Homework 3 Solutions IEORE4004 (Optimization M&M)
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Problem 1

The dual of the problem can be formulated as

$$\begin{array}{llllll}
\min & 40y_1 & +15y_2 & +30y_3 & & \\
\text{s.t.} & & & & & \\
& 3y_1 & +y_2 & +y_3 & \geq & 2 \\
& y_1 & -y_2 & +y_3 & \leq & -1 \\
& y_1 & +2y_2 & -y_3 & = & 1 \\
& y_1 & & & \leq & 0 \\
& & & y_3 & \geq & 0
\end{array}$$

To check for the optimality of the following primal solutions, we first check if it is feasible, then we find a possible dual solution that satisfies the complementary slackness conditions, and if that dual solution is feasible, the the primal and dual solutions are both optimal.

1. $(25, 0, -5)^T$
   $(25, 0, -5)^T$ is a feasible primal solution. As $x_1 > 0$ and $3x_1 + x_2 + x_3 > 40$, to satisfy complementary slackness, the dual solution must satisfy $3y_1 + y_2 + y_3 = 2$ and $y_1 = 0$ along with $y_1 + 2y_2 - y_3 = 1$. This system of equations has a unique solution $(0, 1, 1)^T$. This dual solution does not satisfy $y_1 - y_2 + y_3 \leq -1$, therefore the primal solution is not optimal.

2. $(40, -45, -35)^T$
   $(40, -45, -35)^T$ is a feasible primal solution. As $x_1 > 0$ and $x_2 < 0$, to satisfy complementary slackness, the dual solution must satisfy $3y_1 + y_2 + y_3 = 2$ and $y_1 - y_2 + y_3 = -1$ along with $y_1 + 2y_2 - y_3 = 1$. This system of equations has a unique solution $(-1.5, 3, 3.5)^T$. $(-1.5, 3, 3.5)^T$ is a feasible dual solution. Therefore, $(40, -45, -35)^T$ is an optimal primal solution and $(-1.5, 3, 3.5)^T$ is an optimal dual solution.

3. $(13, 0, 1)^T$
   $(13, 0, 1)^T$ is a feasible primal solution. As $x_1 > 0$ and $x_1 + x_2 - x_3 < 30$, to satisfy complementary slackness, the dual solution must satisfy $3y_1 + y_2 + y_3 = 2$ and $y_3 = 0$ along with $y_1 + 2y_2 - y_3 = 1$. This system of equations has a unique solution $(0.6, 0.2, 0)^T$. This dual solution does not satisfy $y_1 \leq 0$, therefore the primal solution is not optimal.

$(-1.5, 3, 3.5)^T$ is an optimal dual solution as explained in 2. above.

Problem 2

a. The dual of the LP can be formulated as

$$\begin{array}{lllll}
\min & 8y_1 & +10y_2 & & \\
\text{s.t.} & & & & \\
& 2y_1 & +3y_2 & \geq & 3 \\
& 5y_1 & +7y_2 & \geq & 2 \\
& y_1 & & \geq & 0 \\
& & y_2 & \geq & 0
\end{array}$$

The optimal tableau has basis (s_1, x_1), therefore the optimal solution to the primal is $(x_1, x_2, s_1, s_2) = \left(\frac{10}{3}, 0, \frac{4}{3}, 0\right)$. We see that $x_1 > 0$ and $2x_1 + 5x_2 < 8$ in the optimal primal solution, thus from complementary slackness conditions, we get that $2y_1 + 3y_2 = 3$ and $y_1 = 0$ are satisfied in the optimal dual solution. Thus the optimal dual solution is $(y_1, y_2) = (0, 1)$.

b. The current basis remains remains feasible after changing b_2, then it must also be optimal, because the reduced costs $c_N - c_B^T B^{-1} N$ do not depend on b_2. With (s_1, x_1) as basis we get,

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}; \qquad B^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/3 \end{pmatrix}$$

Therefore, the value of the basic variables in the bfs with B as basis is

$$B^{-1}b = \begin{pmatrix} 8 - 2/3b_2 \\ b_2/3 \end{pmatrix}$$

The basis remains feasible as long as $B^{-1}b \geq 0$, which gives us $0 \leq b_2 \leq 12$ as the range of values of b_2 for which the current basis remains optimal. For $b_2 = 5$ we get $B^{-1}b = \begin{pmatrix} 8 - 10/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} 14/3 \\ 5/3 \end{pmatrix}$. Therefore, the new optimal solution is $(x_1, x_2, s_1, s_2) = (5/3, 0, 14/3, 0)$.

c. It can easily be observed that the primal LP is feasible ((0,0) is a feasible solution) and for all feasible primal solutions, $0 \leq x_1 \leq 4$ and $0 \leq x_2 \leq 2$ (from constraint equations). Therefore, the feasible set for the primal problem is bounded. Therefore, the objective value can also not be unbounded. Therefore, the primal is feasible and bounded irrespective of the value of the objective function. Therefore, the dual will also always be feasible and bounded.

