Q2) conditional derivation of distribution

Given priors: $p(\theta) \sim N(\theta_0, \tau_0^2)$ and $p(\tau) \propto \tau^{\frac{\alpha}{2}-1} e^{-\frac{\lambda}{2}\tau}$

Likelihood: $x_i \sim N(\theta, \tau^{-1})$

$$p(x|\theta,\tau)p(\theta)p(\tau)$$

Derivation for $p(\tau|\theta,x)$

Using bayes' rule $p(\tau|\theta,x) \propto p(x|\theta,\tau)p(\tau)$

$$p(x|\theta,\tau) = \prod_{i=1}^{n} N(x_i|\theta,\tau^{-1})$$

Since the Gaussian likelihood is:

$$\prod_{i=1}^{n}\left(\frac{\tau^{1/2}}{(2\pi)^{1/2}}\right)\exp\left(-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$$

then the prior for τ is:

$$p(\tau) \propto \tau^{\alpha/2} e^{-\lambda\tau/2}$$

when multiplied:

$$= \tau^{(n+\alpha)/2} \exp\left(-\frac{\tau}{2}\left(\sum(x_i-\theta)^2+\lambda\right)\right)$$

$\therefore$ It is the gamma distribution

$$\tau|\theta,x \sim \text{Gamma}\left(\frac{n+\alpha}{2}, \frac{\sum(x_i-\theta)^2+\lambda}{2}\right)$$

Derivation for $p(\theta|\tau, x)$

$$p(\theta|\tau,x) \propto p(x|\theta,\tau)\,p(\theta)$$

Given likelihood $p(x|\theta,\tau)$:

$$p(x|\theta,\tau) \propto \exp\left(-\frac{\tau}{2}\sum(x_i-\theta^2)\right)$$

Given prior $p(\theta) \sim N(\theta_0, \tau_0^{-1})$

$$p(\theta) \propto \exp\left(-\frac{\tau_0}{2}(\theta-\theta_0)^2\right)$$

Multiplying the likelihood and prior we get:

$$= \exp\left(-\frac{1}{2}[(n\tau+\tau_0)\theta^2 - 2(n\tau\bar{x}+\tau_0\theta_0)\theta\right)$$

which turns out to be a normal distribution

$$\theta|\tau,x \sim N\left(\frac{n\bar{x}+\tau_0\theta_0}{n+\tau_0}, \frac{1}{(n+\tau_0)\tau}\right)$$

α acts as a pseudo-count of prior observations where a larger α makes the prior contribute more weight to τ, making posterior less sensitive to data.

λ is like a prior sum of squared divisions such that a larger λ means the prior assumes more uncertainty in τ while a smaller λ makes the prior favor higher and larger values of τ $\rightarrow$ lower variance in posterior.