\right); \quad d = \frac{\ln K/F}{\sigma\sqrt{\tau}}$$

- Two mechanisms to generate return non-normality:
    - Use Lévy jumps to generate non-normality for the innovation distribution.
    - Use stochastic volatility to generates non-normality through mixing over multiple periods.
- Over very short maturities (1 period), **only jumps contribute to return non-normalities.**

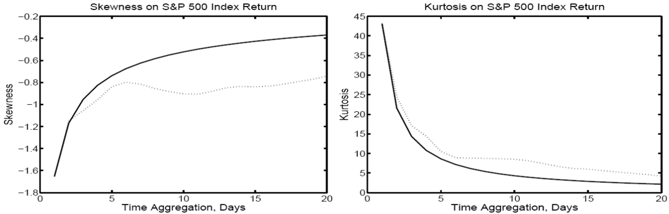
Time decay of short-term OTM options

- As option maturity $\downarrow$ zero, OTM option value $\downarrow$ zero. The speed of decay is exponential $O(e^{c/\tau})$ under pure diffusion, but linear $O(\tau)$ in the presence of jumps.
- Term decay plot: $\ln(OTM/\tau) \sim \ln(\tau)$ at fixed K (SPX; 4/99-3/00, Carr & Wu, 2003)



The impacts of jumps at very long horizons

- Central limit theorem (CLT): Return distribution converge to normal with aggregation under certain conditions (finite return variance,...)
  $\Rightarrow$ As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
- FMLS (Carr and Wu, 2003): Maximum negatively skewed α-stable process.
  - Return variance is infinite $\Rightarrow$ CLT does not apply.
  - Down jumps only $\Rightarrow$ Option has finite value.
- **But CLT seems to hold fine statistically:**



Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL jumps

- Model return innovations under $\mathbb{P}$ by DPL:

$$\pi(x) = \begin{cases} \lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0 \\ \lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x \leq 0 \end{cases}$$

  All return moments are finite with $\beta_\pm > 0$. **CLT applies under $\mathbb{P}$.**

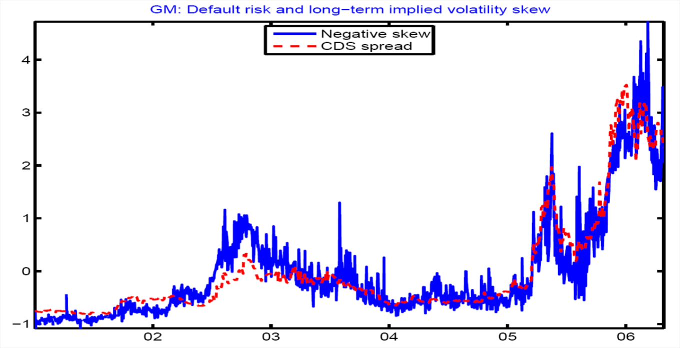
- Market price of jump risk γ: $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \mathcal{E}(-\gamma X)$.
- The return innovation process remains DPL under $\mathbb{Q}$:

$$\pi(x) = \begin{cases} \lambda \exp(-(\beta_+ \gamma) x) x^{-\alpha-1}, & x > 0 \\ \lambda \exp(-(\beta_- \gamma) |x|) |x|^{-\alpha-1}, & x \leq 0 \end{cases}$$

- **To break CLT under $\mathbb{Q}$**, set $\gamma = \beta_-$ so that $\beta^{\mathbb{Q}} = 0$.
- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: Investors charge maximum allowed market price on down jumps.

Default risk & long-term implied vol skew

- When a company defaults, its stock value jumps to zero.
- It generates a steep skew in long-term stock options.
- Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.



Three Lévy jump components in stock returns

I. Market risk (FMLS under $\mathbb{Q}$, DPL under $\mathbb{P}$)
   - The stock index skew is strongly negative at long maturities.

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathbb{Q}$)
   - The smile on single name stocks is not as negatively skewed as that on index at short maturities.

III. Default risk (Compound Poisson jumps).
   - Long-term skew moves together with CDS spreads.

- Information and identification:
  - Identify market risk from stock index options.
  - Identify the credit risk component from the CDS market.
  - Identify the idiosyncratic risk from the single-name stock options.

Lévy jump components in currency returns

- Model currency return as the difference of the log pricing kernels between the two economies.
- Pricing kernel assigns market prices to systematic risks.
- Market risk dominates for exchange rates between two industrialized economies (e.g., dollar-euro).
  - Use a one-sided DPL for each economy (downward jump only).
- Default risk shows up in FX for low-rating economies (say, dollar-peso).
  - Peso drops by a large amount when the country (Mexico) defaults on its foreign debt.
    Peter Carr, and Liuren Wu, Theory and Evidence on the Dynamic Interactions Between Sovereign Credit Default Swaps and Currency Options, Journal of Banking and Finance, 2007, 31(8), 2383–2403.
- When pricing options on exchange rates, it is appropriate to distinguish between world risk versus country-specific risk

Outline

The Delusion of Diffusion

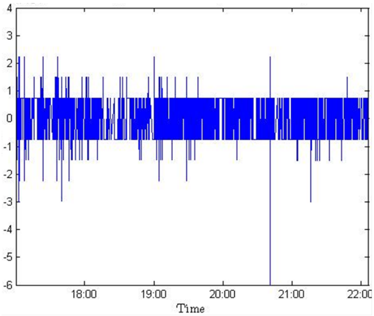
- Starting with Bachelier (1900), diffusion processes have been the most widely used class of stochastic processes used to describe the evolution of asset prices over time.
- The sample paths of diffusion processes show infinite variation.
- If one tried to accurately draw a diffusion sample path, your pen would run out of ink before one second had elapsed.
- If we model an asset price process using a diffusion, then over any finite time interval, there is zero probability that the price does not change.
- Furthermore, the absolute values of price changes over a day (or any other period) sum to infinity.

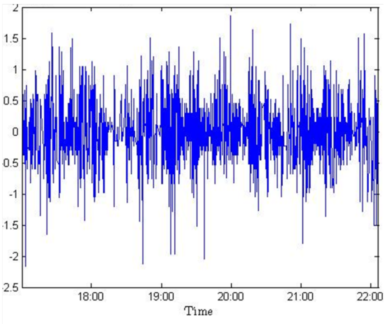
Finite Variation of Financial Markets

- In even the most active markets, one can find small enough time periods over which there is positive probability of no price change.
- Furthermore, if we sum the absolute values of price changes over a day, we get a finite number.
- The failure of diffusions to describe the microscopic behavior of sample paths would not be troubling if financial theory took a more macroscopic view.
- The problem is that the foundations of standard financial theories such as Black Scholes and the intertemporal CAPM rest on the ability of investors to continuously rebalance their portfolios.
- Nobody seriously believes that anyone can trade continuously and even if they could, no one seriously believes that trade sizes can be kept so small that price impact is infinitesimal.

A High-Frequency Perspective on Data

Tick-by-Tick vs 5min Interpolation (USDEUR log returns):





What Lévy Can and Cannot Do

- Lévy processes can generate different iid return innovation distributions.
    - Any distribution you can think of, we can specify a Lévy process, with the increments of the process matching that distribution.
    - **Caveat:** The same type of distribution applies to all time horizons | you may not be able to specify the distribution simultaneously at different time horizons.
- Lévy processes cannot generate distributions that vary over time. Returns modeled by Lévy processes generate implied volatility surfaces that stay the same over time.
- Lévy processes cannot capture the following salient features of the data:
    - Stochastic volatility
    - Stochastic risk reversal (skewness)
    - Stochastic correlation

How to Make Volatility Stochastic

- Discrete-time analog again: $R_{t+1} = \mu_t + \sigma_t \epsilon_{t+1}$
  - ϵ_{t+1} is an iid return innovation, with an arbitrary distribution assumption (Levy analogue)
  - σ_t is the conditional volatility, μ_t is the conditional mean return, both of which can be time-varying and/or stochastic (GARCH)
- In continuous time, we have seen how to model stochastic mean/volatility tractably:
  - If the return innovation is modeled by a Brownian motion, we can let the instantaneous variance to be stochastic and tractable, not volatility (Heston(1993), Bates (1996)).
  - If the return innovation is modeled by a compound Poisson process, we can let the Poisson arrival rate to be stochastic, not the mean jump size, jump distribution variance (Bates(2000), Pan(2002)).
- What we will see: If the return innovation is modeled by a general Lévy process, it is tractable to randomize the time, or something proportional to time.

Time for a Change

- If we confront a mathematical diffusion with real-life sample paths, the modelling question becomes one of finding ways to slow down diffusions to more accurately capture dynamics.
- In 1949, Bochner introduced the notion of time change to stochastic processes, but can actually be traced back to Wolfgang Döblin aka Vincent Doeblin, before WWII.
- In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.
- Mathematically, a clock is just a weakly increasing stochastic process started at zero. When one time changes a stochastic process, this clock is used to index a stochastic process such as a diffusion. Clark suggested that the price process runs on business time, while business time itself increases weakly over calendar time.
- The possibility that business time may not move while calendar time inexorably marches forward is important for our purposes.

Time for a Change

- So far, we know form the Lévy -Khintchine Theorem:

$$\begin{aligned}
\varphi(u) &\equiv \mathbb{E}[e^{iuL_t}] = e^{t\kappa(u)} \\
\kappa(u) &= iu\mu - \frac{1}{2}u^2\sigma^2, \text{ for diffusion with drift } \mu \text{ and variance } \sigma^2, \\
\kappa(u) &= \lambda(e^{iu\mu_J - \frac{1}{2}u^2 v_J} - 1), \text{ for Merton's compound Poisson jump.}
\end{aligned}$$

- The drift μ, the diffusion variance σ^2, and the Poisson arrival rate λ are all proportional to time t.
- We can directly specify $(\mu_t, \sigma_t^2, \lambda_t)$ as following stochastic processes.
- Or we can randomize time $t \to \mathcal{T}_t$ for the same result.

Time for a Change

- We define $\mathcal{T}_t \equiv \int_0^t v_{s\_}\, ds$ as the (stochastic) time change, with v_t being the instantaneous activity rate.
- Depending on the Lévy specification, the activity rate has the same meaning (up to a scale) as a randomized version of the instantaneous drift, instantaneous variance, or instantaneous arrival rate.
- If we apply random time change to $L_t \rightarrow L_{\mathcal{T}_t}$, it is equivalent to assuming that $(\mu_t, \sigma_t^2, \lambda_t)$ are all time varying, but they are all proportional to one common source of variation v_t.
- If we want $(\mu_t, \sigma_t^2, \lambda_t)$ to vary separately, let $L_t^1 \sim (\mu, 0, 0)$, $L_t^2 \sim (0, \sigma^2, 0)$, and $L_t^3 \sim (0, 0, \lambda)$, and then apply separate time changes to the three Lévy processes.

Interpretation

- We can think of t as the calendar time, and $\mathcal{T}_t$ as the business time.
- Business activity accumulates with calendar time, but the speed varies, depending on the business activity.
- At heavy trading hours, one hour on a clock generates two hours worth of business activity ($v_t = 2$).
- At afterhours, one hour generates half hour of activity ($v_t = 1/2$).
- Business activity tends to intensify before earnings announcements, FOMC meeting days... In this sense, v_t captures the intensity of business activity at a certain time t
- We use Lévy processes to model return innovations and stochastic time changes to generate stochastic volatility and higher moments.
- We can think of each Lévy process as capturing one source of economic shock with the time change capturing its intensity.

Classification

- In 1949, Bochner introduced the notion of time change to stochastic processes. In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.
- At present, there are two types of clocks used to model business time:
  1. Continuous clocks have the property that business time is always strictly increasing over calendar time.
  2. Clocks based on increasing jump processes have staircase like paths.
- The first type of business clock can be used to describe stochastic volatility
- The second type of clock can transform a diffusion into a jump process – most Lévy processes considered in the previous section can be generated as changing the clock of a diffusion with an increasing jump process (subordinator).
- All semimartingales can be written as time-changed Brownian motion.

Option Pricing

- To compute the time-0 price of a European option price with maturity at t, we compute the Fourier transform of the log return $\ln S_t/S_0$. Then we compute option value via Fourier inversions.
- The Fourier transform of a time-changed Lévy process:

$$\begin{aligned}\varphi(u) &= \mathbb{E}^{\mathbb{Q}}\left[e^{iuL_{\mathcal{T}_t}}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{iuL_{\mathcal{T}_t}-\kappa(u)\mathcal{T}_t}e^{\kappa(u)\mathcal{T}_t}\right] \\ &= \mathbb{E}^{\mathbb{M}}\left[e^{\kappa(u)\mathcal{T}_t}\right], \quad u \in \mathcal{D} \in \mathbb{C},\end{aligned}$$

  where the new measure $\mathbb{M}$ is defined by the exponential martingale:

$$\left.\frac{d\mathbb{M}}{d\mathbb{Q}}\right| = \exp(iuL_{\mathcal{T}_t} - \kappa(u)\mathcal{T}_t).$$

- Without time-change, $\exp(iuL_{\mathcal{T}_t} - \kappa(u)\mathcal{T}_t)$ is an exponential martingale by Levy-Khintchine Theorem.
- A continuous time change does not change the martingality.
- $\mathbb{M}$ is complex valued.

Complex Valued Measure Change

- When L_t and $\mathcal{T}_t = \int_0^t v_{s-}\, ds$ are independent, we have

$$\begin{aligned}\varphi(u) &= \mathbb{E}^{\mathbb{Q}}\left[e^{iuL_{\mathcal{T}_t}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[e^{iuL_h}] \middle| \mathcal{T}_t = h\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{\kappa(u)\mathcal{T}_t}\right],\end{aligned}$$

  by the law of iterated expectation.

  - No measure change is necessary
  - Similar to Hull and White (1987): The option value of an independent stochastic volatility model is written as the expectation of the BMS formula over the distribution of the integrated variance, $\mathcal{T}_t = \int_0^t v_s ds$.

- When L_t and v_t are correlated, the measure change from $\mathbb{Q}$ to $\mathbb{M}$ hides the correlation under the new measure.

- We may all $\mathbb{M}$ the correlation neutral measure.

Fourier Transform

- The Fourier transform of a time-changed Lévy process:

$$\varphi(u) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuL_{\mathcal{T}_t}}\right] = \mathbb{E}^{\mathbb{M}}\left[e^{\kappa(u)\mathcal{T}_t}\right]$$

  Tractability of $\varphi(u)$ depends on tractability of:

  1. The characteristic exponent of the Lévy process:
     - Tractable Lévy specifications include: Brownian motion, (Compound) Poisson, DPL, NIG, ... (done in previous section)
  2. The Laplace transform of $\mathcal{T}_t$ under $\mathbb{M}$:
     - Tractable Laplace comes from activity rate dynamics: affine, quadratic, Wishart.
     - The measure change from $\mathbb{Q}$ to $\mathbb{M}$ is defined by an exponential martingale.

- The two $(L; \mathcal{T}_t)$ can be chosen separately as building blocks, for different purposes.

Laplace transform of stochastic time $\mathcal{T}_t$

- We have solved the characteristic exponent of the Lévy process (by the Lévy -Khintchine Theorem).
- Now we try to solve the Laplace transform of the stochastic time,

$$\mathcal{L}_{\mathcal{T}}(\kappa) \equiv \mathbb{E}[e^{-\kappa \mathcal{T}_t}] = \mathbb{E}\left[e^{-\kappa \int_0^t v_s ds}\right].$$

- Recall the pricing equation for zero-coupon bonds:

$$P(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s ds}\right].$$

- The two pricing equations look analogous:
  - Both v_t and r_t need to be positive.
  - If we set $r_t = \kappa v_t$, $\mathcal{L}_{\mathcal{T}}(\kappa)$ is essentially the bond price.
- The similarity allows us to borrow the vast literature on bond pricing.

