Q3)

The transition to move from state $x^{(t-1)}$ to state $\tilde{x}^{(t)}$ is:

$$P(x^{(t-1)} \to \tilde{x}^{(t)}) = q(\tilde{x}^{(t)} | x^{(t-1)}) A(x^{(t-1)} \to \tilde{x}^{(t)})$$

with the acceptance probability being

$$A(x^{(t-1)} \to \tilde{x}^{(t)}) = \min\left(1, \frac{p(\tilde{x}^{(t)}) q(x^{(t-1)} | \tilde{x}^{(t)})}{p(x^{(t-1)}) q(\tilde{x}^{(t)} | x^{(t-1)})}\right)$$

The proposal distribution is $q(\tilde{x}^{(t)} | x^{(t-1)}) = \left(\frac{1}{2}\right)^{1-\tilde{x}(t)} \left(\frac{1}{2}\right)^{\tilde{x}^{(t)}}$

with stationary $p(x^{(t)}) = \left(\frac{1}{3}\right)^{1-x^{(t)}} \left(\frac{2}{3}\right)^{x^{(t)}}$.

Since the proposal probabilities in the acceptance probability are symmetric $q(\tilde{x}^{(t)} | x^{(t-1)}) = q(x^{(t-1)} | \tilde{x}^{(t)})$, they cancel out simplifying acceptance to

$$A(x^{(t-1)} \to \tilde{x}^{(t)}) = \min\left(1, \frac{p(\tilde{x}^{(t)})}{p(x^{(t-1)})}\right).$$

To show the transition kernel K we have

$x^{(t-1)} = 0$ and $\bar{x}^{(t)} = 1$

$$A(0 \to 1) = \min\left(1, \frac{p(1)}{p(0)}\right)$$

$$= \min\left(1, \frac{\frac{2}{3}}{\frac{1}{3}}\right)$$

Therefore $P(0 \to 1) = q(1|0)A(0 \to 1) = \left(\frac{1}{2} \times 1\right) = 0.5$

$$P(0 \to 0) = 1 - P(0 \to 1) = 1 - 0.5 = 0.5$$

For transition of $x^{(t-1)} = 1$ to $\bar{x}^{(t)} = 0$

$$A(1 \to 0) = \min\left(1, \frac{p(0)}{p(1)}\right)$$

$$= \min\left(1, \frac{\frac{1}{3}}{\frac{2}{3}}\right) = \min(1, 0.5) = 0.5$$

$$P(1 \to 0) = q(0|1)A(1 \to 0) = \left(\frac{1}{2} \times 0.5\right) = 0.25$$

$$P(1 \to 1) = 1 - P(1 \to 0) = 1 - 0.25 = 0.75$$

$$K = \begin{bmatrix} P(0 \to 0) & P(1 \to 0) \\ P(0 \to 1) & P(1 \to 1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$$