IEOR 4004

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HW#2

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1. $\max z = 2x_1 - x_2 + x_3$

$3x_1 + x_2 + x_3 + x_4 = 60$

$x_1 - x_2 + 2x_3 + x_5 = 10$

$x_1 + x_2 - x_3 + x_6 = 20$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

* first, initial bfs is $x = (0, 0, 0, 60, 10, 20)^T$ with $z = 0$.

* Then, we increase x_1 from 0 to a positive value

⇒ find the constraint: $x_4 = 60 - 3x_1 \geq 0$

$x_5 = 10 - x_1 \geq 0 \quad \rightarrow \quad x_1 \leq 10$

$x_6 = 20 - x_1 \geq 0$

⇒ now the vector becomes $(10, 0, 0, 30, 0, 10)$

with $z = 2 \times 10 - 0 + 0 = 20$

$\max z = 2x_1 - x_2 + x_3 \quad \Rightarrow \quad z = x_2 - 3x_3 - 2x_5 + 20$

$4x_2 - 5x_3 + x_4 - 3x_5 = 30 \qquad ① - 3②$

$x_1 - x_2 + 2x_3 + x_5 = 10$

$2x_2 - 3x_3 - x_5 + x_6 = 10 \qquad ③ - ②$

* Then, we increase x_2 from zero to positive

⇒ find the constraints:

$x_4 = 30 - 4x_2 \geq 0$

$x_1 = 10 + x_2 \geq 0 \quad \rightarrow \quad x_2 \leq 5$

$x_6 = 10 - 2x_2 \geq 0$

⇒ now the vector becomes $(15, 5, 0, 10, 0, 0)$

with $z = 5 - 0 - 0 + 20 = 25$

$\max\ z = -\frac{7}{2}x_3 - \frac{3}{2}x_5 - \frac{1}{2}x_6 + 25$

$x_3 + x_4 - x_5 - 2x_3 = 10$ (1) − 2(3)

$2x_1 + x_3 + x_5 + x_6 = 30$ 2(2) + (3)

$2x_2 - 3x_3 - x_5 + x_6 = 10$

Since the object is $z = -\frac{7}{2}x_3 - \frac{3}{2}x_5 - \frac{1}{2}x_6 + 25$, no feasible change in any variable can improve on $z = 25$ $\Rightarrow$ $(15, 5, 0, 10, 0, 0)^T$ is optimal

Phase I

2. $\max\ z = 2x_1 + 3x_2$ $\Rightarrow$ $\min\ x_5 + x_6$ $\Rightarrow$ $\max\ z = -x_5 - x_6$

$\frac{1}{2}x_1 + \frac{1}{4}x_2 + x_3 = 4$

$x_1 + 3x_2 - x_4 + x_5 = 20$

$x_1 + x_2 + x_6 = 4$

$x_1 \sim x_6 \geq 0$

* Initial bfs $(0, 0, 4, 0, 20, 4)^T$ with $z = -20 - 4 = -24$

Zero out the coefficients corresponding to basic variables

In the object: $z = 2x_1 + 4x_2 - x_4 - 24$

* Then, we increase z by increase x_2

$\Rightarrow$ find the constraints:

$x_3 = 4 - \frac{1}{4}x_2 \geq 0$

$x_5 = 20 - 3x_2 \geq 0$ $\rightarrow$ $x_2 \leq 4$

$x_6 = 4 - x_2 \geq 0$

$\Rightarrow$ Vector becomes $(0, 4, 3, 0, 8, 0)^T$ with $z = 4 \times 4 - 0 - 24 = -8$

$\max\ z = 2x_1 + 4x_2 - x_4 - 24 = 2x_1 + 4(4 - x_1 - x_6) - x_4 - 24 = -2x_1 - x_4 - 4x_6 - 8$

$x_1 + 4x_3 - x_6 = 12$ (1) × 4 − (3)

$2x_1 + x_4 - x_5 + 3x_6 = -8$ 3(3) − (2)

$x_1 + x_2 \quad x_6 = 4$

Since we can no longer optimize the object

and the artificial variable is still non-zero

LP is infeasible * ~~proofed by~~ proved by Gurobi

3 1). Since the current object contains x_1 and x_6

and it is a particular tableau, x_1, x_6 could

not be part of the basis ⇒ the basic variables

can't be 1, 3, 5

2) Based on 3.1), the basic variables should

be among 2, 3, 4, 5. ~~However~~, if 2, 4 are

both basic variables, then they should be

in two different rows. However, for now

x_3 takes up one row, and there's only one

row left for 2, 4 ⇒ 2, 3, 4 could not be

basic variables.