Review: Bond pricing - DTS

A long list of papers propose different dynamic term structure models:

- Specific examples:
  - Vasicek, 1977, JFE: The instantaneous interest rate follows an Ornstein-Uhlenbeck process.
  - Cox, Ingersoll, Ross, 1985, Econometrica: The instantaneous interest rate follows a square-root process.
  - Many multi-factor examples, such as Wishart TSM, ...
- Classifications (back-filling)
  - Duffie, Kan, 1996, Mathematical Finance: Spot rates are affine functions of state variables.
  - Duffie, Pan, Singleton, 2000, Econometrica: Affine with jumps.
  - Duffie, Filipovic, Schachermayer, 2003, Annals of Applied Probability: Super mathematical representation and generalization of affine models.
  - Leippold, Wu, 2002, JFQA: Spot rates are quadratic functions of state variables.
  - Filipovic, 2002, Mathematical Finance: How far can we go?
  - Carr, Gabaix, Wu, 2011: Bond prices are affine.

Example: Heston SV model

- Recall the Heston model:

$$
\begin{aligned}
dS_t/S_t &= (r-q)dt + \sqrt{v_t}dW_t \\
dv_t &= \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^v, \quad dW_t dW_t^v = \rho dt.
\end{aligned}
$$

- We can write the return process as a time-changed Lévy :

$$
lnS_t/S_0 = (r-q)W_{\mathcal{T}_t} - \frac{1}{2}\mathcal{T}_t, \quad \mathcal{T}_t = \int_0^t v_s ds.
$$

- Fourier transform of returns:

$$
\begin{aligned}
\varphi(u) &= \mathbb{E}^{\mathbb{Q}}\left[e^{iu\ln S_t/S_0}\right] = e^{iu(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{iu(W_{\mathcal{T}_t}-\frac{1}{2}\mathcal{T}_t)}\right] \\
&= e^{iu(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{iu(W_{\mathcal{T}_t}-\frac{1}{2}\mathcal{T}_t)-\kappa(u)\mathcal{T}_t+\kappa(u)\mathcal{T}_t}\right] \\
&= e^{iu(r-q)t}\mathbb{E}^{\mathbb{M}}\left[e^{\kappa(u)\mathcal{T}_t}\right],
\end{aligned}
$$

where $\kappa(u) = -\frac{1}{2}(iu + u^2)$ is the characteristic exponent of the Lévy process $W_t - \frac{1}{2}t$.

Example: Heston SV model

- The measure change is given by:
  $\frac{d\mathbb{M}}{d\mathbb{Q}} = \exp(iu(W_{\mathcal{T}_t} - \frac{1}{2}\mathcal{T}_t) - \kappa(u)\mathcal{T}_t)$.
- The v-dynamics under $\mathbb{M}$ are:

$$\begin{aligned} dv_t &= \kappa(\theta - v_t)dt + \mathbb{E}[iudW_{\mathcal{T}_t}, \sigma_v\sqrt{v_t}dW_t^v] + \sigma_v\sqrt{v_t}dW_t^v \\ &= \kappa(\theta - v_t)dt + iu\sigma_v\rho\sqrt{v_t}dt + \sigma_v\sqrt{v_t}dW_t^v \\ &= (\kappa\theta - \kappa^{\mathbb{M}}v_t)dt + \sigma_v\sqrt{v_t}dW_t^v, \end{aligned}$$

  with $\kappa^{\mathbb{M}} = \kappa - iu\sigma_v\rho$.
- Since the v dynamics are affine under $\mathbb{M}$, we have the Laplace transform exponential affine in v,

$$\begin{aligned} \varphi(u) &= \mathbb{E}^{\mathbb{Q}}\left[e^{iu\ln S_t/S_0}\right] = e^{iu(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{iu(W_{\mathcal{T}_t} - \frac{1}{2}\mathcal{T}_t)}\right] \\ &= e^{iu(r-q)t}\mathbb{E}^{\mathbb{M}}\left[e^{\kappa(u)\mathcal{T}_t}\right] = e^{iu(r-q)t + a(t) + b(t)v_0}, \end{aligned}$$

  with $a'(t) = b(t)\kappa\theta$ and $b'(t) = \kappa(u) - \kappa^{\mathbb{M}}b(t) - \frac{1}{2}b(t)^2\sigma_v^2$, starting at $a(0) = b(0) = 0$.

Example: Heston SV model

- The key for tractability is to maintain the v dynamics affine under $\mathbb{M}$.
- Consider $dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{\alpha + \beta v_t}dW_t^v$, which is still affine under $\mathbb{Q}$, but under $\mathbb{M}$:

  $$v_t = \kappa(\theta - v_t)dt + \mathbb{E}[iudW_{\mathcal{T}_t}, \sigma_v\sqrt{\alpha + \beta_v t}dW_t^v] + \sigma_v\sqrt{\alpha + \beta v_t}dW_t^v,$$

  unless we redefine the time change $\mathcal{T}_t = \int_0^t(\alpha + \beta v_s)ds$ or if $\rho = 0$.
- The constant volatility specification does not work either.
- A general specification $v_t = a_v + b_v^\top Z_t$ does not always work for the same reason, but the following two factor specification works.

  $$dv_t = \kappa(m_t - v_t)dt + \sigma_v\sqrt{v_t}dW_t^v, \quad dm_t = \kappa_m(\theta_m - m_t)dt + \sigma_m\sqrt{m_t}dW_t^m,$$

  with $dW_t^m dW_t^v = dW_t^m dW_t = 0$. See, Egloff, Leippold, Wu 2007 on variance swap investing, (click here) for the paper.

Example: Bates 1996

- Bates 1996 uses:

$$\begin{aligned} dS_t/S_t &= (r-q)dt + \sqrt{v_t}dW_t + dJ(\lambda) - \lambda(e^{\mu_J + \frac{1}{2}v_J} - 1)dt \\ dv_t &= \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t, \quad dW_t dW_t^v = \rho dt. \end{aligned}$$

- Conditional on a jump occuring, the jump size in return has a normal distribution (μ_J, v_J).
- We can write returns as a time-changes Lévy process:

$$S_t/S_0 = (r-q)t + [W_{\mathcal{T}_t} - \frac{1}{2}\mathcal{T}_t] + [X - k_X(1)t], \quad \mathcal{T}_t = \int_0^t v_s ds,$$

where X denotes a pure-jump Lévy process with density

$$\nu(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} e^{-\frac{(x-\mu_J)^2}{2v_J}}.$$

The cumulant exponent is

$$k_x(s) = \int_{\mathbb{R}_0} (e^{sx} - 1)\nu(x)dx = \lambda\left(e^{s\mu_J + \frac{1}{2}s^2 v_J} - 1\right).$$

Example: Bates 1996

- The Fourier transform of the return

$$\begin{aligned}\varphi_S(u) &= \mathbb{E}^{\mathbb{Q}}\left[e^{iu\ln S_t/S_0}\right] = e^{iu(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{iu(W_{\mathcal{T}_t}-\frac{1}{2}\mathcal{T}_t)}\right]\mathbb{E}^{\mathbb{Q}}\left[e^{iu(X_t-k_x(1)t)}\right] \\ &= e^{iu(r-q)t}e^{a(t)+b(t)v_0}e^{\varphi_J(u)},\end{aligned}$$

where $[a(t); b(t)]$ come directly from the Heston model and the characteristic exponent of the pure-jump Lévy process $\varphi_J(u)$ is:

$$\varphi_J(u) = \int_{\mathbb{R}_0}(e^{iux}-1)\nu(x)dx - iuk_x(1) = \lambda(e^{iu\mu_J-\frac{1}{2}u^2v_J}-1) - iu\lambda(e^{u_J+\frac{1}{2}v_J}-1)$$

- By definition, jumps are orthogonal to diffusion. Hence, the two components can be processed separately.
- Question: Why just time change diffusion W? Why not also time change the jump X?

Concluding Remarks

- Modeling security returns with (time-changed) Lévy processes enjoys three key virtues:
  - Generality: Lévy process can be made to capture any return innovation distribution; applying time changes can make this distribution vary stochastic over time.
  - Explicit economic mapping: Each Lévy component captures shocks from one economic source. Time changes capture the relative variation of the intensities of these impacts.
  - Tractability: Combining any tractable Lévy process (with tractable $\kappa(u)$) with any tractable activity rate dynamics (with a tractable Laplace) generates a tractable Fourier transform for the time changed Lévy process. The two specifications are separate.
- It is a nice place to start with for generating security return dynamics that are parsimonious, tractable, economically sensible, and statistically performing well.

Financial Engineering - Lecture 10

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

24.11.2011

University of Zurich

What have we been doing?

- **Find model(s)** that can **match European option prices** on the market,
- We want models where it is possible to **hedge**.

Why is it taking so long?

1. Black-Scholes model (log-normal returns, C^0 dynamics, hedging possible)
   $\rightarrow$ **existence of smile :-(** so **log-returns are not normally distributed!**
2. Local vol model (arbitrary distributions, C^0 dynamics, hedging possible)
   $\rightarrow$ **smile is replicated but hedging not satisfactory** so distributions are Ok but dynamics (path properties) are not good.
3. Stochastic vol model (fat tail & skewed returns, C^0 dynamics, hedging possible using more instruments)
   $\rightarrow$ **smile is roughly replicated but the short-end of the smile is much steeper in practice** so dynamics must include jumps.
4. Lévy model (arbitrary returns, possible jumps in dynamics, perfect hedging not possible)
   $\rightarrow$ **smile is replicated but in practice the returns change distributions over time** so we need to include dependence in returns increments, e.g. via time change.

No perfect model but FE is very useful

Since there is no perfect model, it is important to know

- what **stock prices paths** look like (e.g. distribution of returns is heavy tailed, downward jumps, etc.),
- which risks a **derivative** is exposed to (e.g. impact of jumps on derivative depends on T),
- that the quality of a model **depends on the financial product** that needs to be priced (e.g. local vol will price very well European options but misprice path-dependent options),
- which model assumptions make sense / which assumptions are seriously wrong for the product you are pricing/hedging $\rightarrow$ adapt your **model capital reserve**.

Now $\rightarrow$ **Variance swaps**

Motivation

Why variance swaps? Investors want **exposure to volatility!**

Variance/volatility swaps (VS) allow to trade the **realized variance/volatility** of a stock.

- They provide a **direct exposure** to variance/volatility **without the need to delta-hedge** an option.
- Volatility moves are **negatively correlated** to the stock price moves (leverage effect). Volatility assets have a **diversification** benefit.
  VS provide a **hedge against a crash**.

The VIX index

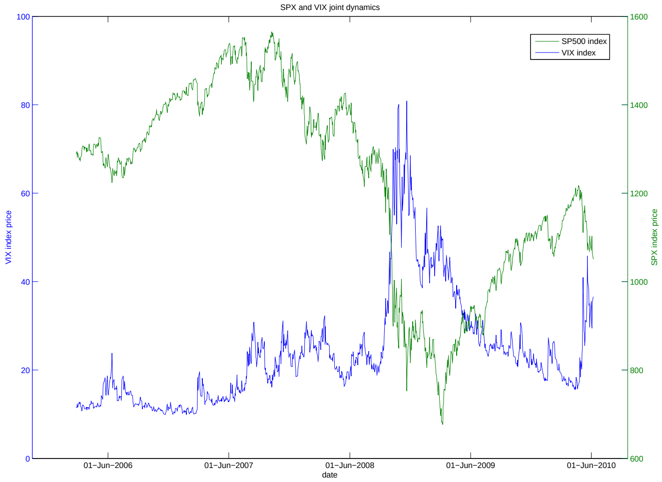
The VIX index (CBOE)

The VIX index represents* the expected future realized volatility of the S&P500 returns over the next 30 days.

There are **liquid futures and options** on the VIX.

*****: under assumptions that we will see when pricing variance swaps.

The SPX and VIX indices



Realized volatility: Definition

Annualized realized volatility

The *annualized realized volatility* over a period $[t, T]$ divided into N equal sized sub-intervals of time (in practice, one time-step is one trading day) is defined as:

$$RV_{t,T}(N) = \sqrt{\frac{252\sum_{i=1}^{N}\left(\ln\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2}{N}}.$$

- **Realized variance** $= RV^2_{t,T}(N)$,
- The average of returns disappeared because **average daily returns = 0**,
- 252 trading days.

RV is potentially unbounded...

Variance swap: Definition

Variance swap

The **holder** of a *variance swap* over $[0, T]$ **receives the realized variance** and pays the fixed leg K of the swap at maturity T. The payoff at T is as follows:

$$H_T^{VS} = M(RV_{0,T}^2(N) - K)$$

where
M: notional in units of variance, RV: realized volatility, K: strike of swap.

In practice,

- **OTC** products,
- **daily** closing prices are used to compute RV,
- K is agreed upon and **fixed at beginning of contract**,
- **No exchange of money at** $t = 0$. At maturity, cash settlement. Holder/seller pay depending on sign of payoff,
- $VIX_t^2 \approx$ **strike of variance swap on SPX over** $[t, t + 30\ days]$.

Variance swap pricing and replication

We write the realized variance in relation to futures prices (to include indices)

Annualized realized variance

The *annualized realized variance* over a period $[0, T]$ divided into N equal sized sub-intervals of time (in practice, one time-step is one trading day) is defined as:

$$RV_{0,T}^2(N) = \frac{252}{N}\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2.$$

Notation:

$$RV_{0,T}^2(N) = U\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2 = U\sum_{i=1}^{N}\left(\ln F_i - \ln F_{i-1}\right)^2.$$

with $U = \frac{252}{N}$

Realized variance replication

Goal: How do we hedge realized variance?
This is what the seller of a variance swap has to do. The **fair strike** of a variance swap is then the **cost of the hedge**.

Idea: Write $\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2$ in terms of hedgeable contingent claims!?
We can hedge futures and any European contingent claim using Breeden-Litzenberger (BL) static hedging formula.

Steps:

1. Get rid of the square in $\left(\ln\frac{F_i}{F_{i-1}}\right)^2$.
   $\rightarrow$ and make the futures prices $(F_k)_{k\geq 0}$ appear and a function $H(F_N)$ of the **terminal** futures price F_N only,
2. Apply the static hedging of European style contingent claims (BL) to $H(F_N)$,
3. Combine both terms and write $\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2$ as a sum of futures and European style options.

Realized variance replication, step 1

Step 1: Transform $\left(\ln \frac{F_i}{F_{i-1}}\right)^2$.

Define simple returns on $[t_{i-1}, t_i]$: $R_i := \frac{F_i - F_{i-1}}{F_{i-1}}$.
Note that $|R_i| << 1 \Rightarrow$ Taylor expansion.

$$\ln \frac{F_i}{F_{i-1}} = \ln\left(1 + \frac{F_i - F_{i-1}}{F_{i-1}}\right) = \ln(1 + R_i) = R_i - R_i^2/2 + O(R_i^3)$$

$$\Longrightarrow \ln^2 \frac{F_i}{F_{i-1}} = R_i^2 + O(R_i^3).$$

Replace R_i^2 in second expression by first expression and get:

$$\ln^2 \frac{F_i}{F_{i-1}} = 2R_i - 2\ln\frac{F_i}{F_{i-1}} + O(R_i^3)$$

$$\ln^2 \frac{F_i}{F_{i-1}} = 2R_i - 2(\ln F_i - \ln F_{i-1}) + O(R_i^3)$$

Realized variance replication, step 1

$$\begin{aligned}
\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2 &= \sum_{i=1}^{N}2R_i - 2\sum_{i=1}^{N}(\ln F_i - \ln F_{i-1}) + O(R_i^3) \\
&= 2\sum_{i=1}^{N}\frac{F_i - F_{i-1}}{F_{i-1}} - 2(\ln F_N - \ln F_0) + \sum_{i=1}^{N}O(R_i^3) \\
&= \underbrace{2\sum_{i=1}^{N}\frac{1}{F_{i-1}}(F_i - F_{i-1})}_{\text{dynamic position in futures}} - 2(\underbrace{\ln F_N}_{=H(F_N)} - \ln F_0) + \sum_{i=1}^{N}O(R_i^3),
\end{aligned}$$

Realized variance replication, step 2

We have written the realized variance $RV^2_{0,T}$ as a sum of **positions in futures** and a **log contract** that pays $\ln(F_T)$ at maturity T.

