Homework 2 Solutions IEORE4004 (Optimization M&M)
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Problem 1

Starting from the basis $(4, 5, 6)$, we have the tableau

$$
\begin{array}{cccccccc}
z & -2x_1 & x_2 & -x_3 & & & & = & 0 \\
\hline
 & 3x_1 & x_2 & x_3 & x_4 & & & = & 60 \\
 & x_1 & -x_2 & 2x_3 & & x_5 & & = & 10 \\
 & x_1 & x_2 & -x_3 & & & x_6 & = & 20
\end{array}
$$

and the bfs is $(0, 0, 0, 60, 10, 20)^T$. x_1 enters the basis and x_5 leaves the basis. The new basis is $(4, 1, 6)$ and the new tableau is:

$$
\begin{array}{ccccccccc}
z & & -x_2 & 3x_3 & & 2x_5 & & = & 20 \\
\hline
 & & 4x_2 & -5x_3 & x_4 & -3x_5 & & = & 30 \\
 & x_1 & -x_2 & 2x_3 & & x_5 & & = & 10 \\
 & & 2x_2 & -3x_3 & & -x_5 & x_6 & = & 10
\end{array}
$$

x_2 enters the basis and x_6 exits the basis. The new basis is $(4, 1, 2)$ and the new tableau is:

$$
\begin{array}{ccccccccc}
z & & & 1.5x_3 & & 1.5x_5 & 0.5x_6 & = & 25 \\
\hline
 & & & x_3 & x_4 & -x_5 & -2x_6 & = & 10 \\
 & x_1 & & 0.5x_3 & & 0.5x_5 & 0.5x_6 & = & 15 \\
 & & x_2 & -1.5x_3 & & -0.5x_5 & 0.5x_6 & = & 5
\end{array}
$$

All the reduced cost are nonpositive, hence the algorithm terminates. The optimal solution is $(15, 5, 0, 10, 0, 0)$ and the optimal objective value is 25.

Problem 2

As seen in class, we can solve the following auxiliary problem to derive a feasible solution of the original LP:

$$
\begin{array}{ll}
\text{minimize} & y_1 + y_2 \\
\text{subject to} & 0.5x_1 + 0.25x_2 + x_3 = 4 \\
 & x_1 + 3x_2 - x_4 + y_1 = 20 \\
 & x_1 + x_2 + y_2 = 4, \\
 & x_i \geq 0, i = 1, 2, 3, 4, y_i \geq 0, i = 1, 2
\end{array}
$$

Recall that y_1, y_2 are called as *artificial variables.* The original problem is feasible if and only if this auxiliary problem has an optimal solution whose objective is 0. In order to apply the theory we developed in class, let's transform it into a maximization problem

$$
\begin{array}{ll}
-\text{maximize} & -y_1 - y_2 \\
\text{subject to} & 0.5x_1 + 0.25x_2 + x_3 = 4 \\
 & x_1 + 3x_2 - x_4 + y_1 = 20 \\
 & x_1 + x_2 + y_2 = 4, \\
 & x_i \geq 0, i = 1, 2, 3, 4, y_i \geq 0, i = 1, 2
\end{array}
$$

We can now apply the classical simplex method as seen in class. Starting from the basis (x_3, y_1, y_2), the corresponding tableau is

$$
\begin{array}{ccccccccc}
z & -2x_1 & -4x_2 & & +x_4 & & & = & -24 \\
\hline
 & 0.5x_1 & 0.25x_2 & x_3 & & & & = & 4 \\
 & x_1 & 3x_2 & & -x_4 & y_1 & & = & 20 \\
 & x_1 & x_2 & & & & y_2 & = & 4
\end{array}
$$

and the bfs is $(0, 0, 4, 0, 20, 4)^T$. x_2 enters the basis and y_2 leaves the basis. The new basis is (x_3, y_1, x_2).

$$
\begin{array}{ccccccccc}
z & 2x_1 & & & +x_4 & & +4y_2 & = & -8 \\
\hline
 & 0.25x_1 & & x_3 & & & -0.25y_2 & = & 3 \\
 & -2x_1 & & & -x_4 & y_1 & -3y_2 & = & 8 \\
 & x_1 & x_2 & & & & y_2 & = & 4
\end{array}
$$

All the reduced cost are nonpositive. So the algorithm terminates with an optimal solution $(0, 4, 3, 0, 8, 0)^T$ for the artificial problem whose objective value is 8 (recall that, when going back from max to min, we need to change the sign of the objective function). Thus, the original problem is infeasible.