3).

z	$-x_1$	0	0	$+x_4$	0	$-x_6$	$= 7$
	0	0	0	$+x_4$	x_5	$+6x_6$	$= 6$
	x_1	0	x_3	$+x_4$	0	$+2x_6$	$= 5$
	x_1	$+x_2$	0	0	0	0	$= 2$
		↑	↑		↑		

RHS
* 右列大于零

* coeff of basis = 0

basis ⇒ vector $(0, 2, 5, 0, 6, 0)$

* $z = x_1 + x_6 + 7$ ⇒ could increase x_6 by 1

4) as argued above, x_5 ~~+~~ must be a basic solution

since the tableau has exactly 3 non-zero entries

(choose 3 from 2, 3, 4, 5, and can't be the case

with only 2, 3, 4) Then, set the first row equal to $ax_5 + bx_6 = c$, where $a, b, c \neq 0$

the particular tableau turns to.

$z \quad -x_1 \qquad 0 \quad -x_6 = 7$

$0 \quad 0 \quad 0 \quad 0 \quad ax_5 \quad +bx_6 = c$

$x_3 \quad 0 \quad +2x_6 = 5$

$x_1 \quad 0 \quad 0 = d$.

Then setting the constraint of x_6

$5 - 2x_6 \geq 0 \Rightarrow x_6 \leq 2.5$

$c - bx_6 \geq 0 \Rightarrow x_6 \leq \frac{c}{b}$

* if a, c have different sign $\Rightarrow x_5 < 0$ then infeasible

$\Rightarrow$ no optimal solution.

* if a, c have the same sign

$\Rightarrow ax_5 = c - bx_6 \Rightarrow x_5 = \frac{c}{a} - \frac{b}{a}x_6 \geq 0$

$\frac{b}{a}x_6 \leq \frac{c}{a}$

* if $a, c > 0$ $bx_6 \leq c$. • $b > 0$ $x_6 \leq \frac{c}{b}$ ← positive

• $b < 0$ $x_6 \geq \frac{c}{b}$ ← negative

* if $a, c < 0$. $bx_6 \geq c$ • $b > 0$ $x_6 \geq \frac{c}{b}$ ← negative

• $b < 0$ $x_6 \leq \frac{c}{b}$ ← positive

Therefore, there always exists an x_6 such

that make the object ~~more~~ greater

$\Rightarrow$ current $\bar{x}$ can't be optimal.

5) Since $\bar{x}$ can't be optimal, $\bar{x}$ couldn't even be the unique optimal solution to the problem.

6) $z \quad -x_1 \qquad -x_6 = 7$

$0 \quad 0 \quad 0 \quad 0 \quad x_5 \quad 0 = -3$ ← since $x_5 = -3 < 0$

$2x_1 \quad 0 \quad x_3 \quad 0 \quad 0 \quad +2x_6 = 5$ infeasible

$x_1 + x_2 \quad 0 \quad 0 \quad 0 \quad 0 = 5$

4. object: max $C^T X - \sum_{i=1}^{m} d_i \lambda_i$

constraints: $A X \le b + \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \le \begin{bmatrix} k_1 \\ \vdots \\ k_m \end{bmatrix}$$

$$X \ge 0$$

5. objects: primary $\max C^T x$
secondary $\max d^T x$

* First, we find the optimal bfs $\vec{x}$ for primary object max $C^T \cdot \vec{x} = m$ where m is the value of the object when plug in $\vec{x}$.
* Then, we move on to the secondary object, our goal turns to:

object $\max d^T x$
constraints $C^T x = m$
$A x = b$

Solving the newly constructed system gives the solution that maximize the primary objective function and achieve the maximum value of the secondary objective function

b. No. If we transform LP into the form $\max c^T x: Ax=b$. it lacks constraints. In this case, the slack / artificial variables don't have strict constraints ⇒ they might be positive or negative. In this way, the original system of inequality functions might not hold when optimizing the object. Therefore, we can't always transform LP in the form $\max c^T x: Ax=b$.