Step 2: The log-contract is a European style contingent claim. It can be **statically hedged using European options!**

Static hedging of the log-contract

Assuming there exists a **continuum of OTM European options** with the same maturity T as the variance swap, we can write the payoff of the log-contract as a infinite sum of weighted payoffs of OTM options.

$$\ln F_N - \ln F_0 = (F_N - F_0)/F_0 - \int_{F_0}^{\infty} \frac{1}{E^2}(F_N - E)^+ dE + \int_0^{F_0} \frac{1}{E^2}(E - F_N)^+ dE$$

Realized variance replication, step 3

Step 3: Add up the dynamic positions in the futures and the static positions in options.

$$
\begin{aligned}
\sum_{i=1}^{N}\left(\ln\frac{F_i}{F_{i-1}}\right)^2 &= 2\sum_{i=1}^{N}\frac{1}{F_{i-1}}(F_i - F_{i-1}) - 2(\ln F_N - \ln F_0) + \sum_{i=1}^{N} O(R_i^3) \\
&= 2\sum_{i=1}^{N}\frac{F_i - F_{i-1}}{F_{i-1}} + \sum_{i=1}^{N} O(R_i^3) \\
&\quad - 2\frac{F_N - F_0}{F_0} + \int_{F_0}^{\infty}\frac{2}{E^2}(F_N - E)^+ dE + \int_0^{F_0}\frac{2}{E^2}(E - F_N)^+ dE \\
&= 2\sum_{i=1}^{N}(F_i - F_{i-1})\left(\frac{1}{F_{i-1}} - \frac{1}{F_0}\right) \\
&\quad + \int_{F_0}^{\infty}\frac{2}{E^2}(F_N - E)^+ dE + \int_0^{F_0}\frac{1}{E^2}(E - F_N)^+ dE + \sum_{i=1}^{N} O(R_i^3).
\end{aligned}
$$

Realized variance replication, step 3

$$\underbrace{RV_{0,T}(N)}_{\text{Payoff we want to replicate}} = \sum_{i=1}^{N} 2U\left(\frac{1}{F_{i-1}} - \frac{1}{F_0}\right)(F_i - F_{i-1}) \quad (1)$$

$$+ \int_{F_0}^{\infty} \frac{2U}{E^2}(F_N - E)^+ dE + \int_0^{F_0} \frac{2U}{E^2}(E - F_N)^+ dE \quad (2)$$

$$+ \sum_{i=1}^{N} O(R_i^3). \quad (3)$$

Realized variance replication:

- **Dynamic part** (1): At each t_{i-1}, one should hold $2U\left(\frac{1}{F_{i-1}} - \frac{1}{F_0}\right)e^{-r(T-t_i)}$ futures contracts,
- **Static part** (2): $\frac{2U\Delta E}{E^2}$ European call options at all strikes $E > F_0$ and $\frac{2U\Delta E}{E^2}$ European put options at all strikes $E < F_0$ where ΔE is the increment between strikes available in practice.
- **Error term** (3) has been neglected in the past.

Variance swap pricing

Find fair strike K of variance swap:

Variance swap is a traded contract.
Fundamental theorem of asset pricing $\Longrightarrow$ discounted payoff of variance swap is a martingale under a risk-neutral measure $\mathbb{Q}$.

No exchange of money at time $t = 0$ means that

$$
\begin{aligned}
K &= \int_{F_0}^{\infty} \frac{2U}{E^2}\mathbb{E}^{\mathbb{Q}}\left[(F_N - E)^+\right] dE + \int_0^{F_0} \frac{2U}{E^2}\mathbb{E}^{\mathbb{Q}}\left[(E - F_N)^+\right] dE \\
&= \int_{F_0}^{\infty} \frac{2U}{E^2} e^{r(T-t_0)} C(t_0, F_0, E, T) dE + \int_0^{F_0} \frac{2U}{E^2} e^{r(T-t_0)} P(t_0, F_0, E, T) dE.
\end{aligned}
$$

Variance swap pricing: remarks

Assumptions:

- **Continuum of strikes**. Not the case in practice, especially for individual stocks. Worse when market dries up. Still possible to calculate a fair (?) strike K making assumption about how the option prices behave when their strike $E \to 0$ and $E \to \infty$.
- **Error term** $\sum_{i=1}^{N} O(R_i^3)$. is neglected to compute the hedge. Carr and Lee (2009) show that this error term $= \sum_{i=1}^{N} O(R_i^3) = -2U/3 \sum_{i=1}^{N} R_i^3 + \sum_{i=1}^{N} O(R_i^4)$.
  $\Longrightarrow$ fair strike K is neglecting the term $\mathbb{E}^{\mathbb{Q}}\left[-2U/3 \sum_{i=1}^{N} R_i^3\right]$ which is related to the **skewness of the returns**. Even though $R_i^3 \to 0$, its expectation $\mathbb{E}^{\mathbb{Q}}[R_i^3]$ does not empirically go to zero. This hedge neglects skewness and kurtosis effects. More precisely since $\mathbb{E}^{\mathbb{Q}}[R_i^3] < 0$, this means that the **hedge underestimates the realized variance**.

The VS disappeared during the financial crisis

The near quintupling of the volatility on some utility stocks and even greater moves in bank stocks has made the product almost impossible to hedge. [...]

They use the options markets to hedge the swaps and a deterioration in liquidity in single stock options hasn't helped.

Wall Street Journal, October 22, 2008, **Volatility kills market for volatility swaps**

The products were developed quickly and aggressively by the industry and transitioned from being an exotic risk to a vanilla one, and hence they got traded as one. Like everything in derivatives, variance swaps carry two or three types of risk; idiosyncratic risk, volatility of volatility risk and liquidity risk. To an extent all of the above were mis-priced.

Head of equity derivative sales at a US bank in London

It is still important to know variance swaps

These are very useful products for hedge funds, in the sense that if they want to take a pure view on realised volatility, variance is the cheapest product and they're easy to keep track of as they only have one payment at maturity. For those reasons there is always going to be demand.

Simon Yates, global head of equity derivatives at Credit Suisse in London

Path-dependent and early exercise options

Path-dependent options

Path-dependent options are options whose payoff (at maturity or upon exercise) **depend on the evolution and past of the underlying stock** before maturity or the exercise moment.

Examples: Barrier options, Asian options, etc.

Early exercise options

Early exercise options can be **settled before their maturity** upon the action of one of the counterparty. The payoff is paid at time of exercise.

Examples: American options, Bermudan options.

Barrier options

Barrier option

A *European barrier option* on the stock S with maturity T is a contract which **behaves like a European option conditionally on whether the barrier level** B **has been touched or not**. An **"out"** option is de-activated if the barrier is hit by the stock price, while an **"in"** option is activated if the barrier is touched. The monitoring of the barrier can be continuous or discrete.

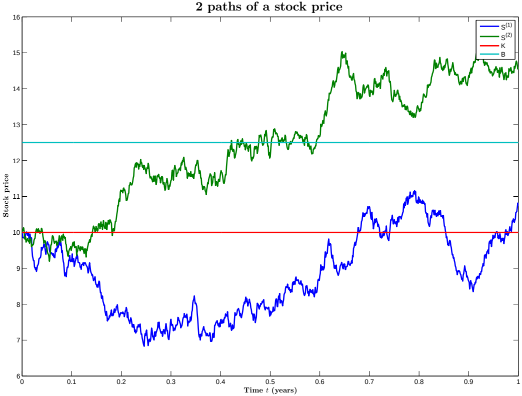
The payoff depends on whether the barrier is touched or not
$\Longrightarrow$ Path-dependent.

Examples: Up-and-out call, Down-and-in put.

Remark: Payoff barrier option $\leq$ payoff corresponding European option
$\Longrightarrow$ **barrier option is cheaper** than corresponding European option.

Barrier options

Up-and-out call. Payoff $= (S_T - K)^+$ **only if** $S_t < B, \forall t$.



scenario $S_T = S^{(1)}$, payoff$=0$,
scenario $S_T = S^{(2)}$, payoff> 0.

Asian options

Asian option

An *Asian option* on the stock S with maturity T and strike K is a contract whose payoff depends on the average underlying price over a fixed period of time.

Example with an arithmetic average:

- Continuous monitoring

$$A([0,T]) = \frac{1}{T}\int_0^T S(t)dt.$$

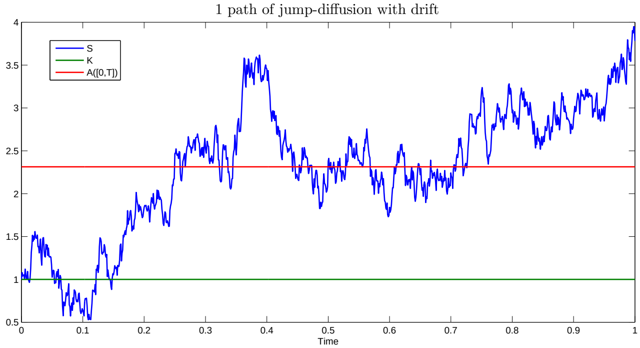
- Discrete monitoring with monitoring dates t_i

$$A([0,T]) = \frac{1}{N}\sum_{i=1}^{N} S(t_i)$$

The **payoff depends on all the values of S in the monitoring period** $\Longrightarrow$ Path-dependent.

Asian options

Asian call with continuous monitoring on $[0, T]$. Payoff $= (A([0,T]) - K)^+$.



American options

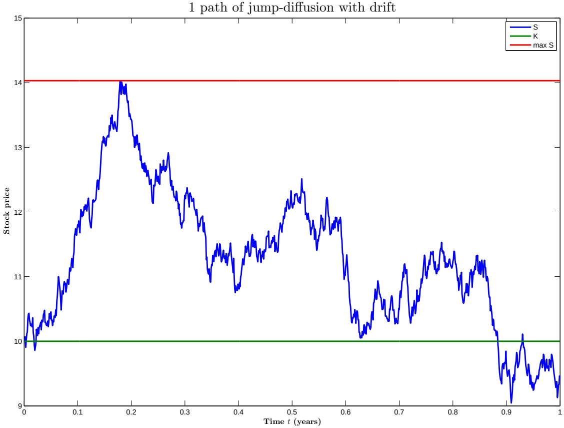
American call option

An *American call option* on the stock S is a contract which gives the holder of the option the right but not the obligation to buy the stock **at any point in time in the future up to maturity** T for a fixed price (strike price) K. (Similar definition for a put).

Can be exercised before maturity $\Longrightarrow$ Early exercise option.

American options

Question: **When is it best to exercise?**



European vs. path-dependent options

	European options	Path-dependent & early exercise options
Exercise	Can only be exercised at maturity	It depends! Ex 1: European Barrier option Ex 2: American option
Pricing challenge	**Reproduce the smile surface at time $t = 0$** which amounts to reproducing the terminal risk-neutral distributions $f(S_T \vert S_0)$ implied by prices (Breeden-Litzenberger formula)	**Reproduce the path properties** of the stock price

American options

Focus of the rest of this lecture: American options. **Most traded options**.

Questions:

- How to know when it is optimal to exercise?
- How to find the "fair" price?

Bounds to the price of an American option

American options are more flexible than European options.

American vs. European option

An American option is always more expensive than a European option:

$$C^A(t_0, S_0, T, K) \geq C^E(t_0, S_0, T, K),$$
$$P^A(t_0, S_0, T, K) \geq P^E(t_0, S_0, T, K).$$

The difference is called **early exercise premium**:

$$C^A(t_0, S_0, T, K) = C^E(t_0, S_0, T, K) + E_C(t_0, S_0, T, K)$$
$$P^A(t_0, S_0, T, K) = P^E(t_0, S_0, T, K) + E_P(t_0, S_0, T, K)$$

No-dividend case

American vs. European option with no dividend

Assuming a **non-dividend paying stock** and non-negative interest rates, an **American call option will never be exercised before maturity, i.e. its price is equal to that of the corresponding European call option**.

Proof:

Idea: We know that $C^A(t_0, S_0, T, K) \geq C^E(t_0, S_0, T, K)$. Show that $C^E(t_0, S_0, T, K) > S_0 - K$.

Consider a portfolio A which contains $C^E(t_0, S_0, T, K)$ and K units of risk-free bonds. At time t_0

$$V_0(A) = C^E(t_0, S_0, T, K) + Ke^{-r(T-t_0)}.$$

At time T:

$$V_T(A) = (S_T - K)^+ + K = \begin{cases} S_T & \text{if } S_T \geq K \\ K > S_T & \text{if } S_T < K \end{cases}$$

No-dividend case

By the no-arbitrage condition:

$$C^E(t_0, S_0, T, K) + Ke^{-r(T-t_0)} > S_{t_0}$$

equivalently:

$$C^E(t_0, S_0, T, K) > S_{t_0} - Ke^{-r(T-t_0)} > S_{t_0} - K.$$

Therefore:

$$C^A(t_0, S_0, T, K) \geq C^E(t_0, S_0, T, K) > S_{t_0} - K.$$

$\Rightarrow$ It is never optimal to exercise the American call!

An American call option never exercised is just a European option:

$$C^A(t_0, S_0, T, K) = C^E(t_0, S_0, T, K).$$

No-dividend case

For American puts: **It can be optimal to exercise**.

Scenario: The stock price goes to 0 at time $\tau < T$. The payoff at time τ if the holder of the option exercises is:

$$H_\tau = K - S_\tau = K.$$

Then he can invest in the bond until maturity and get $Ke^{r(T-\tau)}$.

If the holder waits until maturity, he can get at best $K < Ke^{r(T-\tau)}$.

$\Rightarrow$ Early exercise can increase profits.

So $P^A(t_0, S_0, T, K) > P^E(t_0, S_0, T, K)$ **in general**.

Put-call parity for American options?

Put-call parity inequality

Assuming a non-dividend paying stock and non-negative interest rates, the American call and put options with same maturity and strike satisfy the inequalities

$$S_0 - K \leq C^A(t_0, S_0, T, K) - P^A(t_0, S_0, T, K) \leq S_0 - Ke^{r(T-\tau)}$$

Half-Proof:

Right-inequality:

$$\begin{aligned} C^A(t_0, S_0, T, K) - P^A(t_0, S_0, T, K) &= C^E(t_0, S_0, T, K) - P^A(t_0, S_0, T, K) \\ &\leq \underbrace{C^E(t_0, S_0, T, K) - P^E(t_0, S_0, T, K)}_{=S_0 - Ke^{r(T-\tau)}}. \end{aligned}$$

Left inequality: See script.

Case of dividends

The no dividends assumption is not realistic but it helps understand why dividends are important.

If a company has dividends, it can be optimal to exercise an American call to buy the stock and get the dividend. Prior to the dividend announcement, the value of **the early exercise premium contains information about the expected future dividends** to be paid out.

$\rightarrow$ **Maths formulation**

Mathematical formulation of the early exercise problem

Goal of the holder: Exercise the option so that his expected profits are maximized. Denote by τ the exercise time:

Pricing problem

Find a stopping time τ such that

$$P^A(t_0, S_0, T, K) = \max_{\tau} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau - t_0)}(S_\tau - K)^+ | \mathcal{F}_{t_0}\right],$$

- τ is a **stopping time** w.r.t. filtration $(\mathcal{F}_t)$ generated by S, i.e. the event $\{\tau < t\}$ is known at time t. Natural since random exercise time τ is decided depending on value of S_t,
- FTAP but **payoff at random time**.

