Problem 1

Let: x_1 be the number of 1000 barrels of oil bought; x_2 be the number of barrels of aviation fuel processed in the cracker; x_3 be the number of barrels of heating oil processed (all variables are in unit of 1000 barrels). Then Linear Programming model can be formulated as following to maximize the overall profit.

$$\begin{aligned} &\text{maximize } 60(0.5x_1 - x_2) + 130x_2 + 40(0.5x_1 - x_3) + 90x_3 - 40x_1 \\ &\text{subject to } x_1 \leq 20, \\ &\qquad 0.5x_1 - x_2 \geq 0, 0.5x_1 - x_3 \geq 0 \\ &\qquad x_2 + 0.75x_3 \leq 8 \\ &\qquad x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

Optimal solution would be $(x_1, x_2, x_3) = (20, 8, 0)$, the corresponding maximized profit is 760.

Problem 2

Let x_i be the number of units of process i, $i = 1, 2$. (1 units of process 1 means we buy 1 unit of labor and 2 units of chemicals and get 3 oz of perfume by process 1. Analogously for 2.) Let x_3 denotes the number of hours to hire Jenny. Then the Linear programming can be formulated as,

$$\begin{aligned} &\text{maximize } 3 * 5x_1 + 5 * 5x_2 - (1 * 3 + 2 * 2)x_1 - (2 * 3 + 3 * 2)x_2 - 100x_3 \\ &\text{subject to } x_1 + 2x_2 \leq 20000 \\ &\qquad 2x_1 + 3x_2 \leq 35000 \\ &\qquad 3x_1 + 5x_2 - 200x_3 \leq 1000 \\ &\qquad x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

Optimal solution would be $(x_1, x_2, x_3) = (10000, 500, 270)$, the corresponding maximized profit is 118000.

Problem 3

Since variable x_3 does not appear in the objective function, it is nonnegative, and appears in only one costraint, we can interprete it as a slack variable is the slack variable for the LP. We can therefore reduce to the following 2-dimensional problem:

$$\min\ z = 3x_1 - 2x_2 \ \text{ s.t. }\ 3x_1 + x_2 \leq 12,\ 3x_1 - 2x_2 \geq 12,\ x_1 \geq 2,\ x_1, x_2 \geq 0$$

The optimal solution to the latter problem is $x^* = (4, 0)$. Via the graphic method, we can also observe that this is unique. From the constraint of the original problem we deduce $x^3 = 3x_1 - 2x_2 - 12$. Hence, the set of all optimal solution to the original problem is $\{(4, 0, 0)\}$.

Problem 4

Let $x_i, i = 1, 2, 3$ be the number of barrels refined using method i. Let $y_{1j}, j = 1, 2, 3$ be the number of barrels of grade 6/8/10 oil correspondingly sold as gas and $y_{2j}, j = 1, 2, 3$ be the number of barrels of grade 6/8/10 oil correspondingly sold as heating oil. Let z_1 denote the number of barrels cracked from grade 6 to 8 and z_2 denote the number of barrels cracked from grade 8 to 10. The LP can be formulated as follows,

$$\begin{aligned}
\text{maximize } & 12\sum_{j=1}^{3} y_{1j} + 5\sum_{j=1}^{3} y_{2j} - 3.4x_1 - 3x_2 - 2.6x_3 - z_1 - 1.5z_2 \\
\text{subject to } & 6y_{11} + 8y_{12} + 10y_{13} - 9\sum_{j=1}^{3} y_{1j} \geq 0 \\
& 6y_{21} + 8y_{22} + 10y_{23} - 7\sum_{j=1}^{3} y_{2j} \geq 0 \\
& \sum_{j=1}^{3} y_{1j} \leq 2000, \sum_{j=1}^{3} y_{2j} \leq 600 \\
& 0.3x_1 + 0.4x_2 + 0.1x_3 - y_{11} - y_{21} - z_1 = 0 \\
& 0.5x_1 + 0.2x_2 + 0.3x_3 - y_{12} - y_{22} + z_1 - z_2 = 0 \\
& 0.8x_1 + 0.4x_2 + 0.2x_3 - y_{13} - y_{23} + z_2 = 0 \\
& x_i \geq 0, i = 1, 2, 3, y_{ij} \geq 0, z_i \geq 0, i = 1, 2, j = 1, 2, 3
\end{aligned}$$

$x^* = (1625, 0, 0), y_{1j} = (300, 400, 1300), y_{2j} = (187.5, 412.5.0), z = (0, 0)$. The total revenue is 21475.

Problem 5

Let $x_i, i = 1, 2, ..., 10$ be the shares we sold for stock i. The problem can be formulated as below:

$$\begin{aligned}
\text{maximize } & 36x_1 + 39x_2 + ... + 70x_{10} \\
\text{subject to } & 0.99(30x_1 + 34x_2 + .. + 66x_{10}) - 0.3(10x_1 + 9x_2 + ... + x_{10}) \geq 30000 \\
& x_i \leq 100, i = 1, 2, ..., 10 \\
& x_i \geq 0, i = 1, 2, ..., 10
\end{aligned}$$

The optimum policy is $x_3 = x_4 = x_8 = x_9 = x_{10} = 100, x_6 = 63.75$ and others are zero.

Problem 6

False. Any solutions constructed by convex combination of z_1 and z_2 are still optimal for LP where z_1 and z_2 are two optimal solutions.
True. Example: $\max x_1 + x_2$ with constraints $x_1 + x_2 \leq 1$ and $x_1 \geq 0, x_2 \geq 0$.
False. Because LP can be unbounded.
True. Modify the m constraints into standard form yields extra m constraints and m variables. Modify the n variables into standard form also yields extra $2n$ constraints and n variables. So it has at most $2n + m$ variables and $2n + 2m$ constraints.