Problem 3

1. No, since a basic variables has reduced cost 0, and the reduced cost of x_1 is 1.

2. No, since they are all set to 0 in the first row of the tableau.

3. Yes. Consider the tableau

$$
\begin{array}{ccccccccc}
z & -x_1 & & & & & -x_6 & = & 7 \\
\hline
 & & & & & +x_5 & & = & 1 \\
 & & & x_3 & +x_4 & & +2x_6 & = & 5 \\
 & x_1 & +x_2 & & & & & = & 1
\end{array}
$$

wrt the basis $(2, 3, 5)$. Since the reduced cost of x_1 is positive, x_1 can enter the basis, and x_2 will exit the basis. The tableau corresponding to the new basis $(1, 3, 5)$ is:

$$
\begin{array}{ccccccccc}
z & & +x_2 & & & & -x_6 & = & 8 \\
\hline
 & & & & & +x_5 & & = & 1 \\
 & & & x_3 & +x_4 & & +2x_6 & = & 5 \\
 & x_1 & +x_2 & & & & & = & 1
\end{array}
$$

4-5. If you carefully read the text, then we ask (*) the bfs to have *exactly* 3 *non-zero entries*. In this case both answers are No. If you omit (*), then both answers are Yes[1]. In the following, in (a) we motivate the No answers when (*) is assumed; in (b) the Yes answers when (*) is not assumed.

(a) Notice that x_6 has positive reduced costs, hence we can make it enter the basis. Recall that the value of $\bar{x}_6$ after the pivot is set to $\min_i\{\frac{\bar{b}_i}{\bar{a}_{i,6}} : \bar{a}_{i,6} > 0\}$, if the latter is well-defined, otherwise the problem is unbounded. Since $\bar{b}_2 = 5$ and $\bar{a}_{2,6} = 2$, we have that $\min_i\{\frac{\bar{b}_i}{\bar{a}_{i,6}} : \bar{a}_{i,6} > 0\}$ is well-defined. Moreover, since $\bar{b} > 0$, it is strictly greater than 0.

[1] We give full points to both answers, if correctly motivated.

This means the objective function value after the pivot operation will be $7+\bar{x}_6 > 7$, hence our starting bfs was not optimal.

(b) Consider the following tableau.

$$
\begin{array}{ccccccccc}
z & -x_1 & & & & & -x_6 & = & 7 \\
\hline
 & & & & & +x_5 & +x_6 & = & 0 \\
 & & +x_2 & +x_3 & & & +2x_6 & = & 5 \\
 & x_1 & & x_3 & +x_4 & & & = & 0
\end{array}
$$

Note that the current bfs is $(0,5,0,0,0,0)^T$ with basic variables x_2, x_4, x_5. Summing the objective function to twice the first and twice the last row of the tableau we obtain $z + x_1 + 2x_3 + 2x_4 + 2x_5 + x_6 = 7$. Since $x \geq 0$ and in any feasible solution, we deduce $z \leq 7$. Moreover, since in any other bfs, at least one of x_1, x_3, x_4, x_5, x_6 is strictly more than 0, the current solution is the unique optimal solution.

6. Yes. Consider the tableau

$$
\begin{array}{ccccccccc}
z & -x_1 & & & & & -x_6 & = & 7 \\
\hline
 & & & & & +x_5 & & = & -1 \\
 & & & x_3 & +x_4 & & +2x_6 & = & 5 \\
 & x_1 & +x_2 & & & & & = & 1
\end{array}
$$

wrt the basis $(2,3,5)$. This corresponds to the bs $(0,1,5,0,-1)$, that is not feasible since $\bar{x}_5 < 0$.

Problem 4

Solve

$$\max c^T x - d^T \lambda : \ Ax - \lambda \leq b, \ 0 \leq \lambda \leq k,$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$, $d = (d_1, \ldots, d_m)^T$, $k = (k_1, \ldots, k_m)^T$.

Problem 5

First solve (1) $\max c^T x : x \in P$, and suppose the value of the optimal solution is M. Now add $c^T x = M$ as a constraint to P, as to obtain P'. Note that all and only feasible solutions to P' are the optimal solutions to (1). Last, solve $\max d^T x : x \in P'$ to obtain the required solution.

Problem 6

False. Indeed, problems of the form $\max c^T x : Ax = b$ are very special: if the LP is feasible, either all feasible solutions are optimal, or they are unbounded LPs. Indeed, suppose there exist feasible solutions $\bar{x}, \tilde{x}$ to the problem with $c^T \bar{x} > c^T \tilde{x}$, say $c^T \bar{x} - c^T \tilde{x} = k > 0$. Let $r = \bar{x} - \tilde{x}$ and note that $Ar = A(\bar{x} - \tilde{x}) = b - b = 0$, while $c^T r = c^T \bar{x} - c^T \tilde{x} > 0$. Hence, $x(\alpha) = \bar{x} + \alpha r$ is feasible for each α and has cost $c^T \bar{x} + \alpha k$. As α goes to $+\infty$, so does the cost of $x(\alpha)$.