Mathematical formulation of the early exercise problem

The American put should be exercised when $S = 0$. Can we maximize expected profits by exercising even if $S > 0$? **Is there a value ε^P such that we should exercise put when $S < \varepsilon^P$?**

Exercise boundary for a put option

The *exercise boundary* is the stock price level $\varepsilon^P(t) \in \mathbb{R}_+$ such that it is optimal to exercise the put (resp. call) option when $S_t \leq \varepsilon^P(t)$ (resp. $S_t \geq \varepsilon^C(t)$):

$$\varepsilon^P(t) := \sup\{S > 0,\ P^A(t, S, T, K) = K - S\}.$$

$$\varepsilon^C(t) := \inf\{S > 0,\ C^A(t, S, T, K) = S - K\}.$$

$t \mapsto \varepsilon^P(t)$ is a **function** with values in $[0, K]$,
$t \mapsto \varepsilon^C(t)$ is a **function** with values in $[K, \infty]$,

Intrinsic and time value

Intrinsic value of an option

The *intrinsic value* I_t of an American option is the payoff that the holder would get from exercising this option. We have

$$I_t^P = (K - S_t)^+ \quad ; \quad I_t^C = (S_t - K)^+.$$

Time value of an option

The *time-value* $\mathcal{T}_t$ of the American option is the remaining value of the option, i.e.

$$P^A(t, S, T, K) = I_t^P + \mathcal{T}_t^P$$
$$C^A(t, S, T, K) = I_t^C + \mathcal{T}_t^C$$

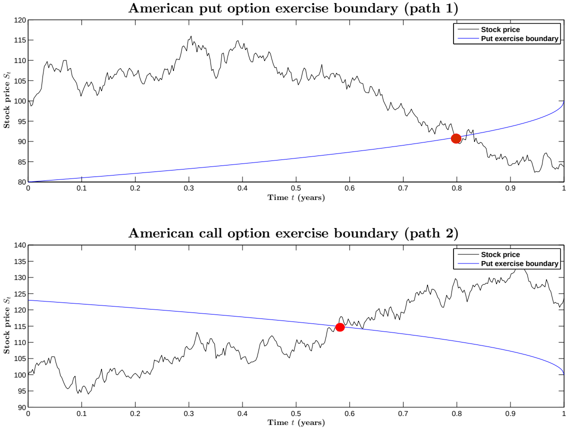
Interpretation: **Time-value = value of waiting to exercise**.

Exercise boundary - Illustration

- The exercise boundary $t \mapsto \varepsilon^P(t)$ is the largest stock price value such that the option value is equal to its intrinsic value (equivalently the time value of the option is zero).
- The region above the exercise boundary (all $S_t > \varepsilon^P(t)$ for all $t < T$) is called the **continuation region**. It contains the values of stock prices S_t for which the holder should continue to hold the option. In this region the time-value of the option is strictly positive, which means that there is value in waiting to exercise the option.
- The region below the exercise boundary (all $S_t \leq \varepsilon^P(t)$ for all $t < T$) is called the **exercise region** and corresponds to the stock prices for which the option should be exercised. In this region, the time value of the option is zero, which means that there is no additional value coming from waiting to exercise.

Exercise boundary - Illustration

GBM, $S_0 = 100$, $K = 100$.



Exercise boundary computation

Remark: **The exercise boundary depends on the model chosen**. It is an unknown function of the parameters of the model. In some models, it is possible to have a semi-closed form expression for ε but **most of the time the exercise boundary is approximated**.

Pricing American options

Assumptions:

- Black-Scholes framework (easily extendable)
- No dividends

Let us consider a put option with strike K and maturity $T \Rightarrow$ **What is its price** $P^A(t, S)$**?**

Pricing American options

We give the **intuition**.

- In the **continuation region**: the option is traded $\Rightarrow$ FTAP: **discounted price is a martingale under** $\mathbb{Q}$.

  $\Rightarrow$ satisfies the Black-Scholes PDE (discounted price has zero drift)

$$\frac{\partial P^A}{\partial t}(t, S) + \mathcal{A}_t P^A(t, S) - rP^A(t, S) = 0,$$

  with

$$\mathcal{A}_t = rS\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}.$$

Pricing American options

- In the **exercise region**: $P^A(t,S) = (K-S)1_{S<K}$.

$$\frac{\partial P^A}{\partial t}(t,S) + \mathcal{A}_t P^A(t,S) - rP^A(t,S) = -rS1_{S<K} + \frac{1}{2}\sigma^2 S^2 \delta_K(S) - r(K-S)1_{S<K}$$
$$= -rK1_{S<K} + \frac{1}{2}\sigma^2 S^2 \delta_K(S) < 0 a.s.$$

$\Rightarrow$ The Black-Scholes PDE is not satisfied and:

$$\frac{\partial P^A}{\partial t}(t,S) + \mathcal{A}_t P^A(t,S) - rP^A(t,S) < 0,$$

Pricing American options

Variational inequality for the American put price

Consider an American put option with maturity T and strike K on a stock paying no dividends. The price of the option $P^A(t, S, T, K)$ for a given stock price S at a given time t satisfies the following inequalities.

$$\begin{cases} \dfrac{\partial P^A}{\partial t}(t,S) + \mathcal{A}_t P^A(t,S) - rP^A(t,S) \leq 0 \\ P^A(t,S) \geq I^P(S) \\ (P^A(t,S) - I^P(S)) \cdot (\dfrac{\partial P^A}{\partial t}(t,S) + \mathcal{A}_t P^A(t,S) - rP^A(t,S)) = 0 \\ P^A(T,S) = I^P(S) \end{cases}$$

for t, S a.e. in $[0, T) \times \mathbb{R}_+$.

Pricing American options

Methods for calculating the price of an American option:

- Discretize the variational inequality: **PDE methods** $\rightarrow$ Linear Complementary Problem,
- **Tree methods**,
- **Monte-Carlo simulations** to solve directly the expectation problem (slow).

Financial Engineering - Lecture 11

Markus Leippold
University of Zurich

Chris Bardgett
University of Zurich

Elise Gourier
University of Zurich

17.11.2011

University of Zurich

Outline

1. Average/Asian Options
2. Trend Options
3. Barrier Options
   - Zero-Rebate Options
   - Barriers with Rebate
   - Static Replication

Asian Options I

- Although Asian options have been discussed in the literature for twenty years, they and their average character are still popular in practice.
- E.g., the Chicago Mercantile Exchange launched trading for three new cash-settled petroleum crack spread average price options contracts in July, 2009. These new average options are the gasoil-Brent crude oil crack spread options, the heating oil-crude oil crack spread options, and the RBOB-crude oil crack spread options.
- The London Metal Exchange (LME) also offers Traded Average Price Options based on the LME Monthly Average Settlement Price for several metals.
- The New York Mercantile Exchange and Intercontinental Exchange offers several average price options linked to energy products, e.g. Brent Average Price Options and WTI Average Price Options.

Asian Options II

- Other examples include commodity-linked bonds on average bond prices and Asian-style catastrophe (CAT) insurance options with payoffs depending on the accumulated catastrophic losses.
- Asian options come in numerous flavors.
    - If the strike price is a fixed quantity, the option is referred to as a fixed-strike Asian option.
    - If instead the strike price is the asset price itself, the contract is called a floating-strike Asian option.
    - If the starting date of averaging is in the future, the option is forward-starting. If instead the averaging period has already begun, the option is in-progress. If the averaging period is exactly the time to maturity, the option is plain-vanilla.

Options on the Geometric Mean I

- Define

$$\bar{S}_T := \bar{S}_T^{t_0,T} := \left(\prod_{i=0}^{m} S_{t_i}\right)^{\frac{1}{m+1}},$$

where $t_m = T$ and $t_i - t_j = \delta$ for all $i, j \in [0, m]$.

- For simplicity (in what follows), we set $t_0 = 0$. Then,

$$\begin{aligned}\bar{S}_T &= \left(S_{t_0} \times S_{t_1} \times \cdots \times S_{t_m}\right)^{\frac{1}{m+1}} \\ &= S_{t_0} \exp\left(\frac{\delta}{m+1}\sum_{i=1}^{m} i\left(r - \frac{1}{2}\sigma^2\right) + \frac{\sigma}{m+1}\sum_{i=1}^{m} W_{t_i}\right).\end{aligned}$$

Options on the Geometric Mean II

- Therefore,

$$\ln \frac{\bar{S}_T}{S_{t_0}} \mid S_{t_0} \sim \mathcal{N}\left(\frac{\delta}{m+1} \sum_{i=1}^{m} i \left(r - \frac{1}{2}\sigma^2 \right); \frac{\sigma^2}{(m+1)^2} \mathrm{Var}(\sum_{i=1}^{m} W_{t_i}) \right).$$

- Since

$$\begin{aligned} \sum_{i=1}^{m} i &= \frac{m(m+1)}{2} \\ \sum_{i=1}^{m} i^2 &= \frac{1}{6}(1+m)m(1+2m), \end{aligned}$$

Options on the Geometric Mean III

and

$$
\begin{aligned}
\sum_{i=1}^{m} W_{t_i} &= mW_{t_1} + (m-1)(W_{t_2} - W_{t-1}) \\
&\quad + (m-2)(W_{t_3} - W_{t-2}) + \cdots + W_{t_m} - W_{t_{m-1}} \\
\mathrm{Var}\left(\sum_{i=1}^{m} W_{t_i}\right) &= \delta m^2 + \delta(m-1)^2 + \cdots + \delta = \delta \sum_{i=1}^{m} i^2,
\end{aligned}
$$

we get

$$
\ln \frac{\bar{S}_T}{S_{t_0}} \mid S_{t_0} \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{T-t_0}{2}; \frac{\sigma^2(1+2m)}{6(1+m)}(T-t_0)\right).
$$

- Furthermore, note that when $m \to \infty$

$$
\ln \frac{\bar{S}_T}{S_{t_0}} \mid S_{t_0} \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{T-t_0}{2}; \frac{\sigma^2}{3}(T-t_0)\right).
$$

Options on the Geometric Mean IV

- For the pricing of European options on the geometric average, we can proceed as follows, with $\tau = T - t_0$:

$$
\begin{aligned}
C_0 &= e^{-r\tau}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(\bar{S}_T - K\right)^+\right] \\
&= e^{-r\tau}\underbrace{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\bar{S}_T\mathbf{1}_{\{\bar{S}_T>K\}}\right]}_{=:(\text{B})} - e^{-r\tau}K\underbrace{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\bar{S}_T>K\}}\right]}_{=:(\text{A})}.
\end{aligned}
$$

- We start by calculating the expression for (A):

$$
\begin{aligned}
(\text{A}) &= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\bar{S}_T>K\}}\right] = \mathbb{Q}(\ln \bar{S}_T > \ln K) \\
&= \mathbb{Q}\left(\frac{\ln \bar{S}_{t_0}/K + \left(r - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}}{\sigma\sqrt{\tau/3}} > \xi\right),
\end{aligned}
$$

where ξ is a standard normal random variable, i.e., $\xi \sim \mathcal{N}(0,1)$.

Options on the Geometric Mean V

- Hence,

$$(\mathrm{A}) = \Phi(d_2),$$

with

$$d_2 = \frac{\ln \bar{S}_{t_0}/K + \left(r - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}}{\sigma\sqrt{\tau/3}}.$$

For the calculation of (B), we introduce the measure $\bar{\mathbb{P}} \sim \mathbb{Q}$ which we define as

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\gamma^2 t + \gamma W_t\right),$$

with a constant parameter γ.

- Hence, we can write (again $t_0 = 0$ for convenience)

$$\begin{aligned}(\mathrm{B}) &= S_{t_0} e^{\left(r-\frac{1}{2}\sigma^2\right)\frac{\tau}{2}} \mathbb{E}_{t_0}^{\mathbb{Q}}\left(e^{\frac{\sigma}{\sqrt{3}}W_T}\mathbf{1}_{\{\bar{S}_T>K\}}\right)\\ &= \mathbb{E}_{t_0}^{\bar{\mathbb{P}}}\left(e^{\frac{\sigma}{\sqrt{3}}W_T+\frac{1}{2}\gamma^2 T-\gamma W_T}\mathbf{1}_{\{\bar{S}_T>K\}}\right).\end{aligned}$$

Options on the Geometric Mean VI

- If we set $\gamma = \sigma/\sqrt{3}$, the above expression simplifies to

$$(\mathrm{B}) = S_{t_0} e^{\left(r-\frac{1}{2}\sigma^2\right)\frac{\tau}{2}+\frac{\sigma^2}{6}\tau}\bar{\mathbb{P}}(\bar{S}_T > K).$$

- For the probability $\bar{\mathbb{P}}(\bar{S}_T > K)$ we get

$$\begin{aligned}\bar{\mathbb{P}}(\bar{S}_T > K) &= \bar{\mathbb{P}}\left(S_{t_0}\exp\left(\left(r-\frac{\sigma^2}{2}\right)\frac{\tau}{2}+\frac{\sigma}{\sqrt{3}}\left(\bar{W}_T+\frac{\sigma}{\sqrt{3}}\tau\right)\right) > K\right) \\ &= \bar{\mathbb{P}}\left(\frac{\ln S_{t_0}/K + \left(r-\frac{\sigma^2}{2}\right)\frac{\tau}{2}+\frac{\sigma^2}{3}\tau}{\sigma\sqrt{\tau/3}} > \xi\right),\end{aligned}$$

where $\xi \sim \mathcal{N}(0,1)$. Therefore,

$$\bar{\mathbb{P}}(\bar{S}_T > K) = \Phi(d_1),$$

Options on the Geometric Mean VII

with

$$d_1 = \frac{\ln S_{t_0}/K + \left(r - \frac{\sigma^2}{2}\right)\frac{\tau}{2} + \frac{\sigma^2}{3}\tau}{\sigma\sqrt{\tau/3}}.$$

- The price of a call option on the geometric average of a stock is therefore, at initiation of the contract, given by

$$\bar{C}_{t_0} = e^{-\left(r+\frac{1}{2}\sigma^2\right)\frac{\tau}{2}+\frac{\sigma^2}{6}\tau} S_{t_0}\Phi(d_1) - e^{-r\tau}K\Phi(d_2).$$

- Note that $\bar{C}_{t_0}$ is the value of the call option at initiation.
- For hedging purposes however, we need to know the value of the option during the lifetime of the contract.
- This value depends on the history of the stock price evolution, as the option's underlying is the geometric average.

Options on the Geometric Mean VIII

- To see this, consider first the case of discretely sampled stock prices. The geometric average $\bar{S}_T$ at a time $t_k \geq t_0$ can be decomposed as follows:

$$\bar{S}_T \big| \bar{S}_{t_k} = \bar{S}_{t_k} \exp\left(\frac{m-k}{m+1} R_{t_k, t_{k+1}} + \cdots + \frac{1}{m+1} R_{t_{m-1}, t_m}\right),$$

where $R_{t_j, t_{j+1}}$ are the returns over the time interval $[t_j, t_{j+1}]$.