Problem 3

a. The dual of the LP can be formulated as

$$\begin{array}{llll} \min & 2y_1 & +8y_2 & \\ \text{s.t.} & & & \\ & 8y_1 & +6y_2 & \geq 4 \\ & 3y_1 & +y_2 & \geq 1 \\ & y_1 & +y_2 & \geq 2 \\ & y_1 & & \geq 0 \\ & & y_2 & \geq 0 \end{array}$$

The optimal tableau has basis (x_3, s_2), therefore the optimal solution to the primal is $(x_1, x_2, x_3, s_1, s_2) = (0, 0, 2, 0, 6)$. We see that $x_3 > 0$ and $6x_1 + x_2 + x_3 < 8$ in the optimal primal solution, thus from complementary slackness conditions, we get that $y_1 + y_2 = 2$ and $y_2 = 0$ are satisfied in the optimal dual solution. Thus the optimal dual solution is $(y_1, y_2) = (2, 0)$.

b. Let the objective function coefficient of x_3 be c_3. As the basis is (x_3, s_2) we get,

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \qquad c_B = \begin{pmatrix} c_3 \\ 0 \end{pmatrix}$$

The basis remains optimal if none of the reduced costs are positive. The reduced costs for the basic variables will always be zero, therefore, we only check the change in the reduced costs of non-basic variables with change in c_3. Since, $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ and $c_B^T B^{-1} = \begin{pmatrix} c_3 & 0 \end{pmatrix}$, we get the following conditions,

$$\bar{c}_1 = c_1 - \begin{pmatrix} c_3 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 4 - 8c_3 \leq 0 \therefore c_3 \geq 1/2$$

$$\bar{c}_2 = c_2 - \begin{pmatrix} c_3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1 - 3c_3 \leq 0 \therefore c_3 \geq 1/3$$

$$\bar{c}_{s_1} = c_{s_1} - \begin{pmatrix} c_3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 - c_3 \leq 0 \therefore c_3 \geq 0$$

Together, these conditions tell us that the current basis remains optimal as long as $c_3 \geq 1/2$

c. As can be seen in the optimal tableau, the reduced cost of x_1, $\bar{c}_1 = c_1 - c_B^T B^{-1} \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ is -12 when $c_1 = 4$. As x_1 is not in the optimal basis, $c_B^T B^{-1} \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ is unaffected by a change in c_1, therefore, $c_B^T B^{-1} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 16$ remains constant for the current basis. Thus, we get $\bar{c}_1 = c_1 - 16$ in the current basis. The current basis remains optimal as long as $\bar{c} \leq 0$. $\bar{c}_1$ is the only reduced cost affected by change in c_1, therefore, the basis remains optimal as long as $\bar{c}_1 \leq 0$, $\therefore c_1 - 16 \leq 0$. $\therefore c_1 \leq 16$ is the range of values of c_1 for which the basis remains optimal.

Problem 4

a. The payoff matrix for Rock-Paper-Scissors, as discussed in class is,

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

With the first, second and third columns (rows) denoting the gains (losses) for cp (rp) playing Paper, Scissors and Rock respectively. cp's optimal strategy, as shown in class, can be calculated using the following LP:

$$\begin{array}{llllll}
\max & v & & & & \\
\text{s.t.} & & & & & \\
 & & x_S & -x_R & \geq & v \\
 & -x_P & & x_R & \geq & v \\
 & x_P & -x_S & & \geq & v \\
 & x_P & +x_R & +x_S & = & 1 \\
 & x_P, & x_R, & x_S & \geq & 0
\end{array}$$

The optimal strategy for cp comes out to be $(x_P, x_S, x_R, v) = (1/3, 1/3, 1/3, 0)$, which is the answer one would logically expect to get for Rock-Paper-Scissors. As the LP for rp's strategy is the dual to cp's LP, the optimal dual solution can be found using complementary slackness. As shown in the notes, the dual for this LP comes out to be:

$$\begin{array}{llllll} \min & u & & & & \\ \text{s.t.} & & & & & \\ & & -y_S & +y_R & \leq & u \\ & y_P & & -y_R & \leq & u \\ & -y_P & +y_S & & \leq & u \\ & y_P & +y_R & +y_S & = & 1 \\ & y_P, & y_R, & y_S & \geq & 0 \end{array}$$