- Note that

$$\begin{aligned} \sum_{i=k}^{m} (m-i) &= \frac{(m-k)(m-k+1)}{2} \\ \sum_{i=k}^{m} (m-i)^2 &= \frac{1}{6}(m-k+1)(2(m-k)+1)(m-k). \end{aligned}$$

Options on the Geometric Mean IX

- For continuously sampled stock prices, i.e., when $m \to \infty$,

$$\int_{t_k}^{T} \frac{T - t_k}{T - t_0} dt_k = \frac{1}{2} \frac{(T - t_k)^2}{T - t_0}$$

$$\int_{t_k}^{T} \left(\frac{T - t_k}{T - t_0} \right)^2 dt_k \quad = \quad \frac{1}{3} \frac{(T - t_k)^3}{(T - t_0)^2}.$$

- Therefore, the conditional distribution of $\ln \bar{S}_T$ is given by

$$\ln \bar{S}_T / S_{t_k} \,\big|\, \bar{S}_{t_k} \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2 \right) \frac{(T - t_k)^2}{2(T - t_0)}, \frac{\sigma^2}{3} \frac{(T - t_k)^3}{(T - t_0)^2} \right).$$

(Remember:)

$$\ln \bar{S}_T / S_{t_0} \,\big|\, \bar{S}_{t_0} \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2 \right) \frac{T - t_k}{2}, \frac{\sigma^2 (T - t_0)}{3} \right).$$

Enhanced Options on the Geometric Mean I

- In principle, we could generalize the geometric averaging to the following weighting scheme:

$$\bar{S}_T^{\{a_i\}}\Big|_{\bar{S}_{t_0}} := \bar{S}_{t_0} \exp\left(a_1 m R_{t_0,t_1} + a_2(m-1)R_{t_1,t_2} + \cdots + a_m R_{t_{m-1},t_m}\right)^{1/(m+1)}$$

- With $a = 1$, we get the standard geometric average as analyzed above. For $a_i = a_m = a$, for all $i = 1, \ldots, m$, we get

$$\bar{S}_T^a \big| \bar{S}_{t_0} \sim \mathcal{N}\left(a\left(r - \frac{1}{2}\sigma^2\right)\frac{T - t_0}{2}m, a^2\frac{\sigma^2(1+2m)}{6(1+m)}(T - t_0)\right) \tag{1}$$

- We know that average or Asian options provide a possibility to diversify away timing risk. This diversification however comes at a price. By inspection of formula (1) the tradeoff becomes clear.

Enhanced Options on the Geometric Mean II

- For instance, we could set $a = 2$. In this case, the mean return of $\bar{S}_T^a$ is equal to the mean return of S_T.
- However, since $\bar{S}_T^a$ provides some time diversification, we might expect the variance of $\bar{S}_T^a$ to be larger than the variance of S_T. Indeed, we have for $a = 2$:

$$\begin{aligned}\mathrm{Var}\left(\ln \bar{S}_T^a / S_{t_0} \,\middle|\, \bar{S}_{t_0}\right) &= 4\frac{m(2m+1)}{6(m+1)}\sigma^2\delta \\ \lim_{\delta\to 0}\mathrm{Var}\left(\ln \bar{S}_T^a / S_{t_0} \,\middle|\, \bar{S}_{t_0}\right) &= \frac{4}{3}\sigma^2(T - t_0).\end{aligned}$$

- Hence, we ‘buy’ time diversification at a higher volatility, a volatility which is one third larger than the volatility of the underlying process.
- On the other hand, we could equalize the variance of the processes S_T^a and S_T. We can achieve this by setting $a = \sqrt{3}$.

Enhanced Options on the Geometric Mean III

- By doing so however, we get a mean return for S_T^a which is smaller than the mean return of the underlying process:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\ln \bar{S}_T^a / S_{t_0} \middle| \bar{S}_{t_0}\right) &= \frac{\sqrt{3}}{2}\left(r - \frac{1}{2}\sigma^2\right)(T - t_0) \\ &< \left(r - \frac{1}{2}\sigma^2\right)(T - t_0).\end{aligned}$$

Exchange Asian Options I

- In principle, we could think of an option which gives the difference of the average and the current level of the index, i.e., we need to price a contract with value

$$
\begin{aligned}
\Pi_{t_0} &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left(\bar{S}^a_T - S_T\right)\\
&= e^{-r\tau}\left(\underbrace{\mathbb{E}^{\mathbb{Q}}_{t_0}\left(\mathbf{1}_{\{\bar{S}^a_T > S_T\}}\bar{S}^a_T\right)}_{=:(A)} - \underbrace{\mathbb{E}^{\mathbb{Q}}_{t_0}\left(\mathbf{1}_{\{\bar{S}^a_T > S_T\}}S_T\right)}_{=:(B)}\right)
\end{aligned}
$$

- To calculate (A), we proceed as follows:

$$
\begin{aligned}
(A) &= \mathbb{E}^{\mathbb{Q}}_{t_0}\left(\mathbf{1}_{\{\bar{S}^a_T > S_T\}}\bar{S}^a_T\right)\\
&= \mathbb{E}^{\tilde{\mathbb{P}}}_{t_0}\left(\mathbf{1}_{\{\bar{S}^a_T > S_T\}}S_{t_0}e^{\frac{a}{2}(r-\frac{1}{2}\sigma^2)\tau+\frac{a}{\sqrt{3}}\sigma W_T+\frac{1}{2}\gamma^2\tau-\gamma W_T}\right).
\end{aligned}
$$

Exchange Asian Options II

- By choosing $\gamma = \sigma a/\sqrt{3}$, we get

$$(\mathrm{A}) \quad = \quad S_{t_0} \exp\left(\frac{a}{2}(r - \frac{1}{2}\sigma^2)\tau + \frac{a^2}{6}\sigma^2\tau\right) \underbrace{\mathbb{E}_{t_0}^{\tilde{\mathbb{P}}}\left(\mathbf{1}_{\{\bar{S}_T^a > S_T\}}\right)}_{:=(\mathrm{A'})}.$$

- To calculate the expectation (A'), we need to perform an intermediate calculation. In particular, we need to know the correlation between the index and its average:

$$\rho_{\bar{S}_T^a, S_T}\Big|\mathcal{F}_{t_0} \quad = \quad \frac{\frac{a}{2}\sigma^2\tau}{\sqrt{\frac{a^2\sigma^2}{3}\tau\sqrt{\sigma^2\tau}}}$$

$$= \quad \sqrt{\frac{3}{4}} := \rho.$$

Exchange Asian Options III

- Hence, the correlation is independent of the multiplier a, which was of course to be expected. With this correlation parameter, we can now calculate the expectation $\mathbb{E}_{t_0}^{\tilde{\mathbb{P}}}\left(\mathbf{1}_{\{\bar{S}_T^a > S_T\}}\right)$ by introducing an appropriate measure change:

$$
\begin{aligned}
\text{(A')} \quad &= \tilde{\mathbb{P}}_{t_0}\Bigg((\frac{a}{2}-1)(r-\frac{\sigma^2}{2})\tau + \frac{a\sigma}{\sqrt{3}}(\tilde{W}_T + \frac{a\sigma}{\sqrt{3}}\tau) \\
&> \sigma\rho(\tilde{W}_T + \frac{a\sigma}{\sqrt{3}}\tau) + \sigma\sqrt{1-\rho^2}W_T^{\perp} \Bigg) \\
&= \tilde{\mathbb{P}}_{t_0}\Bigg((\frac{a}{2}-1)(r-\frac{\sigma^2}{2})\tau + \frac{a\sigma^2}{\sqrt{3}}(\frac{a}{\sqrt{3}}-\rho)\tau \\
&> \sigma(\rho - \frac{a}{\sqrt{3}})\tilde{W}_T + \sigma\sqrt{1-\rho^2}W_T^{\perp} \Bigg)
\end{aligned}
$$

Exchange Asian Options IV

- Since we have

$$\mathrm{Var}_{t_0}\left(\sigma(\rho - \frac{a}{\sqrt{3}})\tilde{W}_T + \sigma\sqrt{1-\rho^2}W_T^{\perp}\right) = \sigma^2\tau\left(\frac{a^2}{3} - a + 1\right),$$

we can continue to simplify the expression for (A') and write

$$(\mathrm{A}') \quad = \quad \tilde{\mathbb{P}}_{t_0}\left(\frac{(\frac{a}{2}-1)(r-\frac{\sigma^2}{2})\tau + \frac{a\sigma^2}{\sqrt{3}}(\frac{a}{\sqrt{3}}-\rho)\tau}{\sigma\sqrt{\tau\left(\frac{a^2}{3}-a+1\right)}} > \xi\right),$$

where $\xi \sim \mathcal{N}(0,1)$. For $a=1$, we get

$$(\mathrm{A}') = \tilde{\mathbb{P}}_{t_0}\left(\frac{-\frac{1}{2}(r-\frac{\sigma^2}{2})\tau - \frac{\sigma^2}{6}\tau}{\sigma\sqrt{\tau/3}} > \xi\right),$$

Exchange Asian Options V

while for $a = 2$, we simply get

$$(\text{A'}) = \tilde{\mathbb{P}}_{t_0}\left(\frac{\sigma\sqrt{\tau}}{\sqrt{3}} > \xi\right).$$

Outline

Trend Derivatives

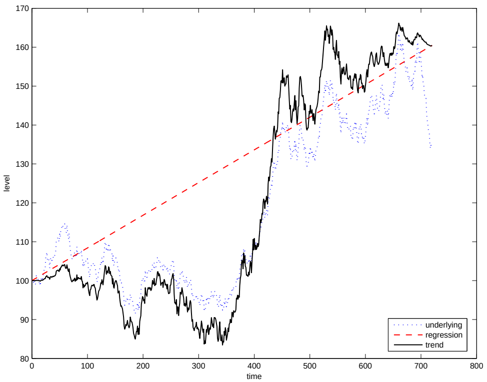


Figure: Linear trend derivative $\mathcal{T}_T^{t_0,T}$ with time-to-maturity $T - t_0$ of 2 years and nominal value of 100. Trajectory of the underlying index (dotted line), Evolution of the trend $T_{t_i}^{t_0,T}$ over time $t_i \in [t_0, T]$ (solid line) and regression line (dashed line). For the simulation, we assume $S_0 = 100$, $r = 0.05$, $\sigma = 0.20$ and a daily sampling for the trend calculation.

Options on Trends I

- The simple trend option pays at expiration T the value of

$$\pi_T = \max\left(\hat{S}_T^{t_0,T} - K, 0\right), \tag{2}$$

where $S_T^{t_0,T}$ is the trend calculated on the stock S over the lifetime of the option, i.e., from time t_0 to T.

- The time-T value of the trend $\hat{S}_T^{t_0,T}$ for the period $[t_0, T]$ is determined as

$$\hat{S}_T^{t_0,T} = S_{t_0} e^{b_{t_0}^T (T-t_0)}, \tag{3}$$

where $T = t_m$ and m is the number of intervals between sampled stock prices starting at t_0 and ending at T.

Options on Trends II

- We determine the term $b_{t_0}^T$ in equation (3) by a least-square principle from the equation

$$\log\left(\frac{S_T}{S_{t_0}}\right) = b_{t_0}^T\,(T - t_0)\,. \tag{4}$$

- We obtain $b_{t_0}^T$ as

$$\begin{aligned} b_{t_0}^T &= \frac{\sum_{i=0}^m \left(t_i - \frac{1}{2}(T - t_0)\right)\left(\log\left(\frac{S_{t_i}}{S_{t_0}}\right) - \frac{1}{m}\sum_{j=0}^m \log\left(\frac{S_{t_j}}{S_{t_0}}\right)\right)}{\sum_{i=0}^m \left(t_i - \frac{1}{2}(T - t_0)\right)^2} \\ &= \frac{12\sum_{i=0}^m \left(i\delta - \frac{1}{2}(T - t_0)\right)}{m(m+2)(m+1)\delta^2}\log\left(\frac{S_{t_i}}{S_{t_0}}\right) \qquad (5) \\ &\equiv \sum_{i=0}^m \alpha_{t_i}\log\left(\frac{S_{t_i}}{S_{t_0}}\right)\,, \end{aligned}$$

Options on Trends III

with the assumption that stock prices are taken from an equidistant grid $[t_0, \dots, t_m = T]$ with $\delta = t_i - t_{i-1}$, for all $i = 1, ..., m$.

- To price the option in equation (2), we first rewrite equation (3) as

$$\hat{S}_T^{t_0,T} = S_{t_0} \exp\left(\left(\alpha_{t_1} R_{t_1} + \alpha_{t_2} R_{t_2} + \ldots + \alpha_{t_m} R_{t_m}\right)(T - t_0)\right), \quad (6)$$

where R_{t_i} is the one-period return for the period $[t_{i-1}, t_i]$ distributed as

$$R_{t_i} \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)\delta, \sigma^2\delta\right),$$

and where we define α_{t_i}, as

$$\alpha_{t_i} \equiv \sum_{j=i}^{m} \beta_j^{t_m}. \quad (7)$$

Options on Trends IV

- For the subsequent derivations, it is essential to know the properties of the cumulative sum of α.
- The term α_{t_i} defined as

$$\alpha_{t_i} \equiv \frac{12 \sum_{j=i}^{m} \left(i\delta - \frac{1}{2}(T - t_0) \right)}{m(m+2)(m+1)\delta^2},$$

has the following two properties:

$$\begin{aligned} \sum_{i=1}^{m} \alpha_{t_i} \delta &= 1, \\ (T - t_0) \sum_{i=1}^{m} \alpha_{t_i}^2 \delta & \quad \begin{cases} = 1 & \text{for } m = 1, 2 \text{ ;} \\ \in]1, 6/5[& \text{for } 2 < \ m < \infty \text{ ;} \\ = 6/5 & \text{for } m = \infty. \end{cases} \end{aligned}$$

Options on Trends V

- We can now derive the distributional properties of the logarithm of the trend. In particular, the term

$$\log \hat{S}_T^{t_0,T} = \log S_{t_0} + \sum_{i=1}^{m} \alpha_{t_i} R_{t_i} (T - t_0)$$

is standard normal distributed with

$$\begin{aligned}\log \hat{S}_T^{t_0,T} &\sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T - t_0)\sum_{i=1}^{m}\alpha_{t_i}\delta, \sigma^2 (T - t_0)^2 \sum_{i=1}^{m}\alpha_{t_i}^2\delta\right],\\ &\sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T - t_0), \sigma^2 (T - t_0)^2 \sum_{i=1}^{m}\alpha_{t_i}^2\delta\right]. \qquad (8)\end{aligned}$$

- In addition, from the second property of α we conclude that the trend $\hat{S}_T^{t_0,T}$ has the same drift but higher volatility than the underlying stock price S_T.

Options on Trends VI

- To simplify notation, we will write $\hat{S}_t$ instead of $\hat{S}_t^{t_0,T}$. No confusion should occur. Furthermore, we define the time-to-maturity variance of the logarithmic trend as

$$v_{t_0}^m = \sigma^2 (T - t_0)^2 \delta \sum_{i=1}^{m} \alpha_{t_i}^2. \tag{9}$$

- We note that, compared to the time-to-maturity volatility of the logarithmic stock price, $\sigma\sqrt{T - t_0}$, the volatility of the logarithmic trend given in equation (9) behaves quite differently. At time t_0, i.e., at initiation of the trend derivative, the volatility of the trend $v_{t_0}^m$ is higher than the volatility of the stock price process $\sigma\sqrt{T - t_0}$.

Options on Trends VII

- This result is obvious from the second property of α, which implies

$$1 < \frac{v_{t_0}^m}{\sigma^2(T - t_0)} \leq \frac{6}{5}, \quad m > 2. \qquad (10)$$

From

$$\mathbb{E}_0\left(\hat{S}_T^{t_0,T}\right) = \mathbb{E}_0\left(\hat{S}_T\right),$$

where $\mathbb{E}_0$ is the conditional time-t_0 expectation under $\mathbb{P}$, together with the relation (10), we can conjecture that, at the initiation of the contract, the price of a plain vanilla call option with strike K will be below the price of a simple trend option with the same strike.

Options on Trends VIII

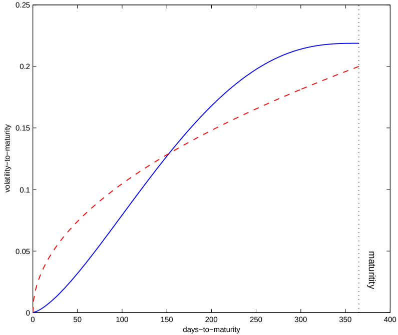


Figure: The time-to-maturity volatilities of the underlying stock price and of its trend, respectively, as a function of time-to-maturity. We fix $\sigma = 0.2$ and calculate the trend by using daily samples, i.e., $m = 360$. The graph plots the volatilities $\sqrt{v_{t_i}^m}$ (solid line) and $\sigma\sqrt{\tau_i}$ (dashed line) as a function of days-to-maturity.

Options on Trends IX

Proposition

At initiation, the price π_{t_0} of the simple trend option with the payoff given in equation (2) and with time-to-maturity $\tau = T - t_0$ is given as

$$\pi_{t_0} = e^{-\frac{1}{2}\left(\sigma^2\tau - v_{t_0}^m\right)} S_{t_0} N\left[d_1\right] - e^{-r\tau} K N\left[d_2\right],$$

where $N(\cdot)$ is the cumulative density of the normal distribution and

$$\begin{aligned} d_1 &= \frac{\log\left(S_{t_0}/K\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau + v_{t_0}^m}{\sqrt{v_{t_0}^m}}, \\ d_2 &= \frac{\log\left(S_{t_0}/K\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sqrt{v_{t_0}^m}}. \end{aligned}$$

Options on Trends X

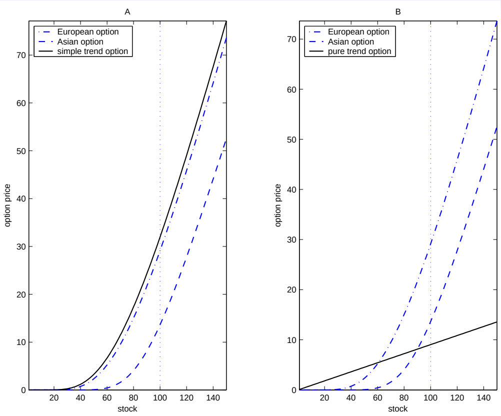


Figure: Panel A compares the time-t_0 option prices of a simple trend option with a European call and an Asian option on the geometric mean. We assume $K = 100$, $\sigma = 0.2$, $r = 0.05$, time-to-maturity $T - t_0 = 5$ years and a daily sampling of the stock price. Panel B compares a pure trend option with the call and the Asian option.

Outline

1. Average/Asian Options
2. Trend Options
3. Barrier Options
   - Zero-Rebate Options
   - Barriers with Rebate
   - Static Replication

Introduction

- Making matters more volatile, dealers say that pitched battles often erupt around knock-out barriers, with traders hollering across the trading floor of looming billion-dollar transactions.
- In three or four minutes it is all over. But in that time every trade gets sucked into the vortex.
- This situation prompted George Soros to say in his book 'Knock-out options relate to ordinary options the way crack relates to cocaine' (1995):

  I would not have said that a few months ago, when I testified before Congress, but we have had a veritable crash in currency markets since then. As I have said before, knock-out options played the same role in the 1995 yen explosion as portfolio insurance did in the stock market crash of 1987, and for the very same reason. Portfolio insurance was subsequently rendered inoperable by the introduction of the so- called circuit breakers. Something similar needs to be done now with knock-out options.

The Image of a Function I

- We work with the standard assumptions of a diffusion process driving the underlying price. We do not consider dividends.

Definition

Consider a payoff function $g(S,t)$ satisfying the Black-Scholes partial differential equation. Then, the image of $g(S,t)$ relative to the Barrier H is defined to be the function:

$$g^*(S,t) = \left(\frac{S}{H}\right)^p g(H^2/S, t).$$

The Image of a Function II

Proposition

For $g(S,t)$ and $g^*(S,t)$ the following properties hold:

1. $(g^*)^* = g$.
2. *If $\mathcal{L}^{BS} g = 0$, then $\mathcal{L}^{BS} g^* = 0$ for any $H > 0$, where $\mathcal{L}^{BS}$ is the Black-Scholes infinitesimal generator.*
3. *When $S = H$, then $g = g^*$.*
4. $y(S,t) = g(S,t) - g^*(S,t)$ *satisfies $\mathcal{L}^{BS} y = 0$ and the boundary condition $y(H,t) = 0$.*

Pricing the Image I

Proposition

Let g be the time T payoff function of a claim on S and define

$$\Pi_t^g = e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left(g(S_T, T)\right) =: e^{-r\tau} f(S, t).$$

Then, the claim with the payoff $g^*(S, T)$ has the arbitrage-free price

$$\Pi_t^{g^*} = e^{-r\tau}\left(\frac{S_t}{H}\right)^p f\left(\frac{H^2}{S_t}, t\right),$$

with $p = 1 - 2r/\sigma^2$.

Pricing the Image II

Proof

Define $Z_t = (S_t/H)^p$. Then,

$$\begin{aligned} dZ_t &= p\frac{S_t^{p-1}}{H^p}dS_t + \frac{1}{2}p(p-1)\frac{S_t^{p-2}}{H^p}(dS_t)^2 \\ &= \left(p\frac{S_t^p}{H^p}r + \frac{1}{2}p(p-1)\frac{S_t^p}{H^p}\sigma^2\right)dt + p\frac{S_t^p}{H^p}\sigma dW_t \end{aligned}$$

with W_t a $\mathbb{Q}$-Brownian motion. Now, our aim is to choose p in such a way that Z_t becomes a martingale. We can achieve this by choosing $p = 1 - 2r/\sigma$. Then,

$$dZ_t = pZ_t\sigma dW_t,$$

which is clearly an exponential $\mathbb{Q}$-martingale.

Pricing the Image III

Proof (con'd)

The process Z_t can now be used to define a measure change as follows:

$$\eta_t\big|_{\mathcal{F}_{t_0}} = \frac{d\mathbb{Q}^Z}{d\mathbb{Q}} = \frac{Z_t}{Z_{t_0}}, \quad \mathbb{Q}^Z \sim \mathbb{Q}.$$

Then,

$$\begin{aligned}
\Pi_t^{g^*} &= e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left(\left(\frac{S_T}{H}\right)^p g\left(\frac{H^2}{S_T}\right)\right) \\
&= e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}^Z}\left(\frac{1}{\eta_T}\left(\frac{S_T}{H}\right)^p g\left(\frac{H^2}{S_T}\right)\right) \\
&= e^{-r\tau}\left(\frac{S_t}{H}\right)^p \mathbb{E}_t^{\mathbb{Q}^Z}\left(g\left(\frac{H^2}{S_T}\right)\right)
\end{aligned}$$

Pricing the Image IV

Proof (con'd)

Girsanov's Theorem tells us that we have

$$dW_t^Z = dW_t - p\sigma dt.$$

Defining $Y_t = H^2/S_t$ and noting that

$$\begin{aligned} dY_t &= -\frac{H^2}{S_t^2} dS_t + \frac{H^2}{S_t^3}(dS_t)^2 \\ &= rY_t dt - \sigma Y_t dW_t^Z. \end{aligned}$$

Hence, under $\mathbb{Q}^Z$ the process Y_t has the same law as the process S_t under the measure $\mathbb{Q}$. Therefore,

$$\mathbb{E}_t^{\mathbb{Q}^Z}(g(Y,T)) = f(Y,t) = f\left(\frac{H^2}{S_t}, t\right).\square$$

Some Extra Calculations I

Proposition

For the maximum M_t of a Wiener process $\{W_t\}_{t_0}$ defined by

$$M_t := \max\{W_s \ : \ 0 \leq s \leq t\},$$

and for the first-passage time $\tau(a)$ defined as follows

$$\tau(a) := \min\{t \ : \ W_t = a\},$$

or $\tau(a) = \infty$ on the event that the process W_t never attains the value a, it holds that for any $a > 0$

$$\mathbb{P}(M_t \geq a) = \mathbb{P}(\tau_a < t) = 2\mathbb{P}(W_t > a) = 2 - 2\Phi(a/\sqrt{t}).$$

Some Extra Calculations II

Proof

- To prove the above proposition, we need the **reflection principle**.
- The reflection principle builds on the point that, given a stopping time $\tau(a) < t$, the value W_t is just as likely to be above the level a as to be below the level a. The proof of this claim requires the use of the Strong Markov property. First, we observe that $\tau(a) \wedge t$ is a stopping time.
- Thus, by the strong Markov property, the post-$\tau(a) \wedge t$ process is a standard Brownian motion independent of the path up to time $\tau(a) \wedge t$, and therefore independent of $\tau(a) \wedge t$. Therefore, for any $s < t$ the conditional distribution $W_t - W_s$ given $\tau(a) = s$ is Gaussian with mean 0 and variance $t - s > 0$.

Some Extra Calculations III

Proof (con'd)

- So by the symmetry of the Gaussian distribution and due to path continuity, we have

$$\begin{aligned}\mathbb{P}\left(W_t - W_s > 0 \middle| \tau(a) = s\right) &= \mathbb{P}\left(W_t - W_s < 0 \middle| \tau(a) = s\right) \\ &= \frac{1}{2}. \qquad (11)\end{aligned}$$

  Now note that

$$\{\tau(a) < t, W_t > a\} = \{W_t > a\},$$

  since by continuity of the Wiener process and by the assumption that $W_0 = 0$, the statement $W_t > a$ implies that $\tau_a \leq t$.

Some Extra Calculations IV

Proof (con'd)

- We can write therefore our quantity of interest, $\mathbb{P}(M_t \geq a) = \mathbb{P}(\tau_a < t)$, as follows:

$$\begin{aligned}\mathbb{P}(\tau_a < t) &= \mathbb{P}(\tau_a < t, W_t < a) + \mathbb{P}(\tau_a < t, W_t > a) \\ &= \mathbb{P}(W_t < a | \tau_a < t)\mathbb{P}(\tau_a < t) + \mathbb{P}(W_t > a).\end{aligned}$$

- Since $W_t \sim \mathcal{N}(0, t)$, the probability $\mathbb{P}(W_t > a)$ is easily calculated as

$$\mathbb{P}(W_t > a) = 1 - \phi(a/\sqrt{t}). \tag{12}$$

- Therefore, combining (12) with the result in equation (11) we can conclude the proof.

Zero-Rebate Options: Adjusted Payoffs I

- Consider a claim on an underlying S with time to maturity $\tau = T - t$ and final value $V(S_T)$.
- We attach to this claim a barrier B as follows:

$$V(S_T)\mathbf{1}_{\{m_T > B\}} \tag{13}$$

and

$$V(S_T)\mathbf{1}_{\{M_T < B\}} \tag{14}$$

with

$$m_T = \min_{n \leq T} S_n, \tag{15}$$

$$M_T = \max_{n \leq T} S_n, \tag{16}$$

i.e., m_T and M_T are the underlying's running minimum and the running maximum, respectively.

Zero-Rebate Options: Adjusted Payoffs II

- Hence, we can call the payoff in (13) a **down-and-out** contract and the payoff in (14) an **up-and-out** contract. We first focus on the down-and-out case.
- To this end, we define the functions

$$g(x) = v(x)\mathbf{1}_{\{x>B\}}$$

and $h = g - g^*$. Note that $h(x) = g(x)$ if $x > B$.

- From the previous analysis, we know that

$$\pi_t^h = e^{-r\tau}\left(f(S_t, t) - \left(\frac{S_t}{B}\right)^p f(B^2/S_t, t)\right).$$

- When the underlying hits the barrier, say at time s, $t \le s \le T$, then obviously $S_s = B$ and

$$\pi_s^h = 0.$$

Zero-Rebate Options: Adjusted Payoffs III

- Hence, in a complete market, we can buy the claim h at time t at a price π_t^h. If S hits the barrier B at a time s before maturity, then $\pi_s^h = 0$ and we can sell this claim.
- If the barrier is not hit until maturity T, we end up with the claim having value

$$\pi_T^h = v(S_T),$$

  i.e., we get the same payoff as the barrier option.
- Therefore, π_t^h must be the price of the barrier claim.

Zero-Rebate Options: Adjusted Payoffs IV

Example

Down and Out Put. This instrument could be constructed via defining the function $g(x)$ as

$$g(x) = (K - x)^+ \mathbf{1}_{\{x>B\}}.$$

Note that the function g is actually not the payoff of the option v, but a B-truncated version of it.

Zero-Rebate Options: Adjusted Payoffs V

Example

Down and Out Call. For this instrument, we define the function $g(x)$ as

$$g(x) = (x - K)^+ \mathbf{1}_{\{x>B\}} = (x - K)^+,$$

since $B \leq K$. Then for $r = 0$ or $r = \delta$, we have

$$g^*(x) = \frac{x}{B}\left(\frac{B^2}{x} - K\right)^+ = \frac{1}{B}\left(B^2 - xK\right)^+ = \frac{K}{B}\left(\frac{B^2}{K} - x\right)^+.$$

Therefore, the knock-out call can be hedged by buying a plain vanilla call and shorting K/B strike-B^2/K puts.

Zero-Rebate: GBM and its minimum. I

- Consider
$$v(x) = \mathbb{1}_{\{x \geq K\}}$$
and put
$$g(x) = v(x)\mathbb{1}_{\{x > B\}},$$
for some B, $0 \leq B \leq \min(K, x)$, i.e., $v(x) = g(x)$.
- Then, with $\tau = T - t$, we can write the price of a claim on S as:
$$f(S_t, t) = \mathbb{E}_t^{\mathbb{Q}}(g(S_T)) = \Phi\left(\frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).$$

Zero-Rebate: GBM and its minimum. II

- Let $h = g - g^*$. Then,

$$\begin{aligned}\pi^h(t) &= e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left(\mathbb{1}_{\{S_T \geq B\}}\mathbb{1}_{\{m_T > B\}}\right) \\ &= e^{-r\tau}\mathbb{Q}\left(S_T \geq K, m_T > B\right),\end{aligned}$$

with $m_T = \min_{t \leq u \leq T} S_u$. Therefore,

$$\mathbb{Q}\left(S_T \geq K, m_T > B\right) = e^{r\tau}\pi^h(t)$$

is the joint distribution of a geometric Brownian motion and its minimum.