As $x_P > 0$, $x_S > 0$ and $x_R > 0$, from complementary slackness we get that $-y_S + y_R = u$, $y_P - y_R = u$, $-y_P + y_S = u$, along with $y_P + y_S + y_R = 0$ in the optimal dual solution. Solving the system of equations gives us $(y_P, y_S, y_R, u) = (1/3, 1/3, 1/3, 0)$ as rp's optimal strategy.

b. We are given that $y_R = 2/5$, $y_S = 1/10$ and $y_P = 1/2$. In this case, cp's optimal strategy will simply be the one that maximizes $[y^T A]_i, i \in \{R, P, S\}$. $[y^T A]_P = y_R - y_S = 3/10$, $[y^T A]_R = y_S - y_P = -4/10$ and $[y^T A]_P = y_P - y_R = 1/10$. Therefore, $\max_{i\in\{R,P,S\}}[y^T A]_i = [y^T A]_P = 3/10$. Therefore, the optimal strategy for cp is $(x_P, x_S, x_R) = (1, 0, 0)$ i.e. to play paper every time.

Problem 5

For general payoff matrix in a zero some game we get the optimal cp strategy by solving for x:

$$\max_x \min_y y^T Ax$$

For the given payoff matrix A we can thus solve the LP:

$$\begin{array}{lllllll} \max & v & & & & & \\ \text{s.t.} & & & & & & \\ & 2x_1 & -x_2 & +3x_3 & -2x_4 & \geq & v \\ & x_1 & +4x_2 & -3x_3 & & \geq & v \\ & & -2x_2 & -x_3 & +3x_4 & \geq & v \\ & x_1 & +x_2 & +x_3 & +x_4 & = & 1 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

The optimal cp strategy (optimal solution of the LP) comes out to be $(x_1, x_2, x_3, x_4) = (9/13, 1/39, 0, 11/39)$ with expected payoff (optimal value) $v = 31/39$. Using complementary slackness, we can see that the optimal dual solution will satisfy $y_1 + y_2 + y_3 = 1$, $2y_1 + y_2 = u$, $-y_1 + 4y_2 - 2y_3 = u$ and $-2y_1 + 3y_3 = u$ as x_1, x_2 and x_4 are positive. Solving these gives us the optimal rp strategy $(y_1, y_2, y_3) = (7/39, 17/39, 5/13)$ with expected payoff (optimal value) $u = 31/39$.

Problem 6

We argued in class that, if there is a negative cost cycle, the label-correction algorithm does not terminate. Conversely, the label-correction algorithm would terminate after scanning $|V|$ times the list of edges. So there is a negative cost cycle if and only if in the $|V| + 1$–th scan there is an update. How do we find such cycle? Take a node u such that d_u has changed during the $|V| + 1$–th scan, and consider nodes $p(u)$, $p(p(u))$, etc. Iterate this: note that if in this process you pass through s, then s must have a predecessor too, otherwise, repeating the argument seen in class to conclude that $|V|$ scans suffice, we would not have changed d_u at round $|V|+1$. Hence, at some point you will visit a node that you already visited, that is, you discovered a cycle in your graph. Why is this a cycle of negative weight? Call $v_1, \ldots, v_k$ the nodes of this cycle, where v_1 is the node whose value d_{v_1} has last changed. This means we can use the edge (v_1, v_2) to decrease the value of d_{v_2}, then use (v_2, v_3) to decrease the value of d_{v_3}, etc., until we can use (v_k, v_1) to decrease the value of d_{v_1}. Hence, the sum of the weights of this cycle must be negative.

Problem 7

In class we saw that the problem can be formulated as the longest path problem with weight as logarithms of the conversion rates. This is equivalent to the shortest path problem with weight as minus logarithms of the conversion rates because

$$-\text{minimize} \sum_{(u,v)\in A} -\log(r_{u,v})x_{u,v} = \text{maximize} \sum_{(u,v)\in A} \log(r_{u,v})x_{u,v}.$$

Note that we can increase our original capital if and only if there exists a negative cost cycle in the digraph (this corresponds to a cycle of conversion rates whose product is more than 1). In fact, if we find such a cycle, there is no limit to the money we can make, since we can keep traversing the cycle as to increase our capital. A cycle of negative weight can be found via the algorithm given at the previous problem. We can then take any path (e.g. the shortest) from the dollar currency to any node of the cycle, iterate currency exchange in the negative cycle as to increase our capital as much as we want, and then take any path (e.g. the shortest) back from the cycle to dollars.