Zero-Rebate: GBM and its minimum. III

- On the other hand, we know

$$
\begin{aligned}
e^{r\tau}\pi^h(t) &= \Phi\left(\frac{\ln\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&\quad -\frac{S_t}{B}^p\Phi\left(\frac{\ln\frac{B^2}{S_tK}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&= \mathbb{Q}\left(S_T \geq K, m_T > B\right).
\end{aligned}
$$

- The joint density of the geometric Brownian motion and its minimum is obtained by differentiating with respect to K and B.
- With this result, we can derive some laws of certain functionals of Brownian motions. Let

$$
X_t = \ln S_t/S_0 = \sigma W_t + vt,\ v = r - \frac{1}{2}\sigma^2,\ k = \ln K,\ b = \ln B,\ m_t = \min_u X_u
$$

Zero-Rebate: GBM and its minimum. IV

- Then,

$$\begin{aligned}\mathbb{Q}\left(X_t \geq k, m_t > b\right) &= \Phi\left(\frac{-k+vt}{\sigma\sqrt{t}}\right) \\ &\quad -e^{2vb\sigma^{-2}}\Phi\left(\frac{2b-k+vt}{\sigma\sqrt{t}}\right).\end{aligned}$$

and

$$\begin{aligned}\mathbb{Q}\left(m_t > b\right) &= \Phi\left(\frac{-b+vt}{\sigma\sqrt{t}}\right) \\ &\quad -e^{2vb\sigma^{-2}}\Phi\left(\frac{b+vt}{\sigma\sqrt{t}}\right).\end{aligned}$$

Zero-Rebate: GBM and its minimum. V

- If we want to focus on the law of the maximum value of X, we first observe that for, say, $y \leq 0$ and u, $0 \leq u \leq t$

$$\begin{aligned}
\mathbb{Q}\left(\min_u X_u \leq y\right) &= \mathbb{Q}\left(\min_u(\sigma W_u + vu) \leq y\right) \\
&= \mathbb{Q}\left(\max_u(-\sigma W_u - vu) \geq -y\right) \\
&= \mathbb{Q}\left(\max_u(\sigma W_u - vu) \geq -y\right) \\
&= \mathbb{Q}\left(\max_u \tilde{X}_u \geq -y\right),
\end{aligned}$$

with $\tilde{X}_u = \sigma W_u - vu$. Setting $\tilde{M}_u = \max_u \tilde{X}_u$, we get

$$\mathbb{Q}\left(m_r < y\right) = \mathbb{Q}\left(\tilde{M}_t \geq -y\right).$$

Zero-Rebate: GBM and its minimum. VI

- Therefore,

$$\begin{aligned}\mathbb{Q}\left(X_t \geq k, M_t < b\right) &= \Phi\left(\frac{b - vt}{\sigma\sqrt{t}}\right) \\ &\quad - e^{2vb\sigma^{-2}}\Phi\left(\frac{-2b + k - vt}{\sigma\sqrt{t}}\right).\end{aligned}$$

Zero-Rebate: Down-and-Out Barrier I

- We now want to explicitly price a down-and-out call with barrier H.
- The payoff of this barrier option can be written as

$$(S_T - K)\mathbb{1}_{\{S_T \geq K\}}\mathbb{1}_{\{m_T > H\}},$$

which we can decompose into

$$\underbrace{S_T \mathbb{1}_{\{S_T \geq K, m_T > H\}}}_{\equiv I} - \underbrace{K \mathbb{1}_{\{S_T \geq K, m_T > H\}}}_{\equiv II}.$$

- For the part I, we need to apply obviously the right change of measure to obtain

$$\begin{aligned} I &= S_t e^{r\tau}\mathbb{P}^*(S_T \geq K, m_T > H) \\ &= S_t e^{r\tau}\mathbb{P}^*(X_T \geq x, m_T > y), \end{aligned}$$

with $x = -\ln S_t/K$ and $y = \ln H/S_t$.

Zero-Rebate: Down-and-Out Barrier II

- Then, with $v = r + \frac{1}{2}\sigma^2$,

$$
\begin{aligned}
I &= S_t e^{r\tau} \left(\Phi\left(\frac{\ln S_t/K + v\tau}{\sigma\sqrt{\tau}} \right) \right. \\
&\quad \left. - \left(\frac{H}{S_t} \right)^{1+\frac{2r}{\sigma^2}} \Phi\left(\frac{\ln H^2/(S_t K) + v\tau}{\sigma\sqrt{\tau}} \right) \right).
\end{aligned}
$$

- Using the same argumentation for II, we get

$$
\begin{aligned}
II &= K \left(\Phi\left(\frac{\ln S_t/K + (v-\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \right. \\
&\quad \left. - \left(\frac{H}{S_t} \right)^{-1+\frac{2r}{\sigma^2}} \Phi\left(\frac{\ln H^2/(S_t K) + (v-\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \right).
\end{aligned}
$$

Zero-Rebate: Down-and-Out Barrier III

Proposition

In the Black-Scholes world, the price of a down-and-out barrier contract with payoff at time T

$$(S_T - K)\mathbb{1}_{\{S_T \geq K\}}\mathbb{1}_{\{m_T > H\}}$$

has the following price at time t

$$\pi_t = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) - \left(\frac{S_t}{H}\right)^{p-2} S_t\Phi(h_1) + \left(\frac{S_t}{H}\right)^{p} Ke^{-r\tau}\Phi(h_2),$$

with

$$\begin{aligned} d_j &= \frac{\ln S_t/K + (r + (-1)^{j-1}\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\ h_j &= \frac{\ln H^2/(S_tK) + (r + (-1)^{j-1}\sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

Barriers with Rebate: Precalculation I

- A precalculation first.
- Consider the $\mathbb{P}$ martingale X. For the hitting time $\tau = \{\inf u | X_u = B\}$ we have for a bounded stopping time $\tau \wedge T$

$$
\begin{aligned}
X_t &= \mathbb{E}^{\mathbb{P}}\left(X_{\tau \wedge T}\right) = \mathbb{E}^{\mathbb{P}}\left(X_\tau \mathbb{1}_{\{\tau < T\}}\right) + \mathbb{E}^{\mathbb{P}}\left(X_\tau \mathbb{1}_{\{\inf_{t \le u \le T} X_u > B\}}\right) \\
&= B\mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau < T\}}\right) + \mathbb{E}^{\mathbb{P}}\left(X_T \mathbb{1}_{\{\inf_{t \le u \le T} X_u > B\}}\right).
\end{aligned}
$$

- Barrier options with rebate R pay at hitting time a fixed amount. The hitting time τ, however, is a stochastic variable. Today's price of the rebate part of the barrier option can be written as

$$
\pi_t = R\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r\tau} \mathbb{1}_{\{\tau < T\}}\right)
$$

with $\tau = \{\inf u | S_u = B\}$.

Barriers with Rebate: Precalculation II

- From the previous result and noting that $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale, we get for a given barrier B:

$$\begin{aligned} S_t &= \mathbb{E}_t^{\mathbb{Q}}\left(S_{\tau\wedge T}e^{-r(\tau\wedge T)}\right) \\ &= B\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r\tau}\mathbb{1}_{\{\tau<T\}}\right) \\ &\quad +\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r(T-t)}S_T\mathbb{1}_{\{\inf_{t\leq u\leq T}S_u e^{-r(u-t)}>Be^{-r(u-t)}\}}\right), \end{aligned}$$

or, respectively:

$$\begin{aligned} &B\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r\tau}\mathbb{1}_{\{\tau<T\}}\right) = S_t \\ &-\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r(T-t)}S_T\mathbb{1}_{\{\inf_{t\leq u\leq T}S_u e^{-r(u-t)}>Be^{-r(u-t)}\}}\right). \end{aligned}$$

Barriers with Rebate: Precalculation III

- Hence, the price of the rebate can be rewritten as

$$\pi_t = \frac{R}{B}\left(S_t - \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r(T-t)} S_T \mathbb{1}_{\{\inf_{t \le u \le T} S_u e^{-r(u-t)} > Be^{-r(u-t)}\}} \right)}_{\equiv A} \right).$$

- The part A equals a strike-0 call with barrier Be^{-rt} on the process $S_t e^{-rt}$.
- Therefore, the barrier call option with barrier B and rebate R corresponds to a long position in the security $S_t e^{-rt}$ (in R/B-units) and R/B units short in a strike-0, rebate-0, barrier-Be^{rt} call on the security $S_t e^{-rt}$.

Barriers with Rebate: Pricing I

- To apply the result we have derived for the $\mathbb{P}$-martingale X, we still face the problem that the calculations so far have been made under the measure $\mathbb{Q}$.
- Therefore, we have to find a transformed process such that we get a $\mathbb{Q}$-martingale for the optimal stopping time decomposition. Let's label the transformed process by X and define

$$X_t = e^{-rt} S_t^a = S_0^a e^{-rt + a(r - \frac{1}{2}\sigma^2)t + a\sigma W_t}.$$

Barriers with Rebate: Pricing II

- Clearly, if a solves

$$a(r - \frac{1}{2}\sigma^2) - r = -a^2\frac{1}{2}\sigma^2,$$

X_t becomes a $\mathbb{Q}$-martingale. The above quadratic equation has two roots:

$$a_\pm = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + \frac{2r}{\sigma^2}}$$

with $p = 1 - \frac{2r}{\sigma^2}$.

- For the down-and-in case, let's look at

$$\begin{aligned} S_t^{a_+} = X_t &= \mathbb{E}_t^{\mathbb{Q}}\left(X_{\tau\wedge T}\right) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-r\tau}B^{a_+}\mathbb{1}_{\{\tau<T\}}\right) \\ &\quad +\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r(T-t)}S_T^{a_+}\mathbb{1}_{\{\inf_{u\le T} S_u^{a_+} > B^{a_+}\}}\right). \end{aligned}$$

Barriers with Rebate: Pricing III

- We get

$$R\mathbb{E}_t^{\mathbb{Q}}\left(e^{-r\tau}\mathbb{1}_{\{\tau<T\}}\right)=\frac{R}{B^{a_+}}\left(S_t^{a_+}-e^{-r(T-t)}\underbrace{\mathbb{E}_t^{\mathbb{Q}}\left(S_T^{a_+}\mathbb{1}_{\{\inf_{t\le u\le T}S_u^{a_+}>B^{a_+}\}}\right)}_{\equiv A}\right)$$

- Therefore, part A can be interpreted as a strike 0 down-and-out call with barrier B^{a_+} and rebate 0 on the transformed process S^{a_+}, i.e., we can explicitly write it with the down-and-out call formula by replacing $\sigma\to a_+\sigma$, $S_t\to S_t^{a_+}$ and $B\to B^{a_+}$.

A General Result on Static Replication I

- Assume that we have a payoff function $f(S_T) \in C^2$ and that we have complete, arbitrage-free markets. Then,

$$\begin{aligned} f(S_T) &= f(S_T)\left(\mathbb{1}_{\{S_T \le K\}} + \mathbb{1}_{\{S_T > K\}}\right) \\ &\quad + f(K) - f(K)\left(\mathbb{1}_{\{S_T \le K\}} + \mathbb{1}_{\{S_T > K\}}\right) \\ &= f(K) - \mathbb{1}_{\{S_T \le K\}}\left(f(K) - f(S_T)\right) + \mathbb{1}_{\{S_T > K\}}\left(f(K) - f(S_T)\right). \end{aligned}$$

- Now, we write $f(K) - f(S_T)$ as integral:

$$\begin{aligned} f(S_T) &= f(K) - \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} f'(u)du + \mathbb{1}_{\{S_T > K\}} \int_{K}^{S_T} f'(u)du \\ &= f(K) - \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} \left(f'(K) + f'(u) - f'(K)\right) du \\ &\quad + \mathbb{1}_{\{S_T > K\}} \int_{K}^{S_T} \left(f'(K) + f'(u) - f'(K)\right) du, \end{aligned}$$

A General Result on Static Replication II

and also write $f'(u) - f'(K)$ as integral:

$$\begin{aligned}
f(S_T) &= f(K) - \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} \left(f'(K) - \int_u^K f''(v)dv \right) du \\
&\quad + \mathbb{1}_{\{S_T > K\}} \int_K^{S_T} \left(f'(K) + \int_K^u f''(v)dv \right) du \\
&= f(K) - \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} f'(K)du + \mathbb{1}_{\{S_T > K\}} \int_K^{S_T} f'(K)du \\
&\quad + \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} \int_u^K f''(v)dvdu + \mathbb{1}_{\{S_T > K\}} \int_K^{S_T} \int_K^u f''(v)dvdu.
\end{aligned}$$

A General Result on Static Replication III

- Change the order of integration and integrate with respect to u to obtain:

$$
\begin{aligned}
f(S_T) &= f(K) + f'(K)(S_T - K) \\
&\quad + \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} \int_{S_T}^{v} f''(v)\,du\,dv \\
&\quad + \mathbb{1}_{\{S_T > K\}} \int_{K}^{S_T} \int_{v}^{S_T} f''(v)\,du\,dv \\
&= f(K) + f'(K)(S_T - K) \\
&\quad + \mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} f''(v)(v - S_T)\,dv \\
&\quad + \mathbb{1}_{\{S_T > K\}} \int_{K}^{S_T} f''(v)(S_T - v)\,dv.
\end{aligned}
$$

A General Result on Static Replication IV

- We observe

$$\mathbb{1}_{\{S_T \le K\}} \int_{S_T}^{K} f''(v)(v - S_T)dv = \begin{cases} \int_{S_T}^{K} f''(v)(v - S_T)dv & \text{if } S_T \le K \\ 0 & \text{if } S_T > K \end{cases}$$
$$= \int_0^K f''(v)(v - S_T)^+ dv.$$

- Using the same argument as above for $\mathbb{1}_{\{S_T > K\}} \int_K^{S_T} f''(v)(S_T - v)dv$, we finally get

$$f(S_T) = f(K) + f'(K)(S_T - K) + \int_0^K f''(v)(v - S_T)^+ dv + \int_0^K f''(v)(S_T - v)^+ dv.$$

- We conclude that we can replicate the payoff $f(S_T)$ with a set of simple instruments:

A General Result on Static Replication V

- zero coupon bonds ($f(K)$),
- forward contracts ($f'(K)$),
- put and call options (∞).

- In particular, at time t the replicating portfolio for a contract with payoff $f(S_T)$ is given by

$$\begin{aligned} V_t &= e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left(f(S_T)\right) = e^{-r\tau}f(K) + f'(K)e^{-r\tau}(\mathbb{E}_t^{\mathbb{Q}}\left(S_T\right) - K) \\ &\quad + e^{-r\tau}\int_0^K f''(v)\mathbb{E}_t^{\mathbb{Q}}\left((v - S_T)^+\right) dv \\ &\quad + e^{-r\tau}\int_0^K f''(v)\mathbb{E}_t^{\mathbb{Q}}\left((S_T - v)^+\right) dv \\ &= f(K)P(t,T) + f'(K)(S_t - P(t,T)) \\ &\quad + \int_0^K f''(v)P(S,v,t,T)dv + \int_0^K f''(v)C(S,v,t,T)dv \end{aligned}$$

Replicating with Vanilla (CC) I

- The Carr and Chou (1997) method allows static hedging of barrier options using plain vanilla options.
- The method is based on the Black-Scholes assumptions and on the reflection principle.
- Consider a position in two European securities with time T payoff function f:

$$g(S_T) = \mathbb{1}_{\{S_T \geq H\}} f(S_T); \quad g^\star(S_T) = \mathbb{1}_{\{S_T < H\}} \left(\frac{S_T^p}{H}\right) f\left(\frac{H^2}{S_T}\right),$$

  with $p = 1 - 2(r-d)/\sigma^2$.
- The portfolio $h(S) = g(S) - g^\star(S)$ corresponds to a European-type down-and-out barrier option with barrier H and payoff $f(S)$.
- Argumentation:

Replicating with Vanilla (CC) II

- Scenario "Barrier Not Breached": payoff will be $f(S_T)$.
- Scenario "Barrier Breached": payoff when $S_\tau = H$, $\tau \in [t, T]$ will be 0, i.e., portfolio value is zero and we can unwind the static hedge portfolio $h(S_\tau) = 0$.

- Using a similar arguments, we get:
  - Down-and-In Option:

$$h(S_T) = \begin{cases} 0 & S_T \in (H, \infty) \\ f(S_T) + \left(\frac{S_T^p}{H}\right) f\left(\frac{H^2}{S_T}\right) & S_T \in (0, H) \end{cases}$$

  - Up-and-Out Option:

$$h(S_T) = \begin{cases} -\left(\frac{S_T^p}{H}\right) f\left(\frac{H^2}{S_T}\right) & S_T \in (H, \infty) \\ f(S_T) & S_T \in (0, H) \end{cases}$$

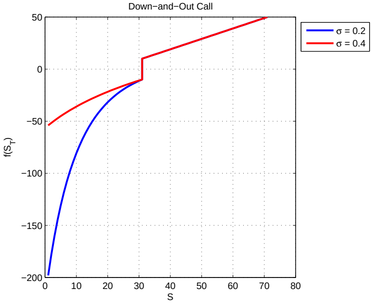
Replicating with Vanilla (CC) III

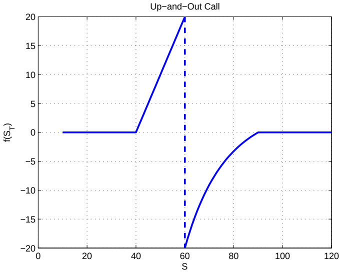
- Up-and-In Option:

$$h(S_T) = \begin{cases} f(S_T) + \left(\frac{S_T^p}{H}\right) f\left(\frac{H^2}{S_T}\right) & S_T \in (H, \infty) \\ 0 & S_T \in (0, H) \end{cases}$$

Replicating with Vanilla (CC) IV

Adjusted Payoffs for Down-and-Out and Up-and-Out Calls





- From Carr & Picron (1999), we know that we can replicate every time-T payoff $f(S_T)$, which is twice differentiable by using:

Replicating with Vanilla (CC) V

- A zero coupon bond with time-to-maturity T.
- A position on a forward on S.
- A continuum of put and call prices of different strikes.

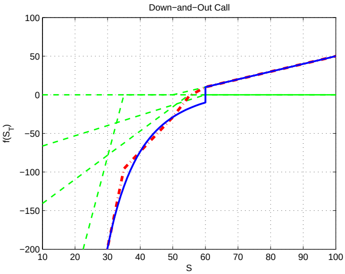
- In particular, we have a time t value for a claim with payoff $f(S_T)$ that can be represented as

$$
\begin{aligned}
V_t &= f(K)B(t,T) + \frac{\partial f(K)}{\partial K}(S_T - KB(t,T)) \\
&\quad + \int_0^K \frac{\partial^2 f(K)}{\partial K^2} P(t,T,v)dv + \int_K^\infty \frac{\partial^2 f(K)}{\partial K^2} C(t,T,v)dv,
\end{aligned}
$$

where $P(t,T,v)$ and $C(t,T,v)$ are the prices of European put and call options with maturity $T-t$ and strike v.

Replicating with Vanilla (CC) VI

Static Replication with 4 Options



Position	K	Type	Price
1.00	50.0	call	3.4444
-15.98	35.0	put	0.0062
-3.12	55.0	put	5.0953
-1.32	60.0	put	9.0299

- Our goal is to replicate the payoff at certain points $S_1, S_2, \cdots$.
- Replication of adjusted payoff above the barrier is simple.

Replicating with Vanilla (CC) VII

- To ensure that the second vanilla option does not interfere with the already matched payoff above the barrier, the second position will be a put option with strike below or at the barrier H.
- The strike of the additional put P_i will have to be greater than S_i. If this were not the case, then the option would have zero value should the final share price be S_i and, hence, no position in this option can be taken to target a certain nonzero payoff at S_i.

We can formalize the arguments:

- Step 0: We take care of the part above the barrier by going long a call option with strike K.
- Step 1:

$$\begin{aligned} f(S_1) &= \alpha_1(K_1 - S_1)^+ + (S_1 - K)^+ \\ &\Rightarrow \alpha_1(K_1 - S_1)^+ = f(S_1) - (S_1 - K)^+ \end{aligned}$$

Replicating with Vanilla (CC) VIII

- Step 2:

$$
\begin{aligned}
f(S_2) &= \alpha_2(K_2 - S_2)^+ \alpha_1(K_1 - S_2)^+ + (S_2 - K)^+ \\
&\Rightarrow \alpha_1(K_1 - S_2)^+ + \alpha_2(K_2 - S_2)^+ = f(S_2) - (S_2 - K)^+
\end{aligned}
$$

- Step n:

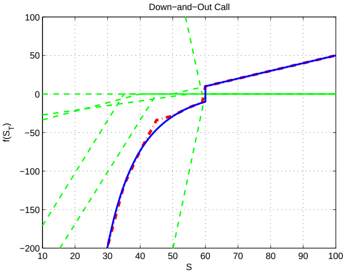
$$
\begin{pmatrix}
(K_1 - S_1)^+ & 0 & 0 & \cdots & 0 \\
(K_1 - S_2)^+ & (K_2 - S_2)^+ & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & & \vdots \\
(K_1 - S_n)^+ & (K_2 - S_n)^+ & (K_3 - S_n)^+ & \cdots & (K_n - S_n)^+
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n
\end{pmatrix}
= \begin{pmatrix}
f(S_1) - (S_1 - K)^+ \\
f(S_2) - (S_2 - K)^+ \\
\vdots \\
f(S_n) - (S_n - K)^+
\end{pmatrix}
$$

Replicating with Vanilla (CC) IX

- After this exercise, we see that the down-ant-out option can be replicated using a portfolio of
  - a single long position in a plain vanilla call with maturity T.
  - n positions of size α_i in puts with strike K_i and maturity T.

Replicating with Vanilla (CC) X

Static Replication with 7 Options

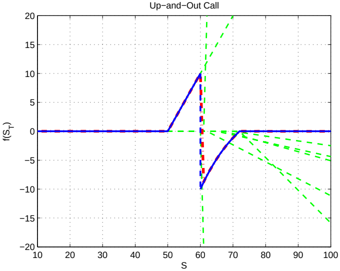


Position	K	Type	Price
1.00	50.0	call	3.4444
-6.83	35.0	put	0.0062
-1.12	40.0	put	0.0997
-6.75	45.0	put	0.6382
-0.60	55.0	put	5.0953
19.58	59.0	put	8.1820
-20.14	60.0	put	9.0299

Replicating with Vanilla (CC) XI

Static Replication with 8 Options

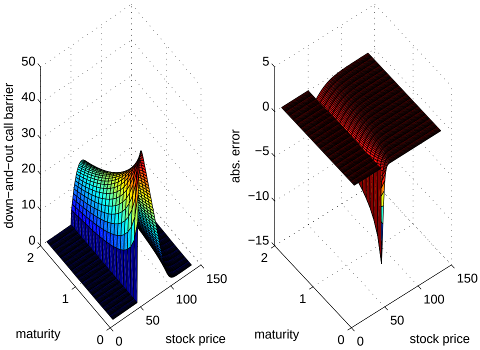
Analogously, we get for an up-and-out call the following scheme:



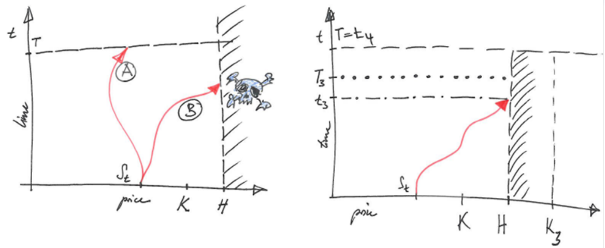
Position	K	Type	Price
1.00	50.0	call	8.0634
-0.56	72.0	call	1.5086
-0.08	70.0	call	1.7840
-0.15	67.5	call	2.1926
-0.12	65.0	call	2.6837
-0.29	62.0	call	3.3997
20.02	61.0	call	3.6726
-19.79	60.0	call	3.9641

Replicating with Vanilla (CC) XII

Static Replication Down and Out Put



Replicating with Vanilla (DEK) I



- Consider an option of the up-and-out type.
- Two mutually exclusive scenarios:
  - Scenario A does not breach the barrier $H = 125$.
  - Scenario B does breach the barrier $H = 125$ before $T = 1$.
- The goal is to develop hedging strategies that hedge both scenarios.

Replicating with Vanilla (DEK) II

- The idea behind the DEK method is that we can represent every claim in terms of a PDE under the appropriate boundary conditions.
- Then, we can discretize the boundary conditions at a finite time grid.
- At each point of this grid, we require that the PDE and the boundary conditions are fulfilled.
- This procedure then gives us the replicating portfolio.
- The advantage of the DEK method, is that it can be used also outside the Black-Scholes framework and with instruments of different maturities!

Replicating with Vanilla (DEK) III

- Justification for the procedure is the so-called superposition principle.

Proposition

Let $\mathcal{A}$ and $\mathcal{L}$ be two linear differential operators. If $u_1, \ldots, u_k$ satisfy the linear partial differential equations $\mathcal{L}(u_i) = 0$ and the boundary conditions $\mathcal{A}(u_i) = 0$ for $i = 1, \ldots, k$ and if $c_1, \ldots, c_k$ are any constants, then

$$u = c_1 u_1 + \ldots + c_k u_k,$$

satisfies

$$\mathcal{L}(u) = 0, \quad \mathcal{A}(u) = 0.$$

- The notion of being able to add known solutions to obtain a new solution to a related linear PDE is known as superposition principle. How can we use this principle for our purpose?

Replicating with Vanilla (DEK) IV

- We know that a plain vanilla call option satisfies:

$$
\begin{aligned}
\mathcal{L}C(S_t, t) &= 0, \quad \text{for } t \le T, S_t \ge 0 \\
C(S_T, T) &= (S_T - K)^+, \quad \text{for } S_T > 0 \\
C(0, t) &= 0, \quad \text{for } t > 0 \\
\lim_{S_t \to \infty} C(S_t, t) &= S_t \quad \text{for } t \le T.
\end{aligned}
$$

- Then, let us construct a portfolio

$$\pi(S_t, t) = \sum_{i=1}^{n} C_i(S_t, t),$$

Replicating with Vanilla (DEK) V

such that

$$\begin{aligned}
\pi(S_T, T) &= (S_T - K)^+, \quad \text{for } S_T < H \\
\pi(0, t) &= 0, \quad \text{for } t \leq T \\
\pi(H, t_j) &= 0 \quad \text{for } t_j < T, j = 1, \ldots, n-1
\end{aligned}$$

- Now, since $\mathcal{C}_i = 0$ for $i = 1, \ldots, n$, we get from the superposition principle

$$\mathcal{L}\pi = 0,$$

and by construction (this will be the next step), π satisfies the boundary conditions above.

Replicating with Vanilla (DEK) VI

- Note that from the boundary conditions, the first two equations are exactly those from an up-and-out call. However, the third condition agrees with the boundary condition of the up-and-out call only at a finite number of points t_j, $j = 1, \ldots, n-1$. Increasing n gives a better approximation to the contract to be replicated.
- To exemplify the construction, we will continue with our up-and-out call with barrier H and strike K.
- We consider two possible outcomes, say A and B. A does not hit the barrier, B does. The goal of the DEK method is to replicate the payoff of the barrier option under the two scenarios using plain vanilla calls and puts.

Replicating with Vanilla (DEK) VII

- Scenario A: The barrier is not hit. Then we can just use a non-barrier version of the barrier option, i.e., in our case a call option with strike K and time-to-maturity T. The replication is achieved for all paths of type A.
- Hence, our first instrument we use is $C(S, t; K, T)$.
- Let's try to match the barrier option, when the stock price hits the barrier at time t_3.
- At time t_3, our current portfolio has value

$$C(S_{t_3} = H, t_3, K, T)$$

- If we take a position, say α_3, in an option with maturity T_3 , we could ensure that the value of the portfolio at the point (H, t_3) is exactly zero.

Replicating with Vanilla (DEK) VIII

Two important remarks:

1. The new option must have a strike $K_3 \geq H$. If this were not the case, a positive payoff may occur at time T, if we have scenario A materializing, hence interfering with the matched payoff under A.
2. The maturity of the option must be strictly greater than t_3, i.e., $T_3 > t_3$. If we would have $T_3 = t_3$, then at point (H, t_3), the option will always have zero value – given $K_3 \geq H$ from the first remark. No matching would be possible.

Replicating with Vanilla (DEK) IX

- In the first step, we would therefore need to solve

$$\alpha_3 C(H, t_3; K_3, T_3) + C(H, t_3, K, T) = 0,$$

which gives

$$\alpha_3 = -\frac{C(H, t_3; K, T)}{C(H, t_3; K_3, T_3}.$$

- Next step: target the value of the up-and-out call at time t_2. Value of current portfolio at point (H, t_2) is

$$\alpha_3 C(H, t_2; K_3, T_3) + C(H, t_2, K, T)$$

We need to take a position in the call option with $K_2 \geq H$, $T_2 < t_2$, and solve

$$\alpha_2 C(H, t_2; K_2, T_2) + \alpha_3 C(H, t_3; K_3, T_3) + C(H, t_3, K, T) = 0.$$

Replicating with Vanilla (DEK) X

- We need to continue the same way. Doing so, we end up with the following equation to be solved:

$$\begin{pmatrix} C(H, t_{n-1}; K_{n-1}, T_{n-1}) & 0 & \cdots & 0 \\ C(H, t_{n-2}; K_{n-1}, T_{n-1}) & C(H, t_{n-2}; K_{n-2}, T_{n-2}) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ C(H, t_0; K_{n-1}, T_{n-1}) & \cdots & \cdots & C(H, t_0; K_0, T_0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$
$$= - \begin{pmatrix} C(H, t_{n-1}; K, T) \\ C(H, t_{n-2}; K, T) \\ \vdots \\ C(H, t_0; K, T) \end{pmatrix}$$

- Note: The difference to the previous method is that here we can use options with different maturities.
- For up-and-out puts, we can use exactly the same arguments as above except that for Scenario A we replace the call with a put option for replication.

Replicating with Vanilla (DEK) XI

Down-and-Out puts/calls

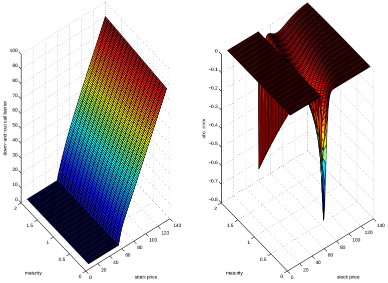
- Payoff under Scenario A: just take a non-barrier version of the payoff.
- Achieve the payoff using puts instead of calls.
- $K_i \leq H$ (down barrier)
- Blow up occurs for α_i, when strike prices are too small.
- $T_i > t_i$.

For in-versions

- Use barrier symmetry:
    - Scenario A: no claim
    - Scenario B: instead of aiming at value zero ,we aim at a value of the non-barrier version of the payoff.

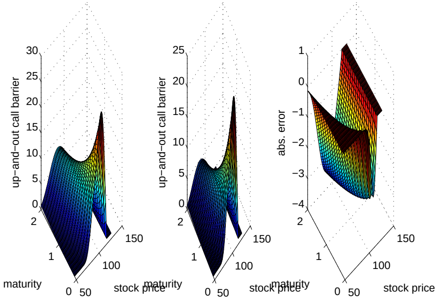
Replicating with Vanilla (DEK) XII

Static Replication Down and Out Call



Replicating with Vanilla (DEK) XIII

Static Replication Up and Out Call



- Basically, there are three possibilities to hedge a position:

Replicating with Vanilla (DEK) XIV

- Dynamic Hedging in the underlying and in options.
- Static Hedging
- Quasi Static Hedging

- Quasi Static Hedging tries to get the best out of dynamic and static hedging.
    - There is some evidence that, e.g., hedging strategies for vanilla options at less liquid strikes with options of liquid strikes results in better stability than pure dynamic hedging strategies.
    - For static hedging, there is no worry about the volatility shape.
- For Quasi Static Hedging we need some optimization criteria.

Comparison of Methods

	Barrier Option Hedging			
Risk exposure	Dynamic hedge	DEK static	Carr/Chou static	Optimization static
Price jump	None	None	None	None
Parallel vol shift	None	High	None	Low
Vol smile shift	None	High	None	Low
Vol skew shift	None	High	High	Low
Dynamic hedging costs	Uncertain	None	None	None
Transaction costs	High+Uncertain	Low and P	Low and P	Low and P
Flexibility for extensions	Easy	Easy	Hard	Easy